International Journal on Emerging Mathematics Education
Vol. 3, No. 1, March 2019, pp. 77-92
P-ISSN: 2549-4996, E-ISSN: 2548-5806, DOI: http://dx.doi.org/10.12928/ijeme.v3i1.11425

Generalization Strategies in the Problem Solving of Derivative and Integral

[1]Nourooz Hashemi, [1]Hamidreza Kashefi, [2]Mohd Salleh Abu
[1]Department of Mathematics Education, Farhangian University, Tehran, Iran
[2]Faculty of Education, Universiti Teknologi Malaysia, Johor, Malaysia
e-mail: hashemi.nourooz@gmail.com

Abstract

This study proposes a learning strategy of derivatives and integrals (LSDI) based on specialized forms of generalization strategies to improve undergraduate students' problem solving of derivative and integral. The main goal of this study is to evaluate the effects of LSDI on students' problem-solving of derivative and integral. The samples of this study were 63 undergraduate students who took Calculus at the Islamic Azad University of Gachsaran, Iran. The students were divided into two groups based on their scores in the pre-test of derivative and integral. The results indicated that there was a significant difference between the achievements of students in experimental and control groups after treatment. Thus, the findings reveal that using generalization strategies improves students' achievements in solving problems of derivative and integral.

Keywords: Derivative, Generalization strategy, Integral, Problem solving, Undergraduate students.

How to Cite: Hashemi, N., Kashefi, H., & Abu, M.S. (2019). Generalization strategies in the problem solving of derivative and integral. *International Journal on Emerging Mathematics Education*, *3*(1), 77-92. http://dx.doi.org/10.12928/ijeme.v3i1.11425

INTRODUCTION

Calculus has determinant effects in the learning of advanced mathematics and other university subjects among undergraduate students (Tall, 1992). The important concepts of calculus at the undergraduate level are derivative and integral (Tall, 1993; Tall, 2002a; Tall, 2002b). However, many studies (Kiat, 2005; Willcox & Bounova, 2004; Metaxas, 2007; Rubio & Chacon, 2011; Pepper, Chasteen, Pollock, & Perkins, 2012; Hashemi, Abu, Kashefi, & Rahimi, 2013; Tarmizi, 2010) indicate that students face serious difficulties in the learning of calculus specifically in derivative and integral.

Students' weakness in problem solving is the most important reason of students' difficulties in the learning of derivative and integral (Tall, 2011; Willcox & Bounova, 2004; Metaxas, 2007; Tarmizi, 2010; Rubio & Chacon, 2011; Pepper *et al.*, 2012; Hashemi *et al.*, 2013). The three main reasons for students' difficulties in solving problems which are related to derivative and integral are the need to have a heuristic and appropriate framework or plan to solve problems, the weakness of recalling previous knowledge and information in new areas, and the inability to solve problems in general form (Tall & Yusof, 1995; Tall, 2001; Tall, 2004a; Kirkley, 2003; Yazdanfar, 2006; Roknabadi, 2007; Aghaee, 2007; Villiers & Garner, 2008; Parhizgar, 2008; Javadi, 2008; Mason, 2010; Tarmizi, 2010; Ghanbari, 2010; Ghanbari, 2012; Azarang, 2012).

Some methods are introduced to support the students to overcome their problem-solving difficulties in the learning of derivative and integral. Researchers endeavour to support students' problem-solving in the learning of calculus by promoting mathematical thinking with or without a computer (Tall, 2008; Rahman, 2009; Kashefi, Ismail, & Yusof, 2012). There are extensive studies in promoting mathematical thinking

Received October 11, 2018; Revised March 2, 2019; Accepted March 20, 2019

to help students' understanding of calculus, especially derivative and integral (Dubinsky, 1991; Schoenfeld, 1992; Tall, 1995; Watson & Mason, 1998; Yusof & Tall, 1999; Gray & Tall, 2001; Mason, 2002; Rahman, 2009; Mason, Stacey, & Burton, 2010).

Mathematical thinking is an active process which improves students' understanding of highly complex activities such as specializing, conjecturing, and generalizing (Tall, 2002b; Yusof & Rahman, 2004; Stacey, 2006; Mason *et al.*, 2010; Kashefi *et al.*, 2012). According to Tall (2004b) and Tall (2008), mathematical thinking process occurs in three worlds of mathematics, such as the embodied world, the symbolic world, and the formal world. The embodied world of sensory meaning and action involve reflection on perception and action. The symbolic world contains computing and manipulating symbols in arithmetic and algebraic forms. The formal world is the world of axioms, formal definitions, and formal proof of theorems (Tall, 2002b). Therefore, in the teaching and learning of calculus specifically derivative and integral, the focus is on embodied and symbolic worlds (Tall, 2011).

Generalization as an important element of mathematical thinking process in problem solving can be supported to overcome students' difficulties in the learning of calculus especially derivative and integral (Polya, 1988; Cruz & Martinon, 1998; Larsen, 1999; Tall, 2002b; Tall, 2004b; Tall, 2011; Sriraman, 2004; Mason *et al.*, 2010; Kabael, 2011). Tall (2002b) asserts that generalization strategies in mathematical thinking worlds are an expansive, reconstructive, and disjunctive generalization. First, expansive generalization happens when the notion expands the applicability range of an existing schema without reconstructing it. Second, reconstructive generalization occurs when the subject reconstructs an existing schema to widen its applicability range. Last, disjunctive generalization occurs when the subject moves from a familiar context to a new one. The subject constructs a new disjoint schema to deal with the new context and adds it to the array of schemas available (Harel & Tall, 1991).

Mason *et al.* (2010) propose the three steps of problem-solving framework namely entry, attack, and review using mathematical thinking activities such as specializing, conjecturing, and generalization. Mason *et al.* (2010) believe that specialization and generalization as the main steps of mathematical thinking processes involve three problem-solving phases. Specialization involves entry and attack, and generalization covers attack and review. The key idea of their framework is a mulling circle between the entry and attack phases. Also, the activity of conjecturing has an important role in connecting specialization to generalization (Mason *et al.*, 2010).

Tall believes that a majority of using generalization happens in the symbolic world as an expansive generalization and this kind of generalization is not able to make the connection between the graphical and symbolical world of mathematics for derivative and integral (Tall, 2002b; Tall, 2004b). Tall (2002b) and Tall (2011) believe that reconstructive generalization enables the making of relationship and connection between concepts of derivative and integral. It is because this kind of generalization changes the contracture of presentation, protects the meaning of concepts and transits it to the new world. Harel & Tall (1991) and Tall (2002b) state that disjunctive generalization also can protect the relationship and connection between concepts, but it has less effect as compared to expansive and reconstructive.

This study adopts a learning strategy of derivatives and integrals (LSDI) which is designed based on generalization strategies and mathematical thinking process through three worlds of mathematics to improve undergraduate students' problem solving of derivative and integral. Furthermore, the effectiveness of LSDI in enhancing students' problem solving is also being evaluated.

LEARNING STRATEGIES FOR DERIVATIVE AND INTEGRAL (LSDI)

This study uses the theory of three worlds of mathematics (Tall, 2004a; Tall, 2004b), viewpoint of Tall about generalization (Tall, 2002a), and the mathematical thinking process framework of Mason *et al.* (2010) in order to design strategies for learning derivative and integral (LSDI) to improve problem solving. In the designing of LSDI, the generalization process in the framework of Mason (specialization, conjecturing and generalization) was being modified along with the three generalization strategies namely; expansive, reconstructive and disjunctive (Tall, 2002b). Then, the modified generalization strategies were merged with three worlds of mathematics to design LSDI. Figure 1 represents the theory and the framework which were established in designing LSDI.

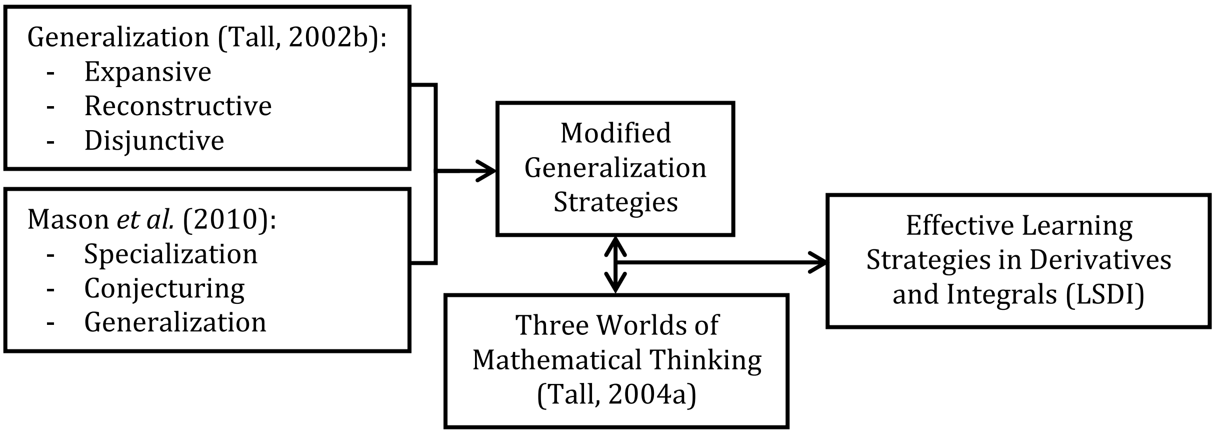


Figure 1. Designed Learning Strategy (LSDI) as Framework of the Study

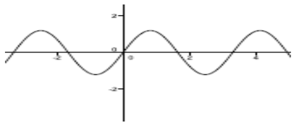
Mathematical thinking worlds were selected because it covers both graphical and symbolical aspects of derivative and integral to overcome students' difficulties in problem-solving. Through mathematical thinking worlds, both graphical and symbolic aspects can be connected by reconstructive generalization. Moreover, expansive generalization can be supported in the embodied world to foster students' problem solving by the graphical aspect. Therefore, based on this study, mathematical thinking and generalization strategies should be considered in designing learning strategies for problem-solving.

Scruggs & Mastropieri (1993) believe that learning strategies are established based on the use of tasks, and they involve how students structure and apply a collection of skills to learn content or to carry out a particular task more effectively and efficiently. Besides, Watson & Mason (1998) assert that learning strategies contain what we think such as planning, realizing and memorizing previous knowledge through doing the problem-solving process. According to Watson & Mason (1998) and Watson (2002), prompts and questions are more appropriate tasks use by teachers as guidance for developing mathematical thinking in the classroom within the problem-solving process.

The questions assist students in focussing on particular strategies and helping them to see patterns and relationships (Mason *et al.*, 2010). These questions provide the establishment of a strong conceptual network. Consequently, the questions can be used as a prompt when students become 'stuck'. Teachers are often tempted to turn these questions into prompts to stimulate thinking and incorporate students in the problem-solving process activities (Watson & Mason, 1998; Watson & Mason, 2006; Mason *et al.*, 2010). Therefore, referring to this study, the contents of designed learning strategies based on prompts and questions should cover generalization strategies, mathematical thinking process, and three worlds of mathematics.

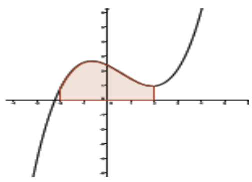
The learning tasks of derivative and integral were prepared based on the LSDI strategies through prompts and questions that are shown in Tables 1 and Table 2. Table 1 presents an example of the designed prompts and questions based on LSDI which was prepared for the derivative task.

Table 1. Designed prompts and questions for derivative based on LSDI

	Specialization	Conjecturing	Generalization
	Entry, Attack	Attack	Attack, Review
Embodied World	Given  - Sketch tangent line for $x = 0, \pi$. - Sketch tangent line for $x = \frac{\pi}{2}, \frac{\pi}{4}$, and $\frac{\pi}{6}$. - Could you give other examples for x negative? - What do you see when you sketch a tangent line from $x = \frac{\pi}{8}\ to\ \pi$? - By comparing examples 1, 2 and 3 what are the differences and the same?	- Give a general example. - What can say about the main property in this example? - Highlight the general rule for sketching the derivative figure in the example.	- What is the general rule for the derivative figure based on the original graph? - Can you show its general form graphically and algebraically?
Symbolic World	Example: Given $x = t^2$. - Find the average velocity from $t_1 = 1$ to $t_2 = 3$. - Find the average velocity from $t_1 = 1$ to $t_3 = 2$. - Find the average velocity from $t_1 = 1$ to $t_4 = 1.5$. - What is the same and the difference in this example? - What information do you need to write general forms of other topics, e.g. function and limit?	- Write general ideas for this example. - Classify the ideas.	- Give the general form and describe it? - When the average velocity gets close to absolute velocity? - Describe this example and the general form with the graph of this example.
Formal World	By using ε and δ, show that the function below are derivable: $f(x) = x^2$ for $x = 3$ ...(i) $g(x) = x^3 - x$ for $x = -3$ ...(ii) - What is the same and different? - Give more examples?	- What is a similar property of (i) and (ii)? - Give a general guess in proving derivability based on ε and δ in these examples.	- Write general form to prove derivability of function with ε and δ. - Check your respond.

In Table 1, prompts and questions were used to design the derivative and integral tasks based on LSDI. These concepts were presented through different worlds of mathematics based on specialization, conjecturing and generalization as components of the mathematical thinking process. Moreover, the three generalization types were considered in the generalization phase of the mathematical thinking process of the problem-solving framework. Table 2 shows that the prompts and questions were designed as learning strategies of integral through mathematical thinking worlds based on specialization, conjecturing and generalization (involved three strategies namely; expansive, reconstructive and disjunctive).

Table 2. Designed prompts and questions for integral based on LSDI

	Specialization	Conjecturing	Generalization
	Entry, Attack	Attack	Attack, Review
Embodied World	Given:  - Find the area between the graph and x-axes from $x = -1$ to $x = 1$ ...(i) - Find the area between the graph and x-axes from $x = -2$ to $x = 2$ ... (ii) - Compare (i) and (ii). - What is the same and what are the differences?	- What is the main property for both of them? - How can you relate the solution of these examples to the algebraic aspect? - Describe how we can find the exact solution of the area in these types of questions.	- Please give a more general example. - Describe how you can connect this example to algebraic form. - Find the area between the graph and x-axes from $x = a$ to $x = b$. - Give another example and find the area between the graph and x-axes from $x = a$ to $x = b$.
Symbolic World	Example 1: Given $f'(x) = x$ - Find $f(x)$ - If $g'(x) = x^2$ then find $g(x)$ - If $h'(x) = x^3 + 4x$ then find $h(x)$ - What is the same? - What is different?	- Describe the procedure to find the original function from the derivative function.	- Give the general form of founded original function symbolically. - Connect the general form to its figure logically.
Formal World	Show that if f is integrable then; (i) $\int_0^2 4x^3 dx = -\int_2^0 4x^3 dx$ (ii) $\int_{-2}^2 4x^3 dx = -\int_2^{-2} 4x^3 dx$ - Compare (i) and (ii), what is the same and what are the differences? - Give more examples.	- Write general ideas based on examples. - Try to categorize the ideas based on examples. - Please give a more general example.	- Prove that if f is integrable then $\int_a^b f(x) dx = -\int_b^a f(x) dx$ - Check your answer.

RESEARCH METHOD

Method of the study involves discussions on research design, sample, instruments, data collection and data analysis which are described sequentially.

Research Design

This experimental study tries to highlight the impacts of using LSDI especially generalization strategies on students' problem solving of derivative and integral. After comparing the scores between two groups of students as control and experimental, the rates of using generalization strategies due to their responds to open-ended problems were compared through pre and post-test of both groups.

Sample

Two classes were randomly selected among 12 classes which offered Calculus I at the University of Gachsaran in Iran. One of the two classes was selected randomly as an experimental group, and it involved 33 students. Another class acted as a control group and consisted of 30 students. Based on the results of pre-test, all students in both groups were in having the same level of problem-solving of derivative and integral. In the control group, derivative and integral were taught by using traditional and common methods used in the teaching of these concepts.

In contrast, derivative and integral taught in experimental class was based on designed strategies of LSDI. The duration of the teaching of derivative and integral in both groups was 8 weeks. A post-test was administered to both classes based on problem-solving of derivative and integral at the end of week 8.

Instrument

This study used two instruments for pre-test and post-test which were designed to understand students' problem-solving abilities of derivative and integral. Three experts who were familiar in this case verified the questions of problem-solving in this research. For pre-test, 9 problems (6 for derivative and 3 for integral) were given to the students, and 11 problems (6 for derivative and 5 for integral) were given in the post-test.

Data Analysis

The goal of the analysis was to see if LSDI affected students' scores and rates of using LSDI specially generalization strategies in the experimental group as compared to the control group. Students' scores were obtained based on their responses to the problems in the pre and post- tests. The students' responses were scored from 20based on Iranian scoring schema. The differences in the mean of those scores were compared within and between groups. This comparison was done with independent samples t-test because the data had the assumption of the t-test. Moreover, the Kappa test was used to see the agreements of students' scores in problem-solving between groups through a pre-test. After implementing LSDI in the experimental group and taking post-test, the two- mixed ANOVA design test and independent samples t-test were applied to highlight the improvement of students' scores in experimental group from the pre to post-test.

After scoring students' answer sheets and comparing them within each group and between two groups, the rates of using LSDI especially generalization strategies were compared within and between groups. A rubric was designed based on LSDI to compare the rates of its application specifically in the application of generalization strategies within and between groups. The rubric contained 13 items and was designed based on the blending of generalization strategies of Tall (2002b) and mathematical thinking process of Mason *et al.* (2010). The rubric and its items are presented in Table 3.

Table 3. The designed rubric based on LSDI

Components		Items
Specialization		S_1. Using more examples related to the problem
		S_2. Categorizing the ideas according to the properties of examples
Conjecturing		C_1. Giving a general guess to solve the problem based on related examples
		C_2. Formulating the ideas based on the examples
		C_3. Choosing the best method to find the answer
Generalization	Expansive	E_1. Finding an answer via using more related examples of the problem
		E_2. Checking the solution
	Reconstructive	R_1. Solving problems by generating ideas in both symbolic and embodied worlds
		R_2. Creating a new schema different from the previous one
		R_3. Extending the solution idea to higher level problems
	Disjunctive	D_1. Finding the answer by using a familiar context
		D_2. Finding an answer by using disconnected pieces of information
		D_3. Generating ideas of the solution in the wrong way

Based on the rubric, the rates of using LSDI particular generalization strategies were investigated based on the students' responses to the open-ended questions. Students' answers to each question were checked to see if they had considered the items of LSDI specifically on a generalization or not. For each item, the score is either 0 or 1. If a student used that item, he or she would be given 1. A Student was given 0 if there was no consideration of the item in responding to that question. To illustrate, if a student considered E1 in his (her) response to the first problem, the given score was 1 for E1. This process was repeated for all of the items given in Table 3. Therefore, students' responses were checked to obtain the scores for the items. Finally, the given scores to all of the items were tabulated to compare within and between groups. The score for each component was a summation of its related items. For example, to know the score of specialization, the given scores of S1, S2 and S3 should be calculated. Therefore, the score for each student in responding to each problem can be from 0 to 3 for specialization. The scores in all components for each student were calculated based on their items and analyzed quantitatively within and between groups.

RESULTS AND DISCUSSION

Findings of Students' Scores

The results of comparing pre-test **scores** indicated there was no difference between the mean of student' **scores** in the control group and experimental group based on independent samples t-test results. For more confidence about balancing of two groups, in the beginning, the Kappa test was done based on students' **scores**. According to the results such as Value=0.786 and Approx. Sig.= 0.001, there were good agreements between students' scores of pre-test in problem-solving of derivative and integral in both groups. Pallant (2010) asserts that if the value of Kappa is bigger than 0.70, there is a good agreement between the data of different groups. Therefore, LSDI could be implemented in an experimental group to see its effects on this group by taking post-test at the end of the teaching process for the concepts.

The data which were collected from students' scores in problem-solving of derivative and integral were normal in the post-test. Also, the Levene's test indicated that there are significant differences between the variances of two groups based on collected data in post-test. Therefore, the parametric tests such as two- mixed ANOVA design and independent samples t-test could be used for comparing the students' scores between and within the groups through pre and post-test. The results of comparing the scores are presented in Table 4.

Table 4. Comparison of students' scores in pre-test and post-tests for PS of derivative and integral

Case	Group	Analysis Method	Sig.
Students' scores in pre-test and post-test	Within- between groups	Two- mixed ANOVA design	0.001
Students' scores in the pre-test	Between control and experimental	Independent Samples Test	0. 204
Students' scores in post-test	Between control and experimental	Independent Samples Test	0.001

The output of two-mixed ANOVA test indicates that students' scores between two groups and changing the scores from pre-test to post-test within groups were statistically significant (p= 0.001). Also, the results of independent samples t-test showed that there was no significant difference for students' scores of pre-test between the control group and the experimental group. Although the mean of scores in the control group was different from the mean of scores in the experimental group in the pre-test, the difference was not significant. However, independent samples t-test showed that there was a significant difference (with p= 0.001) between the mean of students' scores of post-test in the control group and the mean of students' scores of post-test in the experimental group. After analyzing students' scores, the differences in the rates of using LSDI especially generalization strategies were analyzed within and between groups through pre and post-test based on investigating students' responses to the problems.

Rates of Using LSDI in Solving of Derivative and Integral Problems

The components of LSDI involving the framework of Mason and the generalization strategies of Tall based on mathematical thinking such as specialization, conjecturing and generalization strategies (expansive, reconstructive and disjunctive) were investigated in the pre-test and post-test within and between groups. The data for the rates of applying LSDI in solving problems based on students' responses to the problems of derivative and integral were not normal data. Thus, non-parametric tests should be used to see the differences in utilization rate for LSDI within and between control and experimental groups through the pre and post-tests.

The rates of using generalization as the main part of LSDI in pre-test (PrG) and rate of its utilization in post-test (PoG) were compared within groups. The PrG and PoG are the summations of utilization rates of three generalization strategies in the pre and post- tests. Moreover, the differences in using each component of LSDI between the pre-test and post-test were demonstrated within the group. Also, the rates of applying each component of LSDI were also compared between groups for the pre-test and post-test. To indicate differences within the group, the Wilcoxon test was used for non-parametric data, and to see differences between two groups Mann- Whitney test was administered (Pallant, 2010; Kirkpatrick & Feeney, 2012). In Table 5, the results of the comparison of LSDI components specifically generalization are given within each group. It should be noted that Pr is an abbreviation of pre-test and Po is an abbreviation of post-test, the first letter of each component added to Pr and Po. For examples in Tables 5 and 6, PrS means specialization in the pre-test, PoS means specialization in post-test.

Table 5. Utilization of LSDI within the group in PS of derivative and integral

Components	Wilcoxon Test			Group
	Z	Asymp. Sig.	N	
Generalization	-0.495	0.621	30	Control
(PrG- PoG)	-5.019	0.001	33	Experimental
Specialization	-1.554	0.120	30	Control
(PrS- PoS)	-5.029	0.001	33	Experimental
Conjecturing	-1.000	0.317	30	Control
(PrC- PoC)	-5.020	0.001	33	Experimental
Expansive	0.000	1.000	30	Control
(PrE- PoE)	-5.026	0.001	33	Experimental
Reconstructive	0.000	1.000	30	Control
(PrR-PoR)	-5.020	0.001	33	Experimental
Disjunctive	-0.495	0.621	30	Control
(PrD- PoD)	-2.694	0.007	33	Experimental

The results of the Wilcoxon test demonstrate that there is no significant difference in the rate of using components of LSDI especially generalization in the control group between pre-test and post-test. The results of this test such as Wilcoxon N=30and p>0.05 show no difference in the rates of applying generalization strategies between the pre and post- tests. However, the results of the Wilcoxon test in the experimental group show different postures of using items of LSDI. In this group, Wilcoxon N=33and p<0.05 indicate that there significant differences between the pre-test and post-test in the rates of using LSDI in the problem solving of derivative and integral within the experimental group. Furthermore, the rate of using items of LSDI was measured between two groups in the pre-test and post-test by using the Mann-Whitney U test.

Also, the rates of using generalization strategies were compared between control and experimental groups as presented in Table 6.

Table 6. Utilization of LSDI between groups in PS of derivative and integral

Components		Mann-Whitney U	Z	Asymp. Sig. (2-tailed)	Mean of Rank	
					Control Group	Experiment Group
Generalization	PrG	348.500	-1.560	0.119	26.52	28.65
	PoG	0.000	-6.849	0.001	25.50	47.00
Specialization	Prs	495.000	0.000	1.000	32.00	32.00
	Pos	0.000	-6.887	0.001	25.50	47.00
Conjecturing	PrC	477.000	-0.671	0.502	17.60	18.45
	PoC	0.000	-7.179	0.001	15.50	47.00
Expansive	PrE	495.000	0.000	1.000	16.13	16.24
	PoE	0.000	-7.226	0.001	15.50	47.11
Reconstructive	PrR	480.000	-0.953	0.340	15.00	15.50
	PoR	0.000	-7.221	0.001	16.00	47.50
Disjunctive	PrD	381.500	-1.603	0.109	26.78	28.56
	PoD	352.000	-2.007	0.045	27.23	36.33

According to the results which are presented in Table 6, the Mann- Whitney U-Test reveals that the rates of using LSDI especially generalization strategies from control group are not significantly different from the experimental group in the pre-test. The results such as U, Zand two-tailed $p>0.05$ approve this assertion through the pre-test in problem-solving of derivative and integral. Meanwhile, the Mann- Whitney U-test reveals that the rates of all components of LSDI specifically in the experimental group are significantly higher than the rates from the control group in the post-test (U, Zand two- tailed $p<0.05$). Furthermore, the mean of ranks from each group in different tests verifies this result.

Discussion

The results of comparing scores and rates of using generalization strategies are discussed to see if LSDI affected students' scores and their utilization rates of generalization strategies based on the responses to the problems of derivative and integral.

Although there is no significant difference between the mean of students' scores in control group with the mean of scores in the experimental group for the problems solving of derivative and integral in the pre-test, the scenario in responding to the post-test problems is different. The results indicate that the students' scores in the experimental group are better than the students' scores in the control group through solving post-test problems of derivative and integral. It indicates that the scores of students who learned derivative and integral based on LSDI especially generalization strategies were in the high range of qualification according to Iranian scoring rate at the undergraduate level. Also, the two-mixed ANOVA design test indicates that the improvements of students' scores within each group from the pre-test to post-test are dissimilar in different groups.

Based on the results, the improvement of scores from the pre-test to post-test for experimental group students is higher than the improvement of scores in the control group students from the pre-test to post-test in solving problems of derivative and integral. It can be identified that the achievements of the experimental group students who used LSDI especially generalization strategies were better as compared to the other group. Thus, to know the role of LSDI particularly generalization strategies in improving students' scores in problem-solving, the components of LSDI were investigated based on the responses to the problems of derivative and integral.

Generalization strategies as main components of LSDI in problem-solving of derivative and integral were used remarkably in solving the post-test problems among experimental group students. Although students did not display the utilizations of generalization strategies in solving the pre-test problems, they tried to use the strategies many times through solving the post-test problems. Hence, generalization strategies such as expansive, reconstructive and disjunctive were applied in answering questions of problem-solving of derivative and integral. It should be mentioned that specialization and conjecturing as two fundamental bases for generalization were better considered in the post-test than the pre-test among students in the experimental group. However, in the control group, there was no difference in the utilization of specialization, conjecturing and generalization strategies when responding to the pre-test and post-test of derivative and integral.

The control group students did not show applications of generalization strategies in responding to the pre and post- tests. There is no difference in the utilization rates of generalization strategies such as expansive, reconstructive and disjunctive and their bases such as specialization and conjecturing in answering the questions of problem-solving of derivative and integral between the pre-test and post- test in this group. However, the posture of the utilization rate for generalization strategies is different between the two groups.

Although the rates of using specialization, conjecturing and generalization strategies are not meaningfully different between two groups in solving the pre-test problems of derivative and integral, there are significant differences of using these strategies between groups in solving the problem of derivative and integral in the post-test. Based on the rubric and students' responses in their answer sheets, the experimental group students used more examples which were related to the problems when solving them. They tried to find the similar properties and ideas of examples in the entry phase of the problem solving framework. Also, the experimental group students categorized the ideas according to the properties of examples through the attack phase. Moreover, the students attempted to give a general guess to solve the problems based on related examples. Besides, they formulated the ideas based on the examples within the attack phase of Mason's problem-solving framework. They also chose the best method to find the answer for problems of derivative and integral within the attack phase.

The expansive generalization was used by the experimental group students extensively when responding to the post-test. The students found answers to some problems by using more related examples of those problems which belonged to the attack phase of problem-solving. The students in the experimental group tried to check the written solutions for problems in the same cases using review phase. However, control group students did not demonstrate these kinds of solutions when solving problems.

Furthermore, the experimental group students solved problems by generating ideas in both symbolic and embodied worlds of mathematics using properties of reconstructive generalization in the attack phase. Subsequently, they created new schemas which were based on applying the properties of the review phase of the problem solving framework. Also, in a few cases, the students attempted to extend the solution idea to higher level problems. In contrast, usages of these factors were not apparent in solving problems of derivative and integral among the control group students.

The utilization of disjunctive generalization was rated less as compared to other generalization strategies when solving problems by the experimental group. Some students tried to find answers by using familiar contexts. However, the control group students found the answers by using disconnected pieces of information, and also they generated the ideas of the solution in the wrong way in the same cases.

Based on the framework of problem-solving, three phases namely entry, attack and review was well considered among students in the experimental group. It can be concluded that using components of LSDI specifically generalization strategies has remarkable effectiveness in improving students' scores in the experimental group as compared to the control group students.

CONCLUSION

LSDI was implemented based on strategies such as presenting concepts within three worlds, using mathematical thinking process, prompts and questions to teach derivative and integral in the experimental group. Using components of LSDI namely; specialization, conjecturing, expansive generalization, reconstructive generalization and disjunctive generalization based on mathematical thinking worlds play an important role to enhance problem-solving of derivative and integral.

The results indicate that there is an improvement in the problem-solving scores of students who experienced strategies of LSDI when learning derivative and integral. The improvement of scores in the experimental group is better than the control group students' scores when comparing the pre-test and post-test. The components of LSDI are considered remarkable in solving problems among the experimental group students.

Specialization and generalization as main activities of mathematical thinking process were used in solving the problems of derivative and integral in the experimental group. Also, specialization involves two phases of problem-solving framework; entry and attack, and generalization cover attack and review. The utilization of problem-solving framework (entry, attack and review) among the experimental group students were more remarkable. Thus, using LSDI especially generalization strategies based on three worlds of mathematics improves students' problem-solving achievements in the learning of derivative and integral. Moreover, adopting a problem solving framework which involves entry, attack and review help to improve students' performance.

This study recommends evaluating the effectiveness of LSDI on other concepts of calculus for further research. Besides, investigating the process of using LSDI based on students' thinking can be used for future studies. Teachers should put more emphasis on the use of LSDI to teach calculus concepts, especially derivative and integral to enhance their students' performance.

REFERENCES

Aghaee, M. (2007). Investigation of controlling abilities for solving problem in infinite integral. *Thesis*. Kerman: Shahid Bahonar University.

Azarang, Y. (2012). Quality of learning calculus in Iran. *Roshd Mathematics Education Journal*, *27*(1), 24- 30.

Cruz, J. A. G., & Martinon, A. (1998). Levels of generalization in linear patterns. *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 329-336.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95-123). Dordrecht: Kluwer Academic Publishers.

Ghanbari, G. (2010). Looking to change of calculus in Iran. *Roshd Mathematics Education Journal*, *27*(1), 59- 61.

Ghanbari, G. (2012). Drawing figures as a problem solving method. *Roshd Mathematics Education Journal*, *27*(1), 38- 41.

Gray, E. M., & Tall, D. (2001). Relationships between embodied objects and symbolic procepts: An explanatory theory of success and failure in mathematics. *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, *3*, 65-72.

Harel, G., Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, *11*(1), 38-42.

Hashemi, N., Abu, M. S., Kashefi, H., Rahimi, K. (2013). What are difficulties of learning derivative and integral among undergraduate students? *Proceeding of 4th International Graduate Conference on Engineering Science & Humanity*, 1056-1063.

Javadi, M. (2008). Perception of concepts and definition of concept for calculus. *Roshd Mathematics Education Journal*, *27*(2), 23-27.

Kabael, T. (2011). Generalizing single variable functions to two-variable functions, function machine and APOS. *Educational Sciences: Theory and Practice*, *11*(1), 484-499.

Kashefi, H., Ismail, Z., & Yusof, Y. M. (2012). Overcoming students' obstacles in multivariable calculus through blended learning: A *Mathematical Thinking Approach*. *Procedia - Social and Behavioral Sciences*, *56*, 579-586.

Kiat, S. E. (2005). Analysis of students' difficulties in solving integration problems. *The Mathematics Educator*, *9*(1), 39-59.

Kirkley, J. (2003). *Principles for teaching problem solving*. Bloomington: PLATO Learning Inc.

Kirkpatrick, L. A, & Feeney, B. C. (2012). *A Simple Guide to IBM SPSS Statistics*. Boston: Wadsworth Cengage Learning.

Larsen, L. C. (1999). *Problem-Solving Through Problems*. Dordrecht: Springer.

Mason, J. (2002). Generalisation and algebra: Exploiting children's powers. In L. Haggerty (Ed.), *Aspects of Teaching Secondary Mathematics: perspectives on practice* (pp. 105-120). London: Routledge Falmer.

Mason, J. (2010). Attention and intention in learning about teaching through teaching. In R. Leikin & R. Zazkis (Eds.), *Learning Through Teaching Mathematics: Development of Teachers' Knowledge and Expertise in Practice* (pp. 23-47). New York: Springer.

Mason, J., Stacey, K., & Burton, L. (2010). *Thinking mathematically* (2nd ed.). Edinburgh: Pearson.

Metaxas, N. (2007). Difficulties on understanding the indefinite integral. In Woo, J. H., Lew, H. C., Park, K. S., Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (pp. 265-272). Seoul: PME.

Pallant, J. (2010). *SPSS survival manual: A step by step guide to data analysis using SPSS*. Milton Keynes: Open University Press.

Parhizgar, B. (2008). Conceptual understanding of function. *Thesis*. Tehran: University of Shahid Beheshti.

Pepper, R., Chasteen, E. S. V., Pollock, S. J., Perkins, K. K. (2012). Observations on student difficulties with mathematics in upper-division electricity and magnetism. *Physical Review Special Topics- Physics Education Research*, 1-15.

Polya, G. (1988). *How to solve it*. Princeton: Princeton University Press.

Rahman, R. A. (2009). Changing my own and my students' attitudes to calculus through working on Mathematical Thinking. *Dissertation*. Milton Keynes: The Open University.

Roknabadi, H. A. (2007). Varieties of conceptual understanding: Different theories. *Thesis*. Kerman: Shahid Bahonar University.

Rubio, B. S., & Chacón, G. I. M. (2011). Challenges with visualization: The concept of integral with undergraduate students. *Proceedings of the Seventh Congress of European Society for Research in Mathematics Education (CERME-7)*, Poland: University of Rezeszow.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making mathematics. In Grouws, D. (Ed). *Research on Mathematics Teaching and Learning* (pp. 334–370). New York: Macmillan.

Scruggs, T. E., & Mastropieri, M. A. (1993). Special education for the twenty-first century: Integrating learning strategies and thinking skills. *Journal of Learning Disabilities*, *26*, 392–398.

Sriraman, B. (2004). Reflective abstraction, uniframes, and the formulation of generalizations. *The Journal of Mathematical Behavior*, *23*(2): 205–222.

Stacey, K. (2006). *What is mathematical thinking and why is it important?* Melbourne: University of Melbourne.

Tall, D. (1992). Conceptual foundations of the calculus. *Proceedings of the Fourth International Conference on College Mathematics Teaching*, 73- 88.

Tall, D. (1993). Students' difficulties in calculus. *Proceedings of Working Group 3 on Students' Difficulties in Calculus ICME-7*, Québec, 13–28.

Tall, D. (1995). Mathematical growth in elementary and advanced mathematical thinking. *Proceedings of PME*, *19*(1), 61–75.

Tall, D. (2001). Cognitive development in advanced mathematics using technology. *Mathematics Education Research Journal*, *12*(3), 196–218.

Tall, D. (2002a). Differing modes of proof and belief in mathematics. *Proceedings of International Conference on Mathematics: Understanding Proving and Proving to Understand*, Taipei, 91–107.

Tall, D. (2002b). *Advanced Mathematical Thinking* (11 Ed.). London: Kluwer Academic Publisher.

Tall, D. (2004a). Introducing three worlds of mathematics. *For the Learning of Mathematics*, *23*(3), 29–33.

Tall, D. (2004b). Thinking through three worlds of mathematics. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, *4*, 281-288.

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5-24.

Tall, D. (2011). Looking for the bigger picture. *For the Learning of Mathematics*, *31*(2), 17-18.

Tall, D., & Yusof, Y. M. (1995). Professors' perceptions of students' mathematical thinking: do they get what they prefer or what they expect? In L. Meira, D. Carraher, (Eds.), *Proceedings of PME 19* (pp. 170-177). Recife: PME.

Tarmizi, R. A. (2010). Visualizing students' difficulties in learning calculus. *Procedia Social and Behavioral Science*, *8*, 377- 383.

Villiers, M. D., Garner, M. (2008). Problem solving and proving via generalization. *Journal of Learning and Teaching Mathematics*, *5*, 19-25.

Watson, A. (2002). Embodied action, effect, and symbol in mathematical growth. In A. Cockburn, & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (pp. 369- 376). Norwich: PME.

Watson, A., & Mason, J. (1998). *Questions and Prompts for Mathematical Thinking*. Derby: ATM.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning*, *8*(2), 91-111.

Willcox, K., & Bounova, G. (2004). Mathematics in engineering: Identifying, enhancing and linking the implicit mathematics curriculum. *Proceedings of the 2004 American Society for Engineering Education Annual Conference and Exposition*, Washington DC.

Yazdanfar, M. (2006). Investigation of studying skills in calculus for undergraduate. *Thesis*. Kerman: Shahid Bahonar University.

Yusof, Y. M., & Tall, D. (1999). Changing attitudes to university mathematics through problem-solving. *Educational Studies in Mathematics*, *37*, 67-82.

Yusof, Y. M., & Rahman, R. A. (2004). Teaching engineering students to think mathematically. *Proceedings of the Conference on Engineering Education*, Kuala Lumpur.