International Journal on Emerging Mathematics Education (IJEME)
Vol. 1, No. 1, March 2017, pp. 99-120
P-ISSN: 2549-4996, E-ISSN: 2548-5806, DOI: http://dx.doi.org/10.12928/ijeme.v1i1.5760

Motivation Cards to Support Students' Understanding on Fraction Division

[1]**Kamirsyah Wahyu**, [2]**Siti Maghfirotun Amin**, [2]**Agung Lukito**
[1]Institut Agama Islam Negeri (IAIN) Mataram, Jl. Pendidikan, Dasan Agung Baru, Mataram, NTB 83125
[2] Universitas Negeri Surabaya, Jl. Ketintang Baru XII No.34, Ketintang, Gayungan, Surabaya, Jawa Timur 60231
Email: kawahyu@iainmataram.ac.id

Abstrak

Penelitian ini menggunakan *design research* untuk mengembangkan aktivitas pembelajaran yang mendukung pemahaman konsep siswa kelas 5 tentang pembagian pecahan (bilangan asli dibagi oleh pecahan biasa yang menghasilkan bilangan asli, misalnya $3 \div \frac{1}{2}$). Penelitian ini juga menganalisis bagaimana siswa menyelesaikan masalah pembagian pecahan dengan menggunakan lebih dari satu model. Data untuk analisis retrospektif dikumpulkan dari dua eksperimen mengajar dalam bentuk hasil kerja siswa, catatan lapangan, dan beberapa bagian diskusi di kelas. Penemuan penting dalam penelitian ini yaitu: 1) aktivitas pembelajaran yang dikembangkan, dinamakan Kartu Motivasi, mendukung siswa memahami bahwa $3 \div \frac{1}{2}$ berarti ada berapa banyak $\frac{1}{2}$ dalam 3 melalui beberapa model. Namun, jika pembagi bukan pecahan satuan siswa belum bisa secara langsung menghubungkan bagian yang belum diarsir, misalnya pada model luas. 2) model luas sangat tepat digunakan ketika siswa pertama kali menyelesaikan masalah pembagian pecahan. 3) pemahaman siswa terhadap pembagian bilangan asli oleh pecahan biasa membantu siswa memahami pembagian pecahan lain yang melibatkan pecahan biasa sebagai pembagi dan yang dibagi. 4) aktivitas pembelajaran tersebut mendukung pengembangan karakter siswa.

Kata Kunci: pemahaman, pembagian pecahan, multi model, *design research*

Abstract

This design research aims to develop a learning activity which supports the fifth-grade students to understand measurement fraction division problems (A whole number divided by a fraction that result in a whole number answer, e.g. $3 \div \frac{1}{2}$) conceptually. Furthermore, how students solve the fraction division problem using models is also analyzed. Data for the retrospective analysis is collected through two teaching experiments in the form of students' work, field notes, and some part of classroom discussions. The important findings in this research are: 1) the developed learning activity namely Motivation Cards support students understand that $3 \div \frac{1}{2}$ means how many $\frac{1}{2}$s are in 3 through models. However, when the divisor is not a unit fraction they could not directly relate the unshaded part in area model for example. 2) area model is proper model to be firstly introduced when the students work on fraction division. 3) understanding this kind of fraction division help students understand other measurement fraction division where both divisor and dividend are fractions. 4) the learning activity supports the development of character values for students.

Keywords: understanding, measurement fraction division, multiple models, design research

How to Cite: Wahyu, K., Amin, S.M., & Lukito, A. (2017). Motivation cards to support students' understanding on fraction divisions. *International Journal on Emerging Mathematics Education, 1*(1), 99-120. http://dx.doi.org/10.12928/ijeme.v1i1.5760.

INTRODUCTION

Fraction division is a difficult topic in primary and secondary school mathematics. The nature of fractions and division is one of the reasons for the

Received January 30, 2016; Revised February 24, 2017; Accepted February 27, 2017

difficulty. Ma (2010) stated that amongst four arithmetic operations, division is the most complicated one and fractions are considered as the most complex number. Indeed, fraction division can be said as a topic at the summit of arithmetic. Besides that, the way teachers teach and their understanding on fraction division are two main factors which make fraction division as the least understood operation. Some research (Hu & Hsiao, 2013; Siebert, 2002; Streefland, 1991) showed that instruction of fraction division focuses on procedural aspects rather than develop conceptual understanding. Ma (2010) found that only 4% of U.S teachers could provide conceptually correct explanation of the fraction division like $1\frac{3}{4} \div \frac{1}{2}$.

Many researches have been conducting to improve the teaching and learning of fraction division. The researches focus on the students (Aksu, 1997; Bulgar, 2003, 2009; Cramer, Monson, Whitney, Leavitt, & Wyberg, 2010; Gregg & Gregg, 2007; Kribs-Zaleta, 2006, 2008; Li, 2008; Sharp & Adams, 2002; Sharp & Welder, 2014; Warrington, 1997), pre-service teachers (Ervin, 2015; Hu & Hsiao, 2013; Jansen & Hohensee, 2015; Nillas, 2003; Sharon & Swarthout, 2015; Slattery & Fitzmaurice, 2014; Tirosh, 2000; Zembat, 2004), and in-service teachers (Flores, Turner, & Bachman, 2005; Ma, 2010) as the subject with different aims. Intensive researches are conducted on pre-service teachers than the other two subjects. The reason why the research of fraction division focused on pre-service teachers relates to their upcoming role. The researchers provided valuable information for preparing the pre-service teachers to have sound pedagogic and content knowledge of the topic.

The researches of fraction division on students have varied aims, i.e. assessing students' performance (Aksu, 1997), supporting students' understanding with teaching intervention (Bulgar, 2003, 2009; Cengiz & Rathouz, 2011; Cramer et al., 2010; Flores & Priewe, 2014; Gregg & Gregg, 2007; Kribs-Zaleta, 2006, 2008; Sharp & Adams, 2002; Sharp & Welder, 2014; Warrington, 1997), and analysing the learning opportunity provided by textbooks (Feil, 2010; Li, 2008). Aksu (1997) just examined the difference of students' performance in the meaning of fraction, fractions computation and solving word problems involved fractions. Li (2008) conducted examination of the approach that Chinese textbooks use to structure lessons relating to dividing with fractions. Feil (2010) studied the topic of fraction division in selected Chinese and US curricula. Meanwhile, Warrington (1997) and the other cited researches focused on supporting students' understanding of fraction division through varied teaching interventions such as providing a series of tasks or learning activities, problem posing, and the use of models or representations.

This current research includes in the second category, i.e. providing context based learning activities to support students' understanding. The students in groups and individually worked on the developed learning activities and are encouraged to use multiple models. This research is similar to Gregg and Gregg (2007) in term of providing students a sequence of learning activities. However, Gregg and Gregg (2007) developed common denominator and invert-multiply algorithms through the activities. We only focused on supporting students' conceptual understanding.

Cramer et al. (2010) presented story problems based on measurement model of fraction division, i.e. a whole number divided by a fraction (e.g. $3 \div \frac{3}{4}$, $3 \div \frac{2}{3}$), division of two fractions (e.g. $2\frac{1}{4} \div \frac{1}{8}$), and answers with and without remainders (e.g. $\frac{6}{8} \div \frac{1}{4}$, $2\frac{1}{2} \div \frac{3}{4}$). The students constructed their strategies that relied on picture to solve the problems. Students' error were also analyzed. The current research is identical to the work of Cramer et al. except on the use of multiple models, the choice of fraction division, the use of context which support character development, and the background of students. We encouraged the students to use more than one models to solve the designed learning activities. In the case of fraction, we only used whole number divided by a fraction with whole number answers. This choice refers to type 1 of the four types of fraction division problems proposed by Schwartz (2008) and Holiday Bows Task by Bulgar (2009). We used context which fits the Indonesian classroom where character education are highly expected to develop. The background of students will be discussed later.

Schwartz (2008) categorized fraction division problems into four types: 1). A whole number divided by a fraction that result in a whole number answer, e.g. $2 \div \frac{1}{2}$; 2). A fraction divided by a fraction that result in a whole number answer, e.g. $\frac{1}{2} \div \frac{1}{4}$; 3). A fraction divided by a fraction that result in a mixed number answer or known as remainder problems, e.g. $\frac{2}{3} \div \frac{1}{2}$; and 4) A fraction divided by a fraction that result in a fraction as well, e.g. $\frac{1}{3} \div \frac{2}{3}$. Schwartz (2008) underlines that these four types of problems represent increasing levels of difficulty and complexity. This categorization is very helpful and important since there are varied fraction division problems which involve many kinds of fractions such as common and mixed fractions as the dividend and divisor. The questions are: For primary students who learn fraction division at first time, what fraction division problems should be introduced first? Is it easy for them to directly learn $2\frac{1}{4} \div \frac{1}{8}$ or even $\frac{1}{3} \div \frac{2}{3}$ where the divisor is greater than the dividend? In Holiday Bows Task, Bulgar (2009) used a whole number divided by a fraction problems to introduce the fifth-grade students fraction division problems as the starting point.

This research also takes into account the function of multiple models in learning mathematics especially fraction division. NCTM (2000) underlines that representations should be treated as essential elements in supporting students' understanding of mathematical concepts. Duval (2006) stated that representations play a major role in mathematics activities since mathematical objects are not accessible without them. Models are a part of external representations. It inherits the function of representations. Cramer, Wyberg, and Leavitt (2008) asserted that models are needed to support students' understanding of operation with fractions. Thus, we encouraged students to use more than one models.

There are three different types of models that students will interact with, use to solve problems, and use to generalize concepts related to fractions, i.e. area models (regions), set models (sets of objects), and number lines. Using area models involves thinking about part to whole relationships. Area models that students typically interact with include objects or drawings such grids, geo-boards, paper folding and pattern blocks. Using set models involves thinking about a fractional part of a set of objects. Set models that students typically interact with are collections of common objects such as buttons, candies, and marbles. Meanwhile, using a number line involves thinking about the distance traveled on a line or the location of a point on number lines, rulers, or other measurement tools (Petit, Laird, & Marsden, 2010).

Regarding the difficulty of fraction division problems (Ma, 2010) which causes the lack of students' conceptual understanding, the categorization of fraction division problems (Schwartz, 2008) which support students to learn fraction division problems in an easy to difficult sequence, and the importance of using multiple models in mathematics learning especially in fraction computations (Cramer et al., 2008; Duval, 2006). This research aims to develop contextual learning activities which support students' understanding on fraction division problems from a basic problem, i.e. a whole number divided by a fraction, through the use of multiple models. The context used supports students to have good character (care and sympathy) to other students. Through the learning activities, we analyze how students solve fraction division problems using models. This research is important effort to lay a foundation for students to understand other kinds of measurement fraction division problems which involve fractions as the dividend and divisor.

RESEARCH METHOD

Design research is used as the research method. It is the systematic study of designing, developing and evaluating educational interventions (such as programs, teaching-learning strategies and materials, products and system) as solutions for complex problems in educational practice, which also aims at advancing out knowledge about the characteristic of these interventions and the processes of designing and developing them (Plomp, 2010). This research follows three phases of design research, i.e. preparation and design (preparing the experiment), teaching experiment and retrospective analysis (Bakker, 2004; Gravemeijer & Cobb, 2006).

In preparation and design phase, we firstly conducted classroom observation and interview with the fifth-grade mathematics teachers to know how students learn mathematics topics especially fraction. The result of observation and interview reveal that the students did not used to experience a classroom environment that requires them to work with context at the beginning of lesson, develop models, contribute more in learning (such as explaining and justifying their work, arguing about others' work) and practice problem solving (Wahyu, 2015). To support students' knowledge on the different representations of fraction, we introduced (in a short time) multiple models in representing fraction before the teaching experiment take place. The background of students in this research differs from Cramer et al. (2010). In Cramer's research, the sixth graders first completed a four lesson review of fractions. The lessons introduced models to reinforce their understanding of part to whole and the importance of identifying the unit when naming fractions. They also have worked with models in other fraction operations.

The literatures (relevant journals and books) on fraction division problems are intensively read and discussed. The literatures and the result of observation and interview give insight about proper learning activities which support students' understanding on measurement fraction division problems (a whole number divided by a fraction). We developed a hypothetical learning trajectory (HLT) which is made up of starting points, learning goals for students, learning activities and hypothetical learning process (Bakker, 2004; van Eerde, 2013). See Appendix 1.

Teaching experiment involved six fifth grade students (three girls and three boys) at the first phase. They have not learned fraction division formally. The selection of the six students is based on level of students ability in mathematics (low, medium and high). The data of students ability is provided by the teachers. They are considered to have represented the class regarding ability in mathematics. It is important to consider the ability criteria in this selection to see how varied levels of students work with the designed activities. At the second phase, teaching experiment involved 28 students. They were divided into six small groups. The selection of students in each group consider the level of students ability as in the first teaching experiment. We only focused on one group consist of 5 students. Each group in the two teaching experiments consist of same-sex students because the research took place on religion-based school. One of its social norms is to separate students if they learn in group based on their sex.

The two teaching experiments follow the tenets of realistic mathematics education and have same plot i.e. the students in group were given designed contextual activities using worksheet (See Appendix 2) at the beginning of learning to be discussed in group with teacher's guidance, following by classroom discussion and then each student was given individual task. This task is important to determine individual achievement after they work in group. The learning process was videotaped using two cameras. One camera was used to record whole class condition and the another was used to focus on the activity in group. The other data collected in this phase are students' work and field notes.

In retrospective analysis, data from the two teaching experiments are analyzed with the guidance of HLT. We also analyzed the match of HLT and the actual learning process. The analysis revealed the development of students' conceptual understanding on the selected fraction division problem. Some refinements on HLT were made to fit the dynamic of learning. The final refinement will be formulated as the local instructional theory (LIT).

RESULTS AND DISCUSSION

In this session, we are going to provide the result of the two teaching experiments and discuss the findings. To easily trace what happen in each teaching experiment, we discuss them one by one.

The 1st Teaching Experiment

Six students (three boys and girls, their names are pseudonymous) were involved in this teaching experiment. Each two students represent different mathematics ability (low, medium and high). We made them as two separate groups regarding the social norms in school. The boys were given one pink carton card and two yellow carton papers. The girls got one blue carton paper and two green carton papers.

The students started the activity by measuring and cutting the carton papers as the conjectures. Each group shared jobs amongst members. Haik and Nil measured, Ippo cut the cartons. Both groups discussed the worksheet while they were finishing cutting. The cards from each carton could be easily estimated by the students. However, they finished cutting the cards. The boys got six cards made from one pink carton and eight cards from two yellow cartons. The girls got four cards from blue carton papers and six cards from 2 green carton papers.

Both groups discussed the worksheet. However, they were a little bit confused to determine the mathematics sentence. The girls intended to write $\mathbf{1 \div 4}$ since one blue carton was divided into four equal parts. The teacher prompted them as shown in the following transcript.

Transcript 1

Teacher : Why 1 ÷ 4?
Qira : Because we make 4 cards
Teacher : How do you think 1 ÷ 4 = 4 cards?
Students : No. 1 ÷ 4 does not equal to 4
Teacher : Is it divided by 4 or one fourth?
Qira : One fourth
Teacher : Why?
Qira : Each card is one-fourth

The students' work for the first carton paper (pink for the boys and blue for the girls) showed that they could not directly link the activities to division of fractions problem, i.e. make correct mathematics sentence based on the activity. They tend to consider a number of cards made as the divisor. In this case, the teacher plays important role to provide guidance by confronting the students with the fact that 1 ÷ 4 does not equal cards made. This strategy worked well for the second carton. They could directly determine correct mathematics sentence.

In the worksheet, the students were asked to draw area model to represent cards. This aimed at leading them to a formal way of determining the quotient. The boys had known that six cards can be made from one pink carton (Figure 1) where each card represents $\frac{1}{6}$. They also wrote correct mathematics sentence, $1 \div \frac{1}{6}$. So did the girls for the two green carton papers.

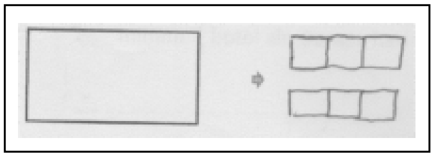

Figure 1. The boys' draw (model)

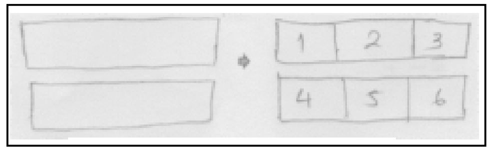

Figure 2. The girls' draw (Model)

It was initially conjectured that the students understand $1 \div \frac{1}{6}$ themselves through model. However, it did not occur. We scaffold them (Transcript 2) by focusing on how many cards made and questioned, "How many one-sixths are in one pink carton paper?" We took one pink and two green carton papers which refer to $1 \div \frac{1}{6}$ and $2 \div \frac{1}{3}$, respectively.

Transcript 2

Teacher : *How many cards made from one pink carton paper?*
Students : *6*
Teacher : *What is the fraction of each card?*
Students : $\frac{1}{6}$
Teacher : *What is the mathematics sentence?*
Students : $1 \div \frac{1}{6}$
Teacher : *For two green papers, one carton paper is divided equally into ...?*
Students : $\frac{1}{3}$
Teacher : *What is the mathematics sentence?*
Students : $2 \div \frac{1}{3}$
Teacher : *How many cards?*
Students : *6 cards*
Teacher : *How many one-sixths are in one pink carton paper? (showing the cards made)*
Haik : *6 (then followed by the girls)*
Teacher : *How many one-thirds in two green papers?*
Students : *6*
Teacher : *Can you conclude what* $2 \div \frac{1}{3}$ *means?*
Students : *... (no answers)*
Qira : *2 carton papers are divided equally into* $\frac{1}{3}$ *which equal to 6 cards*
Teacher : *Is it same to say, there are six one-thirds in 2 carton papers?*

Students *: Yes … (Seem not sure)*
Teacher *: Let's try another example, $1 \div \frac{1}{0}$. How many one- eights are in one?*
Students *: 8*

The conversation above shows that the students are at one step closer to the meaning of a whole number divided by fraction problem. They could determine the quotient through cards or models drawn. A question, "How many $\frac{1}{8}$ s are in one?" plays important role. However, they could not reveal what $1 \div \frac{1}{8}$ means in words yet. We added another example by asking what $2 \div \frac{1}{5}$ means. They could not answer it in words. We drew two circles. Each circle was divided into 5 parts equally. Then we asked, what is the quotient of $2 \div \frac{1}{5}$? They answered 10.

Based on the students' work and classroom discussion, it reveals that the students can draw area model to represent the problem and use them to find the quotient. In other words, they can answer question "How many equal parts are in a whole?" through area model. This finding indicates that the students were able to develop their conceptual understanding on the basic fraction division problem through area model. When students can create and use model, it helps them to make sense of problems involving division of fractions (Bulgar, 2009). However, they cannot directly interpret the meaning of division of fractions yet, e.g. $1 \div \frac{1}{8}$, in words. We think it is important to support the students to develop the meaning of $1 \div \frac{1}{8}$. Once the meaning developed, it will support them understand the other fraction division problems. Regarding this findings, the conjectures will be refined (Table 1)

In the second meeting, we had review session before giving them individual task. Individual task is considered very urgent to give in order to know on what the students achieved. In motivation cards activity, students began to develop their understanding on a whole number divided by fraction problems. Thus, we need to follow up with individual task. In the preview session, we discussed $1 \div \frac{1}{6}$ and $2 \div \frac{1}{3}$. Some students were still difficult in stating the meaning of fraction division problem. The conservation shows it below.

Transcript 3

Teacher *: Yesterday, we had 2 green carton papers. Each was divided equally into three parts. What is the mathematics sentence?*
Students *: $2 \div \frac{1}{3}$*
Teacher *: Good. What does $2 \div \frac{1}{3}$ mean?*
Haik *: There are 2 carton papers, each is divided into 3 parts equally. The result is 6 cards (Same answer to Qira in the previous session)*
Teacher *: Great. Any other opinion?*
Students *: … (no responses)*
Teacher *: How many one-thirds are in 2 carton papers?*
Students *: 6*
Teacher *: That is meaning of $2 \div \frac{1}{3}$. What does $1 \div \frac{1}{6}$ mean?*
Qira *: How many one-sixths are in one carton paper?*
Teacher *: Excellent*

In individual task (Appendix 2), question number 1 of part 1 is challenging. The students were asked to determine the meaning of $1 \div \frac{1}{5}$. At first, the students did not directly answer this problem. They jumped to the next number. Here, the teacher used cards made in the previous meeting to illustrate the question. Qira did it correctly. The other girls got correct answer after giving another explanation by teacher. The boys had different answer to the girls. The teacher asked Haik about his answer (Figure 4). He meant 5 equal parts in 1 carton as learned at the first meeting. Haik's answer was still relying upon cards made in previous meeting. Ippo and Nil had similar answer as Haik, but both of them could not explain their answer.

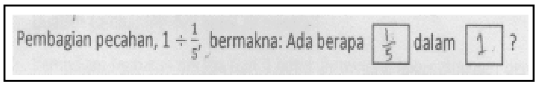

Figure 3. Rina's answer

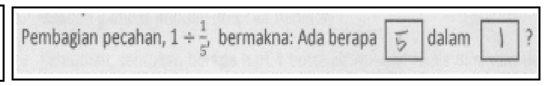

Figure 4. Haik's answer

Question number 2 (part a) uses word 'gambar' because the students were not familiar with word 'model'. We intentionally provided three circles (area model) to trigger students' modeling for $2 \div \frac{2}{5}$. Surprisingly, all students used given area model properly to get correct answer on part (a). They made three equal partitions in each circle then found 9 of $\frac{1}{3}$s in 3 circles.

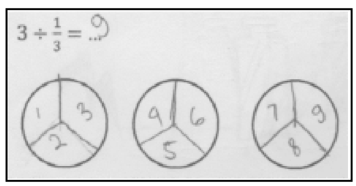

Figure 5. Student's work on part a

On part (b), all students drew rectangle model to solve it. Two rectangles were drawn, each was divided into three equal parts. They shaded $\frac{2}{3}$ in each model and counted them. They got 2 of shaded area with one part left in two model (Figure 6). The students seemed to have no idea about the left parts. In the research of Cramer et al. (2010), before the students learn fractions division they have been given review of fractions to reinforce their understanding of part to whole and the importance of identifying the unit when naming fractions. They could easily determine the remaining parts of 3 when divided by $\frac{3}{4}$. In this current research, the students have not experienced such learning so they could not identify the unit for the drawing model. In this case, the teacher gave hint to add the remaining parts. They were becoming aware of the remaining parts as $\frac{2}{2}$. Cramer et al. (2010) underlines that students' understanding of the part-whole construct was strongly tied to the idea of flexibility of unit.

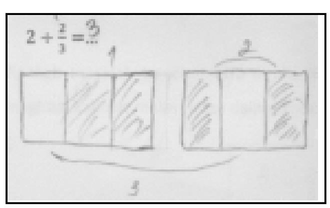

Figure 6. Student's work on part b

Question no. 3 and 4 offered two different models (set model and number line) to solve a whole number divided by fraction problems. All students were able to solve the problem using the two models (Figure 7).

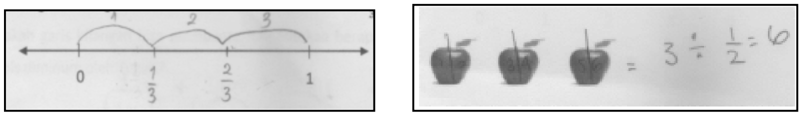

Figure 7. Students' work on number 3 and 4

In part 2 of the individual task, the students are asked to solve the contextual problems through preferred model, find the mathematics sentence and use the number line. The students could easily draw area model of the 4 bottles. However, three students could not directly find the answer. The teacher asked them to focus on the model and make partition based on the problem. When they made partition, they could find the answer (3 days). Question 2 (part b) is very helpful to support the students link the remaining parts in each bottle.

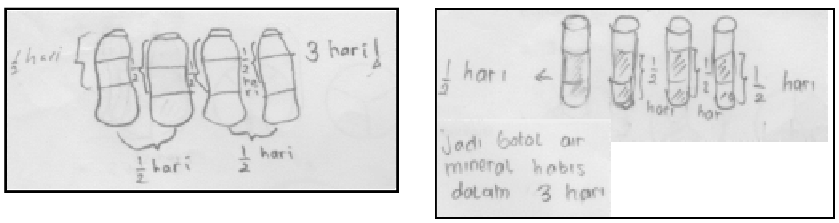

Figure 8. Students' work on part 2 of the individual task (Number 1)

After solving the problem using area model, the students are asked to link the contextual problem with formal fraction division (mathematics sentence). The five students wrote correct mathematics sentence for the problem, $4\div\frac{2}{3}$. One student wrote $4\div\frac{1}{3}$. Students' work on this problem reveals that they could determine a number of parts divided are not the divisor as they did the group activity but the fraction which represents the part. For the quotient, they answered $4\div\frac{2}{3}=3$. We clarified the answer. They meant 3 as 3 days but the answer should be 6. On the last item, two students found no difficulty using number line to solve the problem. The remaining students need help in seeing the number line as the area model. Their difficulty regarding number line is to determine the position of $\frac{2}{3}$. In area model, they could easily represent the bottles and make partition of $\frac{2}{3}$.

The 2nd Teaching Experiment

In this phase, we focused on one group consist of six male students (Dafi, Ali, Wan, Wika, Aher. Their names are pseudonymous). The group was given one green carton paper ($1\div\frac{1}{8}$) and two yellow carton papers ($2\div\frac{1}{2}$). For the green carton paper, the students got 8 cards. They draw the area model to visualize the cards made. In determining the mathematics sentence, they had the same case with the students in the 1st teaching experiment, the divisor was 8. We also confronted them with the fact one divided by eight does not equal to eight. With the prior knowledge of whole number division, the students could easily agree that. For the two yellow carton papers, they had no problem to draw the model, write mathematics sentence and determine the quotient.

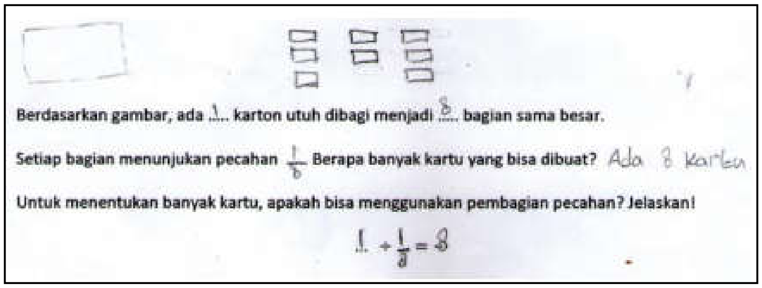

Figure 9. Students' model of the motivation cards

As in the 1st teaching experiment, the students also could not directly reveal the meaning of $1\div\frac{1}{8}$. The following transcripts show it.

Transcript 3

Teacher	*: Who can explain what $1\div\frac{1}{8}$ means?*
Dafi	*: One is divided by one eight*
Teacher	*: The quotient?*
Students	*: 8*
Teacher	*: How many $\frac{1}{8}$s are in one carton paper?*
Students	*: 8*
Teacher	*: Are you aware of the meaning of $1\div\frac{1}{8}$?*
Students	*: Yes*
Teacher	*: What does $1\div\frac{1}{8}$ mean?*
Wika	*: This* is divided into this**. The result is 8 (* points to model of one carton in Figure 9, ** points to the model of 8 cards)*

We tried using the other problem. Some students could directly state the meaning as shown in the following transcript.

Transcript 4

Teacher	*: What $2\div\frac{1}{2}$ means refer to the cards made?*
Dafi	*: How many $\frac{1}{2}$s are in these 2 carton papers?*
Teacher	*: How many?*
Students	*: 4*
Teacher	*: Well, that is the meaning of it. Let's try this. What does $2\div\frac{1}{3}$ mean?*
Dafi, Wika	*: How many $\frac{1}{3}$s are in two?*
Teacher	*: What is the quotient of $2\div\frac{1}{3}$?*
Dafi	*: 6*

The problem of divisor and the meaning of $2\div\frac{1}{3}$ for example also emerged in the other groups. After finishing discussion in each group, the lesson proceeded to whole discussion led by teacher. This whole discussion intended to have mutual understanding on the problem learned. We posed a question, what does $2\div\frac{1}{4}$ mean? The discussion is presented in the following transcript.

Transcript 5

Wika	*: The quotient?*

Teacher : *Ok. No matter*
Wika : *It is 8 (followed by the others)*
Teacher : *Why the quotient is 8?*
Zia : $2\times\frac{4}{1}$
Teacher : *Good. Can anybody use gambar (model)?*

Deka and Dafi came forward to solve the problem using model. Dafi drew two rectangles. Each rectangle was divided into four equal parts. He also wrote the procedure of finding $2\div\frac{1}{4}$ using formal strategy (inverse-multiply). Meanwhile, Deka just drew eight small rectangles. Deka seemed to draw the answer. We had all students look at Dafi's work. The discussion presented in the transcript 6 shows that motivation cards activity contributed positively in supporting students' understanding on a whole number divided by fraction problem. The students can relate back $2\div\frac{1}{4}$ to the activity.

Transcript 6

Teacher : *If we relate* $2\div\frac{1}{4}$ *to cards. What does 2 represent?*
Students : *2 carton papers*
Teacher : *How is each carton paper divided?*
Students : *Four*
Teacher : *What is the fraction of each part?*
Students : $\frac{1}{4}$
Teacher : *How many cards?*
Students : *8*

We also provided all students with individual task. The task is identical to what the students in the 1st teaching experiment did. We analyze all students' work to understand what have they learned from motivation cards activity. On number 1, 27 of 28 students answered it correctly. One student did reversely. She interpreted $1\div\frac{1}{5}$ as how many 1s are in $\frac{1}{5}$? There are some unique answers on this number. Those can be categorized into three kinds (Figure 10a-10c).

Figure 10a. Student directly draw on the model

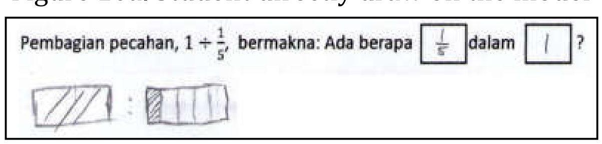

Figure 10b. Student writes answer on the model and draw their model

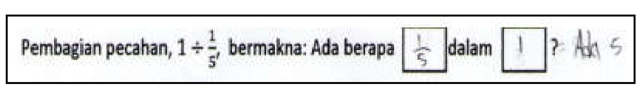

Figure 10c. Student writes answer on the model and the answer as well

On number 2 part (a), 22 students used the model to get correct answers. Meanwhile, the others have varied answer: one has incorrect answer, one could not finish, and the four students just made partition on the model. We found the same difficulty as the students solved part (b). The students drew varied area model (circle, rectangle) to solve the problem. However, they could not identify the remaining part.

We also supported them to link the remaining part in two models to get $\frac{2}{3}$. On number 3, most of the students could make partition on number line and use it to solve the problem. On number 4, only three students could not use the apples to solve $3 \div \frac{1}{2}$. On part 2 (contextual problem), most of the students could write mathematics sentence and solving the problem through area model and number line. In focused group, two students (Dafi and Ali) could directly solve the problem. The other four students needed further help from teacher. Figure 11 shows students' work from the groups.

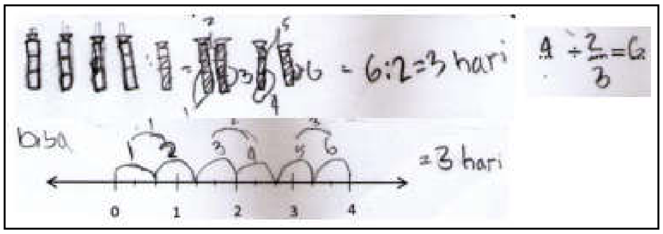

Figure 11a. Dafi's work

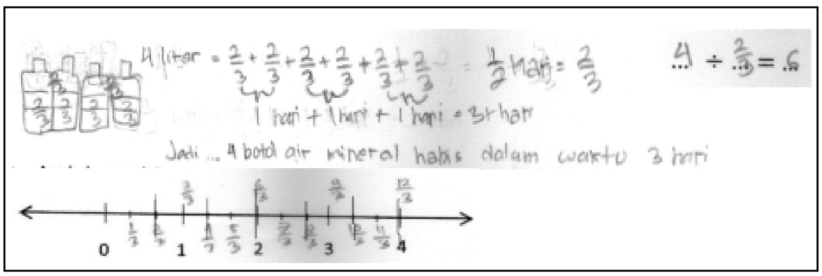

Figure 11b. Zia's work

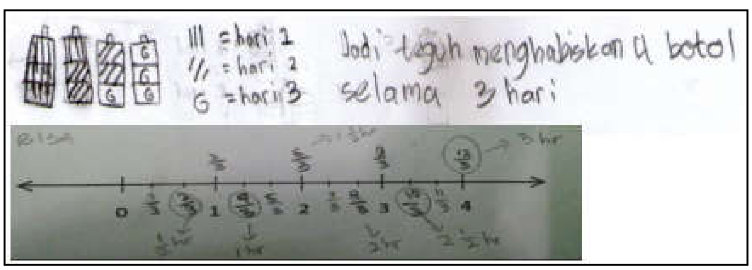

Figure 11c. Izzi's work

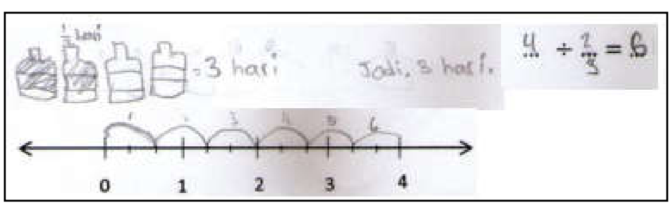

Figure 11d. Ali's work

In this 2nd teaching experiment, the refined conjectures matched the actual learning process. Thus, there will be no refinement for the conjectures. It means that the HLT can be formulated as the LIT on a whole number divided by a fraction as follows.

Table 1. LIT on a whole number divided by a fraction

Learning activity	Tools and practice	Mathematics concept
Motivation cards	**Tools**: rulers, carton papers, scissors, colorful pens **Practice**: *Group activity* Each group is given two different color carton papers, tools, and worksheet. The students in group are asked to make 10 cm × 5 cm from the carton papers. The worksheet will lead them to draw area model and determine how many cards made through the model. Then they are asked to determine correct mathematics sentence and the quotient through model drawn. The conjectures on how the students deal with this activity are: 1. Students start measuring the width and length for a card from the given carton papers. The task might be divided amongst member in group. The size of carton paper exactly fits the amount of cards. 2. Students might easily draw area model to represent the cards. The model shows them that there are four $\frac{1}{4}$s in one carton paper. However, they might be difficult in stating the meaning of a whole number divided by fraction problem through model or cards. To lead the students, the teacher needs to focus on how many cards made from full carton paper and question, "How many one-fourths (part) are in one full carton paper (whole)?" 3. After measuring and cutting each card, a number of cards are found. However, the students are likely not able to directly link the activity to the division of fractions problem. They cannot write a correct mathematics sentence since a number of cards are considered as the divisor. The teacher needs to lead them by focusing on the activity. One carton paper is divided into four parts equally. Each part represents $\frac{1}{4}$. The mathematics sentence will be $1 \div \frac{1}{4}$. If the students come up with $1 \div 4$, the teacher can confront them with a fact that $1 \div 4$ does not equal 4. 4. Students are be able to determine the quotient of $1 \div \frac{1}{4}$ through the area model and cards made *Individual activity (Appendix 2)*	A whole number divided by fraction

The important findings in this research are: 1) the developed learning activity namely Motivation Cards support students understand a whole number divided by a fraction problem. 2) area model is proper model to be firstly introduced when the students work on fraction division. 3) understanding this type of fraction division help students understand other measurement fraction division where both divisor and dividend are fractions. And 4) the learning activity supports the development of character values for students.

The two teaching experiments showed how students draw area model based on the carton paper and cards made. The model helped them to understand $1 \div \frac{1}{4}$. One carton paper represents the dividend (1), the each card represent the divisor ($\frac{1}{4}$), and a number of cards made represent the quotient (4). The way to determine $1 \div \frac{1}{4}$ is to determine how many each card are in one carton paper. Although the students could not directly determine the correct divisor, their prior knowledge confronted them that the divisor should be fraction which represents each card. Students' work on Individual Task confirm what Motivation Cards activity contribute to their understanding. The students can use number line, area model and set model to solve the fraction division problem. The students' difficulty in determining the remaining parts in model while solve $2 \div \frac{2}{3}$ relates to their prior learning with fractions especially about the unit of fraction. Cramer et al. (2010) showed that students' understanding of the part-whole construct is strongly tied to the idea of flexibility of unit. Furthermore, the students could easily model the bottles, make partition based on the problem and solve the problem. Most of them in the 1st and 2nd teaching experiment related the contextual problem into correct fraction division number then determine the quotient through area model and number line. However, some students could not easily determine the equal parts on number line based on the problem.

Measurement fraction division is also called repeated subtraction or equal groups. The equal group is taken from the total repeatedly (NCTM, 2006). This fraction division problem is easy to work with area model since the students can identify the equal parts and whole part (total). Set models involve thinking about a fractional part of a set of objects. Meanwhile, a number line involves thinking about the distance traveled on a line or the location of a point on number lines, rulers, or other measurement tools (Petit et al., 2010). The characteristic of set model and number line does mean that measurement fraction division cannot be solved using them. The use of multiple models shows students' advance understanding on the fractions computation. We need a starting point for students to understand the fraction division problem. The starting point is the use of area model. That is why we did not yet introduce other models in motivation cards activity. Besides, the nature of motivation cards can be visualized easier by using area model.

As we underline in the introduction session, a whole number divided by a fraction is basic fraction division problem. We call it as basic because the role it plays if the students understand it properly. Schwartz (2008) makes it as the type 1 of four types, where these four types of problems represent increasing levels of difficulty and complexity. When students understand that $2 \div \frac{2}{3}$ is to find how many equal parts ($\frac{2}{3}$s) are in total parts (2), they will be easy to understand for example $\frac{2}{3} \div \frac{1}{2}$. That is to find how many $\frac{1}{2}$s are in $\frac{2}{3}$. Thus, we stress that understanding this type of fraction division help students understand other measurement fractions division.

Gregg and Gregg (2007) used the idea of serving sizes using the nutrition facts label to note that the serving size is not always a whole number. Bulgar (2009) used Holiday Bows as a meaningful context. In this research, we used Motivation Cards to fit the Indonesian classroom context and the massive effort to support character education. Although many discourses in the context of character education, we believe that this Motivation Cards activity will support the fifthgrade students to show empathy and care for the sixth-grade students who will face national exams. Empathy and care are two forms of character that schools have been teaching to their students.

CONCLUSION

This research developed learning activities on a whole number divided by fraction problem. The teaching experiments and retrospective analysis show that the developed learning activities can support students understand the fraction division problem, area model is proper model to be firstly introduced when the students work on this fraction division, understanding this type of fraction division help students understand other measurement fraction division, and the learning activity supports the development of character values for students. This research implies that to learn fraction computation especially fraction division, the students should have been introduced the multiple representation of fraction and how to determine unit in fraction. In this research, we just use a whole number divided by common fraction with whole number answer. The further research can study how students solve the fraction division problem where the divisor and the answer include mixed fractions through the use of multiple models.

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Appendix 1

Hypothetical Learning Trajectory
for a whole number divided by fraction problems

A. Description

The contextual learning activities refer to a whole number divided by a fraction problems. Students will work in small groups to discuss and solve problem through worksheet. The worksheet will aid them to achieve the learning goal. After having finished group discussion, the teacher leads whole discussion to share the ideas of group. Afterward, the students are given individual worksheet. The main activity in this lesson is that in group students will make 10 cm × 5 cm motivation cards from carton paper for grade 6 students who will have national exams. Each card will be written with motivation words or quotes.

B. Starting points
1. Students know how to measure width and length using ruler.
2. Students know the representation of fractions in area model (rectangle, circle, etc.)
3. Students can model the carton paper as area model

C. Learning goals

Given the designed learning activities, the students are expected to able to:
1. Write correct mathematics sentence from the contextual problems
2. Solve the fraction division problems (determine the quotient) through multiple models
3. Develop understanding on what a whole number divided by a fraction means

D. Learning activities

Activity 1 (Group activity)

Materials: carton, scissor, ruler, colorful pens

As one of the efforts to support the sixth-grade students who will have national exams, the fifth-grade students (in small groups) will make 10 cm x 5 cm motivation cards from carton papers filled up with motivational quotes or words. The teacher provides different colors and sizes of carton paper for group. The group is asked to find how many motivation cards can be made from each carton paper?

Group	**Carton paper (color and size)**	**Mathematics sentence**	**Number of cards**
1	A green carton paper (4 × 10 cm × 5 cm) Two blue carton papers (each size: 3 × 10 cm × 5 cm)	$1 \div \frac{1}{4}$ $2 \div \frac{1}{3}$	... ...
2	A pink carton paper (5 × 10 cm × 5 cm) Two yellow carton papers (each size: 4 × 10 cm × 5 cm)	$1 \div \frac{1}{5}$ $2 \div \frac{1}{4}$	... ...
3	A green carton paper (6 × 10 cm × 5 cm) Two yellow carton papers (each size: 2 × 10 cm × 5 cm)	$1 \div \frac{1}{6}$ $2 \div \frac{1}{2}$	... ...

Group	Carton paper (color and size)	Mathematics sentence	Number of cards
4	A blue carton paper (7 × 10 cm × 5 cm) Two yellow carton papers (each size: 5 × 10 cm × 5 cm)	$1 \div \frac{1}{7}$ $2 \div \frac{1}{5}$	... ...
5	A yellow carton paper (4 × 10 cm × 5 cm) Three green carton papers (each size: 2 × 10 cm × 5 cm)	$1 \div \frac{1}{4}$ $3 \div \frac{1}{2}$	... ...
6	A pink carton paper (8 × 10 cm × 5 cm) Three blue carton papers (each size: 2 × 10 cm × 5 cm)	$1 \div \frac{1}{8}$ $3 \div \frac{1}{2}$	... ...

Activity 2 (Individual activity, see Appendix 2)

In Activity 2, there are two parts. In part 1, students are asked to use multiple models to solve fraction division problems. Part 2 asks students to solve contextual problems.

E. Conjectures on students' thinking

The following conjectures are for activity 1

1. Students start measuring the width and length for a card from the given carton papers. The task might be divided amongst member in group. The size of carton paper has been set to exactly fits the number of cards.
2. Students might draw area model to represent the cards. This model shows the students along with the carton paper that there are four of one-fourth in one. Eventually, they might conclude what $1 \div \frac{1}{4}$ means and its way of modeling.
3. After measuring and cutting each card, the amount of cards are found. The students might link the activity to the division of fractions problem with the guidance of worksheet. For example, a green carton paper given can be made or divided into 4 cards, each card is one-fourth then mathematics sentence $1 \div \frac{1}{4} = 4$ cards.
4. Students might be able to determine the quotient of $1 \div \frac{1}{4}$ through the area model and cards made

Appendix 2

Student's Worksheet

Membuat Kartu Motivasi Ujian Nasional

Untuk memberikan dukungan kepada siswa kelas VI yang akan mengikuti ujian nasional (UN), akan dibuat kartu motivasi berukuran 10 cm × 5 cm dari karton. Kartu tersebut memuat kata-kata motivasi seperti **"Never Give Up. Pantang Mundur, Terus Belajar untuk Sukses UN"**. Setiap kelompok diberikan 2 karton berwarna dengan ukuran berbeda. Temukan banyak kartu motivasi yang bisa dibuat oleh kelompokmu!

Setelah membuat kartu yang akan dituliskan kata-kata motivasi, diskusikan kegiatan berikut dalam kelompok!

1. Buatlah gambar (model) dari kartu yang sudah dibuat!
   Gambar 1: Karton utuh sebelum dibuat kartu

   Gambar 2: Semua kartu yang sudah dibuat

2. Berdasarkan gambar, ada ... karton utuh dibagi menjadi .... bagian sama besar
   Setiap bagian karton menunjukan pecahan ...
   Kartu yang bisa dibuat sebanyak ....

3. Untuk menentukan banyak kartu, apakah bisa menggunakan pembagian pecahan? Jelaskan!

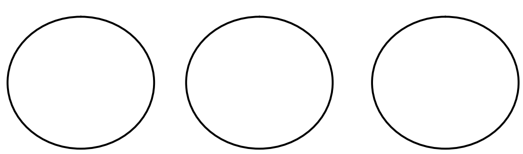
   .... ÷ .... = ...

Individual Task

Bagian 1

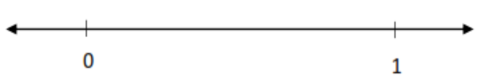
1. Pembagian pecahan $1 \div \frac{1}{5}$ bermakna: Ada berapa □ dalam □?

2. Gunakan gambar untuk menentukan hasil pembagian pecahan berikut!
   a. $3 \div \frac{1}{3} = \cdots$



   b. $2 \div \frac{2}{3} = \cdots$ (Gunakan gambar sendiri!)

3. Gunakan garis bilangan berikut untuk menentukan hasil $1 \div \frac{1}{3}$!



4. Gunakan gambar apel di samping untuk menentukan hasil $3 \div \frac{1}{2}$!

Bagian 2

Teguh mendaki gunung Rinjani. Dia membawa 4 botol air mineral, masing-masing berisi 1 liter. Dalam waktu setengah hari, Teguh minum $\frac{2}{3}$ botol air. Berapa hari 4 botol air mineral habis diminum?

1. Gambarlah botol air mineral yang dibawa Teguh!
   Gunakan gambar tersebut untuk menentukan berapa hari 4 botol air mineral habis diminum!

2. Apakah masalah tersebut bisa ditulis dalam bentuk pembagian pecahan? Jelaskan!

   …. ÷ …. = ….

3. Apakah garis bilangan bisa membantu menentukan berapa hari 4 botol air mineral habis diminum oleh Teguh?