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PROBLEM ON PROPER VALUES FOR DOUBLE-MEASURED SHREDINGER'S OPERATOR WITH RUPTURED COEFFICIENTS

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Abstract. *The work is devoted to the current questions of the problem on proper values for double-measured Shredinger's operator with ruptured coefficients. While researching this problem the author uses the procedure developed by well-known researches L. A. Lyusternik – M. I. Vishik.*

Keywords: boundary conditions, double-measuredoperator shredinger, method by L.A.Lyusternik – M.I.Vishika.

Introduction

In quantum mechanics there are many interesting tasks which do not admit accurate solution. Therefore, the approximate approaches, which are often more useful for understanding the physical phenomena than numerical solutions of the appropriate equations, play a key role.

The principal approximate approaches of quantum mechanics are based on the theory of indignations and variation principles. The critical application of asymptotical method by L .A. Lyusternik – M. I. Vishik has become widely used in many sections of mechanics.

This method enables to research the problems on a program layer, problems on a spectrum, problems on fast oscillation, problems on barriers and many other things. Many works are devoted to the research of asymptotical behavior of solutions, proper numbers and proper functions of differential operators with ruptured coefficients.

In this article the asymptotics of proper numbers and proper functions of double-measured operator Shredinger with strong potential functions were researched.

Asymptotics of proper values and appropriate proper functions for the boundary values is considered which depend on parameter ε such that $\varepsilon \to 0$ coefficients grow in a subarea without limit.

Therefore, the equations which are determined in all spaces of the value E_2 are considered in such a way that the coefficients in subarea D^--are final, and in addition to D^+ when $\varepsilon \to 0$, they grow without limit and let Γ is general border of D^- and D^+.

If u_ε is a solution to the problem, the first approximation to u_ε in the area D^- is a solution of the boundary value in this area with the certain boundary conditions on Γ, and on this basis it appears appropriate to make expansion of scale in a vicinity of Γ on the part of D^+.

During the work the first iterative process passes in the area of D^-, and the second iterative process passes in the area of D^+. In the first approximation we are on a spectrum D^-, where the proper functions can appear in asymptotical decomposition of the task solution.

Let's assume that in x, y-plane the Shredinger's equation with the coefficients, which suffer the rupture on a compact smooth curve Γ the interior of which is designated as D^- , and its addition as $D^+ = E_2 / D^- U \Gamma$.

The coefficients include small parameter ε, where $\varepsilon \to 0$ and in this way the potential function increases without limit.

Besides this, the coefficient at the leading coefficient is also disruptive on Γ. So, now we shall consider a following spectral problem:

$$L_\varepsilon u_\varepsilon \equiv a_\varepsilon \left(\frac{\partial^2 u_\varepsilon}{\partial x^2} + \frac{\partial^2 u_\varepsilon}{\partial y^2} \right) - k_\varepsilon(x) u_\varepsilon = \lambda_\varepsilon u_\varepsilon, \quad (1)$$

where

$$a_\varepsilon = \begin{cases} 1, & x \in D^-, \\ \varepsilon, & x \in D^+, \end{cases} \qquad k_\varepsilon(x) = \begin{cases} k^-(x), & x \in D^-, \\ \dfrac{k^+(x)}{\varepsilon^2}, & x \in D^+, \end{cases} \quad (2)$$

$$k^-(x) \geq 0, \quad k^-(x) \to \infty \text{ when } |x| \to \infty . \quad (3)$$

On the border Γ the factor conditions (conditions of sewing together) are specified:

$$u_\varepsilon\big|_\Gamma^- = u_\varepsilon\big|_\Gamma^+, \qquad (4)$$

$$\frac{\partial u_\varepsilon}{\partial n}\bigg|_\Gamma^- = \frac{\partial u_\varepsilon}{\partial n}\bigg|_\Gamma^+. \qquad (5)$$

Now we'll seek the solution of the problem, limited in all space. Under conditions (3) the problem (1) – (5) has absolutely discrete spectrum. The coefficients L_ε in D^- are supposed to be limited and decaying on degrees.

The aim is to construct the asymptotics on a ε small parameter of proper numbers and appropriate proper functions of the equation (1) on all x, y-plane. In order to construct the asymptotics of proper numbers and proper functions we record the second decomposition of the operator in a vicinity of Γ.

Now we'll add local coordinates (ρ, φ), $\varphi = n$ in a vicinity of Γ and the area D^+, where $\rho > 0$ corresponds to D^+ points.

Then the equation (1) in the area D^+ has a following structure:

$$L_\varepsilon u_\varepsilon \equiv \varepsilon\left[\frac{1}{\rho}\left(\rho\frac{\partial u_\varepsilon}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial u_\varepsilon}{\partial\varphi^2}\right] - \frac{k^+(x)}{\varepsilon^2}u_\varepsilon = \lambda_\varepsilon u_\varepsilon,$$

If $\rho = \varepsilon t$, we have

$$\varepsilon\left[\frac{1}{\varepsilon^2}\frac{\partial^2 u_\varepsilon}{\partial t^2} + \frac{1}{\varepsilon^2 t}\frac{\partial u_\varepsilon}{\partial t} + \frac{1}{\varepsilon^2 t^2}\frac{\partial^2 u_\varepsilon}{\partial\varphi^2} - \frac{k^+(x)}{\varepsilon^2}u_\varepsilon\right] = \lambda_\varepsilon u_\varepsilon$$

or

$$\frac{1}{\varepsilon}\left(\frac{\partial^2 u_\varepsilon}{\partial t^2} + \frac{1}{t}\frac{\partial u_\varepsilon}{\partial t} + \frac{1}{t^2}\frac{\partial^2 u_\varepsilon}{\partial\varphi^2} - k^+(x)u_\varepsilon\right) = \lambda_\varepsilon u_\varepsilon. \qquad (6)$$

Asymptotical decomposition of the proper value λ_ε and appropriate proper function $u_\varepsilon(x)$ the equation (8) in the area D^+ and the equation

$$L_\varepsilon u_\varepsilon \equiv \frac{\partial^2 u_\varepsilon}{\partial x^2} + \frac{\partial^2 u_\varepsilon}{\partial y^2} - k^-(x)u_\varepsilon = \lambda_\varepsilon u_\varepsilon,$$

In the area D^- we'll seek

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots, \qquad (7)$$

$$u_\varepsilon(x) = u_0 + \varepsilon u_1 + \varepsilon^2{}_2 + \ldots, \qquad x \in D^-, \qquad (8)$$

$$u_\varepsilon(x) = u_\varepsilon^+(x) = \vartheta_0 + \varepsilon\vartheta_1 + \ldots, \qquad x \in D^+, \qquad (9)$$

where $u_\varepsilon^+(x)$ has a type of function in the form of a boundary layer, that is they are noticeably distinct from zero only near Γ by means of the smoothing function $\phi(\varsigma)$ continued in zero.

Let's substitute expression λ_ε from (7) and $u_\varepsilon(x)$ where $x \in D^-, x \in D^+$ from (8), (9) in the equation for the proper function

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^-(x) - \left(\lambda_0 + \varepsilon\lambda + \varepsilon^2\lambda_2 + \ldots\right)J\right]\times$$
$$\times\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots\right) = 0, \qquad (10)$$

$$\left[\frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial t^2} + \frac{1}{t}\frac{\partial}{\partial t} + \frac{1}{t^2}\frac{\partial^2}{\partial\varphi^2}\right) - k^+(0) + \right.$$
$$+\left(\varepsilon k_1^+ t + \varepsilon^2 k_2^+ t^2 + \ldots\right) - \qquad (11)$$
$$\left. -\left(\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots\right)J\right]\left(u_0 + \varepsilon u_1 + \ldots\right) = 0,$$

and the factor conditions

$$u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots\big|_\Gamma^- = \vartheta_0 + \varepsilon\vartheta_1 + \varepsilon^2\vartheta_2 + \ldots\big|_\Gamma^+; \qquad (12)$$

$$\frac{\partial u_0}{\partial x} + \varepsilon\frac{\partial u_1}{\partial x} + \varepsilon^2\frac{\partial u_2}{\partial x} + \ldots\bigg|_\Gamma^- = \frac{\partial\vartheta_0}{\partial x} + \varepsilon\frac{\partial\vartheta_1}{\partial x} + \ldots\bigg|_\Gamma^+ =$$
$$= \frac{1}{\varepsilon}\frac{\partial\vartheta_0}{\partial t} + \frac{\partial\vartheta_1}{\partial t} + \varepsilon\frac{\partial\vartheta_2}{\partial t} + \ldots\bigg|_\Gamma^+. \qquad (13)$$

Equating the coefficients at identical degrees on ε, from (10) – (13) we'll receive

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} - k^-(x)u_0 = \lambda_0 u_0, \qquad (14)$$

$$u_0\big|_\Gamma^- = \vartheta_0\big|_\Gamma^+, \qquad (15)$$

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} - k^-(x)u_1 + \lambda_0 u_1 = \lambda_1 u_0, \qquad (16)$$

$$u_1\big|_\Gamma^- = \vartheta_1\big|_\Gamma^+, \qquad (17)$$

$$\frac{\partial^2 u_k}{\partial x^2} + \frac{\partial^2 u_k}{\partial y^2} - k^-(x)u_k + \lambda_0 u_k = \lambda_1 u_{k-1} + \ldots + \lambda_k u_0,$$

$$u_k\big|_\Gamma^- = \vartheta_k\big|_\Gamma^+,$$

$$M_0\vartheta_0 = 0,$$

$$\left.\frac{d\vartheta_0}{dt}\right|_{t=+0} = 0,\ \vartheta_0\big|_{t=+\infty} = 0,$$

$$M_0\vartheta_1 = -M_1\vartheta_0, \qquad (18)$$

$$\left.\frac{d\vartheta_1}{dt}\right|_{t=+0} = 0,\ \left.\frac{du_0}{dx}\right|_\Gamma^-, \qquad (19)$$

$$M_0\vartheta_n = -M_1\vartheta_{n-1} - M_2\vartheta_{n-2}\ldots -$$
$$-M_n\vartheta_0 + \lambda_0\vartheta_{n-2} + \ldots + \lambda_{n-2}\vartheta_0,$$

$$\left.\frac{d\vartheta_n}{dt}\right|_{t=+0} = 0,\ \left.\frac{du_{n-1}}{dx}\right|_\Gamma^-, \vartheta_n\big|_{t=+\infty} = 0.$$

M_i, $(i = 0,1,2,\ldots)$ – are well-known differential expressions.

It is visible, that there is a connection between boundary conditions which is defined as follows: there is a problem for the equation with private derivatives on t, φ:

$$\frac{\partial^2\vartheta_0}{\partial t^2} + \frac{1}{t}\frac{\partial\vartheta_0}{\partial t} + \frac{1}{t^2}\frac{\partial^2\vartheta_0}{\partial\rho^2} - k^+(0)\vartheta_0 = 0, \qquad (20)$$

$$\left.\frac{d\vartheta_0}{dt}\right|_{t=+0} = 0,\ \vartheta_0\big|_{t=+\infty} = 0. \qquad (21)$$

The common solution of the equation (20) is given by the formula

$$\vartheta_0(t) = d_1 J_n(t) + d_2 N_n(t), \qquad t = \frac{\rho}{\varepsilon},$$

where $J_n(t)$ – is Bessel's function;

$N_n(t)$ – is Neumann's function of n kind.

If we consider boundary conditions (21) we'll have $\vartheta_0 \equiv 0$, which has the character of function in the form of boundary layer. Then we'll find the regional conditions of a problem (14), (15) for

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} - k^-(x)u_0 = \lambda_0 u_0, \qquad (22)$$

$$u_0\big|_\Gamma^- = \vartheta_0\big|_{t=+0}. \qquad (23)$$

Thus, we have received a regional condition for the first approximation u_0 in D^- to the accurate solution u_ε. For function u_0 we have received the problems (22), (23) which are outside the area D^-, u_0 is continuously precede by means of the function

$$\vartheta_0(t) = d_1 J_n(t) + d_2 N_n(t),$$

which have a boundary layer.

It is obvious, that above-mentioned asymptotical decomposition of the problems with greater coefficients in the area D^+ shows, that the solution of some boundary values which are in D^+ by the simple functions in the form of a boundary layer serve like their first approximations in the area D^-.

As $k^-(x)$ is real-valued, therefore the operator L_0, is created by the differential expressions (22) and a boundary condition (23). From limitation $k^-(x)$ follows, that the operator L_0 has absolutely discrete spectrum. The proper values of the operator we shall designate like

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots$$

And orthonormal functions through $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x), \ldots$. After the determination of u_0, from (18), (19) we'll solve the problem for ϑ_1,

$$\frac{\partial^2\vartheta_1}{\partial t^2} + \frac{1}{t}\frac{\partial\vartheta_1}{\partial t} + \frac{1}{t^2}\frac{\partial^2\vartheta_1}{\partial\varphi^2} - k^+(0)\vartheta_1 = 0, \qquad (24)$$

$$\left.\frac{d\vartheta_1}{dt}\right|_{t=+0} = 0,\ \left.\frac{du_0}{dx}\right|_\Gamma^-, \vartheta_1\big|_{t=+\infty} = 0. \qquad (25)$$

Solving the equation (24), we receive the common solution in a following type

$$\vartheta_1(t) = d_1 J_n(t) + d_2 N_n(t),$$

Where $\vartheta_1(t)$ have a boundary layer character, and satisfy the condition (25).

Knowing $\vartheta_1(t)$, it is possible to find a boundary condition for $u_1(x)$, i.e. from (16), (17)

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} - k^-(x)u_1 - \lambda_0 u_1 = \lambda_1 u_0, \qquad (26)$$

$$u_1\big|_\Gamma^- = \vartheta_1\big|_{t=+0}. \tag{27}$$

The homogeneous problem which corresponds to the problem (26), (27), is on a spectrum and is not always tractable. Therefore we select λ_1 in order to solve this problem. To solve the non-uniform boundary problem, it is necessary that the right part of a problem (26), (27) is orthogonal to the solution Z_0 of the conjugate homogeneous problem.

Similarly, continuing the process it is possible to found all functions $u_0, u_1, ..., u_{k-1}$, ϑ_0, $\vartheta_1, ..., \vartheta_k$, and the number $\lambda_0, \lambda_1, ..., \lambda_{k-1}$, where ϑ_i, $i = 0,1,...,k-1,k$ is a function of boundary layer.

Apparently, that if ϑ_k are known, it is possible to define the function u_k, as the solution of a following problem in the area D^-

$$L_k u_k \equiv \frac{\partial^2 u_k}{\partial x^2} + \frac{\partial^2 u_k}{\partial y^2} - k^-(x)u_k -$$

$$-\lambda_0 u_k = \lambda_1 u_{k-1} + ... + \lambda_k u_0, \tag{28}$$

$$u_k\big|_\Gamma^- = \vartheta_k\big|_{t=+0}^+. \tag{29}$$

The problem (28), (29) is not always tractable; the appropriate homogeneous problem is on a spectrum. Therefore we choose λ_k in such a way to solve it. In order to solve the problem (28), (29) it is necessary that the right part of a problem is orthogonal to the solution Z_0 of the self-conjugated homogeneous problem. At such choice the problem (28), (29) becomes tractable.

Solving the problem (28), (29), we find its common solution in the form of

$u_k = \tilde{u}_k + e_k u_{k-1}$,

Where $\tilde{u}_k$ - is the partial solution to the problem (28), (29). A constant is chosen to be orthogonal to u_{k-1}, i.e. $(u_k, u_{k-1}) = 0$, from here we find $e_k = (u_k, u_{k-1})$. After such choice, e_k, u_k becomes unequivocal.

Conclusoins

So, continuing the process, it is possible to define other functions of decomposition if the appropriate conditions of smoothness of the given problems are completed. We have the following theorems.

The theorem 1. If $k^-(x) \to \infty$, where $|x| \to \infty$, the problem (1) - (5) has absolutely discrete spectrum, and for every $i, i = 1, 2, ...$ there is a constant b_i that is

$$\left|\lambda_{i\varepsilon} - \lambda_{i0}\right| \le b_i \varepsilon^{1/2},$$

where $\lambda_{i\varepsilon}$ of i-kind proper value of the operator L_ε;

λ_{i0} of i-kind, the proper value of the operator L_0.

The theorem 2. Let the self- conjugated differential operator of the second order (1) on space under the condition of (2), (5) is given. The following asymptotical concepts for i-kind of the proper function $u_{i\varepsilon}$ of this operator look like:

$$\lambda_{i\varepsilon} = \sum_{k=0}^{m} \varepsilon^k \lambda_k + \varepsilon^{m+1}\delta_{m+1}, \quad \delta_{m+1} = O(1),$$

$$u_{i\varepsilon} = u_{i\varepsilon,m} + \varepsilon^{m+1} z_m, \quad \|z_m\| = o(1),$$

where

$$u_{i\varepsilon} = \begin{cases} u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ... + \varepsilon^m u_m + \varepsilon^{m+1}\alpha, & \text{in } D^-, \\ 0 + \varepsilon\vartheta_1 + \varepsilon^2\vartheta_2 + ... + \varepsilon^m\vartheta_m + \varepsilon^{m+1}\vartheta_{m+1} + \varepsilon^{m+2}\vartheta_{m+2}, & \text{in } D^+, \end{cases}$$

where $u_0 = u_{i0}$ is the appropriate proper function of this problem, and the function v_{i0} is in the form of boundary layer.

The theorem 3. Let ε is fixed in (1), for $x \in D^-, k(x, \varepsilon)$ is distinct from zero and for $x \in D^+$ has only one solution, consequently $\lambda = 0$ is not a point of a spectrum of the operator $\lambda = 0$ in the area $\lambda = 0$, where $u\big|_\Gamma = 0$, so the assessment is correct.

$$\|u\|_1 \le \alpha \|L_\varepsilon u\|,$$

where $u\big|_\Gamma = 0$, does not depend on u and ε, i.e. proportional convertibility of the operator L_ε.

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