# Relation Algebras from Cylindric Algebras, I

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**Abstract**

We characterise the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ of subalgebras of relation algebra reducts of $n$-dimensional cylindric algebras (for finite $n \geq 5$) by the notion of a 'hyper-basis', analogous to the cylindric basis of Maddux, and by relativised representations. A corollary is that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n = \mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{Crs}_n) = \mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{G}_n)$. We outline a game-theoretic approximation to the existence of a representation, and how to use it to obtain a recursive axiomatisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

## 1 Introduction

In the nineteenth century there were two main approaches to the formalization of quantification in logic. The first approach, due to de Morgan and taken up by Peirce, led to what we now call relation algebra (see [Mad91b] for an account of the early history of relation algebra); the other approach, due to Frege, became the standard formalism of first-order logic with its explicit universal and existential quantifiers. Both can express quantification, though in different ways — in the algebraic approach to binary relations we use the composition of binary relations. For example, in first-order logic we can say 'there exists a person who is my parent and your sibling', which could be expressed in relation algebra as 'you are either my uncle or my aunt'.

Then, in the twentieth century, first-order logic was given an algebraic setting in the framework of *cylindric algebra* [HMT71, HMT85]. So we now have two main algebraic formalisms for relations of various ranks: relation algebras constitute an algebraization of binary relations and $n$-dimensional cylindric algebras are an algebraization of $n$-ary relations. Ever since these algebras were defined, researchers have investigated the connections between them [Mad91a, for example]. The *relation algebra reduct* is a known way of turning a cylindric algebra into a relation algebra: we extract the essentially binary relations of the cylindric algebra and interpret the relation algebra operations on them by suitable cylindric algebra terms. But the question arises as to when a given relation algebra can be obtained as such a reduct — or at least as a subalgebra of such a reduct. To put it another way, if $\mathcal{C}$ is an $n$-dimensional cylindric algebra and $\mathcal{A}$ is a subalgebra of the relation algebra reduct $\mathfrak{Ra}(\mathcal{C})$ of $\mathcal{C}$, is there a trace purely within $\mathcal{A}$ of its origin as a $n$-dimensional cylindric algebra? This, in essence, is the question that we will investigate in this paper.

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The class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is by definition the class of subalgebras of relation algebras of the form $\mathfrak{Ra}\mathcal{C}$ for sone $n$-dimensional cylindric algebra $\mathcal{C}$. We wish to find an intrinsic characterisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. Maddux has shown that any atomic relation algebra with an *$n$-dimensional cylindric basis,* and hence any subalgebra of such a relation algebra, belongs to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. We will show here that there are (atomic) algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ that have no such basis, though whether every relation algebra in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ embeds in a relation algebra with an $n$-dimensional cylindric basis remains an open problem. On the other hand, the class $\mathsf{RA}_n$ of subalgebras of atomic relation algebras with *$n$-dimensional relational bases* does include $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, but this time we define too big a class. The definitions of these different bases will be given later, and can be found in [Mad89].

In this paper, we introduce the notion of an *$n$-dimensional hyper-basis.* Hyper-bases are very similar to Maddux's cylindric bases, but their elements are *hyper-networks* which carry relations of arity up to $n$, not just two as with cylindric and relational bases. In our main theorem (theorem 1), we show that the class of subalgebras of relation algebras with $n$-dimensional hyper-bases is exactly $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

Further, we provide a representation theory for the algebras of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. These algebras are not always in $\mathsf{RRA}$ and so are not necessarily representable in the classical sense. None the less, they have useful non-classical 'relativised' representations. In [HH97c], we gave relativised '$n$-square' semantics for the algebras of $\mathsf{RA}_n$. An $n$-square representation is 'locally classical' in that without simultaneously considering more than $n$ points, one cannot tell at first sight that it is not classical. Here, we define the related but stronger notion of *$n$-flat relativised representation,* in which detecting its non-classical nature is even harder. We show that a relation algebra has such a representation if and only if it belongs to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. We then give an alternative kind of representation, which we call *$n$-smooth*, and show that the relation algebras with such a representation are again precisely those in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

Game-theoretic approximations to $n$-smooth relativised representations can be used to obtain a recursive axiomatisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, and, using the fact that this class is an equational variety, we can turn such axioms into equations. We will outline how to do this, but we will not go into full detail.

Various other results will be stated in remarks in the text. Proofs are omitted through lack of space.

**Further work** It is easily seen that for $n \geq 4$ (or even 3, if we generalise from relation algebras to non-associative algebras), $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \supseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. In [HHM98], we showed that this inclusion is strict, for each $n$. Using game-theoretic techniques, we will show in part II of this paper [HH99a] that the gap cannot be finitely axiomatised.

## Plan of paper

We will prove:

THEOREM 1 *Let $\mathcal{A}$ be a relation algebra and let $n \geq 5$. Then the following are equivalent:*

1. $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.
2. *The canonical embedding algebra $\mathcal{A}^+$ of $\mathcal{A}$ has an $n$-dimensional hyper-basis.*
3. *$\mathcal{A}$ has an $n$-flat relativised representation.*

4. *$\mathcal{A}$ has an $n$-smooth relativised representation.*

5. *$\mathcal{A}$ satisfies certain explicitly-given equations $\varepsilon_k$ ($k < \omega$).*

Definitions of the terms in this theorem will be given at the appropriate places: in definitions 7, 14, 41, and immediately before theorem 51. The theorem also holds (degenerately) for $n = 4$: see remark 52.

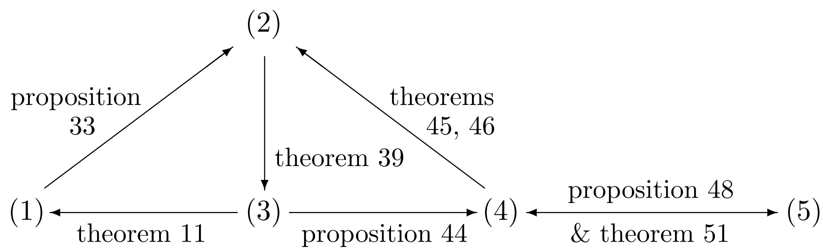


Figure 1: Summary of the proof of theorem 1

The proof will proceed as follows. First, we recall the definition of a relativised representation of a relation algebra, though instead of the usual definition using homomorphisms on algebras, we define our representations as models of a certain first-order theory. In such a representation, $n$-variable first-order formulas can be interpreted, roughly by relativising existential quantifiers to the unit of the relation algebra. We then define an $n$-flat relativised representation to be a relativised representation with the additional property that these relativised quantifiers commute ($\exists x_i \exists x_j \varphi$ is always equivalent to $\exists x_j \exists x_i \varphi$). It follows that the definable sets form an $n$-dimensional cylindric algebra, and so we prove in theorem 11 the implication (3) $\Rightarrow$ (1) of theorem 1.

In section 3, we introduce *hyper-networks* and *hyper-bases*. Hyper-networks are very like the basic matrices of [Mad82, section 4] or the atomic networks of [HH97b], but as well as using atoms to label edges of these hyper-networks, we also have labels for sequences of length greater than two. Hyper-bases correspond approximately to Maddux's cylindric bases, the only difference being that the elements of a hyper-basis are hyper-networks rather than basic matrices. We then develop some results on substitutions in cylindric algebras, based on results of [Tho93] showing that the effect of a string of substitutions in an $n$-dimensional cylindric algebra is determined by an associated map induced on $\{0, 1, \ldots, n-1\}$. This work is used in proposition 33, which proves (1) $\Rightarrow$ (2) of theorem 1.

In section 4, we prove (2) $\Rightarrow$ (3) by showing that any atomic relation algebra with an $n$-dimensional hyper-basis has a (complete) $n$-flat relativised representation (theorem 39). This is done in a 'step-by-step' fashion. This completes the proof of the equivalence of parts (1) to (3) in theorem 1. The proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) has some similarity to that in [AT88].

Another kind of relativised representation for algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, which we call *$n$-smooth*, is introduced in section 5. Its defining property is reminiscent of 'bisimulations', the rough idea being that certain local isomorphisms of size less than $n-1$ can be extended so that their domain includes an arbitrary additional point, provided the extended domain forms a 'clique' in the relativised representation. We show that the existence of an $n$-smooth relativised representation is equivalent to the previous parts of theorem 1 by proving (3) $\Rightarrow$ (4) (proposition 44, using an $\omega$-saturated relativised representation), and (4) $\Rightarrow$ (2) (theorem 45, again using $\omega$-saturation, and theorem 46, by taking a complete $n$-smooth relativised representation and considering the set of all hyper-networks that embed in it).

In section 6, we outline how to use $n$-smooth relativised representations to obtain a recursive equational axiomatisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. We use a two-player game where the players construct better and better 'approximations' to an $n$-smooth relativised representation. Proposition 48 uses these games to find a first-order characterisation of the countable relation algebras with $n$-smooth relativised representations. It states that in the infinite length game, at least for countable relation algebras, a winning strategy for the second player is equivalent to the existence of an $n$-smooth relativised representation, and that a winning strategy in all finite length games is equivalent (by König's tree lemma) to a winning strategy in the infinite length game. Moreover, the existence of a winning strategy for the second player in the game of finite length $k$ can be expressed by a universal sentence $\sigma_k^n$. This gives a recursive set of universal first-order axioms $\sigma_k^n$ $(k < \omega)$ which are true in a countable relation algebra $\mathcal{A}$ iff $\mathcal{A}$ has an $n$-smooth relativised representation. By the other parts of theorem 1, this holds iff $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

In this paper we will only sketch all this, as it is becoming a standard method and also for the following reason. $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is defined to be the class of subalgebras of relation-algebra reducts of $n$-dimensional cylindric algebras, and, as we outlined above, it can also be characterised as either (i) the class of subalgebras of algebras with $n$-dimensional hyperbases, (ii) the class of algebras with $n$-flat relativised representations, or (iii) the class of algebras with $n$-smooth relativised representations. The definition of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ and each of these three characterisations fit the definition of a $PC'_\Delta$ *class* [Hod93, chapters 5, 6]. Roughly, a $PC'_\Delta$ class consists of every $L$-structure (for some first-order signature $L$) that arises in a uniform way as a definable part of a model of some first-order theory in a language extending $L$. In a forthcoming publication it will be shown how to obtain universal axioms for any $PC'_\Delta$ class that is closed under subalgebras. This method of axiomatising a $PC'_\Delta$ class generalises the axiomatisation we provide here.

We continue with a proof (proposition 49) that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety. This eliminates the countability restriction. Further, since $\mathsf{RA}$ is a discriminator class, for each universal formula $\sigma_k^n$ we can find an equation $\varepsilon_k^n$ which is equivalent to $\sigma_k^n$ over simple relation algebras. Since every relation algebra $\mathcal{A}$ is a subdirect product of simple relation algebras which are quotients of $\mathcal{A}$, we deduce in theorem 51 that the equations $\{\varepsilon_k^n : k < \omega\}$, together with the basic Tarski equations for relation algebras, exactly define the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. This completes the proof of theorem 1.

In section 7, we discuss matters arising from the theorem. Section 8 investigates the connections between hyper-bases, cylindric bases and relational bases. We outline how cylindric bases correspond to a kind of homogeneity in a representation (see theorem 60).

NOTATION 2

**Ordinals.** Most ordinals in this paper are finite. For ordinals $m, n$, we write ${}^m n$ for the set of maps from $m$ to $n$, and ${}^{<m}n$ for $\bigcup_{i<m} {}^i n$. We use ${}^{\leq m}n$ to denote ${}^{<m+1}n$.

**Tuples.** We often view a map in ${}^m n$ as an $m$-tuple, and write it as $\bar{a} = (a_0, \ldots, a_{m-1})$ and its length $m$ as $|\bar{a}|$. We will switch between the map view and the tuple view whenever appropriate. We may specify an $m$-tuple $\bar{a}$ be defining the elements $a_i$ for $i < m$.

If $\bar{a}, \bar{b}$ are tuples, we write $\bar{a}\bar{b}$ for their concatenation $(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1})$. Thus, $\bar{a}b$ denotes $\bar{a}$ concatenated with the 1-tuple $b$. For $n$-tuples $\bar{a}, \bar{b}$ and $i_0, \ldots, i_{k-1} < n$, we write $\bar{a} \equiv_{i_0,\ldots,i_{k-1}} \bar{b}$ if $a_j = b_j$ for all $j \in n \setminus \{i_0, \ldots, i_{k-1}\}$. We also write $(\bar{a} \mapsto \bar{b})$ for the relation $\{(a_i, b_i) : i < n\}$; this may or may not be a well-defined map or function.

For any $m$-tuple $\bar{a}$ and any map $\theta$ with $rng(\bar{a}) \subseteq dom(\theta)$, we write $\theta(\bar{a})$ for the $m$-tuple $(\theta(a_0), \ldots, \theta(a_{m-1}))$. As a map, this is $\theta \circ \bar{a}$. On the other side, if $l$ is an ordinal and $\theta : l \to m$ a map, we let $\bar{a} \circ \theta$ denote the $l$-tuple $(a_{\theta(0)}, \ldots, a_{\theta(l-1)})$.

**Structures.** If $L$ is a signature and $M$ an $L$-structure, we write $S^M$ for the interpretation in $M$ of a symbol $S \in L$. For example, $1'^{\mathcal{A}}$ is the identity of the relation algebra $\mathcal{A}$. We usually identify (notationally) a structure with its domain.

# 2 Representation theory

In [Mad82], it was shown that the weakly associative algebras are precisely those that have relativised representations in which the unit is a reflexive and symmetric relation. We will extend this to provide a representation theory for algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$: the unit remains reflexive and symmetric, but the representation is rather more complicated. In [HH97c], we did this for the variety $\mathsf{RA}_n$ of subalgebras of relation algebras with $n$-dimensional relational bases. The approach here is rather similar.

Let $\mathcal{A}$ be a relation algebra. Let $\mathcal{L}(\mathcal{A})$ be the first-order language in a signature consisting of one binary predicate symbol for each element of $\mathcal{A}$. That is, each element of (the domain of) the algebra $\mathcal{A}$ will be regarded as a binary relation symbol. (This will not lead to ambiguity: for $r \in \mathcal{A}$, if we write $r(x, y)$, we are thinking of $r$ as a relation symbol, but if we write simply $r$, we are thinking of $r$ as an element of $\mathcal{A}$.)

## 2.1 Relativised representations

Definition 3

1. $S_{\mathcal{A}}$ is the $\mathcal{L}(\mathcal{A})$-theory consisting of the following axioms:

$$\begin{array}{rl}
\forall xy[1'(x,y) \leftrightarrow (x = y)] & \\
\forall xy[r(x,y) \leftrightarrow s(x,y) \vee t(x,y)] & \text{for all } r,s,t \in \mathcal{A} \text{ with } \mathcal{A} \models r = s + t \\
\forall xy[1(x,y) \to (r(x,y) \leftrightarrow \neg s(x,y))] & \text{for all } r,s \in \mathcal{A} \text{ with } \mathcal{A} \models r = -s \\
\forall xy[r(x,y) \leftrightarrow s(y,x)] & \text{for all } r,s \in \mathcal{A} \text{ with } \mathcal{A} \models r = \breve{s} \\
\forall xy[1(x,y) \to (r(x,y) \leftrightarrow \exists z(s(x,z) \wedge t(z,y)))] & \text{for all } r,s,t \in \mathcal{A} \text{ with } \mathcal{A} \models r = s\,;t \\
\exists xy\, r(x,y) & \text{for all } r \in \mathcal{A} \text{ with } \mathcal{A} \models r \neq 0.
\end{array}$$

2. A *relativised representation* of $\mathcal{A}$ is a model of $S_{\mathcal{A}}$.

3. A *complete relativised representation* of $\mathcal{A}$ is a model of $S_{\mathcal{A}}$ satisfying the (potentially infinitary) axiom

$$\forall xy[(\bigwedge_{s \in S} s(x,y)) \leftrightarrow (\prod S)(x,y)]$$

for each set $S$ of elements of $\mathcal{A}$ whose infimum $\prod S$ exists in $\mathcal{A}$.

The reader should check the next lemma. For the second part, see [HH97b, theorem 5].

Lemma 4 *Let $M$ be a relativised representation of the relation algebra $\mathcal{A}$. Then as boolean algebras, we have*

$$(\mathcal{A}, +, -, 0, 1) \cong (\{r^M : r \in \mathcal{A}\}, \cup, \setminus, \varnothing, 1^M),$$

*where, for $r \in \mathcal{A}$, $r^M$ denotes the interpretation of the binary relation symbol $r$ as a binary relation on $M$ (so $r^M \subseteq {}^2M$). Also, $1^M$ is a reflexive and symmetric relation on $M$.*

*$M$ is a complete relativised representation iff for every $x, y \in M$ with $M \models 1(x, y)$, there is an atom (minimal non-zero element) $\alpha \in \mathcal{A}$ such that $M \models \alpha(x, y)$.*

### 2.2 Flat relativised representations

Until the end of this section we fix $n$ with $3 \leq n < \omega$.

Definition 5 Let $M$ be a relativised representation of the relation algebra $\mathcal{A}$. A *clique* in $M$ is a subset $X \subseteq M$ such that $M \models 1(x, y)$ for all $x, y \in X$. We write $C^n(M)$ for the set $\{\bar{a} \in {}^nM : rng(\bar{a}) \text{ is a clique in } M\}$.

Definition 6 We consider the set $\mathcal{L}^n(\mathcal{A})$ of first-order formulas of $\mathcal{L}(\mathcal{A})$ that are written with the variables $x_0, \ldots, x_{n-1}$ only. Let $M$ be a structure for this language. We define the *clique-relativised semantics* $M \models_C \varphi(\bar{a})$, for $\varphi \in \mathcal{L}^n(\mathcal{A})$ and $\bar{a} \in C^n(M)$ as follows.

- If $\varphi$ is $r(x_i, x_j)$ for $r \in \mathcal{A}$ and $i, j < n$, then $M \models_C \varphi(\bar{a})$ iff $M \models r(a_i, a_j)$.
- If $\varphi$ is $x_i = x_j$ for some $i, j < n$, then $M \models_C \varphi(\bar{a})$ iff $M \models a_i = a_j$.
- $M \models_C \neg\varphi(\bar{a})$ iff $M \not\models_C \varphi(\bar{a})$.
- $M \models_C (\varphi \wedge \psi)(\bar{a})$ iff $M \models_C \varphi(\bar{a})$ and $M \models_C \psi(\bar{a})$, where $\varphi, \psi \in \mathcal{L}^n(\mathcal{A})$.
- For $i < n$, $M \models_C \exists x_i \varphi(\bar{a})$ iff $M \models_C \varphi(\bar{b})$ for some $\bar{b} \in C^n(M)$ with $\bar{b} \equiv_i \bar{a}$.

Notations here are as given in notation 2. We define the abbreviations $\vee, \rightarrow, \forall$ in the usual way.

Definition 7 Let $\mathcal{A}$ be a relation algebra. An *$n$-flat relativised representation of* $\mathcal{A}$ is a relativised representation $M$ of $\mathcal{A}$ with the additional property that for all $\varphi \in \mathcal{L}^n(\mathcal{A})$, all $\bar{a} \in C^n(M)$, and all $i, j < n$, we have

$$M \models_C (\exists x_i \exists x_j \varphi \leftrightarrow \exists x_j \exists x_i \varphi)(\bar{a}).$$

### 2.3 Properties of flat relativised representations

We establish two basic properties of any $n$-flat relativised representation $M$ of a relation algebra $\mathcal{A}$.

Lemma 8 *Let $i_0, \ldots, i_{k-1} < n$ for some $k < n$, and let $\varphi \in \mathcal{L}^n(\mathcal{A})$ and $\bar{a} \in C^n(M)$. Then $M \models_C (\exists x_{i_0} \ldots \exists x_{i_{k-1}} \varphi)(\bar{a})$ iff $M \models_C \varphi(\bar{b})$ for some $\bar{b} \in C^n(M)$ with $\bar{a} \equiv_{i_0, \ldots, i_{k-1}} \bar{b}$.*

Proof:

'$\Rightarrow$' is clear. We prove '$\Leftarrow$' by induction on $k$. If $k = 0$, it is trivial, and if $k = 1$, it holds by definition of $\models_C$. Let $k > 1$, and assume the result for smaller $k$. Since $k < n$, there is $j \in n \setminus \{i_0, \ldots, i_{k-1}\}$. Then $a_j = b_j$. Let $\bar{a}' \in C^n(M)$ be the result of replacing $a_{i_{k-1}}$ by $a_j$ in $\bar{a}$. Define $\bar{b}'$ similarly. Clearly, $\bar{a} \equiv_{i_{k-1}} \bar{a}' \equiv_{i_0,\ldots,i_{k-2}} \bar{b}' \equiv_{i_{k-1}} \bar{b}$. So if $M \models_C \varphi(\bar{b})$, then by three applications of the inductive hypothesis we obtain $M \models_C (\exists x_{i_{k-1}}\varphi)(\bar{b}')$, $M \models_C (\exists x_{i_0} \ldots \exists x_{i_{k-1}}\varphi)(\bar{a}')$, and $M \models_C (\exists x_{i_{k-1}}(\exists x_{i_0} \ldots \exists x_{i_{k-1}}\varphi))(\bar{a})$. Now $M$ is $n$-flat, so by the commutativity of existential quantifiers and a straightforward induction on $k$, we obtain $M \models_C (\exists x_{i_0} \ldots \exists x_{i_{k-1}}\varphi)(\bar{a})$, as required. □

Now we prove that free variables of $\mathcal{L}^n(\mathcal{A})$-formulas behave as we would hope. Cf. [Mad89, lemma 20]. Bear in mind that variables can be 're-used' in $n$-variable formulas, so that $x_0$ occurs both free and bound in $r(x_0, x_1) \wedge \exists x_0\, s(x_0, x_1)$, for example.

Lemma 9 *Let $\varphi \in \mathcal{L}^n(\mathcal{A})$ and let $x_i$ (for some $i < n$) be a variable that does not occur free in $\varphi$. Then $M \models_C (\varphi \leftrightarrow \exists x_i\varphi)(\bar{a})$ for all $\bar{a} \in C^n(M)$.*

Proof:

We show by induction on $\varphi$ that if $x_i$ is not free in $\varphi$, $\bar{a}, \bar{b} \in C^n(M)$, and $\bar{a} \equiv_i \bar{b}$, then $M \models_C \varphi(\bar{a})$ iff $M \models_C \varphi(\bar{b})$. If $\varphi$ is atomic, this is trivial, and the boolean cases are also straightforward. Assume the result for $\varphi$ and consider $\exists x_j\varphi$, assuming that $x_i$ is not free in $\exists x_j\varphi$. If $j = i$, the result follows from the fact that $\equiv_i$ is an equivalence relation on $C^n(M)$. So assume that $j \neq i$. We let $\bar{a} \equiv_i \bar{b}$ and $M \models_C \exists x_j\varphi(\bar{a})$, and check that $M \models_C \exists x_j\varphi(\bar{b})$, also. (The converse is similar.) Plainly, $M \models_C \exists x_i\exists x_j\varphi(\bar{b})$. By $n$-flatness, $M \models_C \exists x_j\exists x_i\varphi(\bar{b})$. So there are $\bar{c}, \bar{d} \in C^n(M)$ with $\bar{b} \equiv_j \bar{c} \equiv_i \bar{d}$ and $M \models_C \varphi(\bar{d})$. Now as $i \neq j$, $x_i$ is not free in $\varphi$. So by the inductive hypothesis, $M \models_C \varphi(\bar{c})$. Hence, $M \models_C \exists x_j\varphi(\bar{b})$, as required. □

## 2.4 From flat representations to RA-reducts

There is a well-known method of obtaining a relation-type algebra $\mathfrak{Ra}(\mathcal{C})$ from an $n$-dimensional cylindric algebra $\mathcal{C}$ (for any $n \geq 3$): $\mathfrak{Ra}(\mathcal{C})$ is constructed by taking the two-dimensional elements of $\mathcal{C}$ and using the spare dimensions to define converse and composition (see [HMT85, 5.3.7]). $\mathfrak{Ra}(\mathcal{C})$ is called the relation algebra reduct of $\mathcal{C}$.

More formally, this is done as follows. Recall that for $i, j < n$, the substitution operator $\mathsf{s}^i_j$ is defined by

$$\mathsf{s}^i_j x = \begin{cases} x, & \text{if } i = j; \\ \mathsf{c}_i(\mathsf{d}_{ij} \cdot x), & \text{otherwise.} \end{cases}$$

Definition 10 [HMT85, 5.3.7] Let $\mathcal{C}$ be any $n$-dimensional cylindric algebra. For $m \leq n$, the *neat $m$-reduct of* $\mathcal{C}$ (in symbols, $\mathfrak{Nr}_m\mathcal{C}$) is the $m$-dimensional cylindric algebra with domain $\{a \in \mathcal{C} : \mathsf{c}_j a = a \text{ for all } m \leq j < n\}$ and with operations $+, -, 0, 1, \mathsf{c}_j, \mathsf{d}_{jk}$ for $j, k < m$ induced from $\mathcal{C}$.

The *relation algebra reduct of* $\mathcal{C}$ — in symbols, $\mathfrak{Ra}(\mathcal{C})$ — is the algebra

$$\langle dom(\mathfrak{Nr}_2\mathcal{C}), +, -, 0, 1, 1', \smile, ; \rangle,$$

where

- $+, -, 0, 1$ are as in $\mathcal{C}$
- $1' = \mathsf{d}_{01}$ $(\in \mathfrak{Nr}_2\mathcal{C})$
- converse is defined by $\breve{r} = \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1 r$, for $r \in \mathfrak{Nr}_2\mathcal{C}$.
- composition is defined by $r_1 \,;\, r_2 = \mathsf{c}_2(\mathsf{s}_2^1 r_1 \cdot \mathsf{s}_2^0 r_2)$, for $r_1, r_2 \in \mathfrak{Nr}_2\mathcal{C}$.

$\mathfrak{Ra}(\mathcal{C})$ can be checked to be closed under these operations. For $n \geq 3$, it is a weakly associative algebra; when $n \geq 4$, it is actually a relation algebra [HMT85, 5.3.8]. For finite $n$, if $\mathcal{C}$ is atomic then so are the algebras $\mathfrak{Nr}_m\mathcal{C}$ $(m < n)$, and hence also $\mathfrak{Ra}\mathcal{C}$. The set $At\,\mathfrak{Nr}_m\mathcal{C}$ of atoms of $\mathfrak{Nr}_m\mathcal{C}$ is $\{\mathsf{c}_m\mathsf{c}_{m+1}\ldots\mathsf{c}_{n-1}x : x \text{ an atom of } \mathcal{C}\}$.

We generally identify notationally the algebras $\mathfrak{Nr}_m\mathcal{C}$, $\mathfrak{Ra}\mathcal{C}$ with their domains.

We now prove our first main result.

Theorem 11 *Let $\mathcal{A}$ be a relation algebra with an $n$-flat relativised representation. Then $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. Indeed, $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{G}_n)$, $\mathsf{G}_n$ being the cylindric relativised set algebras whose unit is closed under substitutions and permutations ('locally cubic').*

Proof:

Let $M$ be an $n$-flat relativised representation of $\mathcal{A}$. For $\varphi \in \mathcal{L}^n(\mathcal{A})$, write

$$\varphi^C = \{\bar{a} \in C^n(M) : M \models_C \varphi(\bar{a})\}.$$

Let $\mathcal{C}$ be the following $\mathsf{CA}_n$-type algebra (its signature is $\{+, -, 0, 1, \mathsf{d}_{ij}, \mathsf{c}_i : i, j < n\}$).

- The domain of $\mathcal{C}$ is the set $\{\varphi^C : \varphi \in \mathcal{L}^n(\mathcal{A})\}$ of all sets definable by $n$-variable formulas in the clique-relativised semantics.
- 0 is interpreted in $\mathcal{C}$ as $\emptyset = 0(x_0, x_1)^C$, and 1 as $C^n(M) = 1(x_0, x_1)^C$.
- $+$ and $-$ are interpreted in $\mathcal{C}$ as follows: $\varphi^C + \psi^C = \varphi^C \cup \psi^C = (\varphi \vee \psi)^C$, and $-(\varphi^C) = C^n(M) \setminus \varphi^C = (\neg\varphi)^C$ (this is plainly well-defined).
- $\mathsf{d}_{ij}$ is interpreted as $\{\bar{a} \in C^n(M) : a_i = a_j\} = (x_i = x_j)^C$.
- $\mathsf{c}_i$ is interpreted by: $\mathsf{c}_i(\varphi^C) = (\exists x_i \varphi)^C$ (this is well-defined).

$\mathcal{C}$ is a subalgebra of the cylindric relativised set algebra $(\mathsf{Crs}_n)$ with domain $\wp(C^n(M))$. The unit $C^n(M)$ of $\mathcal{C}$ is clearly closed under permutations and substitutions, and so $\mathcal{C} \in \mathsf{G}_n$. Also, $n$-flatness implies that $\mathcal{C} \models \forall x(\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x)$ for all $i, j < n$. Hence, $\mathcal{C} \in \mathsf{CA}_n$. We show that $\mathcal{A}$ embeds into $\mathfrak{Ra}\mathcal{C}$. For $r \in \mathcal{A}$, let

$$\iota(r) = r(x_0, x_1)^C \in \mathcal{C}.$$

By lemma 9, $\iota(r) \in \mathfrak{Ra}\mathcal{C}$ for all $r \in \mathcal{A}$. We check that $\iota$ is an algebra embedding of $\mathcal{A}$ into $\mathfrak{Ra}\mathcal{C}$.

$\iota$ plainly preserves the boolean operations, because $M$ is a relativised representation of $\mathcal{A}$. (For negation, note that if $\bar{a} \in C^n(M)$ then $M \models 1(a_0, a_1)$, so $M \models_C (\neg r(x_0, x_1) \leftrightarrow -r(x_0, x_1))(\bar{a})$ by the negation axiom of $S_{\mathcal{A}}$.) Now, to show that $\iota$ is one-one, we only need check that if $r \in \mathcal{A} \setminus \{0\}$ then $\iota(r) \neq 0$ in $\mathcal{C}$. But

$M \models S_{\mathcal{A}}$, so there are $a, b \in M$ with $M \models r(a, b)$. One may check that $\{a, b\}$ is a clique in $M$, so that $\bar{a} = (a, b, b, \ldots, b) \in C^n(M)$. Then $\bar{a} \in \iota(r)$, which is therefore non-empty, as required.

We check the relation algebra operations. By the identity axiom of $S_{\mathcal{A}}$, $M \models \forall xy(1'(x, y) \leftrightarrow x = y)$. So $\iota(1') = (x_0 = x_1)^C = \mathsf{d}_{01}$. For converse, let $r \in \mathcal{A}$: we require $\iota(\breve{r}) = \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\iota(r)$. By definition of $\mathcal{C}$,

$$\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\iota(r) = \Big(\exists x_2(x_2 = x_0 \wedge \exists x_0(x_0 = x_1 \wedge \exists x_1(x_1 = x_2 \wedge r(x_0, x_1))))\Big)^C.$$

It follows that for any $\bar{a} \in C^n(M)$, $\bar{a} \in \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\iota(r)$ iff $M \models r(a_1, a_0)$. By the converse axiom of $S_{\mathcal{A}}$, this is iff $M \models \breve{r}(a_0, a_1)$ — i.e., iff $\bar{a} \in \iota(\breve{r})$, as required.

Finally, we check composition. We require that $\iota(r\,;s) = \mathsf{c}_2(\mathsf{s}_2^1\iota(r) \cdot \mathsf{s}_2^0\iota(s))$ for all $r, s \in \mathcal{A}$. Fix such $r, s$, and let

$$\chi = \exists x_1(x_1 = x_2 \wedge r(x_0, x_1)) \wedge \exists x_0(x_0 = x_2 \wedge s(x_0, x_1)).$$

We require $[r\,;s](x_0, x_1)^C = (\exists x_2\chi)^C$.

First, let $\bar{a} \in (\exists x_2\chi)^C$. So there is $\bar{b} \in C^n(M)$ with $\bar{b} \equiv_2 \bar{a}$ and $M \models r(b_0, b_2) \wedge s(b_2, b_1)$. Because $M \models 1(a_0, a_1)$, it follows by the composition axiom in $S_{\mathcal{A}}$ that $M \models [r\,;s](a_0, a_1)$. So $\bar{a} \in [r\,;s](x_0, x_1)^C$.

Conversely, let $\bar{a} \in [r\,;s](x_0, x_1)^C$. The composition axiom of $S_{\mathcal{A}}$ yields $b \in M$ with $M \models r(a_0, b) \wedge s(b, a_1)$. Other $S_{\mathcal{A}}$-axioms ensure that $\{a_0, a_1, b\}$ is a clique in $M$, so that $\bar{b} = (a_0, a_1, b, b, \ldots, b) \in C^n(M)$. Now, $\bar{b} \equiv_{23\cdots(n-1)} \bar{a}$ and $\bar{b} \in \chi^C$. By lemma 8, $\bar{a} \in (\exists x_3 \ldots \exists x_{n-1}\exists x_2\chi)^C$. By repeated use of lemma 9, $\bar{a} \in (\exists x_2\chi)^C$, as required. □

This proves (3) ⇒ (1) of theorem 1.

# 3 Hyper-networks and hyper-bases

In this section, we will define the terms of part 2 of theorem 1, and prove (1) ⇒ 2) of the theorem. This is a more substantial matter than before. We fix finite $n \geq 4$, an atomic relation algebra $\mathcal{A}$, and a non-empty set $\Lambda$ (of 'labels') disjoint from $At(\mathcal{A})$.

Definition 12 An *$n$-dimensional $\Lambda$-hyper-network over* $\mathcal{A}$ is a map $N : {}^{\leq n}n \to \Lambda \cup At(\mathcal{A})$, such that $N(\bar{a}) \in At\mathcal{A}$ if and only if $|\bar{a}| = 2$, with the following properties.

1. $N(i, i) \leq 1'$ for each $i < n$.
2. For all $i, j, k < n$, $N(i, j)\,; N(j, k) \geq N(i, k)$.
3. For every $\bar{a}, \bar{b} \in {}^{\leq n}n$ of equal length, if $N(a_i, b_i) \leq 1'$ for each $i < |\bar{a}|$, then $N(\bar{a}) = N(\bar{b})$.

As notation, for $\Lambda$-hyper-networks $N, M$ and $i, j < n$, we write $N \equiv_i M$ iff $N(\bar{a}) = M(\bar{a})$ for all $\bar{a} \in {}^{\leq n}(n \setminus \{i\})$, and similarly, $N \equiv_{ij} M$ iff $N(\bar{a}) = M(\bar{a})$ for all $\bar{a} \in {}^{\leq n}(n \setminus \{i, j\})$.

The important labels in hyper-networks are the labels of $n-2$-tuples: see in particular lemma 32. We let hyper-networks have labels on longer and shorter sequences in our definition purely for convenience; these labels have less significance. See, for example, definition 28 where sequences of length $n-1$ and $n$ are labelled by a constant — such labelling carries no information. The following is a basic property of hyper-networks.

Lemma 13 *Let $N$ be an $n$-dimensional $\Lambda$-hyper-network over $\mathcal{A}$, and let $i, j < n$. Then $N(i,j) = N(j,i)^\smile$.*

Proof:

By property 2, $N(i,j) \,;N(j,i) \geq N(i,i)$, so by property 1, $(N(i,j) \,;N(j,i)) \cdot 1' \neq 0$. By the Peircean law in $\mathcal{A}$, $N(i,j) \cdot N(j,i)^\smile \neq 0$. As these are atoms, we obtain $N(i,j) = N(j,i)^\smile$, as required. $\square$

Now we can define the terms used in part 2 of theorem 1.

Definition 14 An *$n$-dimensional $\Lambda$-hyper-basis for $\mathcal{A}$* is a set $\mathcal{H}$ of $n$-dimensional $\Lambda$-hyper-networks over $\mathcal{A}$ satisfying:

1. If $r \in \mathcal{A}$ is non-zero, then there is $N \in \mathcal{H}$ with $N(0,1) \leq r$.
2. If $N \in \mathcal{H}$, $i,j < n$, $k \in n \setminus \{i,j\}$, and $r, s \in \mathcal{A}$ satisfy $N(i,j) \leq r\,; s$, then there is $M \in \mathcal{H}$ with $M \equiv_k N$, $M(i,k) \leq r$, and $M(k,j) \leq s$.
3. If $N, M \in \mathcal{H}$, $i, j < n$, and $N \equiv_{ij} M$, then there is $P \in \mathcal{H}$ with $N \equiv_i P \equiv_j M$.

We drop terms such as '$n$-dimensional', '$\Lambda$', and 'over $\mathcal{A}$' when they are clear from the context. Also, 'an $n$-dimensional hyper-basis' will sometimes mean 'an $n$-dimensional $\Lambda$-hyper-basis for some $\Lambda$'.

In the case where $|\Lambda| = 1$, a $\Lambda$-hyper-basis is essentially a 'cylindric basis' in the sense of Maddux. The conditions above amount to those of a relational basis plus the 'amalgamation' condition of cylindric bases. See [Mad83, Mad89]. We will discuss this further in section 8.

### 3.1 Substitutions in cylindric algebras

We aim to prove that the canonical embedding algebra of any subalgebra of the relation algebra reduct of an $n$-dimensional cylindric algebra has an $n$-dimensional hyper-basis. In this section, we prove some necessary preliminary results about substitutions in cylindric algebras. $n \geq 4$ remains fixed. Recall again that the substitution operator $\mathsf{s}^i_j$ is defined by

$$\mathsf{s}^i_j x = \begin{cases} x, & \text{if } i = j; \\ \mathsf{c}_i(\mathsf{d}_{ij} \cdot x), & \text{otherwise,} \end{cases}$$

for $i, j < n$. As is standard, e.g., in [HMT71], the map $[i/j] : n \to n$ is given by:

$$[i/j](k) = \begin{cases} j, & \text{if } k = i; \\ k, & \text{otherwise.} \end{cases}$$

Of course, this definition depends implicitly on $n$. We write maps on the left, and $\circ$ denotes map composition, so that for example, $([1/2] \circ [2/3])(1) = [1/2]([2/3](1)) = 2$.

#### 3.1.1 s-c-words

DEFINITION 15

1. An *s-word* is a finite string of substitutions ($\mathsf{s}^i_j$), a *c-word* is a finite string of cylindrifications ($\mathsf{c}_k$), and an *s-c-word* is a finite string of substitutions and cylindrifications, all of the signature of $\mathsf{CA}_n$.

2. If $u, w$ are s-c-words, we write simply $uw$ for their concatenation. The length (number of symbols) of $w$ is written $|w|$.

3. With each s-c-word $w$, we associate a partial map $\widehat{w} : n \to n$ by induction on $|w|$:

   - If $w$ is the empty string then $\widehat{w} = Id_n$, the identity map on $n$.
   - $\widehat{w\mathsf{s}^i_j} = \widehat{w} \circ [i/j]$, for $i, j < n$.
   - $\widehat{w\mathsf{c}_i} = \widehat{w}|_{n \setminus \{i\}} = \widehat{w} \circ Id_{n \setminus \{i\}}$, for $i < n$.

Clearly, for any variable $x$ and s-c-word $w$, $wx$ is a term of the signature of $n$-dimensional cylindric algebras. And if $x \in \mathcal{C} \in \mathsf{CA}_n$, then $wx \in \mathcal{C}$.

For familiarisation, the following useful lemma may help.

LEMMA 16 *Let $u, w$ be s-c-words.*

*1.* $\widehat{uw} = \widehat{u} \circ \widehat{w}$.

*2.* $\mathsf{CA}_n \models w\mathsf{d}_{ij} = \mathsf{d}_{\widehat{w}(i)\widehat{w}(j)}$, *for all* $i, j \in dom(\widehat{w})$.

PROOF:

1. We prove $\widehat{uw} = \widehat{u} \circ \widehat{w}$ for all $u$, by a trivial induction on the length of $w$. If this is zero, then $\widehat{w} = Id_n$ and we are done. Assume the result for $w$, and let $i, j < n$. Then $\widehat{uw\mathsf{s}^i_j} = \widehat{uw} \circ [i/j] = (\widehat{u} \circ \widehat{w}) \circ [i/j]$ (by the inductive hypothesis), $= \widehat{u} \circ (\widehat{w} \circ [i/j]) = \widehat{u} \circ \widehat{w\mathsf{s}^i_j}$, as required. The proof for $w\mathsf{c}_i$ is similar.

2. The proof is by induction on the length of $w$. If this is zero, there is nothing to prove. Assume the result for $w$. We first prove it for $u = w\mathsf{s}^k_l$. Let $i, j \in dom(\widehat{u})$. If $i = j$, then in any $n$-dimensional cylindric algebra, $\mathsf{d}_{ij} = 1 = \mathsf{d}_{\widehat{u}(i),\widehat{u}(j)}$. So suppose that $i \neq j$. Now, by [HMT71, theorem 1.5.4(i)] we have $w\mathsf{s}^k_l\mathsf{d}_{ij} = w\mathsf{d}_{[k/l](i),[k/l](j)}$; by assumption, $[k/l](i), [k/l](j) \in dom(\widehat{w})$, so by the induction hypothesis this is $\mathsf{d}_{\widehat{w}([k/l](i)),\widehat{w}([k/l](j))} = \mathsf{d}_{\widehat{u}(i),\widehat{u}(j)}$.

   Next let $u = w\mathsf{c}_k$. Assume that $i, j \in dom(\widehat{u})$ — that is, $i, j \in dom(\widehat{w})$ and $i, j \neq k$. Then $u\mathsf{d}_{ij} = w\mathsf{d}_{ij}$, which by the inductive hypothesis is $\mathsf{d}_{\widehat{w}(i),\widehat{w}(j)}$; and this is clearly equal to $\mathsf{d}_{\widehat{u}(i),\widehat{u}(j)}$, as required.

□

### 3.1.2 Adapting known results on substitutions

We will need some corollaries of the following known results.

FACT 17 *Let $\alpha \geq 2$ be an ordinal.*

1. *If $\theta : \alpha \to \alpha$ is not a permutation of $\alpha$, and $\{i < \alpha : \theta(i) \neq i\}$ is finite, then $\theta$ is either of the form $[i/j]$ (for some $i, j < \alpha$) or a composition of maps of this form.*

   This is a not-too-difficult exercise, or it can be derived from [Jón62]. [How78, theorem 1] proves it for finite $\alpha$, and [Tho93, corollary 1.2] for arbitrary $\alpha$.

2. [Tho93, theorem 3.6] *Let $q, r < \omega$. Assume that $i_1, j_1, \ldots, i_q, j_q, k_1, m_1, \ldots, k_r, m_r < \alpha$ are such that $[i_1/j_1] \circ \cdots \circ [i_q/j_q] = [k_1/m_1] \circ \cdots \circ [k_r/m_r] = f \in {}^\alpha\alpha$ and $|\alpha \setminus rng(f)| \geq 2$. Then*
$$\mathsf{CA}_\alpha \models \forall x(\mathsf{s}^{i_1}_{j_1} \ldots \mathsf{s}^{i_q}_{j_q}(x) = \mathsf{s}^{k_1}_{m_1} \ldots \mathsf{s}^{k_r}_{m_r}(x)).$$

We need to generalise these facts to partial maps and s-c-words. We are only interested in the case $\alpha = n$. Similar results can be obtained using theorem 3.2.52 of [HMT85], but because we deal with partial maps $\theta : n \to n$ whose range is not necessarily contained in $n - 2$, it is more convenient to use fact 17 as a starting point.

DEFINITION 18 Two s-c-words $u, w$ are said to be *congruent* if $\widehat{u} = \widehat{w}$ and $\mathsf{CA}_n \models \forall x(ux = wx)$. We write $u \simeq w$ in this case.

LEMMA 19 *If $v, v'$ are s-c-words and $v \simeq v'$ then $uvw \simeq uv'w$ for all s-c-words $u, w$.*

PROOF:

Assume that $v \simeq v'$. Then $\mathsf{CA}_n \models \forall x(v(wx) = v'(wx))$, so $\mathsf{CA}_n \models \forall x(uvwx = uv'wx)$. By lemma 16.1 and associativity of composition of partial maps, $\widehat{uvw} = \widehat{u} \circ \widehat{v} \circ \widehat{w} = \widehat{u} \circ \widehat{v'} \circ \widehat{w} = \widehat{uv'w}$. So $uvw \simeq uv'w$. □

DEFINITION 20 An s-c-word $w$ is said to be *modest* if whenever $u, v$ are s-c-words, $i < n$, and $w = u\mathsf{c}_i v$, then $|\widehat{v}^{-1}(i)| \leq 1$. Here, $\widehat{v}^{-1}(i) = \{j \in dom(\widehat{v}) : \widehat{v}(j) = i\}$.

EXAMPLES 21 The following are easily checked.

1. $\mathsf{c}_0\mathsf{s}^2_0\mathsf{s}^0_1\mathsf{c}_0\mathsf{s}^1_2\mathsf{c}_2$ is a modest s-c-word.
2. The word $\mathsf{c}_0\mathsf{s}^1_0$ is not modest, as $\widehat{\mathsf{s}^1_0}^{-1}(0) = \{0, 1\}$.
3. If $u$ is an s-word and $v$ a c-word, then $uv$ is modest.
4. More generally, if $u, v$ are modest s-c-words and $\widehat{v}$ is one-one, then $uv$ is also modest.
5. If $u$ is a modest s-c-word, $i < n$, and $i \notin rng(\widehat{u})$, then $\mathsf{c}_i u$ is modest.
6. More generally, if $u, v$ are modest s-c-words and $rng(\widehat{v}) \subseteq dom(\widehat{u})$, then $uv$ is modest.

We will prove:

THEOREM 22

1. *For any partial map $\theta : n \to n$ with $|rng(\theta)| \leq n-1$, there exists a modest s-c-word $w$ with $\widehat{w} = \theta$.*

2. *Let $u, w$ be modest s-c-words with $|rng(\widehat{u})|, |rng(\widehat{w})| \leq n-2$. If $\widehat{u} = \widehat{w}$, then $\mathsf{CA}_n \models \forall x(ux = wx)$, so that $u \simeq w$.*

Part 2 can fail if the words are not modest. For example, $\widehat{\mathsf{c}_0\mathsf{s}_0^1} = \widehat{\mathsf{c}_0\mathsf{c}_1}$, yet $\mathsf{c}_0\mathsf{s}_0^1(-\mathsf{d}_{01}) = 0$ and $\mathsf{c}_0\mathsf{c}_1(-\mathsf{d}_{01}) = -\mathsf{d}_{01} \neq 0$, in general.

PROOF:

The proof of the first part is straightforward. If $\theta$ is the empty map, it is trivial — let $w = \mathsf{c}_0\mathsf{c}_1 \ldots \mathsf{c}_{n-1}$. Assume not; let $n \setminus dom(\theta) = \{i_0, \ldots, i_{k-1}\}$, take $j \in rng(\theta)$, and consider the total map $\theta^+ = \theta \cup \{(i_l, j) : l < k\} : n \to n$. Now $rng(\theta^+) = rng(\theta)$, so by fact 17.1, there is an s-word $u$ with $\widehat{u} = \theta^+$. Then $w = u\mathsf{c}_{i_0} \ldots \mathsf{c}_{i_{k-1}}$ is modest (example 21.3) and $\widehat{w} = \theta^+|_{dom(\theta)} = \theta$.

To prove the second part, we need to be able to move cylindrifications rightwards within s-c-words. Lemma 23 below shows that we can do this. Cf. [HMT85, theorem 3.2.51(vi,vii)].

LEMMA 23 *Let $\mathsf{c}_i w$ be a modest s-c-word, for some s-c-word $w$ and some $i < n$. Then*

$$\mathsf{c}_i w \simeq \begin{cases} w, & \text{if } i \notin rng(\widehat{w}); \\ w\mathsf{c}_l, & \text{if } l < n \text{ and } \widehat{w}(l) = i. \end{cases}$$

*(The 'l' in the second case is unique, as $\mathsf{c}_i w$ is modest. Note that $w$ and $w\mathsf{c}_l$ above are modest.)*

PROOF:

The proof is by induction on $|w|$. If this is zero, there is nothing to prove. Assume the result for $u$. We prove it for modest words $w = \mathsf{c}_j u$ and $w = \mathsf{s}_k^j u$. We use lemma 19 freely in the proof.

First, consider the case of $w = \mathsf{c}_j u$. Assume that $\mathsf{c}_i\mathsf{c}_j u$ is modest. If $i = j$, then $i \notin rng(\widehat{\mathsf{c}_i w})$ and plainly, $\mathsf{c}_i\mathsf{c}_i u \simeq \mathsf{c}_i u$, which is as required. Assume that $i \neq j$. Then $\mathsf{c}_i\mathsf{c}_j u \simeq \mathsf{c}_j\mathsf{c}_i u$. Clearly, $\mathsf{c}_i u$ is modest, so by the inductive hypothesis, $\mathsf{c}_i u \simeq u$ if $i \notin rng(\widehat{u})$, while $\mathsf{c}_i u \simeq u\mathsf{c}_l$ if $\widehat{u}(l) = i$. But as $j \neq i$, for any $l < n$ we have $\widehat{u}(l) = i$ iff $\widehat{\mathsf{c}_j u}(l) = i$. So

$$\mathsf{c}_i w = \mathsf{c}_i\mathsf{c}_j u \simeq \mathsf{c}_j\mathsf{c}_i u \simeq \begin{cases} \mathsf{c}_j u = w, & \text{if } i \notin rng(\widehat{\mathsf{c}_j u}); \\ \mathsf{c}_j u\mathsf{c}_l = w\mathsf{c}_l, & \text{if } \widehat{\mathsf{c}_j u}(l) = i. \end{cases}$$

We pass to the case $w = \mathsf{s}_k^j u$; we can assume that $j \neq k$. Consider $\mathsf{c}_i\mathsf{s}_k^j u$. Here, there are three cases.

**Case** $i \neq j, k$. By [HMT71, theorem 1.5.8(ii)], $\mathsf{c}_i\mathsf{s}^j_k x = \mathsf{s}^j_k\mathsf{c}_i x$ holds in $\mathsf{CA}_n$; and clearly, $\widehat{\mathsf{c}_i\mathsf{s}^j_k} = \widehat{\mathsf{s}^j_k\mathsf{c}_i}$. So $\mathsf{c}_i\mathsf{s}^j_k \simeq \mathsf{s}^j_k\mathsf{c}_i$. Now $\mathsf{c}_i u$ is modest, and by the case assumption, for any $l < n$ we have $i = \widehat{\mathsf{s}^j_k u}(l)$ iff $i = \widehat{u}(l)$. So by the inductive hypothesis,

$$\mathsf{c}_i w = \mathsf{c}_i\mathsf{s}^j_k u \simeq \mathsf{s}^j_k\mathsf{c}_i u \simeq \begin{cases} \mathsf{s}^j_k u = w, & \text{if } i \notin rng(\widehat{\mathsf{s}^j_k u}); \\ \mathsf{s}^j_k u\mathsf{c}_l = w\mathsf{c}_l, & \text{if } \widehat{\mathsf{s}^j_k u}(l) = i. \end{cases}$$

**Case** $i = j$. By [HMT71, theorem 1.5.9(ii)], $\mathsf{c}_j\mathsf{s}^j_k x = \mathsf{s}^j_k x$ holds in $\mathsf{CA}_n$; also, $\widehat{\mathsf{c}_j\mathsf{s}^j_k} = \widehat{\mathsf{s}^j_k}$. So $\mathsf{c}_j\mathsf{s}^j_k \simeq \mathsf{s}^j_k$, whence $\mathsf{c}_j\mathsf{s}^j_k u \simeq \mathsf{s}^j_k u$. Also, $i \notin rng(\widehat{\mathsf{s}^j_k u})$. We are done.

**Case** $i = k$. So we are considering $\mathsf{c}_k\mathsf{s}^j_k u$. Since this is assumed modest, either $j$ or $k$ (or both) is not in $rng(\widehat{u})$. Inductively, therefore, $u \simeq \mathsf{c}_j u$ or $u \simeq \mathsf{c}_k u$, so that $\mathsf{c}_k\mathsf{s}^j_k u \simeq \mathsf{c}_k\mathsf{s}^j_k\mathsf{c}_j u$ or $\mathsf{c}_k\mathsf{s}^j_k u \simeq \mathsf{c}_k\mathsf{s}^j_k\mathsf{c}_k u$. Now by [HMT71, theorems 1.5.8(i), 1.5.9(i)], $\mathsf{s}^j_k\mathsf{c}_j x = \mathsf{c}_j x$ and $\mathsf{c}_k\mathsf{s}^j_k x = \mathsf{c}_j\mathsf{s}^k_j x$ are valid in $\mathsf{CA}_n$. Of course, $\widehat{\mathsf{s}^j_k\mathsf{c}_j} = \widehat{\mathsf{c}_j}$ and $\widehat{\mathsf{c}_k\mathsf{s}^j_k} = \widehat{\mathsf{c}_j\mathsf{s}^k_j}$. So $\mathsf{s}^j_k\mathsf{c}_j \simeq \mathsf{c}_j$ and $\mathsf{c}_k\mathsf{s}^j_k \simeq \mathsf{c}_j\mathsf{s}^k_j$. Hence, $\mathsf{c}_k\mathsf{s}^j_k\mathsf{c}_j \simeq \mathsf{c}_k\mathsf{c}_j \simeq \mathsf{c}_j\mathsf{c}_k \simeq \mathsf{s}^j_k\mathsf{c}_j\mathsf{c}_k$ and $\mathsf{c}_k\mathsf{s}^j_k\mathsf{c}_k \simeq \mathsf{c}_j\mathsf{s}^k_j\mathsf{c}_k \simeq \mathsf{c}_j\mathsf{c}_k \simeq \mathsf{s}^j_k\mathsf{c}_j\mathsf{c}_k$. We conclude that our $\mathsf{c}_k\mathsf{s}^j_k u$ is congruent to $\mathsf{s}^j_k\mathsf{c}_j\mathsf{c}_k u$.

Now $k \notin rng(\widehat{\mathsf{s}^j_k u})$ iff $j, k \notin rng(\widehat{u})$, and $k = \widehat{\mathsf{s}^j_k u}(l)$ iff $\widehat{u}(l) \in \{j, k\}$. The above and two applications of the inductive hypothesis now yield $\mathsf{c}_j\mathsf{c}_k u \simeq u$ in the first case, and $\mathsf{c}_j\mathsf{c}_k u \simeq u\mathsf{c}_l$, in the second. Since $\mathsf{c}_k\mathsf{s}^j_k u \simeq \mathsf{s}^j_k\mathsf{c}_j\mathsf{c}_k u$, the result follows.

□

Definition 24 An s-c-word $w$ is said to be in *normal form* if it has the form $uv$ for some s-word $u$ and some c-word $v$.

By example 21.3, any s-c-word in normal form is modest. Conversely, repeated use of the preceding lemma allows us to push all cylindrifications in a modest s-c-word rightwards until they either disappear or emerge on the right-hand side, all the time preserving modesty. Thus, any modest s-c-word is congruent to one in normal form. Note that if $w = uv$ is in normal form, with $u$ an s-word and $v$ a c-word, then $dom(\widehat{w}) = dom(\widehat{v}) = \{i < n : \mathsf{c}_i \text{ does not occur in } v\}$.

We may now prove the second part of theorem 22. We have to show that if $u, v$ are modest s-c-words with $\widehat{u} = \widehat{v} = \theta$, say, and $|rng(\theta)| \leq n-2$, then $u$ and $v$ are congruent. As outlined above, we may suppose that they are in normal form; and since $\widehat{u}, \widehat{v}$ have the same domain, we can suppose $u = u'w$ and $v = v'w$ for some s-words $u', v'$ and some fixed c-word $w$ (with $dom(\widehat{w}) = dom(\theta)$).

Let $n \setminus dom(\theta) = \{i_0, \ldots, i_{k-1}\}$. Choose $j \in dom(\theta)$, if $\theta$ is non-empty; otherwise, choose $j < n$ arbitrarily. Let $t = \mathsf{s}^{i_0}_j \ldots \mathsf{s}^{i_{k-1}}_j$. Then clearly, $\widehat{u't} = \widehat{v't} = \theta^+$, say. $\theta^+$ is a total map $: n \to n$, and $|rng(\theta^+)| \leq n-2$. So by fact 17.1, the s-words

$u't$ and $v't$ are congruent. By [HMT71, theorem 1.5.8(i)] (to wit, $\mathsf{s}^i_j\mathsf{c}_i x = \mathsf{c}_i x$) and induction on $k$, $tw$ and $w$ are also congruent. So

$$u = u'w \simeq u'tw \simeq v'tw \simeq v'w = v.$$

This completes the proof of theorem 22. $\square$

We will need the following corollary.

Corollary 25 *Let $w$ be a modest s-c-word, let $m \leq n$, and suppose that $m \subseteq dom(\widehat{w})$. Let $\mathcal{C} \in \mathsf{CA}_n$. Then the map $x \mapsto wx$, for $x \in \mathfrak{Nr}_m\mathcal{C}$, is a homomorphism from the boolean reduct of $\mathfrak{Nr}_m\mathcal{C}$ into the boolean reduct of $\mathcal{C}$.*

Proof:

Let $x, y \in \mathfrak{Nr}_m\mathcal{C}$. Given that substitutions and cylindrifications are additive operators on cylindric algebras and preserve 0 and 1 (for substitutions, see [HMT71, theorem 1.5.3]), a trivial induction on the length of $w$ shows that $w0 = 0$, $w1 = 1$, and $w(x + y) = wx + wy$ for all $x, y \in \mathcal{C}$. We check that $w(-x) = -wx$, for $x \in \mathfrak{Nr}_m\mathcal{C}$. We may suppose that $w$ is in normal form, so has the form $uv$ for an s-word $u$ and a c-word $v$. If $\mathsf{c}_i$ occurs in $v$, then $i \notin dom(\widehat{w})$, so $i \geq m$. As $x, -x \in \mathfrak{Nr}_m\mathcal{C}$, we have $\mathsf{c}_i x = x$ and similarly for $-x$. Thus, we obtain $wx = ux$ and $w(-x) = u(-x)$. But by [HMT71, theorem 1.5.3(ii)], $\mathsf{s}^i_j(-x) = -\mathsf{s}^i_j x$ for any $i, j < n$ and any $x \in \mathcal{C}$. So by induction on $|u|$, we obtain

$$w(-x) = u(-x) = -ux = -wx,$$

as required. $\square$

### 3.2 A hyper-basis from a cylindric algebra

Now fix finite $n \geq 5$. (All but one of our results go through unchanged if $n = 4$; and even the one (lemma 32) that requires $n \geq 5$ can be generalised to cover the case $n = 4$ at the cost of complicating the definition of hyper-basis.) We will now prove (1) $\Rightarrow$ (2) of theorem 1: that the canonical embedding algebra $\mathcal{A}^+$ of any $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ has an $n$-dimensional hyper-basis. We assume familiarity with canonical embedding algebras of relation algebras and cylindric algebras; cf. [JT51]. In particular, we identify elements of $\mathcal{A}$ with elements of $\mathcal{A}^+$ in the natural way, so that $\mathcal{A} \subseteq \mathcal{A}^+$, and similarly for cylindric algebras.

So suppose that $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. So there is $\mathcal{B} \in \mathsf{CA}_n$ with $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{B}$. Let $\mathcal{C} = \mathcal{B}^+$, the canonical embedding algebra of $\mathcal{B}$. Then $\mathcal{C}$ is atomic, and $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{C}$. Most of the time we will work in $\mathcal{C}$. We generally write $r, s, \ldots$ for elements of $\mathcal{A}$, $f, g, \ldots$ for filters on $\mathcal{A}$, and $x, y, \ldots$ for atoms of $\mathcal{C}$.

Notation 26 If $\bar{a} \in {}^{<n-1}n$, we write $\mathsf{s}_{\bar{a}}$ for an arbitrarily-chosen modest s-c-word $w$ with $\widehat{w} = \bar{a}$. Such a word exists by theorem 22.1; any two such are congruent (theorem 22.2), so $\mathsf{s}_{\bar{a}}x$ for $x \in \mathcal{C}$ is independent of the choice of $\mathsf{s}_{\bar{a}}$. As a slight abbreviation of this, we may write $\mathsf{s}_{a_0a_1\ldots a_{l-1}}$ for $\mathsf{s}_{\bar{a}}$, where $\bar{a} = (a_0, \ldots, a_{l-1})$. In this way, we can write $\mathsf{s}_{ijk}$ instead of, say, $\mathsf{s}_{(i,j,k)}$. It will help to remember that $\widehat{\mathsf{s}_{ijk}}$ is the partial map from $n$ to $n$ that takes 0 to $i$, 1 to $j$, 2 to $k$, and is undefined on larger numbers.

The definition of relation algebra composition in $\mathfrak{Ra}\mathcal{C}$ is in terms of the indices 0, 1, and 2. These indices can be 'moved', using substitutions.

LEMMA 27 *Let $i, j, k < n$ with $k \neq i, j$. Then $\mathsf{s}_{ij}(r\,;s) = \mathsf{c}_k(\mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s)$ for all $r, s \in \mathcal{A}$.*

PROOF:

First, as $k \neq i, j$, the s-c-word $\mathsf{c}_k\mathsf{s}_{ijk}$ is modest, $\widehat{\mathsf{c}_k\mathsf{s}_{ijk}} = \widehat{\mathsf{s}_{ij}\mathsf{c}_2}$, and clearly this map has range of size at most $n-2$. As $\mathsf{s}_{ij}\mathsf{c}_2$ is also modest, we obtain $\mathsf{c}_k\mathsf{s}_{ijk} \simeq \mathsf{s}_{ij}\mathsf{c}_2$ by theorem 22.2. Similarly, $\mathsf{s}_{ik} \simeq \mathsf{s}_{ijk}\mathsf{s}^1_2\mathsf{c}_2$ and $\mathsf{s}_{kj} \simeq \mathsf{s}_{ijk}\mathsf{s}^0_2\mathsf{c}_2$.

Second, if $3 \leq i < n$ then by [HMT71, theorem 1.5.8(ii)], $\mathsf{c}_i\mathsf{s}^1_2 r = \mathsf{s}^1_2\mathsf{c}_i r = \mathsf{s}^1_2 r$ (as $r \in \mathfrak{Nr}_2\mathcal{C}$), and similarly, $\mathsf{c}_i\mathsf{s}^0_2 s = \mathsf{s}^0_2\mathsf{c}_i s = \mathsf{s}^0_2 s$. So $\mathsf{s}^1_2 r, \mathsf{s}^0_2 s \in \mathfrak{Nr}_3\mathcal{C}$, and $3 \subseteq dom(\widehat{\mathsf{s}_{ijk}})$.

Now,

$$\begin{array}{rcll} \mathsf{s}_{ij}(r\,;s) & = & \mathsf{s}_{ij}\mathsf{c}_2(\mathsf{s}^1_2 r \cdot \mathsf{s}^0_2 s) & \text{by definition of composition} \\ & = & \mathsf{c}_k\mathsf{s}_{ijk}(\mathsf{s}^1_2 r \cdot \mathsf{s}^0_2 s) & \text{as } \mathsf{c}_k\mathsf{s}_{ijk} \simeq \mathsf{s}_{ij}\mathsf{c}_2 \\ & = & \mathsf{c}_k(\mathsf{s}_{ijk}\mathsf{s}^1_2 r \cdot \mathsf{s}_{ijk}\mathsf{s}^0_2 s) & \text{by corollary 25, as } \mathsf{s}^1_2 r, \mathsf{s}^0_2 s \in \mathfrak{Nr}_3\mathcal{C} \\ & = & \mathsf{c}_k(\mathsf{s}_{ijk}\mathsf{s}^1_2\mathsf{c}_2 r \cdot \mathsf{s}_{ijk}\mathsf{s}^0_2\mathsf{c}_2 s) & \text{as } r, s \in \mathfrak{Nr}_2\mathcal{C} \\ & = & \mathsf{c}_k(\mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s) & \text{as } \mathsf{s}_{ijk}\mathsf{s}^1_2\mathsf{c}_2 \simeq \mathsf{s}_{ik} \text{ and } \mathsf{s}_{ijk}\mathsf{s}^0_2\mathsf{c}_2 \simeq \mathsf{s}_{kj}, \end{array}$$

as required. □

For $m < n-1$ let $\Lambda_m$ be the set of ultrafilters of $\mathfrak{Nr}_m\mathcal{C}$, or up to a natural identification the set of atoms of its canonical embedding algebra $(\mathfrak{Nr}_m\mathcal{C})^+$, and let $\lambda$ be any fixed element disjoint from $\bigcup_{m<n-1}\Lambda_m$. Let $\Lambda = \bigcup_{m<n-1}\Lambda_m$.

DEFINITION 28 For each atom $x \in \mathcal{C}$, we define an $n$-dimensional $\Lambda$-hyper-network $N_x$ over $\mathcal{A}^+$, as follows. For $\bar{a} \in {}^{\leq n}n$, we let:

$$N_x(\bar{a}) = \begin{cases} \{r \in \mathcal{A} : x \leq \mathsf{s}_{\bar{a}}r\}, & \text{if } |\bar{a}| = 2; \\ \lambda, & \text{if } |\bar{a}| \in \{n-1, n\}; \\ \{r \in \mathfrak{Nr}_{|\bar{a}|}\mathcal{C} : x \leq \mathsf{s}_{\bar{a}}r\}, & \text{otherwise (i.e. } 2 \neq |\bar{a}| < n-1). \end{cases}$$

LEMMA 29 *Let $x \in At\mathcal{C}$ and $\bar{a} \in {}^{<n-1}n$.*

1. *If $|\bar{a}| = 2$, $N_x(\bar{a})$ is an ultrafilter of $\mathcal{A}$, and so an atom of $\mathcal{A}^+$.*
2. *If $|\bar{a}| \neq 2$, $N_x(\bar{a})$ is an ultrafilter of $\mathfrak{Nr}_{|\bar{a}|}\mathcal{C}$, and so an atom of $(\mathfrak{Nr}_{|\bar{a}|}\mathcal{C})^+$.*

In the second case, it can be shown that $N_x(\bar{a})$ is actually a principal ultrafilter.

PROOF:

Assume that $|\bar{a}| = m \neq 2$. Trivially, $m \subseteq dom(\widehat{\mathsf{s}_{\bar{a}}})$. So by corollary 25, the map $\psi : \mathfrak{Nr}_m\mathcal{C} \to \mathcal{C}$ given by $\psi(r) = \mathsf{s}_{\bar{a}}r$ is a boolean homomorphism. Let $f = \{y \in \mathcal{C} : y \geq x\}$, an ultrafilter of $\mathcal{C}$. Then $N_x(\bar{a})$ is the inverse image under $\psi$ of $f$, and hence is an ultrafilter of $\mathfrak{Nr}_m(\mathcal{C})$. This proves the second part. The proof of the first part is similar, using $\psi : \mathcal{A} \to \mathcal{C}$ given by $\psi(r) = \mathsf{s}_{\bar{a}}r$; this is a boolean homomorphism as $\mathcal{A} \subseteq \mathfrak{Nr}_2\mathcal{C}$. □

Lemma 30 *Let $x \in At\mathcal{C}$. Then $N_x$ is an $n$-dimensional $\Lambda$-hyper-network over $\mathcal{A}^+$.*

Proof:

Bear in mind that $n \geq 4$. We first check that $\mathcal{A}^+ \models N_x(i,i) \leq 1'$ for each $i < n$. That is, we must show that $1'^{\mathcal{A}} \in N_x(i,i)$, or, that $\mathcal{C} \models x \leq \mathsf{s}_{ii}\mathsf{d}_{01}$. By lemma 16.2, $\mathsf{s}_{ii}\mathsf{d}_{01} = \mathsf{d}_{ii} = 1$ in $\mathcal{C}$, and we are done.

Next, we let $i, j, k < n$ and check that $N_x(i,k) \,;N_x(k,j) \geq N_x(i,j)$ in $\mathcal{A}^+$. We require $r\,;s \in N_x(i,j)$ whenever $r \in N_x(i,k)$ and $s \in N_x(k,j)$. So we let $r, s \in \mathcal{A}$, assume that $x \leq \mathsf{s}_{ik}r$ and $x \leq \mathsf{s}_{kj}s$, and prove that $x \leq \mathsf{s}_{ij}(r\,;s)$. But by lemma 27, $x \leq \mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s \leq \mathsf{c}_k(\mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s) = \mathsf{s}_{ij}(r\,;s)$, and we are done.

Finally, we have to check that if $\bar{a}, \bar{b} \in {}^{\leq n}n$ are of equal length $l \leq n$ and satisfy $N_x(a_i, b_i) \leq 1'$ for each $i < l$, then $N_x(\bar{a}) = N_x(\bar{b})$. If $|\bar{a}| = |\bar{b}| \in \{n-1, n\}$ then $N_x(\bar{a}) = N_x(\bar{b}) = \lambda$, so suppose $|\bar{a}| = |\bar{b}| = l$ (say) $< n-1$. By lemma 16.2 again, the condition is equivalent to $x \leq \mathsf{d}_{a_ib_i}$ for each $i < l$. Let $d(\bar{a}, \bar{b}) = |\{i < l : a_i \neq b_i\}|$. The proof is by induction on $d(\bar{a}, \bar{b})$.

If $d(\bar{a}, \bar{b}) = 0$, then $\bar{a} = \bar{b}$ and there is nothing to prove. Assume that $d(\bar{a}, \bar{b}) = 1$. Let $i < l$ be the index with $a_i \neq b_i$. Now $|rng(\bar{a}) \cup rng(\bar{b})| \leq n-1$, so we may choose $j < n$ with $j \notin rng(\bar{a}) \cup rng(\bar{b})$. Let $\bar{c} \in {}^l n$ be given by $\bar{c} \equiv_i \bar{a}$, $c_i = j$. Then $\widehat{\mathsf{s}_{\bar{a}}} = (\mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}})\widehat{\ }$ and $\widehat{\mathsf{s}_{\bar{b}}} = (\mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}})\widehat{\ }$; and clearly, the words $\mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}}$ and $\mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}}$ are modest. So by theorem 22.2, $\mathsf{s}_{\bar{a}} \simeq \mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}}$ and $\mathsf{s}_{\bar{b}} \simeq \mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}}$.

Now we show $N_x(\bar{a}) \subseteq N_x(\bar{b})$; the converse is similar (and indeed is not needed as $N_x(\bar{a}), N_x(\bar{b})$ are ultrafilters). Let $r \in N_x(\bar{a})$. Then $x \leq \mathsf{s}_{\bar{a}}r$. Also, by assumption, $x \leq \mathsf{d}_{a_ib_i}$. So

$$
\begin{array}{rcll}
x & \leq & \mathsf{d}_{a_ib_i} \cdot \mathsf{s}_{\bar{a}}r & \text{by assumption} \\
 & = & \mathsf{d}_{a_ib_i} \cdot \mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}}r & \text{as } \mathsf{s}_{\bar{a}} \simeq \mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}} \\
 & = & \mathsf{d}_{a_ib_i} \cdot \mathsf{c}_j(\mathsf{d}_{j,a_i} \cdot \mathsf{s}_{\bar{c}}r) & \text{by definition of } \mathsf{s}^j_{a_i} \\
 & = & \mathsf{c}_j\mathsf{d}_{a_ib_i} \cdot \mathsf{c}_j(\mathsf{d}_{j,a_i} \cdot \mathsf{s}_{\bar{c}}r) & \text{by a } \mathsf{CA}_n \text{ axiom, as } j \neq a_i, b_i \\
 & = & \mathsf{c}_j(\mathsf{c}_j\mathsf{d}_{a_ib_i} \cdot (\mathsf{d}_{j,a_i} \cdot \mathsf{s}_{\bar{c}}r)) & \text{by a } \mathsf{CA}_n \text{ axiom} \\
 & = & \mathsf{c}_j(\mathsf{d}_{a_ib_i} \cdot \mathsf{d}_{j,a_i} \cdot \mathsf{s}_{\bar{c}}r) & \text{as 2 lines above} \\
 & \leq & \mathsf{c}_j(\mathsf{d}_{j,b_i} \cdot \mathsf{s}_{\bar{c}}r) & \text{by another } \mathsf{CA}_n \text{ axiom} \\
 & = & \mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}}r & \text{by definition of } \mathsf{s}^j_{b_i} \\
 & = & \mathsf{s}_{\bar{b}}r & \text{as } \mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}} \simeq \mathsf{s}_{\bar{b}}.
\end{array}
$$

Hence, $r \in N_x(\bar{b})$, as required.

Now let $d(\bar{a}, \bar{b}) \geq 2$ and assume the result for smaller $d$. Choose $i < l$ with $a_i \neq b_i$, and let $\bar{c} \in {}^l n$ be given by $\bar{c} \equiv_i \bar{a}$, $c_i = b_i$. Then clearly, $d(\bar{a}, \bar{c}), d(\bar{c}, \bar{b}) < d(\bar{a}, \bar{b})$, and $x \leq \mathsf{d}_{a_jc_j}$, $x \leq \mathsf{d}_{c_jb_j}$ for all $j < l$. By the inductive hypothesis, $N_x(\bar{a}) = N_x(\bar{c}) = N_x(\bar{b})$, which completes the induction and the proof. $\Box$

The next two lemmas relate atoms to hyper-networks. The second is in some way the converse of the first.

Lemma 31 *Let $x, y \in At\mathcal{C}$, $i < n$, and suppose that $\mathsf{c}_ix = \mathsf{c}_iy$. Then $N_x \equiv_i N_y$.*

Proof:

Let $\bar{a} \in {}^{\leq n}(n \setminus \{i\})$. We show $N_x(\bar{a}) = N_y(\bar{a})$. If $|\bar{a}| \in \{n-1, n\}$ then $N_x(\bar{a}) = N_y(\bar{a}) = \lambda$, so we assume $|\bar{a}| < n-1$. As $N_x(\bar{a}), N_y(\bar{b})$ are ultrafilters, it suffices to show $N_x(\bar{a}) \subseteq N_y(\bar{a})$. Let $r \in N_x(\bar{a})$, so that $x \leq \mathsf{s}_{\bar{a}}r$. Then $y \leq \mathsf{c}_i x \leq \mathsf{c}_i \mathsf{s}_{\bar{a}} r$. But $i \notin rng(\bar{a})$, so $\mathsf{c}_i \mathsf{s}_{\bar{a}}$ is modest, and $\mathsf{c}_i \mathsf{s}_{\bar{a}} \simeq \mathsf{s}_{\bar{a}}$ by lemma 23. Hence, $\mathsf{c}_i \mathsf{s}_{\bar{a}} r = \mathsf{s}_{\bar{a}} r$, so that $y \leq \mathsf{s}_{\bar{a}} r$ and $r \in N_y(\bar{a})$. Hence, $N_x(\bar{a}) \subseteq N_y(\bar{a})$, as required. $\square$

LEMMA 32 *Let $x, y \in At\mathcal{C}$, let $i, j < n$ be distinct, and suppose that $N_x \equiv_{ij} N_y$. Then $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_i \mathsf{c}_j y$.*

PROOF:

Pick $\bar{a} \in {}^{n-2}n$ with $rng(\bar{a}) = n \setminus \{i, j\}$. Then $N_x(\bar{a}) = N_y(\bar{a})$.

Let $w$ be a modest s-c-word with $\widehat{w} = (\bar{a})^{-1}$, so that $rng(\widehat{w}) = n-2$ and $\bar{a} \circ \widehat{w} = Id_{n \setminus \{i,j\}}$. Such a $w$ exists by theorem 22.1 and because $\bar{a}$ is one-one. By lemma 16.1, $\widehat{\mathsf{s}_{\bar{a}} w} = \bar{a} \circ \widehat{w} = \widehat{\mathsf{c}_i \mathsf{c}_j}$. The word $\mathsf{s}_{\bar{a}} w$ is modest, because $\mathsf{s}_{\bar{a}}$ and $w$ are, and $\widehat{w}$ is one-one (see example 21.4). As $\mathsf{c}_i \mathsf{c}_j$ is certainly modest, and $|rng(\widehat{\mathsf{c}_i \mathsf{c}_j})| = n-2$, we obtain $\mathsf{s}_{\bar{a}} w \simeq \mathsf{c}_i \mathsf{c}_j$ by theorem 22.2.

Thus, $x \leq \mathsf{c}_i \mathsf{c}_j x = \mathsf{s}_{\bar{a}} w x$.

Now $n-2, n-1 \notin rng(\widehat{w})$, so $\mathsf{c}_{n-2} w$ is modest and (by lemma 23) $\mathsf{c}_{n-2} w \simeq w$, and similarly for $\mathsf{c}_{n-1}$. Hence, $wx \in \mathfrak{Nr}_{n-2}\mathcal{C}$. As $n-2 \neq 2$ (here, we use our assumption that $n \geq 5$ for the only time) we obtain $wx \in N_x(\bar{a})$. But $N_x(\bar{a}) = N_y(\bar{a})$, so $y \leq \mathsf{s}_{\bar{a}} w x = \mathsf{c}_i \mathsf{c}_j x$. Because $y$ is an atom of $\mathcal{C}$, we obtain $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_i \mathsf{c}_j y$. $\square$

Now let $\mathcal{H} = \{N_x : x \in At\mathcal{C}\}$.

PROPOSITION 33 *$\mathcal{H}$ is an $n$-dimensional $\Lambda$-hyper-basis for $\mathcal{A}^+$.*

PROOF:

Recall that we assumed $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{B}$ for some $\mathcal{B} \in \mathsf{CA}_n$, and $\mathcal{B}^+ = \mathcal{C}$. We will use the following properties of $\mathcal{C}$:

FACT 34

1. If $S \subseteq \mathcal{B}$ is a filter base (that is, for every finite $S' \subseteq S$ there is $s \in S$ with $0 < s \leq s'$ for every $s' \in S'$), then there is an atom $x \in \mathcal{C}$ with $\mathcal{C} \models x \leq s$ for every $s \in S$.

   For, the upward closure $\{b \in \mathcal{B} : \exists s \in S(s \leq b)\}$ of $S$ in $\mathcal{B}$ is a proper filter in $\mathcal{B}$ and so extends to an ultrafilter of $\mathcal{C}$. This ultrafilter corresponds to an atom $x$ of $\mathcal{C}$ which lies beneath every element of $S$.

2. If $x \in At\mathcal{C}$ and $k < n$, then in $\mathcal{C}$ we have $\mathsf{c}_k x = \prod_{b \in \mathcal{B},\, b \geq x} \mathsf{c}_k b$. See [JT51, definition 2.14, theorem 2.15].

We now check that $\mathcal{H}$ has the properties of the definition of hyper-basis. First, let $f \in \mathcal{A}^+$ be non-zero. We must find $N \in \mathcal{H}$ with $\mathcal{A}^+ \models N(0,1) \leq f$. Since $\mathcal{A}^+$ is atomic, we may suppose that $f$ is an atom of $\mathcal{A}^+$. Regarding $f$ as an ultrafilter on $\mathcal{A}$, it is a filter base in $\mathcal{B}$, so fact 34.1 gives an atom $x$ of $\mathcal{C}$ with

$\mathcal{C} \models x \leq r$ for every $r \in f$. Let $r \in f$. As $\mathcal{C} \models x \leq \mathsf{s}_{01}r$ — for example, by taking $\mathsf{s}_{01} = \mathsf{c}_2\mathsf{c}_3 \ldots \mathsf{c}_{n-1}$ — we have $r \in N_x(0,1)$. In terms of $\mathcal{A}^+$, this says that $\mathcal{A}^+ \models N_x(0,1) \leq f$, as required.

Second, we let $N_x \in \mathcal{H}$, $i, j < n$, $k \in n \setminus \{i, j\}$, and $f, g \in \mathcal{A}^+$, and assume that $\mathcal{A}^+ \models N_x(i,j) \leq f\,;g$. We must find $N_y \in \mathcal{H}$ with $N_y \equiv_k N_x$, $N_y(i,k) \leq f$, and $N_y(k,j) \leq g$. Since $N_x(i,j)$ is an atom of $\mathcal{A}^+$, by complete additivity of ';' in $\mathcal{A}^+$ we can suppose that $f, g \in At\mathcal{A}^+$, also. That is, $f$ and $g$ are ultrafilters of $\mathcal{A}$.

We know that $\mathcal{A}^+ \models N_x(i,j) \leq f\,;g$. So for every $r \in f$ and $s \in g$, we have $r\,;s \in N_x(i,j)$, whence $x \leq \mathsf{s}_{ij}(r\,;s)$. By lemma 27, $x \leq \mathsf{c}_k(\mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s)$. Define:

$$\begin{array}{rcl} S & = & \{\mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s : r \in f,\ s \in g\} \\ X & = & \{b \in \mathcal{B} : \mathcal{C} \models x \leq b\} \\ Y & = & \{\mathsf{c}_k b \cdot s : b \in X, s \in S\}. \end{array}$$

Note that each is a subset of $\mathcal{B}$. Then $\mathsf{c}_k x \cdot s \neq 0$ for all $s \in S$, so that every element of $Y$ is non-zero. Since both $S$ and $X$ are filter bases in $\mathcal{B}$ (e.g., use corollary 25 for $S$), it follows that $Y$ is also a filter base in $\mathcal{B}$, so by fact 34.1, there is an ultrafilter $y$ of $\mathcal{B}$ containing $Y$. Then $y \in At\mathcal{C}$, so $N_y \in \mathcal{H}$. Also, $\mathcal{C} \models y \leq \mathsf{c}_k b$ for all $b \in X$. By fact 34.2, in $\mathcal{C}$ we have $y \leq \prod_{b \in X} \mathsf{c}_k b = \mathsf{c}_k x$, so $\mathsf{c}_k x = \mathsf{c}_k y$. By lemma 31, $N_x \equiv_k N_y$. Finally, $\mathcal{C} \models y \leq s$ for every $s \in S$, so that $r \in N_y(i,k)$ and $s \in N_y(k,j)$ for every $r \in f$, $s \in g$. In terms of $\mathcal{A}^+$, this says that $N_y(i,k) = f$ and $N_y(k,j) = g$, as we wanted.

Finally, let $N_x, N_y \in \mathcal{H}$, let $i, j < n$ be distinct, and suppose that $N_x \equiv_{ij} N_y$. By lemma 32, $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_i\mathsf{c}_j y$. Hence, by additivity of cylindrifications, there is an atom $z \in \mathcal{C}$ with $\mathsf{c}_i x = \mathsf{c}_i z$ and $\mathsf{c}_j z = \mathsf{c}_j y$. Then $N_z \in \mathcal{H}$, and by lemma 31, $N_x \equiv_i N_z \equiv_j N_y$. □

This completes the proof of (1) ⇒ (2) of theorem 1. The same argument shows that the atomic relation algebra $\mathfrak{Ra}\mathcal{C}$ also has an $n$-dimensional $\Lambda$-hyper-basis.

# 4 From hyper-basis to relativised representation

In this section, we show that (2) ⇒ (3) in theorem 1. First, we need to make our hyper-bases more symmetrical.

## 4.1 Symmetric hyper-bases

Definition 35 If $N$ is a hyper-network and $\sigma : n \to n$ is any map, we write $N\sigma$ for the hyper-network defined by $(N\sigma)(\bar{a}) = N(\sigma(\bar{a}))$, for all $\bar{a} \in {}^{\leq n}n$.

It is easy to check that $N\sigma$ is indeed a hyper-network. In particular, suppose that $\bar{a}, \bar{b} \in {}^{\leq n}n$ have equal length $l$, and $(N\sigma)(a_i, b_i) \leq 1'$ for each $i < l$. We require $(N\sigma)(\bar{a}) = (N\sigma)(\bar{b})$. Write $\bar{a}'$ for the $l$-tuple $\sigma(\bar{a}) = (\sigma(a_0), \ldots, \sigma(a_{l-1}))$, and define $\bar{b}'$ similarly. Then $N(a'_i, b'_i) = (N\sigma)(a_i, b_i) \leq 1'$ for each $i$, so $(N\sigma)(\bar{a}) = N(\bar{a}') = N(\bar{b}') = (N\sigma)(\bar{b})$, as required.

Lemma 36 *Let $M, N$ be hyper-networks, and let $i, j < n$ be distinct. Then $M = N[i/j]$ iff $M \equiv_i N$ and $M(i,j) \leq 1'$.*

*Let $\mathcal{H}$ be a hyper-basis, let $N \in \mathcal{H}$, and let $i, j < n$. Then $N[i/j] \in \mathcal{H}$.*

Proof:

Left-to-right of the first part is clear. For the converse, suppose that $M \equiv_i N$ and $M(i,j) \leq 1'$. For each $k < n$, let $k' = [i/j](k)$. Let $\bar{a} \in {}^{\leq n}n$, and define $\bar{a}' = [i/j](\bar{a})$. Then $N[i/j](\bar{a}) = N(\bar{a}') = M(\bar{a}')$, since $\bar{a}' \in {}^{\leq n}(n \setminus \{i\})$. Now because $M(i,j) \leq 1'$, it follows that $M(a'_m, a_m) \leq 1'$ for each $m < |\bar{a}|$. So $M(\bar{a}') = M(\bar{a})$. It follows that $N[i/j] = M$.

If $i = j$, the second part is trivial. Assume otherwise. Let $N(j,j) = r$. Then $r \leq 1'$, and it follows in relation algebras that $r = r\,;r$. By the second defining property of hyper-bases, there is $M \in \mathcal{H}$ with $N \equiv_i M$ and $M(j,i) = M(i,j) = r$. By the first part, $M = N[i/j]$. □

Definition 37 A hyper-basis $\mathcal{H}$ for a relation algebra $\mathcal{A}$ is said to be *symmetric* if whenever $N \in \mathcal{H}$ and $\sigma : n \to n$ then $N\sigma \in \mathcal{H}$.

Lemma 38 *If an atomic relation algebra has an $n$-dimensional hyper-basis, then it has a symmetric $n$-dimensional hyper-basis.*

Proof:

Let $\mathcal{H}$ be an $n$-dimensional hyper-basis for the atomic relation algebra $\mathcal{A}$. Let $\mathcal{H}^+ = \{N\sigma : N \in \mathcal{H}, \sigma : n \to n\}$. We show that $\mathcal{H}^+$ is also a hyper-basis.

The elements of $\mathcal{H}^+$ are certainly hyper-networks, and $\mathcal{H}^+$ is certainly symmetric. For any non-zero $r \in \mathcal{A}$, there is $N \in \mathcal{H}^+$ with $N(0,1) \leq r$, because $\mathcal{H}^+ \supseteq \mathcal{H}$.

We check the second defining property of hyper-bases. Let $N \in \mathcal{H}$ and $\sigma : n \to n$, let $i, j, k < n$ with $k \neq i, j$, and let $r, s \in \mathcal{A}$ with $N\sigma(i,j) \leq r\,; s$. We seek $P \in \mathcal{H}^+$ with $P \equiv_k N\sigma$, $P(i,k) \leq r$, and $P(k,j) \leq s$.

Well, $N(\sigma(i), \sigma(j)) \leq r\,; s$. Pick $l \in n \setminus \{\sigma(m) : m < n, m \neq k\}$. As $N \in \mathcal{H}$, there is $M \in \mathcal{H}$ with $M \equiv_l N$, $M(\sigma(i), l) \leq r$, and $M(l, \sigma(j)) \leq s$. Define $\tau : n \to n$ by $\tau(k) = l$ and $\tau(m) = \sigma(m)$ for $m \neq k$. Then $M\tau \in \mathcal{H}^+$. If $\bar{a} \in {}^{\leq n}(n \setminus \{k\})$, then $M\tau(\bar{a}) = M(\tau(\bar{a})) = M(\sigma(\bar{a}))$ (because $k \notin rng(\bar{a})$), $= N(\sigma(\bar{a}))$ (because $l \notin rng(\sigma(\bar{a})$ and $M \equiv_l N$), $= N\sigma(\bar{a})$. So $M\tau \equiv_k N\sigma$. Furthermore, $M\tau(i,k) = M(\tau(i), \tau(k)) = M(\sigma(i), l) \leq r$, and (similarly) $M\tau(k,j) \leq s$. This is as required.

Lastly, we check that if $N, M \in \mathcal{H}$, $\sigma, \tau : n \to n$, and $N\sigma \equiv_{ij} M\tau$ for some distinct $i, j < n$, then $N\sigma \equiv_i P \equiv_j M\tau$ for some $P \in \mathcal{H}^+$. Then $(N\sigma)[i/j] \equiv_i N\sigma \equiv_{ij} M\tau \equiv_j (M\tau)[j/i]$, so that $(N\sigma)[i/j] \equiv_{ij} (M\tau)[j/i]$. Now $(N\sigma)[i/j] = N(\sigma \circ [i/j])$. By fact 17.1, $\sigma \circ [i/j]$ is a product of substitutions, so by lemma 36, $(N\sigma)[i/j] \in \mathcal{H}$. Similarly, $(M\tau)[j/i] \in \mathcal{H}$. As $\mathcal{H}$ is a hyper-basis, there is $P \in \mathcal{H}$ with $(N\sigma)[i/j] \equiv_i P \equiv_j (M\tau)[j/i]$. So $P \in \mathcal{H}^+$ and $N\sigma \equiv_i P \equiv_j M\tau$, as required.
□

## 4.2 Constructing a relativised representation from a hyper-basis

Theorem 39 *Let $\mathcal{A}$ be an atomic relation algebra with an $n$-dimensional hyper-basis (where $n \geq 4$). Then $\mathcal{A}$ has a (complete) $n$-flat relativised representation.*

Proof:

Let $\mathcal{H}$ be an $n$-dimensional $\Lambda$-hyper-basis for $\mathcal{A}$, for some set $\Lambda$. By lemma 38, we may suppose that $\mathcal{H}$ is symmetric. We will extract a (complete) $n$-flat relativised representation of $\mathcal{A}$ directly from $\mathcal{H}$. We will build a chain of possibly uncountable labelled, directed hyper-graphs $M_t$ ($t < \omega$); they will not be complete hyper-graphs. Their union, $M_\omega$, will essentially be the representation we seek. Each $M_t$ will have edges (ordered pairs) labelled by atoms of $\mathcal{A}$, and hyper-edges ($l$-tuples for $l \leq n$, $l \neq 2$) labelled by elements of $\Lambda$. No non-edges or non-hyper-edges are labelled. We will require (inductively) that for each $t < \omega$, $M_t$ satisfies conditions 1–4 below:

1. The set of edges forms a reflexive and symmetric binary relation on $M_t$.
2. Each directed edge $(x, y)$ of $M_t$ is labelled by an atom of $\mathcal{A}$, written $M_t(x, y)$.
3. If $(x, y)$ is an edge of $M_t$, then $M_t(x, y) \leq 1'$ iff $x = y$.

For such a graph $M_t$, and a hyper-network $N \in \mathcal{H}$, a map $\nu : N \to M_t$ (formally a map from $n$ into $dom(M_t)$) is said to be an *embedding* if whenever $i, j < n$ then $(\nu(i), \nu(j))$ is an edge of $M_t$ and $M_t(\nu(i), \nu(j)) = N(i, j)$; and whenever $\bar{a} \in {}^{\leq n}n$ with $|\bar{a}| \neq 2$, then $\nu(\bar{a})$ is a hyper-edge of $M_t$ and is labelled with $N(\bar{a})$. Note that despite their name, embeddings need not be one-to-one, but they do preserve atoms under $1'$. Say (as usual in graph theory) that a subset $C$ of $M_t$ is a *clique* if $(x, y)$ is an edge of $M_t$ for all $x, y \in C$. We further require of $M_t$:

4. Any clique in $M_t$ is contained in $rng(\nu)$ for some $N \in \mathcal{H}$ and some embedding $\nu : N \to M_t$.

Let $N$ be a hyper-network and let $S \subseteq n$. We say that the labelled hypergraph $N|_S$ induced by $N$ on $S$ is *strict* if for all distinct $i, j \in S$ we have $N(i, j) \cdot 1' = 0$. $N|_S$ is *maximal strict* if it is strict and for all $S \subset T \subseteq n$, $N|_T$ is not strict. Let $M_0$ be the disjoint union of all maximal strict labelled hypergraphs $N|_S$, where $N \in \mathcal{H}$ and $S \subseteq n$. Thus, $M_0$ satisfies requirements 1–3 above. If $N|_S$ is maximal strict, then for all $i < n$ there is unique $s_i \in S$ such that $N(i, s_i) \leq 1'$. The map $\nu = \{(i, s_i) : i < n\}$ is an embedding of $N$ onto $N|_S$. So $M_0$ satisfies requirement 4, too.

Assume inductively that $M_t$ is defined for some $t < \omega$. Then we define the extension $M_{t+1}$ of $M_t$ so that for every quadruple $(N, \nu, k, N')$, where $N, N' \in \mathcal{H}$, $k < n$, $N \equiv_k N'$, and $\nu : N \to M_t$ is an embedding, the restriction $\nu|_{n \setminus \{k\}}$ of $\nu$ extends to an embedding $\nu' : N' \to M_{t+1}$. We do this as follows.

- If $N'(k, i) \leq 1'$ for some $i \neq k$, then we may (must) set $\nu'(k) = \nu(i)$. This is well-defined if there are several such $i$, and is an embedding. No change to $M_t$ is made for these $(N, \nu, k, N')$.
- For each other $(N, \nu, k, N')$, we adjoin a new point $\pi = \pi_{(t,N,\nu,k,N')}$ to $M_t$. We add just the following new edges: $(\pi, \pi)$, and $(\pi, \nu(i)), (\nu(i), \pi)$ for each $i \in n \setminus \{k\}$.
- The new edges are labelled by atoms as follows: $(\pi, \pi)$ is labelled by $N'(k, k)$, $(\pi, \nu(i))$ by $N'(k, i)$, and $(\nu(i), \pi)$ by $N'(i, k)$.
- We may extend $\nu|_{n \setminus \{k\}}$ to $\nu'$ defined on $k$, by setting $\nu'(k) = \pi$.

- We also add a new hyper-edge $\nu'(\bar{a})$ for every $\bar{a} \in {}^{\leq n}n$ of length $\neq 2$ with $k \in rng(\bar{a})$. We label it by $N'(\bar{a})$. This is well-defined.

  Because $N' \equiv_k N$, $\nu'$ is an embedding $: N' \to M_{t+1}$.
- For distinct $(N, \nu, k, N')$, the new points $\pi_{(t,N,\nu,k,N')}$ are distinct.
- $M_{t+1}$ will consist of $M_t$, with its old labels and edges, together with all these new points, edges, and labels.

It is easy to check that the properties 1–4 above are preserved by these actions. For 4, note that any clique in $M_{t+1}$ is either a clique in $M_t$, for which we have the result inductively, or else it contains a new point $\pi_{(t,N,\nu,k,N')}$, in which case it is contained in $rng(\nu')$, since the only edges involving $\pi_{(t,N,\nu,k,N')}$ lie in this set.

Let $M = M_\omega = \bigcup_{t<\omega} M_t$. Clearly, $M$ satisfies properties 1–4 (for the last, observe that any clique $C$ in $M_\omega$ is finite of size at most $n$, hence $C \subseteq M_t$ for some $t < \omega$). It also has a further property:

5. If $k < n$, $N \equiv_k N'$ in $\mathcal{H}$, and $\nu : N \to M$ is an embedding, then $\nu|_{n\setminus\{k\}}$ extends to an embedding $\nu' : N' \to M$.

Now define $M$ as an $\mathcal{L}(\mathcal{A})$-structure by:

$$M \models r(x,y) \iff \exists t < \omega\ ((x,y) \text{ is an edge of } M_t \wedge M_t(x,y) \leq r),$$

for each $r \in \mathcal{A}$ and $x, y \in M$.

Lemma 40 *$M$ is an $n$-flat relativised representation of $\mathcal{A}$.*

Proof:

First, we show that $M \models S_{\mathcal{A}}$ (see definition 3). To see that $M \models \forall xy(1'(x,y) \leftrightarrow x = y)$, use properties 1 and 3 above. The boolean clauses are easy to check.

We check the axiom for converse, $\forall xy(r(x,y) \leftrightarrow \breve{r}(y,x))$. Suppose that $M \models r(x,y)$. By property 1, $\{x,y\}$ is a clique in $M$. By property 4, there are $N \in \mathcal{H}$, an embedding $\nu : N \to M$, and $i, j < n$ with $\nu(i) = x$ and $\nu(j) = y$. Then $N(i,j) = M(x,y) \leq r$, and by lemma 13 and as the map $r \mapsto \breve{r}$ preserves $\leq$, we have $M(y,x) = N(j,i) = N(i,j)^\smile \leq \breve{r}$. So $M \models \breve{r}(y,x)$, as required. The other direction is similar.

Now consider the composition axiom. Let $x, y \in M$ with $M \models 1(x,y)$. First, suppose that $M \models r(x,z) \wedge s(z,y)$ for some $z \in M$. We require $M \models [r\,;s](x,y)$. By property 1, $\{x,y,z\}$ is a clique in $M$, so by property 4 there are $N \in \mathcal{H}$, an embedding $\nu : N \to M$, and $i, j, k < n$ with $\nu(i) = x$, $\nu(j) = y$, $\nu(k) = z$. Then $M(x,y) = N(i,j) \leq N(i,k)\,;N(k,j) = M(x,z)\,;M(z,y) \leq r\,;s$. So $M \models [r\,;s](x,y)$.

Conversely, suppose that $M \models [r\,;s](x,y)$. Since $\{x,y\}$ is a clique in $M$, there is $N \in \mathcal{H}$ and an embedding $\nu : N \to M$ with $x, y \in rng(\nu)$ — say, $x = \nu(i), y = \nu(j)$. Let $k < n$ with $k \neq i, j$. Clearly, $N(i,j) \leq r\,;s$, so as $\mathcal{H}$ is a hyper-basis, there is $P \in \mathcal{H}$ with $N \equiv_k P$, $P(i,k) \leq r$, and $P(k,j) \leq s$. By property 5, there is an embedding $\pi : P \to M$

extending $\nu|_{n\setminus\{k\}}$. Let $z = \pi(k)$. Then $M(x,z) = P(i,k) \leq r$ and $M(z,y) = P(k,j) \leq s$, as required.

The axiom $\exists xy\, r(x,y)$, for $r \neq 0$ in $\mathcal{A}$, holds because for any atom $a \leq r$ there is $N \in \mathcal{H}$ with $N(0,1) = a$ (as $\mathcal{H}$ is an $n$-dimensional hyper-basis for $\mathcal{A}$). Note that if $a \leq 1'$ then $a\,;a = \breve{a} = a$, so $N(0,0) = a$, too. Choose $S \subseteq n$ containing 0 and, if possible, 1, such that $N|_S$ is maximal strict. Since $N|_S \subseteq M_0$, there is evidently an edge $(x,y)$ of $M_0$ with $M_0(x,y) = a$ (using the above if $a \leq 1'$). Hence, $M \models r(x,y)$.

So $M$ is a relativised representation of $\mathcal{A}$. It remains to show that $M$ is $n$-flat. Recall (definition 5) that $C^n(M) = \{\bar{a} \in {}^nM : rng(\bar{a})$ is a clique in $M\}$. Let $\bar{a} \in C^n(M)$, $\varphi \in \mathcal{L}^n(\mathcal{A})$, and $i,j < n$. We have to show that $M \models_C (\exists x_i \exists x_j \varphi \leftrightarrow \exists x_j \exists x_i \varphi)(\bar{a})$ (see definitions 6, 7). We may assume that $i \neq j$. We begin with two claims.

Claim 1. Let $\bar{a} \in C^n(M)$. Then there is $N \in \mathcal{H}$ and an embedding $\nu : N \to M$ with $\nu(i) = a_i$ for each $i < n$.

Proof of claim. By property 4, there is $N \in \mathcal{H}$ and an embedding $\nu : N \to M$ with $rng(\bar{a}) \subseteq rng(\nu)$. For each $i < n$, choose $i' < n$ with $\nu(i') = a_i$, and let $\sigma$ be the map $\{(i,i') : i < n\} : n \to n$. As $\mathcal{H}$ is symmetric, $N\sigma \in \mathcal{H}$; and $\nu \circ \sigma : N\sigma \to M$ is an embedding with $\nu \circ \sigma(i) = a_i$ for each $i < n$.

Claim 2. Let $N \in \mathcal{H}$ and let $\mu, \nu : N \to M$ be embeddings. Let $\bar{a}, \bar{b} \in C^n(M)$ be given by $a_i = \mu(i)$, $b_i = \nu(i)$, for each $i < n$. Then $M \models_C \varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$, for all $\varphi \in \mathcal{L}^n(\mathcal{A})$.

Proof of claim. By induction on $\varphi$. If $\varphi$ is atomic, of the form $r(x_i, x_j)$, the result follows because embeddings preserve labels on edges of hyper-networks; and if $\varphi$ is $x_i = x_j$ then $\varphi$ is equivalent in $M$ to $1'(x_i, x_j)$, so the preceding case gives the result here. The boolean cases are easy and we omit them. Consider $\exists x_i \varphi$. If $M \models_C \exists x_i \varphi(\bar{a})$, then there is $\bar{a}' \in C^n(M)$ with $\bar{a}' \equiv_i \bar{a}$ and $M \models_C \varphi(\bar{a}')$. By claim 1, there is $N' \in \mathcal{H}$ and an embedding $\mu' : N' \to M$ with $\mu'(l) = a'_l$ for all $l < n$. It follows that $N' \equiv_i N$. By property 5 of $M$, $\nu|_{n\setminus\{i\}}$ extends to an embedding $\nu' : N' \to M$. Let $\bar{b}' \in C^n(M)$ be given by $b'_l = \nu'(l)$, for $l < n$. By the inductive hypothesis, $M \models_C \varphi(\bar{b}')$. But $\bar{b}' \equiv_i \bar{b}$, so $M \models_C \exists x_i \varphi(\bar{b})$. The converse is similar. This proves the claim.

Now assume that $M \models_C \exists x_i \exists x_j \varphi(\bar{a})$ for some $\bar{a} \in C^n(M)$, $\varphi \in \mathcal{L}^n(\mathcal{A})$, and distinct $i,j < n$. So there are $\bar{b}, \bar{c} \in C^n(M)$ with $\bar{a} \equiv_i \bar{b} \equiv_j \bar{c}$ and $M \models_C \varphi(\bar{c})$. As $\bar{a}, \bar{c} \in C^n(M)$, claim 1 provides $P, Q \in \mathcal{H}$ and embeddings $\pi : P \to M$, $\psi : Q \to M$ with $\pi(l) = a_l$ and $\psi(l) = c_l$ for all $l < n$. Thus, $\pi(l) = \psi(l)$ if $l \neq i,j$. As $\pi, \psi$ are embeddings, $P \equiv_{ij} Q$. As $\mathcal{H}$ is a hyper-basis, there is $R \in \mathcal{H}$ with $P \equiv_j R \equiv_i Q$. By property 5 of $M$, $\pi|_{n\setminus\{j\}}$ extends to an embedding $\rho : R \to M$, and $\rho|_{n\setminus\{i\}}$ extends to an embedding $\psi' : Q \to M$. Let $d_l = \rho(l)$ and $e_l = \psi'(l)$, for $l < n$. Then $\bar{a} \equiv_j \bar{d} \equiv_i \bar{e}$ in $C^n(M)$; and by claim 2, $M \models_C \varphi(\bar{e})$. Hence, $M \models_C \exists x_i \varphi(\bar{d})$ and $M \models_C \exists x_j \exists x_i \varphi(\bar{a})$, as required.

Thus, $M$ is $n$-flat. This proves lemma 40. $\square$

Thus $M$ is an $n$-flat relativised representation of $\mathcal{A}$. By lemma 4, since every edge of $M$ is labelled by an atom, it has the further property of being *complete* — respecting all meets and joins that exist in $\mathcal{A}$. $\Box$

Applying this theorem to the canonical extension $\mathcal{A}^+$ of an arbitrary relation algebra $\mathcal{A}$, we see that if $\mathcal{A}^+$ has an $n$-dimensional hyper-basis then it has an $n$-flat relativised representation, $M^+$, say. $M^+$ is a $\mathcal{L}(\mathcal{A}^+)$-structure. It follows from the definitions that the reduct of $M^+$ to the language $\mathcal{L}(\mathcal{A})$ is an $n$-flat relativised representation of $\mathcal{A}$. We obtain theorem 1, (2) $\Rightarrow$ (3).

So the equivalence of parts 1–3 of theorem 1 has now been shown.

## 5 Smooth relativised representations

There is an alternative approach to representations of algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ — what we call *$n$-smooth relativised representations.* Essentially, we make an $n$-flat relativised representation $n$-smooth by dropping explicit mention of the formulas $\varphi$ of $\mathcal{L}^n(\mathcal{A})$, and stating instead by means of equivalence relations which $n$-tuples of a relativised representation agree on all these formulas with respect to $\models_C$. We can axiomatise the properties required for quantifiers to commute, by stating that the equivalence relations should have certain '$n$-back-and-forth' properties. The reader may consult [DLW95] for similar work in finite model theory, showing that the equivalence relations can be taken to be definable in fixed-point logic. $n$-smooth representations have the disadvantage (over $n$-flat ones) that one must expand a relativised representation $M$ by adding further relations, but the advantage that the infinitely many conditions $M \models_C (\exists x_i \exists x_j \varphi \leftrightarrow \exists x_j \exists x_i \varphi)(\bar{a})$, for all formulas $\varphi$, reduce to a single one. In section 6 we will use them to obtain an equational axiomatisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

We fix $n \geq 3$ here. Recall from notation 2 that if $\bar{x}, \bar{y}$ are $m$-tuples, we write $(\bar{x} \mapsto \bar{y})$ for $\{(x_i, y_i) : i < |\bar{x}|\}$; this may or may not be a well-defined map. The concatenation of tuples $\bar{x}, \bar{y}$ is denoted $\bar{x}\bar{y}$.

Definition 41 Let $M$ be a relativised representation of the relation algebra $\mathcal{A}$. Recall that a clique in $M$ is a subset $C \subseteq M$ with $M \models 1(x, y)$ for all $x, y \in C$. For $m \leq n$, $C^m(M) = \{\bar{x} \in {}^mM : rng(\bar{x}) \text{ is a clique in } M\}$.

1. $M$ is said to be *$n$-square* if for any clique $C$ in $M$ with $|C| < n$, if $r, s \in \mathcal{A}$, $x, y \in C$, and $M \models (r\,;s)(x, y)$, then there is a point $z \in M$ such that $C \cup \{z\}$ is a clique in $M$ and $M \models r(x, z) \wedge s(z, y)$.

2. $M$ is said to be an *$n$-smooth relativised representation* if it is $n$-square,[1] and also, for each $0 < m \leq n$, there is an equivalence relation $E^m$ on $C^m(M)$ such that:

   - if $0 < l, m \leq n$, $(\bar{x}, \bar{y}) \in E^m$, and $\theta : l \to m$, then $(\bar{x} \circ \theta, \bar{y} \circ \theta) = ((x_{\theta(0)}, \ldots, x_{\theta(l-1)}), (y_{\theta(0)}, \ldots, y_{\theta(l-1)})) \in E^l$,
   - for any $(\bar{x}, \bar{y}) \in E^m$, $r \in \mathcal{A}$, and $i, j < m$, if $M \models r(x_i, x_j)$ then $M \models r(y_i, y_j)$ (i.e., $(\bar{x} \mapsto \bar{y})$ is a well-defined local isomorphism of $M$), and
   - if $(\bar{x}, \bar{y}) \in E^{n-2}$ and $rng(\bar{x}x)$ and $rng(\bar{y}y)$ are cliques in $M$, then there exists a point $z \in M$ such that $(\bar{x}x, \bar{y}z) \in E^{n-1}$ and $rng(\bar{y}yz)$ is a clique.

[1] This condition is not needed, as it follows from the others; but it is easier to add it explicitly.

3. We can extend the theory $S_{\mathcal{A}}$ to an $\mathcal{L}(\mathcal{A})$-theory $Sq^n(\mathcal{A})$ whose models (if any) are precisely the $n$-square relativised representations of $\mathcal{A}$; and we can extend the language $\mathcal{L}(\mathcal{A})$ to the language $\mathcal{L}^+(\mathcal{A})$ by adding $n$ extra symbols $E^m$, for $0 < m \leq n$, where $E^m$ is a $2m$-ary predicate symbol, and extend the theory $Sq^n(\mathcal{A})$ to a $\mathcal{L}^+(\mathcal{A})$-theory $Sm^n(\mathcal{A})$ whose models are precisely the $n$-smooth relativised representations of $\mathcal{A}$. So, for example, the statement '$E^m$ is an equivalence relation on $C^m(M)$' translates to the $\mathcal{L}^+(\mathcal{A})$-sentence

$$\forall \bar{x}, \bar{y}, \bar{z}\ [(Clique(\bar{x}) \leftrightarrow E^m(\bar{x}, \bar{x})) \wedge (E^m(\bar{x}, \bar{y}) \wedge E^m(\bar{y}, \bar{z}) \rightarrow E^m(\bar{z}, \bar{x}))],$$

where $\bar{x}, \bar{y}, \bar{z}$ are sequences of $m$ distinct variables and $Clique(\bar{x})$ abbreviates the formula $\bigwedge_{j,k<m} 1(x_j, x_k)$.

Remark 42 For $n$-smooth $M$, the set $\Theta = \{(\bar{x} \mapsto \bar{y}) : (\bar{x}, \bar{y}) \in \bigcup_{m \leq n} E^m\}$ is a certain kind of $n$-back-and-forth system of local isomorphisms of $M$. Each map $\theta \in \Theta$ with $0 < |dom(\theta)| \leq n-2$ can be extended within $\Theta$ to be defined on a new point $a$, so long as $dom(\theta) \cup \{a\}$ is a clique. Moreover, the extension can be chosen so that its range extends to a clique containing some other new point $b$, so long as $rng(\theta) \cup \{b\}$ was already a clique. Such $n$-back-and-forth systems offer an alternative definition of $n$-smooth.

Remark 43 In [HH97c], we showed that $n$-square relativised representations correspond to subalgebras of relation algebras with $n$-dimensional relational bases: a relation algebra $\mathcal{A}$ has such a representation iff $\mathcal{A} \in \mathsf{RA}_n$, in the notation of [Mad89]. A relativised representation is $n$-square iff all formulas of the form $1(x_i, x_j) \wedge \exists x_k(r(x_i, x_k) \wedge s(x_k, x_j))$, where $i, j, k < n$, $k \neq i, j$, and $r, s \in \mathcal{A}$, have the same meaning whether evaluated classically or in the relativised semantics $\models_C$. We already gave an argument (in theorem 11) that shows that any $n$-flat relativised representation is $n$-square. Theorem 1 will show that $n$-smooth relativised representations correspond exactly to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

We now show that (3) $\Rightarrow$ (4) in theorem 1.

Proposition 44 *Let $\mathcal{A}$ be a relation algebra with an $n$-flat relativised representation, for some $n \geq 3$. Then $\mathcal{A}$ has an $n$-smooth relativised representation.*

Proof [sketch]:

Let $M$ be an $n$-flat relativised representation of $\mathcal{A}$. By replacing $M$ with a suitable elementary extension, we may assume it is $\omega$-saturated. This preserves $n$-flatness, as $C^n(M)$ is first-order definable. Using lemmas 8 and 9, it can be checked that $M$ is $n$-square; this is similar to the last part of the proof of theorem 11, and is left to the reader.

A partial map $\rho : M \rightarrow M$ is said to be *$n$-elementary* if whenever $\bar{a} \in C^n(M)$ and $rng(\bar{a}) \subseteq dom(\rho)$, then $M \models_C \varphi(\bar{a})$ iff $M \models_C \varphi(\rho(\bar{a}))$ for all formulas $\varphi$ of $\mathcal{L}^n(\mathcal{A})$. Now, for $\bar{x}, \bar{y} \in C^m(M)$ $(m \leq n)$, let $E^m(\bar{x}, \bar{y})$ hold iff $(\bar{x} \mapsto \bar{y})$ is a well-defined $n$-elementary map.

We check that the $E^m$ meet the $n$-smoothness conditions. First, an $n$-elementary map must preserve edge relations, as these are just the atomic formulas of $\mathcal{L}^n(\mathcal{A})$. Second, if $(\bar{x}, \bar{y}) \in E^m$ and $\theta : l \rightarrow m$ is any map, we must check that

$(\bar{x} \circ \theta, \bar{y} \circ \theta) \in E^l$ — i.e., $(\bar{x} \circ \theta \mapsto \bar{y} \circ \theta)$ is a well-defined $n$-elementary map. But this is clear, since it is a restriction of $(\bar{x} \mapsto \bar{y})$.

Finally, suppose that $(\bar{x}, \bar{y}) \in E^{n-2}$ and $rng(\bar{x}x)$, $rng(\bar{y}y)$ are cliques in $M$. So the map $(\bar{x} \mapsto \bar{y})$ is well-defined and $n$-elementary. Let

$$\begin{array}{rcl} \bar{a} & = & (x, x_0, x_1, \ldots, x_{n-3}, x_0) \\ \bar{a}' & = & (x_0, x_0, x_1, \ldots, x_{n-3}, x_0) \\ \bar{b}' & = & (y_0, y_0, y_1, \ldots, y_{n-3}, y_0) \\ \bar{b} & = & (y_0, y_0, y_1, \ldots, y_{n-3}, y). \end{array}$$

These lie in $C^n(M)$. For each $\varphi \in \mathcal{L}^n(\mathcal{A})$ such that $M \models_C \varphi(\bar{a})$, we have $M \models_C \exists x_0 \varphi(\bar{a}')$, so (by $n$-elementarity) $M \models_C \exists x_0 \varphi(\bar{b}')$ and $M \models_C \exists x_{n-1} \exists x_0 \varphi(\bar{b})$. By $n$-flatness, $M \models_C \exists x_0 \exists x_{n-1} \varphi(\bar{b})$, for all such $\varphi$. So using $\omega$-saturation twice, we can find $\bar{b} \equiv_0 \bar{c} \equiv_{n-1} \bar{d}$ in $C^n(M)$ with $M \models \varphi(\bar{d})$ for all these $\varphi$. So

$$\begin{array}{rcl} \bar{c} & = & (z, y_0, y_1, \ldots, y_{n-3}, y) \\ \bar{d} & = & (z, y_0, y_1, \ldots, y_{n-3}, y_0), \end{array}$$

for some $z \in M$. (We have $d_{n-1} = d_1$ because $M \models_C (x_{n-1} = x_1)(\bar{a})$.) Then $rng(\bar{y}yz) = rng(\bar{c})$ is a clique. Further, $(\bar{a} \mapsto \bar{d}) = (\bar{x}x \mapsto \bar{y}z)$ is a well-defined $n$-elementary map. So $(\bar{x}x, \bar{y}z) \in E^{n-1}$, as required. $\square$

The following two theorems show (4) $\Rightarrow$ (2) of theorem 1, by applying theorem 46 to $\mathcal{A}^+$.

THEOREM 45 *If $\mathcal{A}$ has an $n$-smooth relativised representation then $\mathcal{A}^+$ has a complete $n$-smooth relativised representation.*

Cf. Monk's theorem (reported in [McK66], theorem 2.12) that if a relation algebra $\mathcal{A}$ is representable then its canonical extension $\mathcal{A}^+$ has a complete representation.

PROOF [SKETCH]:

To say that $\mathcal{A}$ has an $n$-smooth relativised representation is to say that the first-order theory $Sm^n$, defined in definition 41 part 3, is consistent. Let $M$ be an $\omega$-saturated model of $Sm^n$. The proof of [HH97d, theorem 22] or [HH97c, lemma 26] shows how to check that $M$ is a complete, $n$-square relativised representation of $\mathcal{A}^+$. Since $M \models Sm^n$, it follows that $M$ is $n$-smooth. $\square$

THEOREM 46 *If the atomic relation algebra $\mathcal{A}$ has a complete $n$-smooth relativised representation then $\mathcal{A}$ has an $n$-dimensional hyper-basis.*

PROOF:

Let $M$ be a complete $n$-smooth relativised representation of $\mathcal{A}$. For $m \leq n$, $E^m$ is an equivalence relation on $C^m(M)$. So the union $E = \bigcup_{0<m\leq n} E^m$ is an equivalence relation on $\bigcup_{0<m\leq n} C^m(M)$. Let $\Lambda$ be the set of $E$-equivalence classes. For any $n$-dimensional $\Lambda$-hyper-network $N$ over $\mathcal{A}$ and map $\nu : n \to M$, we say that $\nu$ is an *embedding* of $N$ into $M$ if (i) for $r \in At(\mathcal{A})$ and $i, j < n$, we have $N(i,j) = r$ iff $M \models r(\nu(i), \nu(j))$, and (ii) for any $\bar{a} \in {}^{\leq n}n$ with $|\bar{a}| \neq 2$, $\nu(\bar{a})$ is a member of the equivalence class $N(\bar{a}) \in \Lambda$.

We let $\mathcal{H}$ be the set of all $n$-dimensional $\Lambda$-hyper-networks over $\mathcal{A}$ that embed into $M$, and check that $\mathcal{H}$ is a hyper-basis. The first two properties of hyper-bases are easy to verify, using $n$-squareness. We check the 'amalgamation' condition for $\mathcal{H}$. So take $P, Q \in \mathcal{H}$ and distinct $i, j < n$ with $P \equiv_{ij} Q$. We seek a hyper-network $R \in \mathcal{H}$ with $P \equiv_i R \equiv_j Q$.

Let $\bar{b}$ be any $(n-2)$-tuple enumerating $n \setminus \{i, j\}$. Since $P, Q \in \mathcal{H}$, there are embeddings $\pi : P \to M$ and $\psi : Q \to M$. Since $P \equiv_{ij} Q$, we know that $P(\bar{b}) = Q(\bar{b}) \in \Lambda$, so that $(\pi(\bar{b}), \psi(\bar{b})) \in E$. Now $rng(\pi|_{n\setminus\{i\}})$ and $rng(\psi|_{n\setminus\{j\}})$ are both cliques, so since $M$ is $n$-smooth, there is a point $z \in M$ such that $rng(\psi|_{n\setminus\{j\}}) \cup \{z\}$ is a clique and $((\pi(j), \pi(b_0), \ldots, \pi(b_{n-3})), (z, \psi(b_0), \ldots, \psi(b_{n-3}))) \in E$. Let $\rho : n \to M$ be defined by: $\rho(k) = \psi(k)$ for $k \neq j$, and $\rho(j) = z$. By the first part of the definition of $n$-smooth, for any $\bar{a} \in {}^{\leq n}(n \setminus \{i\})$ we have $(\pi(\bar{a}), \rho(\bar{a})) \in E$.

We use $\rho$ to define the required hyper-network $R \in \mathcal{H}$ in the obvious way: for $k, l < n$, $R(k, l)$ is the atom $\alpha$ of $\mathcal{A}$ satisfying $M \models \alpha(\rho(k), \rho(l))$, and for $\bar{a} \in {}^{\leq n}n$ of length $\neq 2$, $R(\bar{a})$ is the $E$-class of $\rho(\bar{a})$. Since $\psi$ and $\rho$ agree on all points except perhaps $j$, it follows that $R \equiv_j Q$. For any $\bar{a} \in {}^{\leq n}(n \setminus \{i\})$, because $E(\pi(\bar{a}), \rho(\bar{a}))$, we have $P(\bar{a}) = R(\bar{a})$. Hence $R \equiv_i P$, as required. $\square$

Now we have proved the equivalence of parts (1) to (4) of theorem 1.

Remark 47 Adjustments to the equivalence relations $E^m$ yield rather different classes of algebras. So at one extreme, if for each $m \leq n$ we make $E^m$ as small as possible (subject to being an equivalence relation) we get the identity relation $(\bar{x}, \bar{y}) \in E^m \Leftrightarrow \bar{x} = \bar{y}$. From this, if $M$ expanded by the $E^m$ is $n$-smooth, we can show that $\{(x, y) : M \models 1(x, y)\}$ is transitive. So this gives a classical representation and thus we obtain the class $\mathsf{RRA}$. This would be too strong a requirement to characterise $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

At the other extreme, suppose that each $E^m$ is as big as possible (subject to preserving all edge relations). That is, $E^m(\bar{x}, \bar{y})$ holds iff $(\bar{x} \mapsto \bar{y})$ is a well-defined local isomorphism of $M$. Under this condition, an $n$-smooth relativised representation becomes *$n$-homogenous.* To explain what this is, we first define a *local isomorphism* of a relativised representation $M$ to be a partial, finite map $\iota : M \to M$ such that $dom(\iota), rng(\iota)$ are cliques and if $x, y \in dom(\iota)$ then $M \models a(x, y) \iff M \models a(\iota(x), \iota(y))$, for all $a \in \mathcal{A}$. We now say that $M$ is an $n$-homogeneous relativised representation iff for all local isomorphisms $\iota$ with $0 < |\iota| \leq n-2$ and for all $x, y \in M$ with $\{x\} \cup dom(\iota)$ and $\{y\} \cup rng(\iota)$ both cliques, there is $z \in M$ such that $\iota \cup \{(x, z)\}$ is a local isomorphism and $\{y, z\} \cup rng(\iota)$ is a clique (cf. remark 42). Then we can show, in much the same way as earlier, that for any relation algebra $\mathcal{A}$:

1. if $\mathcal{A}$ has an $n$-flat relativised representation with quantifier elimination with respect to the semantics $\models_C$, then it has an $n$-homogeneous relativised representation;

2. if $\mathcal{A}$ has an $n$-homogeneous relativised representation $M$, then $M$ 'is' a complete $n$-homogeneous relativised representation of the atomic relation algebra $\mathcal{B} \supseteq \mathcal{A}$ consisting of unions of isomorphism types of elements of $1^M$.

3. if $cA$ is atomic, then it has a complete $n$-homogeneous relativised representation iff it has an $n$-dimensional cylindric basis (also cf. theorem 60 below).

Again, these conditions are too strong for $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, but in a different direction (see theorem 57).

## 6 Axiomatising $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$

We can use theorem 1 to construct a recursive axiomatisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ by defining a game to determine whether a relation algebra has an $n$-smooth relativised representation. This method was used in [HH97d, section 9] to axiomatise the representable relation algebras following earlier axiomatisations in [Lyn56, Mon69, HMT85]. A similar game-theoretic method was outlined in [HH97c, section 4.3], to axiomatise the class $\mathsf{RA}_n$. A summary of various game-theoretic axiomatisations appeared in [HH97a]. In [HMV99] it will be shown how to axiomatise the class of complex algebras over any given variety; and in another forthcoming paper it will be shown how to obtain an explicit universal axiomatisation of any $PC'_\Delta$ class that is closed under subalgebras. As these methods have been used before and will be made available in a general form soon, it does not seem necessary to go through the axiomatisation in detail here. Instead, we only sketch an outline of the method.

We can define a two-player game $G^n_\omega(\mathcal{A})$ over a relation algebra $\mathcal{A}$, and show, for countable $\mathcal{A}$, that a winning strategy for the second player ('$\exists$') is equivalent to the existence of an $n$-smooth relativised representation of $\mathcal{A}$. This game has countably many rounds played over finite structures that are intended to provide better and better approximations to a genuine $n$-smooth relativized representation. The first player, '$\forall$', picks defects in the current approximation, and the second player, $\exists$, tries to repair the defect by refining the approximation.

These approximations are finite structures $(X, h, E)$ where $X$ is a finite set, $h$ is a partial labelling $h : {}^{\leq n}X \to \mathcal{A}$ such that $dom(h) \cap (X \times X)$ is reflexive and symmetric, and $E$ is an equivalence relation over $\{\bar{x} \in {}^{\leq n}X : (x_i, x_j) \in dom(h),\ i, j < |\bar{x}|\}$ satisfying, if we may speak loosely, all the universal conditions for an $n$-smooth relativised representation. So, (i) for $x \in X$ we have $h(x, x) \leq 1'$, (ii) if $(x, y), (y, z), (x, z) \in dom(h)$ then $h(x, y) \,;h(y, z) \cdot h(x, z) \neq 0$, (iii) for $(\bar{x}, \bar{y}) \in E$ if $|\bar{x}| = |\bar{y}| = l$, say, and $\theta : l \to m$ (some $m \leq n$) then $(\bar{x} \circ \theta, \bar{y} \circ \theta) \in E$, and (iv), for $(\bar{x}, \bar{y}) \in E$ and $i, j < l$ we have $h(x_i, x_j) = h(y_i, y_j)$.

An approximation $(X, h, E)$ determines a structure $M = M(X, h, E)$ with domain $X$ and defined by $M \models a(x, y)$ iff $h(x, y) \leq a$, for $x, y \in X$ and $a \in \mathcal{A}$. $M$ may fail to be an $n$-smooth relativised representation for four reasons.

1. There could be a non-zero $a \in A$ but no edge $(x, y) \in dom(h)$ with $h(x, y) \leq a$.

2. The approximation might not carry enough information: there could be an edge $(x, y) \in dom(h)$ and an element $a \in \mathcal{A}$ such that $h(x, y) \not\leq a$ and $h(x, y) \not\leq (-a)$. In a genuine $n$-smooth relativised representation, for any $a \in \mathcal{A}$ and for any labelled edge in the relativised representation, we know that either $a$ or $-a$ holds on that pair.

3. There might be an edge $(x, y) \in dom(h)$ with $x, y \in C$ for some clique $C$ of the approximation with $|C| < n$ and $r, s \in \mathcal{A}$ such that $h(x, y) \leq r; s$. This is a defect if there is no witness $z$ in the approximation such that $\{z\} \cup C$ is a clique, $(x, z), (z, y) \in dom(h)$ and $h(x, z) \leq r,\ h(z, y) \leq s$.

4. Finally there are 'hyper-defects'. A hyper-defect is a pair $\bar{x}, \bar{y} \in {}^{\leq n-2}X$ with $(\bar{x}, \bar{y}) \in E$ and two points $x, y \in X$ such that $rng(\bar{x}x), rng(\bar{y}y)$ are cliques but there is no witness $z \in X$ such that $rng(\bar{y}yz)$ is a clique of $h$ and $(\bar{x}x, \bar{y}z) \in E$.

We can define a game played on approximations in which the first player ('$\forall$') will pick inaccuracies of these four types in the current approximation, and player $\exists$ will try to repair them by refining it to a better one. If she suceeds in finding the required approximation in every round of the game, $\exists$ wins; if not, $\forall$ wins.

Crucially, for each type of defect, $\exists$ can narrow her set of possible responses to a finite number and still win the game, if she has a winning strategy at all.

The first kind of defect, 'lack of faithfulness', is dealt with in the initial round of the game. In this round, $\forall$ is allowed to pick any non-zero $a \in \mathcal{A}$, and $\exists$ must respond with an approximation $(X, h, E)$ containing a labelled edge $(x, y)$ such that $h(x, y) \leq a$. If such an approximation exists then there is a unique approximation that she can play which gives the least possible information and we assume that she plays this.

For case 2, when presented with a labelled edge $(x, y) \in dom(h)$ and $a \in \mathcal{A}$, $\exists$ either 'accepts' by resetting the label on $(x, y)$ to $h(x, y) \cdot a$ in the new approximation, or she 'rejects' by resetting the label on $(x, y)$ to $h(x, y) \cdot (-a)$; in either case other labels and the relation $E$ are unaltered. These two responses repair the inaccuracy in the approximation but give no more information than is required.

Given a defect $((x, y), C, r, s)$ of type 3, $\exists$ chooses a set $C' \subseteq X$ containing $C$, adds a new point $z$ to $X$, labels all edges $(z, c), (c, z) : c \in C'$ with 1 except that $(x, z), (z, y)$ are labelled by $r, s$ respectively, labels $(z, z)$ with $1'$, and leaves other labels unaltered. She also chooses one of the finitely many ways of exending $E$ to an equivalence relation over ${}^{\leq n}(X \cup \{z\})$.

Finally, if presented with a hyper-defect $(\bar{x}, \bar{y}, x, y)$, $\exists$ adds a new point $z$ to $X$, extends the domain of $h$ so that $rng(\bar{y}yz)$ is a clique, leaving old labels unchanged and labelling new edges with 1, and chooses an extension of $E$ containing the tuple $(\bar{x}x, \bar{y}z)$.

These games, which have countably many rounds, are designed to test membership of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ in that the algebra belongs to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ if and only if it has an $n$-smooth relativised representation which is equivalent, at least for countable algebras, to the existence of a winning strategy in the game for $\exists$. We can define approximations to the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ by curtailing the games to finitely many rounds. We write $G^n_k(\mathcal{A})$ for the game with $k$ rounds. As before, if $\exists$ successfully plays the required refinement in each of the $k$ rounds she wins, otherwise $\forall$ wins.

We now summarise the main results on these games. Recall that a relation algebra is *simple* iff $1 ; r ; 1 = 1$ for all non-zero elements $r$ of the algebra.

Proposition 48

1. *For any relation algebra $\mathcal{A}$, $\exists$ has a winning strategy in $G^n_k(\mathcal{A})$ (for all $k < \omega$) if and only if she has a winning strategy in $G^n_\omega(\mathcal{A})$.*

2. *There is a universal first-order sentence $\sigma^n_k$, effectively constructible from $k, n$, such that for any relation algebra $\mathcal{A}$, $\mathcal{A} \models \sigma^n_k$ if and only if $\exists$ has a winning strategy in $G^n_k(\mathcal{A})$.*

3. *A countable simple relation algebra $\mathcal{A}$ has an $n$-smooth relativised representation if and only if $\exists$ has a winning strategy in $G^n_\omega(\mathcal{A})$.*

The first part, which uses a version of König's tree lemma, depends on the fact that $\exists$ has only finitely many choices for her moves. The sentences in the second part are universal because $\exists$ is never required to choose an element of the algebra. In the third part, if $\exists$ has a winning strategy then, as $\mathcal{A}$ is countable, there is a play of the game in which $\forall$ picks all possible defects. The 'limit' of such a play will determine an $n$-smooth relativised representation of $\mathcal{A}$, if $\mathcal{A}$ is simple.

Thus, for countable simple $\mathcal{A}$, we have $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \iff \mathcal{A} \models \{\sigma_k^n : k < \omega\}$.

The final step is to replace the universal axioms $\sigma_k^n$ by equations and to remove the assumptions of countability and simplicity. This is a fairly standard argument. The analogue of the following lemma for neat reducts of cylindric algebras was proved by Monk [Mon61].

Proposition 49 *For $n \geq 4$, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety contained in* $\mathsf{RA}$.

Proof:

We show that $\mathbf{HSP}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. Evidently, if $\mathcal{B}_i \in \mathsf{CA}_n$, $i \in I$, then $\prod_{i \in I} \mathfrak{Ra}\mathcal{B}_i = \mathfrak{Ra}\prod_{i \in I} \mathcal{B}_i \in \mathfrak{Ra}\mathsf{CA}_n$. So $\mathbf{P}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, and it suffices to check that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is closed under homomorphic images. Let $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{B}$ for $\mathcal{B} \in \mathsf{CA}_n$, and let $I$ be an ideal of $\mathcal{A}$. Plainly, $I$ is a subset of $\mathcal{B}$. Let $J$ be the ideal of $\mathcal{B}$ generated by $I$. By [HMT71, theorem 2.3.8], $J = \{b \in \mathcal{B} : b \leq \mathsf{c}_{i_0} \ldots \mathsf{c}_{i_l}(x_0 + \cdots + x_{k-1})$ for some $i_0, \ldots, i_l < n$ and $x_0, \ldots, x_{k-1} \in I\}$. Now since $I$ is an ideal of $\mathcal{A}$, it is closed under $+$. Further, if $x \in I$ then $1 ; x \in I$. By [HMT71, 1.5.9(i),1.5.8(i)] and the fact that $x$ is 2-dimensional, that is:

$$1 ; x = \mathsf{c}_2(\mathsf{s}_2^1 1 \cdot \mathsf{s}_2^0 x) = \mathsf{c}_2(1 \cdot \mathsf{s}_2^0 \mathsf{c}_2 x) = \mathsf{c}_2 \mathsf{s}_2^0 \mathsf{c}_2 x = \mathsf{c}_0 \mathsf{s}_0^2 \mathsf{c}_2 x = \mathsf{c}_0 \mathsf{c}_2 x = \mathsf{c}_0 x \in I.$$

Similarly, $\mathsf{c}_1 x \in I$. So the above expression simplifies to $J = \{b \in \mathcal{B} : b \leq x$ for some $x \in I\}$. It follows that $J \cap \mathcal{A} = I$.

Now define a homomorphism from $\mathcal{A}/I$ into $\mathfrak{Ra}(\mathcal{B}/J)$ by $a/I \mapsto a/J$ (for $a \in \mathcal{A}$). As $J \cap \mathcal{A} = I$, this map is one-one. Since $\mathcal{B}/J \in \mathsf{CA}_n$, we have $\mathcal{A}/I \in \mathbf{S}\mathfrak{Ra}\{\mathcal{B}/J\} \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

So $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is closed under $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$. By Birkhoff's theorem, this shows that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety and can be equationally axiomatised. It is known (see proposition 58) that $\mathsf{RA} = \mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 \supseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_5 \supseteq \cdots$, which completes the proof. □

Lemma 50 *Let $\psi(\bar{x})$ be any quantifier-free formula of the language of relation algebras. Then there is an equation, of the form $s = 0$ for some relation algebra term $s(\bar{x})$, that is equivalent in any simple relation algebra to $\psi$ and which can be obtained effectively from $\psi$.*

Proof:

By induction on $\psi$. The equation $t = u$ is equivalent in any relation algebra to $(t \cdot -u) + (u \cdot -t) = 0$. Assume inductively that $\psi$ is equivalent to $t = 0$, and $\chi$ to $u = 0$. Then $\neg\psi$ is equivalent to $\neg(t = 0)$ and so (in simple relation algebras) to $1 ; t ; 1 = 1$, and so to $-(1 ; t ; 1) = 0$. Clearly, $\psi \wedge \chi$ is equivalent in any relation algebra to $t + u = 0$. □

Each universal sentence $\sigma_k^n$ $(k < \omega)$ from proposition 48 can be put in prenex form in an effective manner. It then follows from lemma 50 that each $\sigma_k^n$ is equivalent in simple relation algebras to an equation $\varepsilon_k^n$ which can be obtained effectively from $k$.

We can now prove the final part of theorem 1, which provides a recursive axiomatisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

Theorem 51 *For $n \geq 5$, the variety $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is axiomatised by the equations defining* $\mathsf{RA}$ *together with the equations $\varepsilon_k^n$ for $k < \omega$.*

PROOF:

It suffices to show that a relation algebra $\mathcal{A}$ is in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ iff $\mathcal{A} \models \varepsilon_k^n$ for each $k$. As $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is elementary (by proposition 49), we may suppose that $\mathcal{A}$ is countable. Now by [JT52, theorem 4.15], $\mathcal{A}$ is embeddable in a direct product of simple algebras $\mathcal{A}_i$ ($i \in I$), where each $\mathcal{A}_i$ is a homomorphic image of $\mathcal{A}$. The $\mathcal{A}_i$ are clearly countable also. Since $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety (proposition 49), $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ iff $\mathcal{A}_i \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ for all $i \in I$; and since the $\varepsilon_k^n$ are equations, $\mathcal{A} \models \varepsilon_k^n$ (all $k < \omega$) iff $\mathcal{A}_i \models \varepsilon_k^n$ (all $k$) for each $i \in I$. So we may further assume that $\mathcal{A}$ is a simple relation algebra. But now, $\mathcal{A} \models \{\varepsilon_k^n : k < \omega\}$ iff $\mathcal{A} \models \{\sigma_k^n : k < \omega\}$, iff $\mathcal{A}$ has an $n$-smooth relativised representation (by proposition 48), iff $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ (by the equivalence of parts (1) and (4) of theorem 1). $\square$

The proof of theorem 1 is now complete.

# 7 Remarks on the theorem

Here, we present some observations and questions.

REMARK 52 Theorem 1 also holds degenerately for $n = 4$ — all five parts are true for any relation algebra in this case. For part 1, it is well-known that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 = \mathsf{RA}$ [HMT85, theorems 5.3.8, 5.3.17]. Maddux shows in [Mad83] that the set of all 4-dimensional atomic networks, for any atomic relation algebra, is a 4-dimensional 'relational basis', and in dimension 4, such a basis is essentially a $\Lambda$-hyper-basis for $|\Lambda| = 1$ (the 'hyper-labels' carry no information). Since $\mathsf{RA}$ is a canonical variety, it follows that if $\mathcal{A} \in \mathsf{RA}$ then $\mathcal{A}^+ \in \mathsf{RA}$ and hence that $\mathcal{A}^+$ has a 4-dimensional hyper-basis. As shown in [HH97c], any relation algebra has a '4-square' relativised representation, and the notions of 4-square and 4-flat coincide; or one may simply apply theorem 39. So parts 2 and 3 also hold for any relation algebra. Part 4 holds, by proposition 44. Since $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 = \mathsf{RA}$, by proposition 48 we obtain that every countable relation algebra satisfies $\{\sigma_k^4 : k < \omega\}$ and hence an arbitrary relation algebra satisfies $\{\varepsilon_k^4 : k < \omega\}$, proving part 5. Of course, we are not very interested in obtaining a recursive axiomatisation of $\mathsf{RA}$.

REMARK 53 We saw in proposition 49 that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety, for $n \geq 4$. We remark at this point that it is a canonical variety (closed under the map $\mathcal{A} \mapsto \mathcal{A}^+$). This is clear for $n = 4$, as $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 = \mathsf{RA}$. Let $n \geq 5$, and let $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. By theorem 1, $\mathcal{A}^+$ has an $n$-dimensional hyper-basis. By theorem 39, $\mathcal{A}^+$ has an $n$-flat relativised representation. By theorem 1 again, $\mathcal{A}^+ \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

REMARK 54 In [HH99b] it was shown that the problem of whether a finite relation algebra is representable is undecidable. The same techniques will show that for finite $n \geq 5$, it is undecidable whether a finite relation algebra is in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. The set of isomorphism types of finite algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is co-r.e., since we have a recursive axiomatisation of it. So if $\mathcal{A}$ is a finite relation algebra in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, it does not follow that $\mathcal{A}$ has a *finite* $n$-dimensional hyper-basis or a finite $n$-flat relativised representation, nor that $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{C}$ for a finite $\mathcal{C} \in \mathsf{CA}_n$, as if any of these implications held, it would mean that the set of isomorphism types of finite algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ would be r.e. and hence recursive, a contradiction.

By modifying the definition of $n$-flat relativised representation so that the unit $C^n(M)$ is replaced by a $n$-ary relation contained in it, it can be shown that a finite relation algebra is a subalgebra of $\mathfrak{Ra}\mathcal{C}$ for a finite $n$-dimensional cylindric algebra $\mathcal{C}$ iff it has a finite (modified) $n$-flat relativised representation. The proof uses techniques of [AHN98].

Remark 55 In [Mad89, §6], a discussion of $n$-variable proof theory is given. Using $n$-dimensional cylindric bases, an algebraic semantics for $n$-variable formulas is introduced; it is shown that each axiom of a Hilbert system for $n$-variable first-order logic taken from [TG87] is valid in this semantics, and that the proof rules preserve validity. We note that $n$-flat relativised representations form an alternative to this algebraic semantics, with the advantage of being quite close to classical Tarskian semantics for first-order logic. One may hope to prove a soundness and completeness theorem for $n$-flat relativised representations with respect to $n$-variable proof theory, but a full treatment would involve cylindric algebras and we do not pursue it further here.

# 8 Hyper-bases, cylindric bases, and relational bases

Hyper-bases are related to the relational and cylindric bases of Maddux. In this section we investigate the connections.

Let $\mathcal{A}$ be an atomic relation algebra, and fix $n \geq 4$. An *$n$-dimensional (atomic) network* over $\mathcal{A}$ is similar to a $n$-dimensional hyper-network $N$ over $\mathcal{A}$, except that $N(\bar{a})$ is only defined if $|\bar{a}| = 2$. The same information is carried by an $\Lambda$-hyper-network $N$ over $\mathcal{A}$ if we insist that $|\Lambda| = 1$.

- An *$n$-dimensional relational basis* is then a set of $n$-dimensional networks over $\mathcal{A}$, satisfying conditions 1 and 2 of definition 14.
- An *$n$-dimensional cylindric basis* is a set of $n$-dimensional networks over $\mathcal{A}$, satisfying all three conditions of definition 14.

The first definition is the same as that of Maddux [Mad83], and the second is equivalent to Maddux's definition [Mad89].

## 8.1 Direct connections between bases

Proposition 56 *Let $\mathcal{A}$ be an atomic relation algebra, with an $n$-dimensional $\Lambda$-hyper-basis for some set $\Lambda$ and some $n \geq 4$. Then $\mathcal{A}$ has a $n$-dimensional relational basis.*

Proof:

Let $\mathcal{H}$ be an $n$-dimensional $\Lambda$-hyper-basis for $\mathcal{A}$. If $N \in \mathcal{H}$, write $N^\dagger$ for the $n$-dimensional network over $\mathcal{A}$ obtained by throwing away the labels $N(\bar{a})$ for any $\bar{a}$ of length not equal to 2. We may regard $N^\dagger$ as a reduct of $N$. Write $\mathcal{H}^\dagger$ for $\{N^\dagger : N \in \mathcal{H}\}$.

We claim that $\mathcal{H}^\dagger$ is an $n$-dimensional relational basis for $\mathcal{A}$. We check the two properties required.

1. Let $r \in \mathcal{A}$ be non-zero. As $\mathcal{H}$ is a hyper-basis, we may choose $N \in \mathcal{H}$ with $N(0,1) \leq r$. Then certainly, $N^\dagger(0,1) \leq r$, also.

2. Let $N^\dagger \in \mathcal{H}^\dagger$, let $i, j, k < n$ with $k \neq i, j$, let $r, s \in \mathcal{A}$ with $N^\dagger(i,j) \leq r\,;s$. Then $N(i,j) \leq r\,;s$, so there is $M \in \mathcal{H}$ with $N \equiv_k M$, $M(i,k) \leq r$, and $M(k,j) \leq s$. Clearly, $M^\dagger \in \mathcal{H}^\dagger$, $N^\dagger \equiv_k M^\dagger$, $M^\dagger(i,k) \leq r$, and $M^\dagger(k,j) \leq s$, as required.

□

$\mathcal{H}^\dagger$ in the proof need not be an $n$-dimensional *cylindric* basis for $\mathcal{A}$. Indeed, we will now see that $\mathcal{A}$ may not have such a basis.

Take $p$ to be a whole number, at least 2. The *Lyndon algebra* $\mathcal{A}_p$ is finite, with $p+2$ atoms, say $1', a_0, \ldots, a_p$. It is defined by:

- $a_i\,; a_i = a_i + 1'$ if $p \geq 3$, and $a_i; a_i = 1'$ if $p = 2$.
- $a_i\,; a_j = \sum_{k \neq i,j} a_k$ if $i \neq j$,
- (necessarily) $\breve{a}_i = a_i$,

where $i, j, k \leq p$. On arbitrary elements of $\mathcal{A}$, ';' can be calculated from this using distributivity over $+$. So can '$\smile$': we have $\breve{r} = r$ for all $r \in \mathcal{A}_p$.

It can be shown that any given Lyndon algebra $\mathcal{A}_p$ is representable iff there is a projective plane of order $p$. In particular, infinitely many $\mathcal{A}_p$ are representable [Lyn61, theorem 1].

THEOREM 57 *There exists a (finite) relation algebra with an $n$-dimensional hyper-basis for all finite $n \geq 5$, but with no 5-dimensional cylindric basis.*

PROOF:

Choose $p \geq 4$ such that $\mathcal{A}_p$ is representable. By [HMT85, theorem 5.3.16], $\mathcal{A}_p \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ for all finite $n$. By theorem 1, $\mathcal{A}_p^+$ has an $n$-dimensional hyper-basis for all finite $n \geq 5$. Since $\mathcal{A}_p$ is finite, $\mathcal{A}_p = \mathcal{A}_p^+$, so the same holds for $\mathcal{A}_p$.

It remains to show that $\mathcal{A}_p$ does not have a 5-dimensional cylindric basis. Assume for contradiction that $\mathcal{M}$ is such a basis. For convenience, let $a, b, c, d, e$ be distinct diversity atoms ($\neq 1'$) of $\mathcal{A}_p$. By property 1 of the definition of cylindric basis, there is a network $N_0 \in \mathcal{M}$ with $N_0(0,1) = a$. Now $N_0(0,1) \leq b\,;c$, so by property 2 there are $N_3, N_4 \in \mathcal{M}$ with

- $N_3 \equiv_3 N_0$, $N_3(0,3) = b$, and $N_3(3,1) = c$, and
- $N_4 \equiv_4 N_0$, $N_4(0,4) = b$, and $N_4(4,1) = c$.

As $c \leq d\,;a$ and $c \leq e\,;a$, there are $N_3', N_4' \in \mathcal{M}$ with

- $N_3' \equiv_2 N_3$, $N_3'(3,2) = d$, and $N_3'(2,1) = a$, and
- $N_4' \equiv_2 N_4$, $N_4'(4,2) = e$, and $N_4'(2,1) = a$.

Notice that $N_3'(0,2) \leq (N_3'(0,1)\,; N_3'(1,2)) \cdot (N_3'(0,3)\,; N_3'(3,2)) = (a\,;a) \cdot (b\,;d) = a$, so $N_3'(0,2) = a$. Similarly, $N_4'(0,2) = a$. See figure 2.

Thus, $N_3' \equiv_{34} N_4'$. By property 3 of the definition of cylindric basis, there is $P \in \mathcal{M}$ with $N_3' \equiv_4 P \equiv_3 N_4'$. But now, $P(3,4) \leq P(3,0)\,;P(0,4) = N_3'(3,0)\,;N_4'(0,4) = b\,;b$. Similarly, using 1 and 2 instead of 0, we obtain $P(3,4) \leq c\,;c$ and $P(3,4) \leq d\,;e$. So $P(3,4) \leq b\,;b \cdot c\,;c \cdot d\,;e = 0$, a contradiction. □

So theorem 1 fails if we replace 'hyper-basis' by 'cylindric basis': for all $n \geq 5$, $\mathcal{A}_p \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ but $\mathcal{A}_p^+ = \mathcal{A}_p$ has no $n$-dimensional cylindric basis.

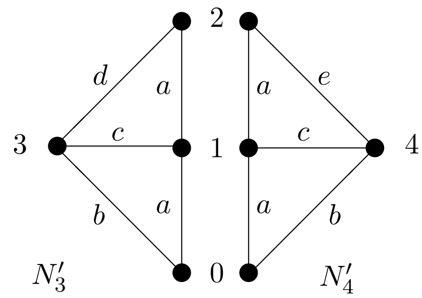


Figure 2: The networks $N'_3, N'_4$

### 8.2 Varieties generated by bases

However, better results may be expected if we close under taking subalgebras. Recall from [Mad89] that $\mathsf{RA}_n$ is the class of all relation-type algebras that embed into a (complete) atomic algebra with an $n$-dimensional relational basis. The paper [HH97c] studied these varieties. Also define $\mathsf{CB}_n$ analogously, but using cylindric bases instead of relational bases.

PROPOSITION 58

1. $\mathsf{CB}_4 = \mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 = \mathsf{RA}_4 = \mathsf{RA}$.
2. *For* $n \geq 4$, *we have* $\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathsf{RA}_n$.
3. $\bigcap_{n<\omega} \mathsf{CB}_n = \bigcap_{n<\omega} \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n = \bigcap_{n<\omega} \mathsf{RA}_n = \mathsf{RRA}$.

PROOF:

1. As Maddux proved that $\mathsf{RA}_4 = \mathsf{RA}$, (1) follows from (2) if we show that $\mathsf{RA}_4 \subseteq \mathsf{CB}_4$. But it is easily verified that any 4-dimensional relational basis is also a cylindric basis.
2. From any cylindric basis $\mathcal{M}$ we may obtain a $\{0\}$-hyper-basis, by letting $N(\bar{a}) = 0$ for all $\bar{a} \in {}^{\leq n}n$ of length $\neq 2$ and all $N \in \mathcal{M}$. So by theorem 1, $\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.
   By proposition 56, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathsf{RA}_n$.
3. By [Mad83, theorems 6, 10], $\bigcap_{n<\omega} \mathsf{RA}_n = \mathsf{RRA}$. So using the second part, we only need verify that $\mathsf{RRA} \subseteq \mathsf{CB}_n$ for all $n$. Let $\mathcal{A} \in \mathsf{RRA}$, and let $h : \mathcal{A} \to \wp(X \times X)$ be a representation of $\mathcal{A}$. Then $h$ is an embedding of $\mathcal{A}$ into the proper relation algebra $\mathcal{B}$ with domain $\wp(h(1))$. For any $n$, let $\Phi$ be the set of all $n$-tuples $\bar{a} = (a_0, \ldots, a_{n-1})$ of elements of $X$ such that $(a_i, a_j) \in h(1)$ for each $i, j < n$. For each $\bar{a} \in \Phi$, define a network $N_{\bar{a}}$ by $N_{\bar{a}}(i,j) = \{(a_i, a_j)\} \in At\mathcal{B}$. It can be checked that $\{N_{\bar{a}} : \bar{a} \in \Phi\}$ is an $n$-dimensional cylindric basis for $\mathcal{B}$.

$\square$

Hence, $\mathcal{A}_p$ as in theorem 57 is a *subalgebra* of a (finite) relation algebra with a 5-dimensional cylindric basis. Indeed, any representable relation algebra has this property. This shows an important distinction between $\mathsf{RA}_n$ and $\mathsf{CB}_n$. It is known that $\mathcal{A} \in \mathsf{RA}_n$ iff the canonical extension $\mathcal{A}^+$ has an $n$-dimensional relational basis. But the algebra $\mathcal{A}_p$ of the theorem belongs to $\mathsf{CB}_n$ yet its canonical extension $\mathcal{A}_p^+ \cong \mathcal{A}_p$ has no $n$-dimensional cylindric basis. We do not know if the inclusion $\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is proper. An example in [HHM98] shows that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subset \mathsf{RA}_n$ for all finite $n \geq 5$.

Problem 1 *For each finite $n \geq 5$, is the inclusion $\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ proper? That is, does any algebra $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ embed in a relation algebra with an $n$-dimensional cylindric basis?*

*For each $n < \omega$, is there an $m < \omega$ such that $\mathsf{RA}_m \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$?*

*For each $n < \omega$, is there an $m < \omega$ such that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_m \subseteq \mathsf{CB}_n$?*

### 8.3 Cylindric bases and homogeneous representations

A (classical) representation of a relation algebra $\mathcal{A}$ can be equivalently regarded as a relativised representation of $\mathcal{A}$ in the sense of definition 3 with the property that $1^M$ is an equivalence relation on $M$. Such a representation is said to be *homogeneous* if it is 'ultra-homogeneous' in the model-theoretic sense: every partial isomorphism of $M$ with finite domain is induced by a full automorphism of $M$.

For finite relation algebras, an argument in the style of [Fra54] shows that cylindric bases and homogeneous representations 'coincide'. The $n$-dimensional analogue was discussed in remark 47.

Lemma 59 *If $\mathcal{M}_n$ is an $n$-dimensional cylindric basis for $\mathcal{A}$ then the set $\mathcal{M}_n^+ = \{N\sigma : N \in \mathcal{M}_n,\ \sigma : n \to n\}$ is also an $n$-dimensional cylindric basis.*

Proof:

Copy the proof of lemma 38. □

Theorem 60 *Let $\mathcal{A}$ be a finite relation algebra. Then $\mathcal{A}$ has an $n$-dimensional cylindric basis for all finite $n$, iff $\mathcal{A}$ has a homogeneous representation.*

Proof [sketch]:

(Cf. theorem 39.) If $M$ is a homogeneous representation of $\mathcal{A}$, then consider the set of all atomic networks of dimension $n$ that embed in $M$, as in proposition 58.3. This is an $n$-dimensional cylindric basis for $\mathcal{A}$.

To prove the converse to the theorem, suppose that $\mathcal{A}$ has a $n$-dimensional cylindric basis $\mathcal{M}_n$ for all finite $n$. For $m < n$, write $\mathcal{M}_n|_m$ for the set $\{N|_m : N \in \mathcal{M}_n\}$ of $m$-dimensional networks, where for a network $N \in \mathcal{M}_n$, $N|_m$ denotes the $m$-dimensional network obtained by restricting $N$ to indices $< m$. Then it can be checked that $\mathcal{M}_n|_m$ is an $m$-dimensional cylindric basis for $\mathcal{A}$.

Now there are only finitely many $m$-dimensional networks over $\mathcal{A}$, for any $m < \omega$. So by König's tree lemma, we may suppose without loss of generality that $\mathcal{M}_n|_m = \mathcal{M}_m$ for all $m < n < \omega$. By lemma 59 we may suppose that $\mathcal{M}_n$ is symmetric (closed under permutations), for each $n < \omega$.

We are going to build a homogeneous representation of $\mathcal{A}$ by a game. The following lemma will be used to show that it is homogeneous. First, a definition.

Definition 61 Let $q \leq p < \omega$, $P \in \mathcal{M}_p$, and $Q \in \mathcal{M}_q$. Let $\theta : q \to p$ be a one-one map. We say that $\theta$ is an *embedding* from $Q$ to $P$ if $Q(i,j) = P(\theta(i), \theta(j))$ for all $i, j < q$.

Lemma 62 *Let $p, q < \omega$, $P \in \mathcal{M}_p$, $Q \in \mathcal{M}_{q+1}$, and let $\theta : Q|_q \to P$ be an embedding. Then there is $P^+ \in \mathcal{M}_{p+1}$ with $P^+|_p = P$, and an embedding $\theta^+ : Q \to P^+$ extending $\theta$.*

Proof:

Clearly, $p \geq q$. The proof is by induction on $p - q$. If this is zero, extend $\theta$ to a permutation $\theta^+$ of $q+1$ in the unique way. By lemma 59 and our assumption that our $\mathcal{M}_{q+1}$ is symmetric, $Q^{\theta^+} \in \mathcal{M}_{q+1}$. Clearly, $\theta^+ : Q \to Q^{\theta^+}$ is an embedding, and $Q^{\theta^+}|_p = P$. So we may take $P^+ = Q^{\theta^+}$.

Let $p - q \geq 1$ and assume the result for smaller $p - q$. Take $P, Q, \theta$ as in the lemma. By lemma 59, we may suppose that $p-1 \notin rng(\theta)$, so that $\theta$ is an embedding : $Q \to P|_{p-1}$. By the inductive hypothesis, there is $P' \in \mathcal{M}_p$ with $P'|_{p-1} = P|_{p-1}$, and an embedding $\theta' : Q \to P'$ extending $\theta$. So also, there is $R \in \mathcal{M}_{p+1}$ with $R|_{p-1} = P|_{p-1}$, and an embedding $\rho : Q \to R$ extending $\theta$ (e.g., any $R \in \mathcal{M}_{p+1}$ with $R|_p = P'$). Now, by lemma 59 again, we can assume that $\rho(q) = p$. Clearly, $P^* \equiv_{p-1,p} R$ in $\mathcal{M}_{p+1}$. Let $S \in \mathcal{M}_{p+1}$ with $P^* \equiv_p S \equiv_{p-1} R$. So $S|_p = P$, and $\rho : Q \to S$ is an embedding extending $\theta$. $\square$

Two players, $\forall$ (male) and $\exists$ (female), now play a game to build a representation of $\mathcal{A}$. The game has $\omega$ rounds, numbered $1, 2, \ldots, t, \ldots$ ($2 \leq t < \omega$). In round 1, $\forall$ picks non-zero $r \in \mathcal{A}$, and $\exists$ responds with some $N_2 \in \mathcal{M}_2$ with $N_2(0,1) \leq r$; she can find such a network because $\mathcal{M}_2$ is a cylindric basis. In each subsequent round $t \geq 2$, if the current network is $N_t \in \mathcal{M}_t$, then $\forall$ can make one of two types of move:

**Triangle move:** he chooses $i, j < t$ and $r, s \in \mathcal{A}$ with $N_t(i,j) \leq r\,;s$. $\exists$ must respond with some $N_{t+1} \in \mathcal{M}_{t+1}$ such that $N_{t+1}(i,t) \leq r$ and $N_{t+1}(t,j) \leq s$. She can do so because there is $N' \in \mathcal{M}_{t+1}$ with $N'|_t = N_t$, so as $\mathcal{M}_{t+1}$ is a cylindric basis, there is $N_{t+1} \in \mathcal{M}_{t+1}$ with $N_{t+1} \equiv_t N'$, $N_{t+1}(i,t) \leq r$ and $N_{t+1}(t,j) \leq s$.

**Amalgamation move:** he chooses $Q \in \mathcal{M}_{q+1}$ for some $q \leq t$, and an embedding $\theta : Q|_q \to N_t$. $\exists$ must respond with $N_{t+1} \in \mathcal{M}_{t+1}$ such that $\theta$ extends to an embedding $\theta^+ : Q \to N_{t+1}$. Lemma 62 shows that she can do this.

Consider a play $N_2, N_3, \ldots$ of the game in which $\forall$ plays $r_0 \in \mathcal{A} \setminus \{0\}$ and then makes every possible move in some round. That is, if $N_t(i,j) \leq r\,;s$ for some $t, i, j, r, s$, then he plays a triangle move $i, j, r, s$ in some round $u \geq t$; and if $t \geq 1$, $q \leq t$, $Q \in \mathcal{M}_{q+1}$, and $\theta : Q|_q \to N_t$ is an embedding, then $\forall$ plays $Q, \theta$ in some round $u \geq t$ (note that $\theta : Q \to N_u$ will also be an embedding). He can do all this because $\mathcal{A}$ and each $\mathcal{M}_n$ are countable, and he makes countably many moves during the game. Since $N_u|_t = N_t$ for each $t \leq u < \omega$, the play has a well-defined limit $N_\omega$, a network of dimension $\omega$, where for each $i, j < \omega$, $N_\omega(i,j) = N_k(i,j)$ for any $k > i, j$. $N_\omega$ has the following properties:

1. $N_\omega(0,1) \le r_0$ (by $\forall$'s first move).
2. $N_\omega$ is a network: $N_\omega(i,i) \le 1'$ and $N(i,k) \le N(i,j)\,;N(j,k)$ for all $i,j,k < \omega$ (because $N_\omega$ is the 'limit' of networks).
3. If $N_\omega(i,j) \le r\,;s$ for some $i,j < \omega$ and $r,s \in \mathcal{A}$, there is $k < \omega$ with $N_\omega(i,k) \le r$ and $N_\omega(k,j) \le s$. (Take $k > i,j$ such that $\forall$ played the triangle move $i,j,r,s$ in round $k$.)

A *partial isomorphism* of $N_\omega$ is a partial one-one map $\theta : \omega \to \omega$ with finite domain, such that $N_\omega(i,j) = N_\omega(\theta(i),\theta(j))$ for all $i,j \in dom(\theta)$.

4. If $\phi : N_\omega \to N_\omega$ is a partial isomorphism, and $i < \omega$, then $\phi$ extends to a partial isomorphism defined on $i$.

   We may suppose $i \notin dom(\phi)$. Let $u < \omega$ be such that $\{i\} \cup dom(\phi) \cup rng(\phi) \subseteq u$. Let $|\phi| = q$, and choose $\theta : q+1 \to u$ with $rng(\theta|_q) = dom(\phi)$ and $\theta(q) = i$. Then there is $Q \in \mathcal{M}_{q+1}$ such that $\theta : Q \to N_u$ is an embedding. (Using lemma 59, take $Q = ((N_u)^\psi)|_{q+1} \in \mathcal{M}_{q+1}$, where $\psi$ is a permutation of $u$ extending $\theta^{-1}$.) Clearly, $\phi \circ \theta : Q|_q \to N_t$ is also an embedding, for all $t \ge u$. We may choose $t \ge u$ such that $\forall$ made the amalgamation move $Q, \phi \circ \theta$. So there is an embedding $\chi : Q \to N_{t+1}$ extending $\phi \circ \theta$. Then $\chi \circ \theta^{-1}$ is a partial isomorphism of $M_\omega$ extending $\phi$ and defined on $i$.
5. Any partial isomorphism of $N_\omega$ is induced by a partial isomorphism that is actually a permutation of $\omega$. (As $N_\omega$ is countable, repeated 'smooth' application of property 4 shows this.)

Define a binary relation $\sim$ on $\omega$ by $i \sim j$ iff $N_\omega(i,j) \le 1'$. It is easily checked that $\sim$ is an equivalence relation, and indeed a *congruence,* in that if $i \sim i'$ and $j \sim j'$ then $N_\omega(i,j) = N_\omega(i',j')$. Write $\omega/\!\sim$ for the set of equivalence classes, and $i^\sim$ for the equivalence class of $i$ ($i < \omega$). Let $M_{r_0}$ be the structure for the language of the theory $S_\mathcal{A}$ defining relativised representations, with domain $\omega/\!\sim$, given by

$$M_{r_0} \models r(i^\sim, j^\sim) \text{ iff } N_\omega(i,j) \le r,$$

for $r \in \mathcal{A}$. (We make explicit the dependence on $\forall$'s first move $r_0$ here.) There is such a structure $M_{r_0}$ for each non-zero $r_0 \in \mathcal{A}$. Property 5 ensures that each $M_{r_0}$ is homogeneous.

Let $r \approx s$ in $At\mathcal{A}$ iff $r \le 1\,;s\,;1$. Then $\approx$ is an equivalence relation on $At\mathcal{A}$, and, as can be checked, $r \approx s$ iff $M_r \models \exists xy\, s(x,y)$, for all atoms $r,s$. Take a set $E$ of representatives for the $\approx$-classes, and let $M$ be the disjoint union of the structures $M_e$ for $e \in E$. Then $M \models \exists xy\, r(x,y)$ for all non-zero $r \in \mathcal{A}$. It can be checked routinely that $M$ is a representation of $\mathcal{A}$; see [HH97d] for more details. Moreover, if $\theta$ is any finite partial isomorphism of $M$, then take $x \in dom(\theta) \cap M_e$. There is $a \in At\mathcal{A}$ such that $M \models a(x,x)$. If $\theta(x) \in M_{e'}$, then since $M \models a(\theta(x),\theta(x))$, then $e \approx a \approx e'$, so $e = e'$. So $\theta$ is a union of partial isomorphisms of the $M_e$. Each one extends to an automorphism of $M_e$, and their union is an extension of $\theta$ to an automorphism of $M$. So $M$ is homogeneous. □

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