Atom structures of cylindric algebras and relation algebras

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Abstract

For any finite $n \geq 3$ there are two atomic n-dimensional cylindric algebras with the same atom structure, with one representable, the other, not.

Hence, the complex algebra of the atom structure of a representable atomic cylindric algebra is not always representable, so that the class RCA_n of representable n-dimensional cylindric algebras is not closed under completions. Further, it follows by an argument of Venema that RCA_n is not axiomatisable by Sahlqvist equations, and hence nor by equations where negation can only occur in constant terms.

Similar results hold for relation algebras.

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1 Introduction

Algebraic logic is the study of algebraic theories corresponding to logical systems. Perhaps the oldest case is boolean algebra, which corresponds closely to propositional logic, or the logic of unary relations. In this paper we are concerned with the analogous systems for n-ary relations and binary relations, namely, cylindric algebras and relation algebras. As with boolean

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algebra, the notion of a cylindric algebra (or relation algebra) is defined axiomatically. The task is then to show (if possible) that any model of these axioms is isomorphic to a concrete algebra whose elements are n-ary relations on some set and whose operations are defined set-theoretically in terms of these relations. This is known as the representation problem: a general model of the axioms is a cylindric algebra, and one isomorphic to a concrete algebra is called a representable cylindric algebra, the isomorphism itself being a representation. The logical analogue of an algebraic representation result is a completeness theorem.

For boolean algebras, the representation problem found a successful solution in work of Stone [S]. Given a boolean algebra $\mathcal{B}$, Stone constructed a certain 'perfect' or 'canonical' extension $\mathcal{B}^*$ of it. $\mathcal{B}^*$ is isomorphic to a concrete algebra of unary relations, and so suffices to represent $\mathcal{B}$, but it can be characterised abstractly up to isomorphism over $\mathcal{B}$ by its topological properties. It is complete (closed under arbitrary joins, or sums) and atomic.

For cylindric algebras, the representation problem is not so easily resolved. In [JT], Jónsson and Tarski extended the canonical extension construction to cylindric algebras and relation algebras (and to BAOs: boolean algebras enriched with arbitrary additive operators), but this could not be used to show that a cylindric algebra $\mathcal{C}$ was representable because its canonical extension $\mathcal{C}^*$ was only isomorphic to an algebra of unary relations, and not, perhaps, of n-ary ones. The situation for relation algebras was similar. As it turned out, not every relation algebra is representable (Lyndon, [L]), and, indeed, the representable relation algebras are not finitely axiomatisable (Monk, [Mo1]); the same goes for cylindric algebras. However, Monk did show (reported in theorem 2.12 of [McK]) that the canonical extension of a representable algebra was also representable. For this and other reasons, canonical extensions became an important tool in algebraic logic and also in modal logic.

Another important kind of extension of an algebra $\mathcal{A}$ is its *completion,* which in essence is its smallest complete extension. More correctly, it is a complete algebra extending $\mathcal{A}$ and in which $\mathcal{A}$ is dense; this characterises it up to isomorphism over $\mathcal{A}$. Although the canonical extension $\mathcal{A}^*$ is also complete, in general it is not the same as the completion of $\mathcal{A}$. For example, the completion is only atomic when $\mathcal{A}$ is. Also, unlike canonical extensions, completions preserve all joins that exist in the original algebra. Monk [Mo3] extended the known notion of completion of a boolean algebra to completely additive BAOs, including the cylindric algebras and relation algebras, and showed that the completion of a cylindric algebra *is* a cylindric algebra (and similarly for relation algebras). However, the analogue for completions of the preservation of representability by canonical extensions could not be established. In this paper, we prove that representability is not always preserved by completions.

Our main result is:

Theorem 1.1 *For any finite $n \geq 3$ there are two atomic n-dimensional cylindric algebras $\mathcal{A}_n, \mathcal{C}_n$ with the same atom structure*[1]*, with $\mathcal{A}_n$ representable and $\mathcal{C}_n$ not representable.*

There are also two atomic relation algebras with the same atom structure, with one representable, the other, not.

We may replace $\mathcal{C}_n$ in the theorem by the full complex (or power set) algebra[2] over its atom structure, as this will also be non-representable (in fact, $\mathcal{C}_n$ is obtained that way anyway). $\mathcal{A}_n$ being atomic, it is evidently dense in $\mathcal{C}_n$, and $\mathcal{C}_n$ is clearly complete. So the completion of $\mathcal{A}_n$ is isomorphic to $\mathcal{C}_n$. Hence the following is completely equivalent to theorem 1.1:

Corollary 1.2 *For any finite $n \geq 3$, there exists a representable atomic n-dimensional cylindric algebra $\mathcal{A}_n$ whose completion $\mathcal{C}_n$ is not representable.*

There also exists a representable atomic relation algebra whose completion is not representable.

Moreover, $\mathcal{A}_n$ is dense in $\mathcal{C}_n$, since both are atomic and have the same atoms. So we obtain:

Corollary 1.3 *For each finite $n \geq 3$, there exists a non-representable atomic n-dimensional cylindric algebra $\mathcal{C}_n$ with a representable dense subalgebra, and similarly for relation algebras.*

This answers negatively a question posed in [AGMNS], namely whether a cylindric algebra with a representable dense subalgebra is necessarily representable itself. As the authors point out, this is equivalent to asking whether representability of cylindric algebras (and relation algebras) is preserved by completions, so corollary 1.3 is also equivalent to theorem 1.1.

We derive one further consequence of theorem 1.1 in corollary 1.7 below.

It is striking that, taking the boolean algebra structure of $\mathcal{A}_n$ and $\mathcal{C}_n$ as given, their cylindric algebra structure is determined by the way the diagonal and cylindrification operations behave on their atoms — i.e., by their atom structure. Of course, they have the *same* atom structure. Now the difficulties in finding representations for cylindric algebras mostly arise from their cylindric structure — as we saw, it is easy to find representations of a boolean algebra, while the representable cylindric algebras are not finitely axiomatisable. So one would think that these problems could be pinned down to the atom structure, in the case of atomic algebras. That is, representability of an atomic cylindric algebra should presumably depend only on its atom structure. Theorem 1.1 shows that this is not so: there is more to the issue than that.

[1]This will be defined formally below.

[2]This will also be defined formally below.

1.1 Varieties of BAOs

Let us now consider this in more detail, from the point of view of boolean algebras with operators (BAOs). First, some terminology. We write the boolean operations as $+, \cdot, -$. Let V be any variety of BAOs. So V is equationally axiomatised, and each of its non-boolean operations f (n-ary, say) is *normal* (meaning $\mathsf{V} \models f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ for each $1 \leq i \leq n$) and *additive* (i.e., $\mathsf{V} \models f(x_1, \ldots, x_{i-1}, y+z, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$ for each $1 \leq i \leq n$; the variables are implicitly universally quantified here).

Definition 1.4

1. In this context, an *atom structure* is a structure in the signature consisting of an $(n+1)$-ary relation symbol R_f for every n-ary function symbol $f \in Sig\mathsf{V}$, the non-boolean part of the signature of V.

2. Let $\mathcal{A} \in \mathsf{V}$, and suppose that $\mathcal{A}$ is atomic (more properly, the boolean reduct of $\mathcal{A}$ is an atomic boolean algebra). The *atom structure of* $\mathcal{A}$, written $At\mathcal{A}$, is the atom structure with domain the set of all atoms of $\mathcal{A}$ and with the relation symbol R_f (for n-ary $f \in Sig\mathsf{V}$) being interpreted so:

$$At\mathcal{A} \models R_f(a, b_1, \ldots, b_n) \iff \mathcal{A} \models a \leq f(b_1, \ldots, b_n), \quad \text{for all } a, b_1, \ldots, b_n \in At\mathcal{A}.$$

3. We write $At\mathsf{V}$ for the class $\{At\mathcal{A} : \mathcal{A} \in \mathsf{V}, \mathcal{A} \text{ atomic}\}$ of atom structures of atomic algebras in V.

4. Given an atom structure $\mathcal{F}$, the *complex algebra* over $\mathcal{F}$ is defined to be the algebra $Cm\mathcal{F} = (\wp(\mathcal{F}), -, \cap, \emptyset, \mathcal{F}, f)_{f \in Sig\mathsf{V}}$, where $(\wp(\mathcal{F}), -, \cap, \emptyset, \mathcal{F})$ is the boolean algebra of subsets of $\mathcal{F}$, and for each n-ary $f \in Sig\mathsf{V}$ and $X_1, \ldots, X_n \in \wp(\mathcal{F})$,

$$f(X_1, \ldots, X_n) = \{a \in \mathcal{F} : \mathcal{F} \models R_f(a, b_1, \ldots, b_n) \text{ for some } b_1 \in X_1, \ldots, b_n \in X_n\}.$$

   Of course, the atom structure of $Cm\mathcal{F}$ is isomorphic to $\mathcal{F}$.

5. By *complex algebra* we mean simply the complex algebra of some atom structure.

6. Write RCA_n for the variety of representable n-dimensional cylindric algebras ($n \geq 3$) and RRA for the variety of representable relation algebras.

Intuitively, the harder it is to determine whether an algebra is in V, the more complicated V is. So one measure of the complexity of V is the difficulty in distinguishing two algebras, one in V and the other not. Proving that V is not finitely axiomatisable, for example, shows that no first-order sentence serves to make the distinction. Similarly, if it can be shown that no equations generated by schemata of a certain type will axiomatise V, then these schemata do not capture the full nature of V. Results of these kinds have indeed been proved for the important varieties RCA_n and RRA — e.g., [Mo1,Mo2,A]. (The results of the current paper add to them somewhat; see corollary 1.7 below.)

1.2 Completely additive varieties

Atomic algebras provide another measure of the complexity of V in the same vein, as one can ask whether, for an atomic algebra, its membership of V is determined by its atom structure: whether if $\mathcal{A}, \mathcal{B}$ are atomic algebras of the signature of V, and $At\mathcal{A} \cong At\mathcal{B}$, then $\mathcal{A} \in \mathsf{V} \iff \mathcal{B} \in \mathsf{V}$. If not, it indicates again that V is rather complicated. But if so, then as we can often recover an atomic algebra from its atom structure, the study of at least the atomic algebras in V will reduce to the study of $At\mathsf{V}$. In modal logic, this corresponds to working on the 'frame' level, and it has the advantage of allowing the use of modal-logical techniques.

Let us see how this recovery works. An algebra is said to be *completely additive* if the operations of $Sig\mathsf{V}$ distribute over all joins that exist in the algebra. Formally, $\mathcal{A}$ is completely additive if for any n-ary $f \in Sig\mathsf{V}$, $r_1, \ldots, r_n \in \mathcal{A}$, $1 \leq i \leq n$, $S \subseteq \mathcal{A}$, we have

$$r_i = \bigvee S \quad \Rightarrow \quad f(r_1, \ldots, r_n) = \bigvee_{s \in S} f(r_1, \ldots, r_{i-1}, s, r_{i+1}, \ldots, r_n).$$

If $\mathcal{A}$ is atomic and completely additive, we have

$$f(r_1, \ldots, r_n) = r \iff$$
$$r = \bigvee \{a : a, b_1, \ldots, b_n \in At\mathcal{A},\ b_i \leq r_i \, (1 \leq i \leq n),\ At\mathcal{A} \models R_f(a, b_1, \ldots, b_n)\}$$

for all $r, r_1, \ldots, r_n \in \mathcal{A}$ and n-ary $f \in Sig\mathsf{V}$. So the full structure of $\mathcal{A}$ is recoverable from its atom structure. This is not to say that there is always a unique algebra with a given atom structure; there is not. We only mean that the non-boolean structure of an atomic algebra is recoverable from its boolean structure together with its atom structure.

We say that V is completely additive if every algebra in V is so. One might expect such varieties to be rare, as complete additivity appears unlikely to be axiomatisable in first-order logic. But the boolean meet and join are already completely additive, and this often transfers to the non-boolean operations on V. This is because many common varieties are *conjugated:* for any n-ary $f \in Sig\mathsf{V}$ and $1 \leq i \leq n$, there is a term $t_i^f(x_1, \ldots, x_n)$ in the signature of V such that for any $\mathcal{A} \in \mathsf{V}$ and $a_1, \ldots, a_n, b \in \mathcal{A}$, we have

$b \cdot f(a_1, \ldots, a_n) = 0$ iff $a_i \cdot t_i^f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) = 0$. It is an exercise to show that any conjugated variety is completely additive (see [JT]). The varieties RRA and RCA_n are conjugated, and so are completely additive.

In the completely additive case we can tighten the connection between V and $At\mathsf{V}$, as Venema has shown. Let $\mathcal{F}$ be an atom structure, and write $Tm\mathcal{F}$ for the subalgebra of $Cm\mathcal{F}$ generated by the atoms of $Cm\mathcal{F}$. $Tm\mathcal{F}$ is atomic and its atom structure is $\mathcal{F}$. We call it the *term algebra* over $\mathcal{F}$, since every element of it is the value of some V-term with atoms as parameters. Because in the completely additive context the structure of an atomic algebra is determined by its atom structure, if $\mathcal{A}$ is *any* atomic algebra in V with atom structure $\mathcal{F}$ then the subalgebra of $\mathcal{A}$ generated by its atoms is isomorphic to $Tm\mathcal{F}$. Since V is closed under isomorphism and taking subalgebras, we have

$$(*) \qquad \mathcal{F} \in At\mathsf{V} \iff Tm\mathcal{F} \in \mathsf{V}, \text{ for all atom structures } \mathcal{F}.$$

Clearly, $Tm\mathcal{F}$ is completely additive. It follows that for each $t \in Tm\mathcal{F}$, the set of atoms lying beneath t is definable in $\mathcal{F}$ by a first-order formula with parameters in $\mathcal{F}$. By 'substituting' these formulas into the equations defining V, we arrive at a set Σ_V of first-order sentences expressing in terms of $\mathcal{F}$ that $Tm\mathcal{F}$ satisfies the equations of V. By $(*)$,

Theorem 1.5 (Venema, [V2]) *If V is completely additive then $At\mathsf{V}$ is elementary and is explicitly axiomatised by Σ_V, a first-order theory that can be constructed effectively from an equational axiomatisation of V.*[3]

It follows from this that $At\mathsf{RCA}_n$ and $At\mathsf{RRA}$ are elementary classes. (It can be shown that they are not finitely axiomatisable in first-order logic, nor in the infinitary logic $L^\omega_{\infty\omega}$.)

1.3 Sahlqvist axiomatisations

Given a completely additive variety V, we know that we can recover an atomic algebra in V from its atom structure. Theorem 1.5 makes us think that we have at least as tight a grasp on $At\mathsf{V}$ as on V. To complete the picture, it would be satisfactory to show that an atomic algebra's membership of V is determined by its atom structure: i.e., for atomic $\mathcal{A}, \mathcal{B}$, if $\mathcal{A} \in \mathsf{V}$ and $At\mathcal{A} \cong At\mathcal{B}$ then $\mathcal{B} \in \mathsf{V}$. Then, as we said, the study of V could in large measure be carried out on $At\mathsf{V}$.

Unfortunately, things are not so simple. In [V1], Venema shows that conjugated Sahlqvist varieties do behave like this:

[3] An earlier draft of the present paper included a proof of this result for V the variety RRA of representable relation algebras; it has been superseded by Venema's result and so no longer appears.

Fact 1.6 (Venema, [V1]) *If* V *is a conjugated variety and is axiomatisable by Sahlqvist equations,* $\mathcal{A}, \mathcal{B}$, *are atomic, and* $At\mathcal{A} \cong At\mathcal{B}$, *then* $\mathcal{A} \in \mathsf{V} \iff \mathcal{B} \in \mathsf{V}$.

But in [V2] he shows that not all varieties do. Theorem 1.1 above shows that the representable cylindric algebras are also badly-behaved in this regard, since, in the notation of that theorem, we have $At\mathcal{A}_n = At\mathcal{C}_n$, $\mathcal{A}_n \in \mathsf{RCA}_n$, but $\mathcal{C}_n \notin \mathsf{RCA}_n$. So the idea of studying RCA_n via $At\mathsf{RCA}_n$ is problematic. The situation for RRA is similar, by the relation algebra part of theorem 1.1.

Note that since RCA_n and RRA are conjugated, we deduce from theorem 1.1 and fact 1.6 the following:

Corollary 1.7 RCA_n *is not axiomatisable by Sahlqvist equations. Neither is* RRA.

This strengthens a result of Andréka [A] that RCA_n cannot be axiomatised by positive equations. It also solves a problem raised in [HMT], namely whether RCA_n can be axiomatized with 'positive-in-the-wider-sense' formulas — i.e., complementation can occur in constant terms. Andréka's proof does not settle that case. But corollary 1.7 shows that no Sahlqvist, hence no positive-in-the-wider-sense axiomatization, is possible.

1.4 Complex algebras

In many important cases, V is the variety generated by its complex algebras. If so, we might hope to study V by studying the class $Str\mathsf{V}$ of atom structures of complex algebras in V, or perhaps some other class of atom structures whose complex algebras generate V, as an alternative to the now compromised plan to use $At\mathsf{V}$.

One such case is when V is *canonical* — closed under the map taking an algebra $\mathcal{A}$ to its canonical extension. Then, V is actually the class of all algebras that embed into $Cm\mathcal{F}$ for some $\mathcal{F} \in Str\mathsf{V}$ (in the standard notation, $\mathsf{V} = \mathsf{S}CmStr\mathsf{V}$), so the connection is even tighter. We saw above the result of Monk that RRA and RCA_n are canonical varieties.

Goldblatt [G] discusses this approach. We conclude with some questions related to it. Not many of them are due to us.

1. Is $Str\mathsf{RRA}$ an elementary class? (This was asked by Maddux [M2].) Is it closed under elementary equivalence? Is it set-theoretically absolute? We make two remarks here. (1) It can be shown that $Str\mathsf{RRA}$ is not finitely axiomatisable in first-order logic, nor in $L^{\omega}_{\infty\omega}$. (2) In [L], Lyndon gave an infinite set of first-order conditions which axiomatise the finite representable relation algebras. All quantifiers in these conditions are already relativised to atoms, so they can be rewritten easily as conditions on relation algebra atom structures. It can be

shown that any relation algebra atom structure satisfying these conditions is in $Str\mathsf{RRA}$. The converse fails: a counterexample can be found in [M1, p.154ff], where it is used differently, to show that there are relation algebras with an n-dimensional 'cylindric basis' but no such $(n+1)$-dimensional basis.

2. What is the corresponding situation for $Str\mathsf{RCA}_n$ $(3 \leq n < \omega)$?

3. Note that $At\mathsf{V} = \{$atom structures $\mathcal{F} : \mathcal{A} \in \mathsf{V}$ for *some* $\mathcal{A}$ with $At\mathcal{A} = \mathcal{F}\}$, while if V is completely additive, $Str\mathsf{V} = \{$atom structures $\mathcal{F} : \mathcal{A} \in \mathsf{V}$ for *all* $\mathcal{A}$ with $At\mathcal{A} = \mathcal{F}\}$. So it is of interest whether results such as theorem 1.5 carry over to $Str\mathsf{V}$. We therefore ask for which V is $Str\mathsf{V}$ elementary.

4. More generally, is every canonical variety generated by the complex algebras of an elementary class of frames? This is an important question in modal logic, equivalent to asking whether any canonical modal logic (one validated by its canonical frame) is characterised by an elementary class of Kripke frames. (The converse was proved by van Benthem [Be].) Goldblatt [G] showed it to be true when V also satisfies $At\mathsf{V} = Str\mathsf{V}$; a proof can be obtained using the methods of theorem 1.5 above.

1.5 The proof

The underlying reason why an atomic algebra $\mathcal{A}$ can be representable and its completion $\mathcal{C}$ not representable is that $\mathcal{C}$ generally has more elements than $\mathcal{A}$. These would have to be represented properly in a representation of $\mathcal{C}$, which means that their boolean and cylindric properties in $\mathcal{C}$ must be mirrored by the relations they become. For certain $\mathcal{A}$, deadlocks occur however one tries to find suitable relations for the extra elements in $\mathcal{C}$. Crudely, $\mathcal{A}$ has few relations so a representation of it can sweep potential problems under the carpet. Adding the new relations in $\mathcal{C}$ forces the problems to the surface.

Let us outline the proof of theorem 1.1. We said that n-dimensional cylindric algebra is intended as an algebraic analogue of the logic of n-ary relations, so it is fitting that the algebra $\mathcal{A}_n$ of theorem 1.1 is obtained as the algebra of those sets of n-tuples of a certain structure M that are definable in n-variable first-order logic, L^n. $\mathcal{A}_n$ is by definition a concrete algebra, so is representable. We also require it to be atomic, which roughly we achieve by finding M which is 'n-homogeneous'; there are some subtleties here, which we will go into later.

$\mathcal{C}_n$ is the completion of $\mathcal{A}_n$. So one might think that it should be the sets of n-tuples of M definable in n-variable infinitary logic, where arbitrary conjunctions and disjunctions of formulas can be taken. This would only be so if the given representation of $\mathcal{A}_n$ is *complete* — that is, it respects all joins that exist in $\mathcal{A}_n$ — whereas in fact no $\mathcal{A}_n$ as in theorem 1.1 can have a complete representation. But there is available another kind of representation of $\mathcal{A}_n$, obtained by *relativising* to the union of the atoms of $\mathcal{A}_n$. This union is a set of n-tuples of M but not the set of all n-tuples. This relativised representation is complete, and we can now obtain $\mathcal{C}_n$ from it by closing under union. By giving n-variable logic itself a relativised semantics, the two notions match, and $\mathcal{C}_n$ is expressed in terms of infinitary n-variable logic, as expected.[4]

The final step is to choose M so that $\mathcal{C}_n$ is not representable. Let us sketch how this is done for relation algebras, which are also covered in theorem 1.1; the argument for cylindric algebras is essentially the same. We have a representable atomic relation algebra $\mathcal{A}$ and its elements can be taken to be all binary relations definable on M in 3-variable first-order logic. Its operations are of course the boolean functions, identity (equality), converse, and relational composition ';'. Write $\mathcal{C}$ for the completion of $\mathcal{A}$.

In this case, the signature of M consists of binary relation symbols, and, roughly speaking, their interpretations in M are the atoms of $\mathcal{A}$. For this sketch we treat the atoms as being symmetric. Broadly, we can view M as a complete undirected graph whose edges are coloured, the colours being the relations in its signature. The most important colours are the shades of red, r^i_{jk} for $i < \omega$ and $j < k < 3$, and these can be regarded as atoms of $\mathcal{C}$. There are further atoms of $\mathcal{C}$, coloured white, green, etc, but we will not discuss these now. There is also a special shade of red, ρ, which comes from a relation outside the signature and is not an atom of $\mathcal{C}$. Nonetheless, M does have ρ-coloured edges.

The critical part of the structure of M is the red part. M will have an infinite set of points with ρ-edges between any two of them. Because $\mathcal{C}$ arises from M, any representation of $\mathcal{C}$ would also have an infinite set of points, say a_n $(n < \omega)$, the relation between any two being 'red'. We may call this set a 'red clique'. More formally, the representation is an isomorphism from $\mathcal{C}$ to a concrete relation algebra. Under this isomorphism, the join in $\mathcal{C}$ of all the red atoms corresponds to a binary relation 'red' which holds on any pair (a_n, a_m) for distinct $n, m < \omega$. Remember that the red atoms are the r^i_{jk} only, and do not include ρ.

The difficulty we find in representing $\mathcal{C}$ is that it is complete, so that the join of every set of red atoms is available as an element of the algebra. This means that the representation must 'decide' whether (the relation corresponding to) any given join of reds should hold or fail between any two distinct a_n in the clique. Consider the three joins $\bigvee_{i<\omega} r^i_{jk} = R_{jk}$, for each $j < k < 3$. We know that the join R of all red atoms holds between every pair a_n, a_m. R is finitely partitioned by $R_{01} \vee R_{02} \vee R_{12}$, so exactly one R_{jk}

[4] It therefore seems more natural to *define* $\mathcal{A}_n$ in terms of this relativised representation, and this is the approach we take in the text.

holds between any two a_n. By the pigeonhole principle, we can find three points, say a_0, a_1, a_2, with the same R_{jk} holding between a_0, a_1 and between a_0, a_2.

However, M is designed so that no red triangle of the form $(r^i_{jk}, r^{i'}_{jk}, r^{i''}_{j'k'})$, with two equal lower pairs of indices, embeds into it. Hence, $(r^i_{jk}; r^{i'}_{jk}) \cdot r^{i''}_{j'k'} = 0$ in $\mathcal{C}$. By complete additivity, $(R_{jk}; R_{jk}) \cdot R = 0$ in $\mathcal{C}$ also. So no representation of $\mathcal{C}$ can contain a triangle of the form (a_0, a_1, a_2) above. This is a contradiction and we deduce that $\mathcal{C}$ has no representation.

Why was this not a problem in representing $\mathcal{A}$? Simply because the joins $R_{jk} = \bigvee_{i<\omega} r^i_{jk}$ do not exist in $\mathcal{A}$, so a representation is not forced to decide which of them holds on an edge in a red clique. In fact, the only joins of red atoms that exist in $\mathcal{A}$ are joins of finitely and of cofinitely many reds. In the representation given by M, only the cofinite joins of reds hold on edges coloured by ρ. No inconsistency is created by this. Thus, ρ functions in a sense as a non-standard red colour, corresponding to the non-principal ultrafilter of $\mathcal{A}$ generated by all cofinite sets of red atoms.

Clearly, all this hinges on the construction of M. This is done in proposition 2.6 below, and we will try to explain the idea informally in section 2.2. The method was developed by Hirsch in [Hi], where it was used to show that the class of relation algebras having a complete representation is not elementary. The method was extended to give the corresponding result for both relation algebras and cylindric algebras in [HH2]. More recently it has been used to prove that it is undecidable whether a finite relation algebra is representable [HH3], and that the variety generated by the relation algebras having an n-dimensional basis (cf. [M3]) is not finitely axiomatisable [HH4].

This paper is in the area of algebraic logic, but we have hopes of reaching a wider audience. We believe that the use of model theory may help to do this, as well as being appropriate to the material. To the same end, we have tried to make the work self-contained; we hope that algebraic logicians in particular will bear with us if we appear to be repeating some of their standard arguments, as in lemma 5.5, for example.

Outline of the paper

In the next section we construct the coloured graph M discussed above. Some discussion of it takes place in section 3, preceded by some model-theoretic definitions. The sets of n-tuples of M that are definable by formulas with n variables will be used in sections 4 and 5 to provide algebras with the required properties for theorem 1.1. In section 4 we also recall the requisite facts on cylindric algebras. Section 6 briefly discusses the relation algebra case of theorem 1.1.

Notation

Our notation is mostly standard. We usually use the same notation for a structure, graph, or algebra as for its domain or universe; this is standard model-theoretic and algebraic practice, though it is admittedly not common in algebraic logic. An ordinal is the set of all smaller ordinals: so for $n < \omega$, $n = \{0, 1, \ldots, n-1\}$. Throughout, maps are regarded formally, as sets of ordered pairs. Thus, if θ is a map, we write $|\theta|$ for the cardinality of the set that is θ. We write $dom(\theta), rng(\theta)$ for the domain and range of θ, respectively. We write Id_X for the identity map on a set X. $\wp(X)$ denotes the power set of X.

We write $\bar{a}, \bar{x}$, etc., for sequences. A sequence (or tuple) $\bar{a}$ of elements of a set X, of length n, is formally an element of the set ${}^nX$ of maps from n to X. We write a_i for the ith element of this sequence ($i < n$), and $rng(\bar{a})$ for $\{a_0, \ldots, a_{n-1}\}$. We may write $\bar{a}$ as $(a_0, \ldots, a_{n-1})$. If $\theta : X \to Y$ is a map, we write $\theta(\bar{a})$ for the sequence $(\theta(a_0), \ldots, \theta(a_{n-1})) \in {}^nY$. If $\bar{a}, \bar{b}$ are n-sequences, we write $(\bar{a} \mapsto \bar{b})$ for the map $\{(a_i, b_i) : i < n\}$. For $i < n$, we write $\bar{a} \equiv_i \bar{b}$ if $a_j = b_j$ for all $j < n$ with $j \neq i$.

2 Coloured graphs

We are now going to deal with the cylindric algebra case, and we fix the dimension $3 \leq n < \omega$ of our cylindric algebras. The first aim is to construct a certain 'coloured graph' ('M', of proposition 2.6, as discussed in section 1.5). We will discuss the construction informally in section 2.2.

2.1 Definitions

Let us first set down what a coloured graph is.

Definition 2.1 A *coloured graph* is an undirected graph Γ such that every edge (*unordered* pair of nodes) of Γ is coloured (or labelled) by a unique *edge colour* (below), and some *ordered* $(n-1)$-tuples have unique colours, too. The edge colours are:

- greens: g_i ($i = 1, \ldots, n-2$) and g_0^i ($i < \omega$);
- whites: w_i ($i = 0, \ldots, n-2$);
- reds: r_{jk}^i ($i < \omega, j < k < n$), and ρ.

The colours for $(n-1)$-tuples are:

- yellows: y_S ($S \subseteq \omega$, $S = \omega$ or S finite[5].)

[5]For the construction of proposition 2.6, it suffices if $S = \omega$ or $|S| < n$.

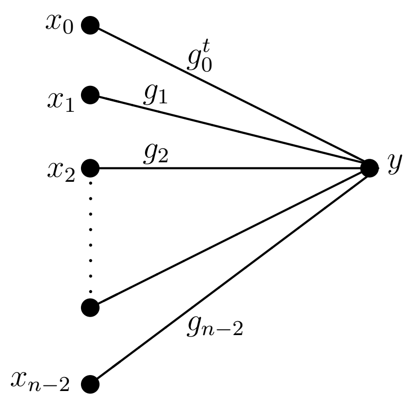

Figure 1: a t-cone

Notation 2.2 We will sometimes write $\Gamma(x, y)$ for the colour of an edge (x, y) in the coloured graph Γ. Note that these may not always be defined: for example, $\Gamma(x, x)$ is not.

If Γ is a coloured graph, and $D \subseteq \Gamma$, we write $\Gamma\lceil D$ for the induced subgraph of Γ on the set D (it inherits the edges and colours of Γ, on its domain D). We write $\Delta \subseteq \Gamma$ if Δ is an induced subgraph of Γ in this sense.

Definition 2.3 Let Γ, Δ be coloured graphs, and $\theta : \Gamma \to \Delta$ be a map. θ is said to be a *coloured graph embedding,* or simply an *embedding*, if it is injective and preserves all edges, and all colours, where defined, in both directions. An *isomorphism* is a bijective embedding.

Definition 2.4 Let Γ be a coloured graph consisting of n nodes, $x_0, \ldots, x_{n-2}, y$, such that (x_j, y) is an edge of Γ for each $j < n-1$. Let $t < \omega$. We call Γ a *t-cone* if for each $j < n-1$, the edge (x_j, y) is coloured g_j if $j > 0$, and g_0^t if $j = 0$, and no other edges of Γ (if any) are coloured green. See figure 1. The *apex* of the cone is y, its *base* $\{x_0, \ldots, x_{n-2}\}$. The *tint* of the cone is t. These are well-defined, as any Γ can be viewed as a cone in at most one way. Notice that a cone induces a linear ordering on its base, namely, $x_0, \ldots, x_{n-2}$.

Now we define a class $\mathcal{G}$ of certain coloured graphs.

Definition 2.5 The class $\mathcal{G}$ consists of all coloured graphs Γ (possibly the empty graph) with the following properties.

1. Γ is a complete graph (all possible edges are present)
2. Γ contains no triangles of the following types:

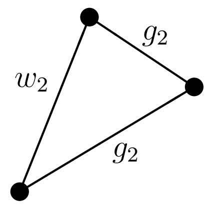

Figure 2: a triangle of the form (g_2, g_2, w_2)

- (g, g', g^*) for any green colours g, g', g^*
- (g_i, g_i, w_i) for any $i = 1, \dots n-2$
- (g_0^j, g_0^k, w_0) for any $j, k < \omega$
- $(r_{jk}^i, r_{j'k'}^{i'}, r_{j^*k^*}^{i^*})$ unless $i = i' = i^*$ and $|\{(j,k), (j',k'), (j^*,k^*)\}| = 3$
- $(r_{jk}^i, r_{j'k'}^{i'}, \rho)$ for any i, j, k, i', j', k'
- (r_{jk}^i, ρ, ρ) for any i, j, k

Roughly (ignoring yellows), this means that no coloured graph of the form shown in figure 2, for example, embeds into Γ. More formally, there do not exist $x, y, z \in \Gamma$ with $\Gamma(x,y) = \Gamma(y,z) = g_2$ and $\Gamma(x,z) = w_2$.

3. If $a_0, \dots, a_{n-2} \in \Gamma$ are distinct, and no edge (a_i, a_j) $(i < j < n-1)$ is coloured green, then the tuple $\langle a_0, \dots, a_{n-2} \rangle$ is coloured a unique shade of yellow. No other $(n-1)$-tuples are coloured yellow.

4. If $D = \{d_0, \dots, d_{n-2}, \delta\} \subseteq \Gamma$, and $\Gamma \lceil D$ (the coloured graph induced on D) is a t-cone with apex δ, inducing the ordering $d_0, \dots, d_{n-2}$ on its base, and the tuple $\langle d_0, \dots, d_{n-2} \rangle$ is coloured y_S, then $t \in S$.

Clearly, $\mathcal{G}$ is closed under isomorphism and under induced subgraphs. $\mathcal{G}$ depends on n.

2.2 Remarks

The idea behind the definition is roughly as follows. In proposition 2.6 below, we construct a countably infinite graph $M \in \mathcal{G}$ which will be 'n-homogeneous', in the sense that the *context* of any subgraph $\Delta \subseteq M$ of size $< n$ — the ways in which Δ can be extended in M to a subgraph of size n — depend only on the isomorphism type of Δ, and not on the 'location' of Δ within M. We achieve this by building M as the union of a chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ of finite graphs in $\mathcal{G}$, in ω stages. The stages will be used to pad out the contexts of any two copies of any Δ to be the same. For this to work, the rules defining $\mathcal{G}$ must make it easy to 'glue on' (or amalgamate) a context to any Γ_i. This is the role of the white and yellow colours. Triangles with a white side are uncommon in definition 2.5(2), so white allows a context to be glued on fairly freely. Where white cannot be used, ρ can be; but only if it fits the existing context. Yellow helps here, as it prohibits certain inconvenient contexts from occurring at all, by coding the ones that are allowed.

Roughly, the n-homogeneity of M will allow us to construct an *atomic* n-dimensional cylindric algebra $\mathcal{A}$ from M. The atoms of $\mathcal{A}$ will be essentially the subgraphs of M of size $\leq n$ with no ρ-edge. To show that every non-zero element of $\mathcal{A}$ contains an atom will require blurring the distinction between ρ and the r^i_{jk}; but $\mathcal{G}$ is fairly even-handed between these, and the homogeneity of M is sufficient to cope. The machinery that makes it work is introduced in section 3, where we will discuss it further; the process is completed in section 4.

The green colours are not to do with homogeneity. They create the 'red clique' of the introduction (Section 1.5), yielding non-representability of the algebra $\mathcal{C}_n$ of theorem 1.1.

2.3 The main construction

Proposition 2.6 *There is a countable coloured graph $M \in \mathcal{G}$ with the following property:*

- *If $\Delta \subseteq \Delta' \in \mathcal{G}$, $|\Delta'| \leq n$, and $\theta : \Delta \to M$ is an embedding, then θ extends to an embedding $\theta' : \Delta' \to M$.*

Proof. Two players, $\forall$ and $\exists$, play a game to build a coloured graph M. They play by choosing a chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ of finite graphs in $\mathcal{G}$; the union of the chain will be the graph M.

There are ω rounds. In each round, $\forall$ and $\exists$ do the following. Let $\Gamma \in \mathcal{G}$ be the graph constructed up to this point in the game. $\forall$ chooses $\Delta \in \mathcal{G}$ of size $< n$, and an embedding $\theta : \Delta \to \Gamma$. He then chooses an extension $\Delta \subseteq \Delta^+ \in \mathcal{G}$, where $|\Delta^+ \setminus \Delta| \leq 1$. These choices, $(\Delta, \theta, \Delta^+)$, constitute his move. $\exists$ must respond with an extension $\Gamma \subseteq \Gamma^+ \in \mathcal{G}$ such that θ extends to an embedding $\theta^+ : \Delta^+ \to \Gamma^+$. Her response ends the round.

The starting graph $\Gamma_0 \in \mathcal{G}$ is arbitrary but we will take it to be the empty graph in $\mathcal{G}$.

Lemma 2.7 *$\exists$ never gets stuck — she can always find a suitable extension $\Gamma^+ \in \mathcal{G}$.*

Proof. Let $\Gamma \in \mathcal{G}$ be the graph built at some stage, and let $\forall$ choose the graphs $\Delta \subseteq \Delta^+ \in \mathcal{G}$ and the embedding $\theta : \Delta \to \Gamma$. Thus, his move is $(\Delta, \theta, \Delta^+)$.

We now describe $\exists$'s response. If Γ is empty, she may simply play Δ^+. Otherwise, *she plays $\Gamma^+ = \Gamma$ if she can* — i.e., if $\Delta^+ = \Delta$, if Δ is empty

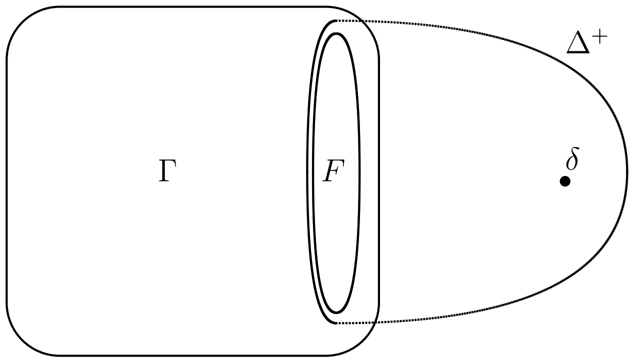

Figure 3: $\forall$'s move — the graph Γ^*

and Γ is not, or if $|\Delta^+ \setminus \Delta| = 1$, $\Delta^+ \setminus \Delta = \{\delta\}$, and there is already a node $\gamma \in \Gamma$ such that $\theta \cup \{(\delta, \gamma)\}$ is a coloured graph embedding from Δ^+ into Γ.

So assume that she can't play $\Gamma^+ = \Gamma$ (we will use this assumption later). Let $F = rng(\theta) \subseteq \Gamma$. (So $|F| < n$.) Since Δ and $\Gamma\lceil F$ are isomorphic coloured graphs (via θ), and $\mathcal{G}$ is closed under isomorphism, we may assume with no loss of generality that $\forall$ actually played $(\Gamma\lceil F, Id_F, \Delta^+)$, where $\Gamma\lceil F \subseteq \Delta^+ \in \mathcal{G}$, $\Delta^+ \setminus F = \{\delta\}$, and $\delta \notin \Gamma$. We may view $\forall$'s move as building a coloured graph $\Gamma^* \supseteq \Gamma$, whose nodes are those of Γ together with δ, and whose edges are the edges of Γ together with edges from δ to every node of F. The coloured graph structure on Γ^* is given by

- Γ is an induced subgraph of Γ^* (i.e., $\Gamma \subseteq \Gamma^*$)
- $\Gamma^*\lceil(F \cup \{\delta\}) = \Delta^+$.

See figure 3. Colours of edges and $(n-1)$-tuples in Δ^+ but not in Γ are determined by $\forall$'s move, so we regard him as having chosen them. Note that no $(n-1)$-tuple containing both δ and elements of $\Gamma \setminus F$ has a colour in Γ^*.

Now $\exists$ must extend Γ^* to a complete graph on the same nodes and complete the colouring, yielding a graph $\Gamma^+ \in \mathcal{G}$. Thus, she has to define the colour $\Gamma^+(\beta, \delta)$ for all nodes $\beta \in \Gamma \setminus F$, and also select appropriate yellow colours for $(n-1)$-tuples of nodes of Γ^* where necessary, in such a way as to meet the conditions of definition 2.5. She does this as follows.

1. If there is no $f \in F$ such that $\Gamma^*(\beta, f)$, $\Gamma^*(\delta, f)$ are coloured g_0^t and g_0^u for some t, u, respectively, then $\exists$ defines the colour $\Gamma^+(\beta, \delta)$ to be w_0.

2. Otherwise, if for some i with $0 < i < n-1$, there is no $f \in F$ such that $\Gamma^*(\beta, f)$, $\Gamma^*(\delta, f)$ are both coloured g_i, then $\exists$ defines the colour $\Gamma^+(\beta, \delta)$ to be w_i for any such i (say, the least such).

3. Otherwise, δ and β are both the apexes of cones on F in Γ^* that induce the same linear ordering on F. (There are no green edges in F because $\Delta^+ \in \mathcal{G}$, so it has no green triangles.) Now $\exists$ has no choice but to pick a red colour for $\Gamma^+(\beta, \delta)$ — a green label is impossible because then (β, δ, f) (any $f \in F$) would be an all-green triangle, contrary to definition 2.5(2); w_i for $i > 0$ is also impossible, because there is $f \in F$ with (β, f) and (δ, f) both labelled g_i; and a similar problem occurs with w_0.

   The colour she chooses is ρ.

This defines all edge colours of Γ^+. Notice that $\exists$ only chooses red or white colours for her edges. She never uses green.

4. Finally, for each tuple of distinct elements $\bar{a} = (a_0, \dots, a_{n-2}) \in {}^{n-1}(\Gamma^+)$ such that $\bar{a} \notin {}^{n-1}\Gamma \cup {}^{n-1}\Delta^+$ and with no edge (a_i, a_j) coloured green in Γ^+, $\exists$ colours $\bar{a}$ by y_S, where

$$S = \{t < \omega : \text{there is a } t\text{-cone in } \Gamma^* \text{ with base } a_0, \dots, a_{n-2}, \text{ in the induced ordering}\}.$$

   Clearly, S is finite. We remark that it can be shown that $|S| \leq |F| < n$.

This completes the definition of Γ^+.

It remains to check that this strategy works — that conditions 2, 3, and 4 from the definition of $\mathcal{G}$ (definition 2.5) are met.

First we check that $(n-1)$-tuples are labelled appropriately, by yellow colours. Condition 3 of definition 2.5 is trivially satisfied. Consider condition 4. Let D be a set of n nodes of Γ^+, and suppose that $\Gamma^+\lceil D$ is a t-cone with base $\{d_0, \dots, d_{n-2}\}$, say, and that the tuple $\bar{d} = (d_0, \dots, d_{n-2})$ (in the induced ordering) is labelled y_S in Γ^+. We must show that $t \in S$. Note first that if $D \subseteq \Gamma$ then as the graph Γ constructed so far is in $\mathcal{G}$, we do have $t \in S$. If $D \subseteq \Delta^+$, then as $\forall$ chose Δ^+ in $\mathcal{G}$ we get $t \in S$ similarly. If neither holds, then D contains δ and also some node $\beta \in \Gamma \setminus F$. $\exists$ has just chosen the colour $\Gamma^+(\beta, \delta)$, and her strategy ensures that it is not green. Hence, neither β nor δ can be the apex of the cone $\Gamma^+\lceil D$, so they must both lie in the base, $\bar{d}$. This implies that $\bar{d}$ is not yet labelled in Γ^*; so $\exists$ has just applied her strategy to choose the colour y_S to label $\bar{d}$ in Γ^+. But the strategy will have chosen S containing t, since $\Gamma^*\lceil D$ is already a cone in Γ^* — $\exists$ never chooses a green edge, so all green edges of Γ^+ lie in Γ^*. This is satisfactory, and we are done.

We now check condition 2, about edge colours of triangles. The new triangles — those in Γ^+ but not in Γ^* — come in two kinds: those of the form (β, δ, f) for some $f \in F$ and $\beta \in \Gamma \setminus F$, and those of the form (β, β', δ) for distinct $\beta, \beta' \in \Gamma \setminus F$. See figure 4.

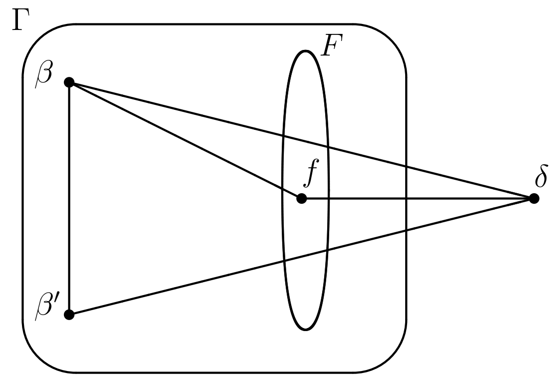

Figure 4: the new triangles

For the first kind, note that if $\exists$ coloured (β, δ) with ρ then both (β, f) and (δ, f) must be green, so there can be no clash with definition 2.5(2). If she used white (w_i, say) to colour (β, δ), the only problem with definition 2.5(2) would be if (β, f) and (δ, f) were both coloured by a green with lower index i. Her strategy avoids using w_i in precisely this case.

Consider now the second kind of triangle, (β, β', δ). If $\exists$ coloured (β, δ) white then there can be no problem, since she didn't colour (β', δ) green. Similar reasoning applies if (β', δ) is white. If $\exists$ coloured (β, δ) and (β', δ) red, and (β, β') is not red, then again there is no clash with definition 2.5.

That leaves one hard case, where $\exists$ colours both (β, δ) and (β', δ) red (with ρ), and the old edge (β, β') has already been coloured red, earlier in the game.

We claim that (β, β') was coloured by $\exists$. As we assume inductively that $\exists$ used the given strategy throughout the game so far, she will have also used ρ to colour (β, β') — so there is no problem with definition 2.5(2).

So suppose, for a contradiction, that (β, β') was coloured by $\forall$. Since $\exists$ has just chosen red colours for (β, δ) and (β', δ), it must be the case that there are cones in Γ^* with apexes β, β', δ and the same base, F, each inducing the same linear ordering $\bar{f} = f_0, \ldots, f_{n-2}$, say, on F. Of course, the tints of these cones may all be different (see figure 5).

We know that no edge in F is labelled green, as no cone base can contain green edges. Since $\Gamma \in \mathcal{G}$, it obeys condition 3 of definition 2.5, so $\bar{f}$ must be labelled by some yellow colour, y_S, say. Since $\Delta^+ \in \mathcal{G}$, it obeys condition 4 of definition 2.5, so the tint t (say) of the cone from δ to $\bar{f}$ must lie in S.

Suppose that λ was the last node of $F \cup \{\beta, \beta'\}$ to be created, as the game proceeded. As $|F \cup \{\beta, \beta'\}| = n + 1$, we see that $\exists$ must have chosen the colour of at least one edge in this set: say, (λ, μ). Now all edges from β into F are green, and so coloured by $\forall$, and the edge (β, β') was also coloured by him. The same holds for edges from β' to F. Hence $\lambda, \mu \in F$.

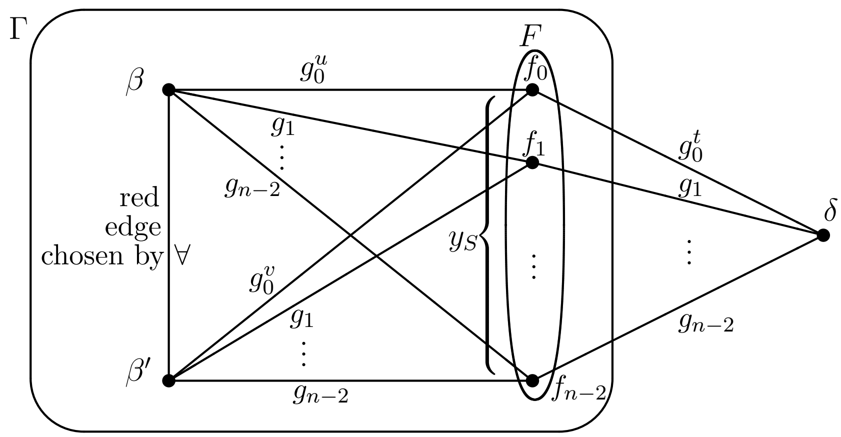

Figure 5: the hard case

We can now see that it was $\exists$ who chose the colour y_S of $\bar{f}$. For y_S was chosen in the round when F's last node, λ, was created. It could only have been chosen by $\forall$ if he also picked the colour of every edge in F involving λ. This is not so, as the edge (λ, μ) was coloured by $\exists$, and lies in F.

As $t \in S$, it follows from the definition of $\exists$'s strategy that at the time when λ was added, there was already a t-cone with base $\bar{f}$ in the same induced order $f_0, \ldots, f_{n-2}$, and apex γ, say. Thus, $\Gamma\lceil(F \cup \{\gamma\})$ is a t-cone in Γ.

Let $\theta' = Id_F \cup \{(\delta, \gamma)\}$. Now the only $(n-1)$-tuples of either $F \cup \{\delta\}$ or $F \cup \{\gamma\}$ with a yellow colour in Δ^+, Γ (respectively) are in F, since all others involve a green edge. It follows that $\theta' : \Delta^+ \to \Gamma$ is a coloured graph embedding.

But this means that $\exists$ could have taken $\Gamma^+ = \Gamma$ in the current round, and not extended the graph. This is contrary to our original assumption, and completes the proof. $\dashv$

Now there are only countably many finite graphs in $\mathcal{G}$, up to isomorphism, and each of the graphs built during the game is finite. Hence $\forall$ may arrange to play every possible $(\Delta, \theta, \Delta^+)$ (up to isomorphism) at some round in the game. Suppose he does this, and let M be the union of the graphs played in the game. We check that M is as required. Certainly, $M \in \mathcal{G}$, since $\mathcal{G}$ is clearly closed under unions of chains. Also, let $\Delta \subseteq \Delta' \in \mathcal{G}$ with $|\Delta'| \leq n$, and $\theta : \Delta \to M$ be an embedding. We prove that θ extends to Δ', by induction on $d = |\Delta' \setminus \Delta|$. If this is 0, there is nothing to prove. Assume the result for smaller d. Choose $a \in \Delta' \setminus \Delta$ and let $\Delta^+ = \Delta'\lceil(\Delta \cup \{a\}) \in \mathcal{G}$. As $|\Delta| < n$, at some round in the game, at which the graph built so far was Γ, say, $\forall$ would have played $(\Delta, \theta, \Delta^+)$ (or some isomorphic triple). Hence, if $\exists$ constructed Γ^+ in that round, there is an embedding $\theta^+ : \Delta^+ \to \Gamma^+$ extending θ. As $\Gamma^+ \subseteq M$, θ^+ is also an embedding $: \Delta^+ \to M$. Since

$|\Delta' \setminus \Delta^+| < d$, θ^+ extends inductively to an embedding $\theta' : \Delta' \to M$, as required. $\dashv$

3 Model theory of M

Here we establish the main properties of the graph M of proposition 2.6. To do so, we will need some (fairly) standard notions from model theory, and we discuss these first. A good modern reference is [Hodges].

Let L be a signature without function or constant symbols, and let A be an L-structure.

3.1 Classical semantics

Definition 3.1 Recall the definition of the infinitary language $L^n_{\infty\omega}$. The atomic formulas are $x_i = x_j$ for any $i, j < n$, and $R(\bar{x})$ for any k-ary $R \in L$ and any k-tuple $\bar{x}$ of variables taken from $x_0, \ldots, x_{n-1}$. If φ is an $L^n_{\infty\omega}$-formula then so are $\neg\varphi$ and $\exists x_i \varphi$ for $i < n$; and if Φ is a set of $L^n_{\infty\omega}$-formulas then $\bigwedge \Phi$ and $\bigvee \Phi$ are also $L^n_{\infty\omega}$-formulas. Of course, we write $\bigwedge\{\varphi, \psi\}$ as $\varphi \wedge \psi$, etc.

The logic $L^n_{\infty\omega}$ is given semantics in A in the usual way, defining $A \models \varphi(\bar{a})$ for an n-tuple $\bar{a}$ of elements of A by induction on the formula φ. Note that not all of $x_0, \ldots, x_{n-1}$ need occur free in φ: so, for example, $A \models (x_3 = x_2)(a_0, \ldots, a_{n-1})$ iff $a_3 = a_2$. We generally use the notation $A \models \varphi(\bar{a})$ only when $\bar{a}$ is an n-tuple, though if $R \in L$ has arity k we do write $A \models R(a_1, \ldots, a_k)$ if $(a_1, \ldots, a_k)$ stands in the relation defined by R in A. A similar convention holds for '$A \models a = b$'. In lemma 6.1 we will use both notations.

Definition 3.2 Let L^n denote the first-order fragment of $L^n_{\infty\omega}$.

Definition 3.3 An *n-back-and-forth system* on A is a set Θ of one-to-one partial maps : $A \to A$ such that:

1. if $\theta \in \Theta$ then $|\theta| \leq n$
2. if $\theta' \subseteq \theta \in \Theta$ then $\theta' \in \Theta$
3. if $\theta \in \Theta$, $|\theta| < n$, and $a \in A$, then there is $\theta' \supseteq \theta$ in Θ with $a \in dom(\theta')$ ('forth')
4. if $\theta \in \Theta$, $|\theta| < n$, and $a \in A$, then there is $\theta' \supseteq \theta$ in Θ with $a \in rng(\theta')$ ('back').

We could require that Θ is non-empty, but this will always be so in the applications in any case.

Definition 3.4 Recall that a *partial isomorphism* of A is a partial map $\theta : A \to A$ that preserves all quantifier-free L-formulas.

Fact 3.5 (Barwise, [Ba]) *Let Θ be an n-back-and-forth system of partial isomorphisms on A, let $\bar{a}, \bar{b} \in {}^nA$, and suppose that $\theta = (\bar{a} \mapsto \bar{b})$ is a map in Θ. Then $A \models \varphi(\bar{a})$ iff $A \models \varphi(\bar{b})$, for any formula φ of $L^n_{\infty\omega}$.*

Proof. By induction on the structure of φ. If φ is quantifier-free, the result is immediate because θ is a partial isomorphism of A. The boolean cases are also evident. If the result holds inductively for φ, consider $\exists x_i\varphi$. If $A \models \exists x_i\varphi(\bar{a})$ then for some $\bar{a}' \in {}^nA$ with $\bar{a}' \equiv_i \bar{a}$, we have $A \models \varphi(\bar{a}')$. Let $\theta^- = \theta\lceil\{a_j : j \neq i\}$. Then $\theta^- \in \Theta$ and $|\theta^-| < n$. Using the 'forth' property of Θ, take $\theta' \in \theta$ extending θ^- and defined on a_i'. Let $\bar{b}' = \theta'(\bar{a}')$. By the inductive hypothesis, $A \models \varphi(\bar{b}')$. Since $\bar{b}' \equiv_i \bar{b}$, we have $A \models \exists x_i\varphi(\bar{b})$. The converse is similar, using the 'back' property of Θ. ⊣

3.2 Relativised semantics

Suppose that $W \subseteq {}^nA$ is a given non-empty set. We can relativise quantifiers to W, giving a new semantics '$\models_W$' for $L^n_{\infty\omega}$, which has been intensively studied in recent times (see, e.g., [ABN1,ABN2]). If $\bar{a} \in W$:

- for atomic φ, $A \models_W \varphi(\bar{a})$ iff $A \models \varphi(\bar{a})$
- the boolean clauses are as expected
- for $i < n$, $A \models_W \exists x_i\varphi(\bar{a})$ iff $A \models_W \varphi(\bar{a}')$ for some $\bar{a}' \in W$ with $\bar{a}' \equiv_i \bar{a}$.

Corollary 3.6 *If W is $L^n_{\infty\omega}$-definable, Θ is an n-back-and-forth system of partial isomorphisms on A, $\bar{a}, \bar{b} \in W$, and $(\bar{a} \mapsto \bar{b}) \in \Theta$, then $A \models_W \varphi(\bar{a})$ iff $A \models_W \varphi(\bar{b})$ for any formula φ of $L^n_{\infty\omega}$.*

Proof. Assume that W is definable by the $L^n_{\infty\omega}$-formula ψ, so that $W = \{\bar{a} \in {}^nA : A \models \psi(\bar{a})\}$. We may relativise the quantifiers of $L^n_{\infty\omega}$-formulas to ψ. For each $L^n_{\infty\omega}$-formula φ we obtain a relativised one, φ^ψ, by induction, the main clause in the definition being:

- $(\exists x_i\varphi)^\psi = \exists x_i(\psi \wedge \varphi^\psi)$.

Then clearly, $A \models_W \varphi(\bar{a})$ iff $A \models \varphi^\psi(\bar{a})$, for all $\bar{a} \in W$. The corollary now follows from fact 3.5. ⊣

3.3 Coloured graphs and model theory

We wish to view the graph M of proposition 2.6 as a classical structure.

Definition 3.7 Let L^+ be the signature consisting of the binary relation symbols g_i $(i = 1, \dots n-2)$, g_0^i $(i < \omega)$, w_i $(i < n-1)$, r^i_{jk} $(i < \omega, j < k < n)$, and ρ, and the $(n-1)$-ary relation symbols y_S $(S \subseteq \omega,\ S = \omega$ or S finite).

Let $L = L^+ \setminus \{\rho\}$. From now on, the logics L^n, $L^n_{\infty\omega}$ are taken in this signature.

We may regard any non-empty coloured graph equally as an L^+-structure, in the obvious way.

The 'n-homogeneity' built into M by its construction would suggest that the set of all partial isomorphisms of M of cardinality at most n forms an n-back-and-forth system. This is indeed true, but we can go further. The rules defining $\mathcal{G}$ in definition 2.5(2) treat each of the reds r^i_{jk} for $i < \omega$ in the same way. Even ρ is not dissimilar, for a clique of at most n points of M with all edges between them labelled by ρ behaves very like a subset of M of the same size with the edges labelled by r^i_{jk} for fixed i and distinct pairs (j, k) $(j < k < n)$ — there are just enough pairs (j, k) to go round, so such a set does exist in M. Thus, the one-to-one maps of size $\leq n$ defined on M that preserve all colours modulo a suitable permutation of the red colours will also form an n-back-and-forth system. This is the content of lemma 3.10 below.

Definition 3.8 Let χ be a permutation of the set $\omega \cup \{\rho\}$. Let $\Gamma, \Delta \in \mathcal{G}$ have the same size, and let $\theta : \Gamma \to \Delta$ be a bijection. We say that θ is a *χ-isomorphism* from Γ to Δ if for each $\bar{a} \in {}^{n-1}\Gamma$, $\theta(\bar{a})$ is coloured y_S in Δ iff $\bar{a}$ is coloured y_S in Γ, and for each distinct $x, y \in \Gamma$,

- If $\Gamma(x, y) = r^i_{jk}$, then $\Delta(\theta(x), \theta(y)) = \begin{cases} r^{\chi(i)}_{jk}, & \text{if } \chi(i) \neq \rho \\ \rho, & \text{otherwise.} \end{cases}$
- If $\Gamma(x, y) = \rho$, then
$$\Delta(\theta(x), \theta(y)) = \begin{cases} r^{\chi(\rho)}_{jk} \text{ for some } j, k < n, & \text{if } \chi(\rho) \neq \rho \\ \rho, & \text{otherwise.} \end{cases}$$
- If $\Gamma(x, y)$ is not red, then $\Delta(\theta(x), \theta(y)) = \Gamma(x, y)$.

Definition 3.9 For any permutation χ of $\omega \cup \{\rho\}$, Θ^χ is the class of partial one-to-one maps from M to M of size at most n that are χ-isomorphisms on their domains. We write Θ for $\Theta^{Id_{\omega\cup\{\rho\}}}$.

Lemma 3.10 *For any permutation χ of $\omega \cup \{\rho\}$, Θ^χ is an n-back-and-forth system on M.*

Proof. Clearly, Θ^χ is closed under restrictions. We check the 'forth' property. Let $\theta \in \Theta^\chi$ have size $t < n$. Enumerate $dom(\theta), rng(\theta)$ respectively as $\{a_0, \ldots, a_{t-1}\}, \{b_0, \ldots, b_{t-1}\}$, with $\theta(a_i) = b_i$ for $i < t$. Let $a_t \in M$ be arbitrary, let $b_t \notin M$ be a new element, and define a complete coloured graph $\Delta \supseteq M\lceil\{b_0, \ldots, b_{t-1}\}$ with nodes $\{b_0, \ldots, b_t\}$ as follows.

Consider the possible lower indices (j, k) $(j < k < n)$ of red colours. Since $|\Delta| \leq n$, there are at least as many of them as there are edges in Δ, so we may choose distinct indices (j_s, k_s) for each $s < t$ such that no $r^i_{j_s k_s}$ labels any edge in $M\lceil\{b_0, \ldots, b_{t-1}\}$. We can now define the colour of edges (b_s, b_t) of Δ for $s < t$.

- If $M(a_s, a_t)$ is not red, then $\Delta(b_s, b_t) = M(a_s, a_t)$.
- If $M(a_s, a_t) = r^i_{jk}$, then $\Delta(b_s, b_t) = \begin{cases} r^{\chi(i)}_{jk}, & \text{if } \chi(i) \neq \rho \\ \rho, & \text{otherwise.} \end{cases}$
- If $M(a_s, a_t) = \rho$, then $\Delta(b_s, b_t) = \begin{cases} r^{\chi(\rho)}_{j_s k_s}, & \text{if } \chi(\rho) \neq \rho \\ \rho, & \text{otherwise.} \end{cases}$

If $t \geq n - 2$ we need to deal with the yellow colours as well. This is easy: if $\eta : (n-1) \to (t+1)$ is one-to-one and $t \in rng(\eta)$, then $(b_{\eta(0)}, \ldots, b_{\eta(n-2)})$ is coloured y_S in Δ iff $(a_{\eta(0)}, \ldots, a_{\eta(n-2)})$ is coloured y_S in M. This completes the definition of Δ.

We check that $\Delta \in \mathcal{G}$. Since Δ differs from $M\lceil\{a_0, \ldots, a_t\}$ only on red labels, it is enough to confirm that any all-red triangle $(b_s, b_{s'}, b_t)$ in Δ meets the restrictions in definition 2.5(2). Now the corresponding triangle $(a_s, a_{s'}, a_t)$ on the other side is also red, so it has the form

I $(r^i_{jk}, r^i_{j'k'}, r^i_{j^*k^*})$ with all lower indices distinct, or

II (ρ, ρ, ρ).

Case I. If the former, then $(b_s, b_{s'}, b_t)$ has the form $(r^{\chi(i)}_{jk}, r^{\chi(i)}_{j'k'}, r^{\chi(i)}_{j^*k^*})$, if $\chi(i) \neq \rho$, or (ρ, ρ, ρ), otherwise — and these are OK.

It may be worth mentioning here that we are using in an essential way the fact that a triangle of the form $(r^i_{jk}, r^{i'}_{j'k'}, r^{i^*}_{j^*k^*})$ cannot embed into M if i, i', i^* are not all equal. For if the triangle $(a_s, a_{s'}, a_t)$ had the form $(r^i_{jk}, r^{i'}_{j'k'}, r^{i^*}_{j^*k^*})$ and we had $\chi(i) = \rho$ and $\chi(i') = l \neq \rho$, say, then we would be forced to label the triangle $(b_s, b_{s'}, b_t)$ by $(\rho, r^l_{j'k'}, r)$ for some red r. This triangle is in conflict with definition 2.5(2).

This is not a minor technical point. If we weakened definition 2.5(2) to allow any triangle $(r^i_{jk}, r^{i'}_{j'k'}, r^{i^*}_{j^*k^*})$ where $(j, k), (j', k'), (j^*, k^*)$ are distinct, the joins R_{jk} mentioned in section 1.5 would exist in the algebra $\mathcal{A}$.

Case IIa. If the latter, and if $\chi(\rho) = \rho$, $(b_s, b_{s'}, b_t)$ also has the form (ρ, ρ, ρ), which is OK.

Case IIb. If the latter, and if $\chi(\rho) \neq \rho$, we have $\Delta(b_s, b_t) = r^{\chi(\rho)}_{j_s k_s}$, $\Delta(b_{s'}, b_t) = r^{\chi(\rho)}_{j_{s'} k_{s'}}$, and (as $\theta \in \Theta^\chi$) $\Delta(b_s, b_{s'}) = M(b_s, b_{s'}) = r^{\chi(\rho)}_{jk}$ for some j, k. But $r^{\chi(\rho)}_{jk}$ labels the edge $(b_s, b_{s'})$ in $M\lceil\{b_0, \ldots, b_{t-1}\}$, so by choice of the j_s, k_s, all three lower indices here are distinct.

So in all cases there is no conflict with definition 2.5(2), and $\Delta \in \mathcal{G}$, as we wanted.

Hence, by proposition 2.6, there is a graph embedding $\phi : \Delta \to M$ extending the map $Id_{\{b_0,\ldots,b_{t-1}\}}$. Note that $\phi(b_t) \notin rng(\theta)$. So the map $\theta^+ = \theta \cup \{(a_t, \phi(b_t))\}$ is injective, and it is easily seen to be a χ-isomorphism in Θ^χ and defined on a_t.

The converse, 'back' property is similarly proved (or by symmetry, using the fact that the inverses of maps in Θ are χ^{-1}-isomorphisms). ⊣

As a special case, we obtain:

Corollary 3.11 *The class $\Theta = \Theta^{Id_{\omega\cup\{\rho\}}}$ of partial L^+-isomorphisms of M (partial isomorphisms of M regarded as an L^+-structure) of size at most n is an n-back-and-forth system on M.*

But we can also derive a connection between classical and relativised semantics in M, over the following set W:

Definition 3.12 Let $W = \{\bar{a} \in {}^nM : M \models (\bigwedge_{i<j<n} \neg\rho(x_i, x_j))(\bar{a})\}$.

W is simply the set of tuples in ${}^nM$ not involving a label ρ. Lemma 3.10 allows us to replace ρ-labels by suitable r^i_{jk}-labels within an n-back-and-forth system. Thus, we may arrange that the system maps a tuple $\bar{b} \in {}^nM \setminus W$ to a tuple $\bar{c} \in W$, and by fact 3.5, this will preserve any formula containing no relation symbols r^i_{jk} that are 'moved' by the system. The next proposition uses this idea to show that the classical and W-relativised semantics agree.

Proposition 3.13 *$M \models_W \varphi(\bar{a})$ iff $M \models \varphi(\bar{a})$, for all $\bar{a} \in W$ and all L^n-formulas φ.*

Proof. The proof is by induction on φ. If φ is atomic, the result is clear; and the boolean cases are simple.

Let $i < n$ and consider $\exists x_i\varphi$. If $M \models_W \exists x_i\varphi(\bar{a})$, then there is $\bar{b} \in W$ with $\bar{b} \equiv_i \bar{a}$ and $M \models_W \varphi(\bar{b})$. Inductively, $M \models \varphi(\bar{b})$, so clearly, $M \models \exists x_i\varphi(\bar{a})$.

For the (more interesting) converse, suppose that $M \models \exists x_i\varphi(\bar{a})$. Then there is $\bar{b} \in {}^nM$ with $\bar{b} \equiv_i \bar{a}$ and $M \models \varphi(\bar{b})$. Take $L_{\varphi,\bar{b}}$ to be any finite subsignature of L containing all the symbols from L that occur in φ or as a label in $M\lceil rng(\bar{b})$. (Here we use the fact that φ is first-order. The result may fail for infinitary formulas involving infinitely many red predicates.) Choose a permutation χ of $\omega \cup \{\rho\}$ fixing any i' such that some $r^{i'}_{jk}$ occurs in $L_{\varphi,\bar{b}}$, and moving ρ.

Let $\theta = Id_{\{a_m : m \neq i\}}$. Take any distinct $l, m \in n \setminus \{i\}$. If $M(a_l, a_m) = r^{i'}_{jk}$, then $M(b_l, b_m) = r^{i'}_{jk}$ because $\bar{a} \equiv_i \bar{b}$, so $r^{i'}_{jk} \in L_{\varphi,\bar{b}}$ by definition of $L_{\varphi,\bar{b}}$. So $\chi(i') = i'$ by definition of χ. Also, $M(a_l, a_m) \neq \rho$ because $\bar{a} \in W$. It now follows that θ is a χ-isomorphism on its domain, so that $\theta \in \Theta^\chi$.

Extend θ to $\theta' \in \Theta^\chi$ defined on b_i, using the 'forth' property of Θ^χ (lemma 3.10). Let $\bar{c} = \theta'(\bar{b})$. Now by choice of χ, no labels on edges of the subgraph of M with domain $rng(\bar{c})$ are ρ. Hence, $\bar{c} \in W$. Moreover, each map in Θ^χ is evidently a partial isomorphism of the reduct of M to the signature $L_{\varphi,\bar{b}}$. Hence by fact 3.5 applied to $L_{\varphi,\bar{b}}$, and lemma 3.10, we have $M \models \varphi(\bar{b})$ iff $M \models \varphi(\bar{c})$. So $M \models \varphi(\bar{c})$. Inductively, $M \models_W \varphi(\bar{c})$. Since $\bar{c} \equiv_i \bar{a}$, we have $M \models_W \exists x_i \varphi(\bar{a})$ by definition of the relativised semantics. This completes the induction. $\dashv$

4 The algebra of L^n-definable subsets of ${}^nM$

We can now extract from the coloured graph M of proposition 2.6 a relativised set algebra $\mathcal{A}$, which will turn out to be representable (hence a cylindric algebra) and atomic. (In section 5 we will study the complex algebra over its atom structure.)

First, we recall some relevant facts about cylindric algebras.

4.1 Cylindric algebras

We do not wish to give a comprehensive introduction to these (those who want one may read the standard reference [HMT]), but we feel we should list those of their features that are relevant here.

Let n be an ordinal (finite, in this paper). An n-dimensional *cylindric algebra* is an algebra $\mathcal{A}$ in the signature consisting of the boolean operations $-, \cdot, 0, 1$, constants d_{ij} for $i, j < n$ ('diagonals') and unary functions c_i for $i < n$ ('cylindrifications'), and satisfying certain equations which can be found in [HMT] and which we will not go into here. We only need to know that every cylindric algebra is a boolean algebra with operators, and that the complex algebra of the atom structure of any atomic cylindric algebra is also a cylindric algebra.

We generally write $d^{\mathcal{A}}_{ij}, c^{\mathcal{A}}_i$, etc, for the interpretations of the respective operations in $\mathcal{A}$.

An n-dimensional *set algebra* is an algebra of n-ary relations of the form

$$\mathcal{A} = (A, -, \cap, \emptyset, W, d^{\mathcal{A}}_{ij}, c^{\mathcal{A}}_i)_{i,j<n},$$

where W is of the form ${}^nU$ for some non-empty set U, $(A, -, \cap, \emptyset, W)$ is a boolean subalgebra of the boolean algebra $(\wp(W), -, \cap, \emptyset, W)$, $d_{ij}^{\mathcal{A}} = \{\bar{a} \in W : a_i = a_j\}$, and for $X \in A$, $c_i^{\mathcal{A}} X = \{\bar{a} \in W : \bar{a} \equiv_i \bar{b}$ for some $\bar{b} \in X\}$. The set W is called the *unit* of $\mathcal{A}$. Set algebras are automatically cylindric algebras, but not conversely, even up to isomorphism. A *relativised set algebra* is similar, but has a weaker condition on 'W': we only require that $W \subseteq {}^nU$ for some set U. Relativised set algebras are not necessarily cylindric algebras.

Let $\mathcal{A}$ be an algebra of the similarity type of cylindric algebras. A *representation* of $\mathcal{A}$ is an algebra embedding h from $\mathcal{A}$ into a direct product of set algebras, and $\mathcal{A}$ is said to be *representable* if there is such a representation. Note that because the class of cylindric algebras is a variety and so closed under taking products and subalgebras, any representable algebra — for example, a representable relativised set algebra — is a cylindric algebra.

4.2 Definition of $\mathcal{A}$

Θ will continue to denote the set of all partial L^+-isomorphisms of M of size $\leq n$; it is an n-back-and-forth system on M. W remains as in definition 3.12.

Definition 4.1

1. For an $L^n_{\infty\omega}$-formula φ, we define φ^W to be the set $\{\bar{a} \in W : M \models_W \varphi(\bar{a})\}$. Here we use the *relativised semantics* of section 3.2.
2. We define $\mathcal{A}$ to be the relativised set algebra with domain $\{\varphi^W : \varphi$ a first-order L^n-formula$\}$ and unit W, endowed with the algebraic operations d_{ij}, c_i, etc., in the standard way (see the passage on cylindric algebras above).

Note that $\mathcal{A}$ is indeed closed under the operations and so is a bona fide relativised set algebra. For, reading off from the definitions of the standard operations and the relativised semantics, we see that for all L^n-formulas φ, ψ,

- $-^{\mathcal{A}}(\varphi^W) = (\neg\varphi)^W$
- $\varphi^W \cdot^{\mathcal{A}} \psi^W = (\varphi \wedge \psi)^W$
- $d_{ij}^{\mathcal{A}} = (x_i = x_j)^W$ for all $i, j < n$
- $c_i^{\mathcal{A}}(\varphi^W) = (\exists x_i \varphi)^W$ for all $i < n$.

4.3 $\mathcal{A}$ is representable

Proposition 4.2 *$\mathcal{A}$ is representable. Hence, $\mathcal{A}$ is a cylindric algebra.*

Proof. Let $\mathcal{S}$ be the set algebra with domain $\wp({}^nM)$ and unit ${}^nM$. Then by proposition 3.13, the map $h : \mathcal{A} \to \mathcal{S}$ given by $h : \varphi^W \mapsto \{\bar{a} \in {}^nM : M \models \varphi(\bar{a})\}$ is a well-defined representation of $\mathcal{A}$. $\dashv$

4.4 Atoms of $\mathcal{A}$

Here we show that $\mathcal{A}$ is atomic.

Definition 4.3 A formula α of L^n is said to be MCA ('maximal conjunction of atomic formulas') if (i) $M \models \exists x_0 \dots x_{n-1}\alpha$ and (ii) α is of the form

$$\bigwedge_{i \neq j < n} \alpha_{ij}(x_i, x_j) \quad \wedge \bigwedge_{\substack{\eta:(n-1)\to n \\ \eta \text{ one-one}}} \gamma_\eta(x_{\eta(0)}, \dots, x_{\eta(n-2)}),$$

where:

- for each i, j, α_{ij} is either $x_i = x_j$, or $R(x_i, x_j)$ for some binary relation symbol R of L;
- for each one-to-one map $\eta : (n-1) \to n$, γ_η is either $y_S(x_{\eta(0)}, \dots, x_{\eta(n-2)})$ for some $y_S \in L$, if for all distinct $i, j < n$, $\alpha_{\eta(i)\eta(j)}$ is not equality or green, or else $x_0 = x_0$, otherwise.

The rough idea is that a formula α being MCA says that the set it defines in ${}^nM$ is non-empty, and that if $M \models \alpha(\bar{a})$ then the graph $M\lceil rng(\bar{a})$ is determined up to isomorphism and has no edge labelled ρ. Hence, any two tuples satisfying α are isomorphic and one is mapped to the other by the n-back-and-forth system Θ. By fact 3.5, no $L^n_{\infty\omega}$-formula can distinguish them. So α defines an atom of $\mathcal{A}$ — it is literally indivisible. Since the MCA-formulas clearly 'cover' W, the atoms defined by them are dense in $\mathcal{A}$. So $\mathcal{A}$ is atomic, as required. This, informally, is the content of the next two results.

Lemma 4.4 *Let φ be any $L^n_{\infty\omega}$-formula, and α any MCA-formula. If $\varphi^W \cap \alpha^W \neq \emptyset$, then $\alpha^W \subseteq \varphi^W$.*

Proof. Take $\bar{a} \in \varphi^W \cap \alpha^W$. Let $\bar{b} \in \alpha^W$ be arbitrary. Clearly, the map $(\bar{a} \mapsto \bar{b})$ is in Θ. Also, W is $L^n_{\infty\omega}$-definable in M, since we have

$$W = \{\bar{a} \in {}^nM : M \models (\bigwedge_{i<j<n} (x_i = x_j \vee \bigvee_{R \in L} R(x_i, x_j)))(\bar{a})\}.$$

By corollaries 3.6 and 3.11, we have $M \models_W \varphi(\bar{a})$ iff $M \models_W \varphi(\bar{b})$. Since $M \models_W \varphi(\bar{a})$, we have $M \models_W \varphi(\bar{b})$. Hence, $\alpha^W \subseteq \varphi^W$. $\dashv$

Definition 4.5 Let $F = \{\alpha^W : \alpha \text{ is an MCA } L^n\text{-formula}\} \subseteq \mathcal{A}$.

Evidently, $W = \bigcup F$.

Proposition 4.6 *$\mathcal{A}$ is an atomic algebra, with F as its set of atoms.*

Proof. First, we show that any non-empty element φ^W of $\mathcal{A}$ contains an element of F. Take $\bar{a} \in W$ with $M \models_W \varphi(\bar{a})$. Since $\bar{a} \in W$, there is an MCA-formula α such that $M \models_W \alpha(\bar{a})$. By lemma 4.4, $\alpha^W \subseteq \varphi^W$.

Now by the same lemma, if α is an MCA-formula, φ an L^n-formula, and $\emptyset \neq \varphi^W \subseteq \alpha^W$, then $\varphi^W = \alpha^W$. It follows that each α^W (for MCA α) is an atom of $\mathcal{A}$. ⊣

Remark 4.7 It follows from the foregoing that the identity map on $\mathcal{A}$ is a *complete relativised* representation of $\mathcal{A}$ — an isomorphism from $\mathcal{A}$ onto a relativised set algebra that preserves infinite meets and joins where defined. By arguments of [HH2] and proposition 5.4 below, $\mathcal{A}$ has no non-relativised complete representation.

In any event, $\mathcal{A}$ has an atom structure, which we write as $At\mathcal{A}$ as usual.

5 The complex algebra over $At\mathcal{A}$

Here we study the complex algebra over the atom structure of $\mathcal{A}$, aiming at a non-representability proof.

5.1 The complex algebra as a relativised set algebra

Definition 5.1 Define $\mathcal{C}$ to be the complex algebra over $At\mathcal{A}$, the atom structure of $\mathcal{A}$.

So formally, the domain of $\mathcal{C}$ is $\wp(At\mathcal{A})$. The diagonal d_{ij} is interpreted in $\mathcal{C}$ as the set of all $S \in At\mathcal{A}$ with $a_i = a_j$ for some (equivalently, all) $\bar{a} \in S$. The cylindrification c_i is interpreted in $\mathcal{C}$ by $c_i^{\mathcal{C}} X = \{S \in At\mathcal{A} : S \subseteq c_i^{\mathcal{A}}(S')$ for some $S' \in X\}$, for $X \subseteq At\mathcal{A}$.

However, there is a more concrete way of viewing $\mathcal{C}$, as we will now see.

Definition 5.2 We let $\mathcal{D}$ be the relativised set algebra with domain $\{\varphi^W : \varphi$ an $L^n_{\infty\omega}$-formula$\}$ and unit W.

As before, by definition of relativised set algebras and relativised semantics, for all $L^n_{\infty\omega}$-formulas φ, ψ:

- $-^{\mathcal{D}}(\varphi^W) = (\neg\varphi)^W$
- $\varphi^W \cdot^{\mathcal{D}} \psi^W = (\varphi \wedge \psi)^W$

- $d_{ij}^{\mathcal{D}} = (x_i = x_j)^W$ for all $i, j < n$
- $c_i^{\mathcal{D}}(\varphi^W) = (\exists x_i \varphi)^W$ for all $i < n$.

Thus, $\mathcal{D}$ is indeed closed under the operations. Of course, $\mathcal{A}$ is a subalgebra of $\mathcal{D}$. In fact, $\mathcal{D}$ is isomorphic to the complex algebra over the atom structure of $\mathcal{A}$:

Lemma 5.3 *$\mathcal{C} \cong \mathcal{D}$, via the map given by $X \mapsto \bigcup X$.*

Proof. The map is evidently injective. It is also surjective, since by lemma 4.4 we have $\varphi^W = \bigcup\{\alpha^W : \alpha \text{ an MCA-formula}, \alpha^W \subseteq \varphi^W\}$ for any $L^n_{\infty\omega}$-formula φ. Preservation of boolean operations and diagonals is clear. We check preservation of cylindrifications. We require that for any $X \subseteq At\mathcal{A}$, we have $\bigcup c_i^{\mathcal{C}} X = c_i^{\mathcal{D}}(\bigcup X)$ — that is,

$$\begin{aligned} & \bigcup\{S \in At\mathcal{A} : S \subseteq c_i^{\mathcal{A}}(S') \text{ for some } S' \in X\} \\ = \quad & \{\bar{a} \in W : \bar{a} \equiv_i \bar{a}' \text{ for some } \bar{a}' \in \bigcup X\}. \end{aligned}$$

For '$\subseteq$', let $\bar{a} \in S \subseteq c_i^{\mathcal{A}}(S')$, where $S' \in X$. So there is $\bar{a}' \equiv_i \bar{a}$ with $\bar{a}' \in S'$ — and so $\bar{a}' \in \bigcup X$.

For the converse, let $\bar{a} \in W$ with $\bar{a} \equiv_i \bar{a}'$ for some $\bar{a}' \in \bigcup X$. Choose $S \in At\mathcal{A}$, $S' \in X$ with $\bar{a} \in S$, $\bar{a}' \in S'$. We may choose MCA formulas α, α' with $S = \alpha^W$ and $S' = \alpha'^W$. Then $\bar{a} \in \alpha^W \cap (\exists x_i \alpha')^W$, so by lemma 4.4, $\alpha^W \subseteq (\exists x_i \alpha')^W$, or $S \subseteq c_i^{\mathcal{A}}(S')$. Thus, $\bar{a}$ is in the left-hand side. ⊣

5.2 The complex algebra is not representable

The proof of theorem 1.1 will be completed with:

Proposition 5.4 *The complex algebra $\mathcal{C}$ of $At\mathcal{A}$ is not representable.*

This says that although the $L^n_{\infty\omega}$-formulas have relativised semantics, one cannot give them classical semantics without changing which formulas are equivalent to which!

Proof. Suppose for contradiction that $\mathcal{C}$ is representable.

Lemma 5.5 *$\mathcal{D}$ is isomorphic to a set algebra.*[6]

Proof. By lemma 5.3, $\mathcal{C} \cong \mathcal{D}$, so there is an algebra embedding g from $\mathcal{D}$ into $\prod_{l\in I} \mathcal{B}_l = \prod_{l\in I}(B_l, -, \cap, \emptyset, ^n U_l, d_{ij}^{\mathcal{B}_l}, c_i^{\mathcal{B}_l})$, a product of set algebras. Because $|\mathcal{D}| \geq 2$, the index set I of the product is non-empty. Take any U_l, and consider the projection $\pi_l : \prod_{l\in I} \mathcal{B}_l \to B_l$. Let $h = \pi \circ g : \mathcal{D} \to \mathcal{B}_l$. Then h preserves all the algebra operations (it is a homomorphism).

[6] We are proving that $\mathcal{D}$ is simple (see [HMT] for the meaning).

To prove the lemma, it remains to establish that h is injective. Because h preserves the boolean operations, it suffices to check that if α is an MCA formula then $h(\alpha^W) \neq 0^{\mathcal{B}_l}$.

Let $\bar{a} \in W$ satisfy $M \models_W \alpha(\bar{a})$. By proposition 3.13 we have $M \models \alpha(\bar{a})$ classically, so $M \models (\exists x_0 \ldots x_{n-1}\alpha)(\bar{b})$ for all $\bar{b} \in {}^nM$. By proposition 3.13 again, $M \models_W (\exists x_0 \ldots x_{n-1}\alpha)(\bar{b})$ for all $\bar{b} \in W$, so $(\exists x_0 \ldots x_{n-1}\alpha)^W = W = 1^{\mathcal{D}}$. Hence we have

$$c_0^{\mathcal{B}_l} \ldots c_{n-1}^{\mathcal{B}_l} h(\alpha^W) = h(c_0^{\mathcal{D}} \ldots c_{n-1}^{\mathcal{D}}(\alpha^W)) = h((\exists x_0 \ldots x_{n-1}\alpha)^W) = h(1^{\mathcal{D}}) = {}^nU_l.$$

So certainly, $h(\alpha^W) \neq \emptyset = 0^{\mathcal{B}_l}$. ⊣

Thus, $\mathcal{D}$ embeds via a map h into the set algebra based on $\wp({}^nN)$, for some non-empty set $N(= U_l)$. We have

Proposition 5.6 *For all $L^n_{\infty\omega}$-formulas φ, ψ:*

- $\varphi^W = \emptyset$ *iff* $h(\varphi^W) = \emptyset$, *and* $\varphi^W = W$ *iff* $h(\varphi^W) = {}^nN$
- $h((\neg\varphi)^W) = {}^nN \setminus h(\varphi^W)$
- $h((\varphi \wedge \psi)^W) = h(\varphi^W) \cap h(\psi^W)$ *(but there is no analogue for infinitary conjunctions)*
- $h((x_i = x_j)^W) = \{\bar{a} \in {}^nN : a_i = a_j\}$, *for all* $i, j < n$
- $h((\exists x_i\varphi)^W) = \{\bar{a} \in {}^nN : \bar{a} \equiv_i \bar{a}'$ *for some* $\bar{a}' \in h(\varphi^W)\}$, *for all* $i < n$.

Now we can get down to business. We will find an infinite red clique in N, so obtaining a contradiction as described in section 1.5. The clique will be forced by an $(n-1)$-tuple $\bar{b}$ in the relation y_ω: its nodes will be the apexes of cones with base $\bar{b}$. We will use proposition 5.6 frequently in the proofs, sometimes without explicit mention.

Lemma 5.7 *There are $b_0, \ldots, b_{n-1} \in N$ with $(b_0, \ldots, b_{n-1}) \in h(y_\omega(x_0, \ldots, x_{n-2})^W)$.*

Proof. By proposition 2.6, $y_\omega(x_0, \ldots, x_{n-2})^W \neq \emptyset$, so the result is immediate by proposition 5.6. ⊣

Fix $b_0, \ldots, b_{n-1}$ as in the lemma.

Lemma 5.8 *For any $t < \omega$, there is $c_t \in N$ such that*

$$\bar{b}_t =_{\text{def.}} (b_0, \ldots, b_{n-2}, c_t)$$

lies in $h(g_0^t(x_0, x_{n-1})^W)$ and in $h(g_i(x_i, x_{n-1})^W)$ for each i with $1 \leq i \leq n-2$.

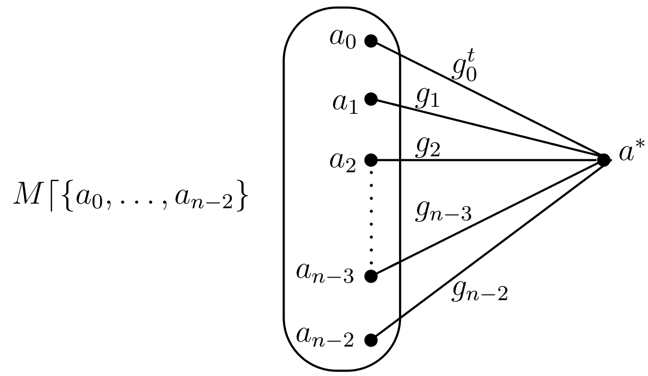

Figure 6: a t-cone on $\{a_0,\ldots,a_{n-2}\}$

Proof. Let

$$\varphi_t = y_\omega(x_0,\ldots,x_{n-2}) \to \exists x_{n-1}\Big(g_0^t(x_0,x_{n-1}) \wedge \bigwedge_{1\le i\le n-2} g_i(x_i,x_{n-1})\Big).$$

Claim. $(\varphi_t)^W = W$.

Proof of claim. Let $\bar a \in W$, and suppose that $M \models_W (y_\omega(x_0,\ldots,x_{n-2}))(\bar a)$. Then the graph shown in figure 6 is in $\mathcal{G}$, as there are no green edges in $M\lceil\{a_0,\ldots,a_{n-2}\}$ and the condition of definition 2.5(4) is obviously met. So by proposition 2.6, the identity map on the set $\{a_0,\ldots,a_{n-2}\}$ extends to a graph embedding $\theta : \{a_0,\ldots,a_{n-2},a^*\} \to M$. Clearly, $\bar a' = (a_0,\ldots,a_{n-2},\theta(a^*)) \in W$, $\bar a' \equiv_{n-1} \bar a$, and $M \models_W (g_0^t(x_0,x_{n-1}) \wedge \bigwedge_{1\le i\le n-2} g_i(x_i,x_{n-1}))(\bar a')$. This proves the claim.

Now by proposition 5.6, $h(\varphi_t^W) = {}^nN$, so $(b_0,\ldots,b_{n-1}) \in h(\varphi_t^W)$. By choice of $b_0,\ldots,b_{n-1}$, there is $c_t \in N$ with $(b_0,\ldots,b_{n-2},c_t) \in h((g_0^t(x_0,x_{n-1}) \wedge \bigwedge_{1\le i\le n-2} g_i(x_i,x_{n-1}))^W)$, and the lemma follows. ⊣

Pick $c_t \in N$ $(t<\omega)$ as in the lemma, let $\bar b_t$ also be as in the lemma, and for each $s<t<\omega$ define $\bar c_{st}$ to be the sequence $(c_s,b_1,\ldots,b_{n-2},c_t) \in {}^nN$. Fix $s<t<\omega$.

Lemma 5.9 *$\bar c_{st} \notin h(w_i(x_0,x_{n-1})^W)$ for each i with $1 \le i \le n-2$.*

Proof. Consider the L^n-formula

$$\psi = \exists x_0 g_i(x_i,x_{n-1}) \wedge \exists x_{n-1}(x_{n-1}=x_0 \wedge \exists x_0 g_i(x_i,x_{n-1})).$$

Clearly, ψ is classically equivalent to $g_i(x_i,x_{n-1}) \wedge g_i(x_i,x_0)$ (we use the assumption $n \ge 3$ here). Now in M we have no triangles of the form (g_i,g_i,w_i) (see definition 2.5(2)). It follows that $M \models \Big(\psi \to \neg w_i(x_0,x_{n-1})\Big)(\bar a)$ for all $\bar a \in {}^nM$. By proposition 3.13, we obtain $(\psi \to \neg w_i(x_0,x_{n-1}))^W = W$. Hence, by proposition 5.6, $\bar c_{st} \in h((\psi \to \neg w_i(x_0,x_{n-1}))^W)$.

Now by the same proposition again, and the choice of the $\bar b_t$, we have

1. $\bar{b}_t = (b_0, \ldots, b_{n-2}, c_t) \in h(g_i(x_i, x_{n-1})^W)$, so that $\bar{c}_{st} \in h((\exists x_0 g_i(x_i, x_{n-1}))^W)$.

2. $\bar{b}_s = (b_0, \ldots, b_{n-2}, c_s) \in h(g_i(x_i, x_{n-1})^W)$, so that $(c_s, b_1, \ldots, b_{n-2}, c_s) \in h((x_{n-1} = x_0 \wedge \exists x_0 g_i(x_i, x_{n-1})))^W)$ and $\bar{c}_{st} \in h((\exists x_{n-1}(x_{n-1} = x_0 \wedge \exists x_0 g_i(x_i, x_{n-1})))^W)$.

So $\bar{c}_{st} \in h(\psi^W)$. Hence $\bar{c}_{st} \notin h(w_i(x_0, x_{n-1})^W)$. ⊣

Let γ be the $L^n_{\infty\omega}$-formula

$$x_0 = x_{n-1} \vee w_0(x_0, x_{n-1}) \vee \bigvee_{g \in L \text{ green}} g(x_0, x_{n-1}).$$

Lemma 5.10 $\bar{c}_{st} \notin h(\gamma^W)$.

Proof. This is similar but more complicated. Inspection of definition 2.5(2) shows that $M \models_W (\exists x_1(g_0^s(x_1, x_0) \wedge g_0^t(x_1, x_{n-1})) \to \neg\gamma)(\bar{a})$ for all $\bar{a} \in W$. Consider the L^n-formula

$$\delta = \exists x_1 \Big(x_1 = x_0 \wedge \exists x_0 \Big(\exists x_1 g_0^t(x_0, x_{n-1}) \\ \wedge \exists x_{n-1}(x_{n-1} = x_1 \wedge \exists x_1 g_0^s(x_0, x_{n-1})) \Big) \Big).$$

Now δ and $\exists x_1(g_0^s(x_1, x_0) \wedge g_0^t(x_1, x_{n-1}))$ are classically equivalent first-order L^n-formulas, so by proposition 3.13 they are equivalent in the relativised semantics $\models_W$ too. Hence, $(\delta \to \neg\gamma)^W = W$. So by proposition 5.6, $\bar{c}_{st} \in h((\delta \to \neg\gamma)^W)$. Now $\bar{b}_s \in h(g_0^s(x_0, x_{n-1})^W)$, and $\bar{b}_t \in h(g_0^t(x_0, x_{n-1})^W)$. Working through the definitions of $\delta, \bar{b}_s, \bar{b}_t$, and $\bar{c}_{st}$ shows that $\bar{c}_{st} \in h(\delta^W)$. So $\bar{c}_{st} \notin h(\gamma^W)$, as required. ⊣

For each $j < k < n$ define R_{jk} to be the $L^n_{\infty\omega}$-formula $\bigvee_{i<\omega} r^i_{jk}(x_0, x_{n-1})$.

Lemma 5.11 *For each $s < t < \omega$ there are $j < k < n$ with $\bar{c}_{st} \in h(R^W_{jk})$.*

Proof. As $M \in \mathcal{G}$, and no label in the range of a tuple in W is ρ, we have

$$\Big(\gamma \vee \bigvee_{1 \le i \le n-2} w_i(x_0, x_{n-1}) \vee \bigvee_{j<k<n} R_{jk}\Big)^W = W.$$

Pick $s < t < \omega$. By lemma 5.9, $\bar{c}_{st} \notin h((\bigvee_{1 \le i \le n-2} w_i(x_0, x_{n-1}))^W)$. By lemma 5.10, $\bar{c}_{st} \notin h(\gamma^W)$. So by proposition 5.6, there are $j < k < n$ with $\bar{c}_{st} \in h(R^W_{jk})$. ⊣

By lemma 5.11 and the pigeon-hole principle, there are $s < t < \omega$ and $j < k < n$ with $\bar{c}_{0s}, \bar{c}_{0t} \in h(R^W_{jk})$. Since the lemma also gives $\bar{c}_{st} \in h(R^W_{j'k'})$ for some j', k', we see (using proposition 5.6 throughout) that the sequence $(c_0, c_s, b_2, \ldots, b_{n-2}, c_t)$ is in $h(\chi^W)$, where

$$\chi = (\exists x_1 R_{jk}) \wedge (\exists x_{n-1}(x_{n-1} = x_1 \wedge \exists x_1 R_{jk})) \wedge (\exists x_0(x_0 = x_1 \wedge \exists x_1 R_{j'k'})).$$

So $\chi^W \neq \emptyset$. Let $\bar{a} \in \chi^W$. Then $M \models_W R_{jk}(a_0, a_{n-1}) \wedge R_{jk}(a_0, a_1) \wedge R_{j'k'}(a_1, a_{n-1})$. Hence there are $i, i', i'' < \omega$ with $M \models_W r^i_{jk}(a_0, a_{n-1}) \wedge r^{i'}_{jk}(a_0, a_1) \wedge r^{i''}_{j'k'}(a_1, a_{n-1})$.

But $\mathcal{M} \in \mathcal{G}$, so by definition 2.5(2) it can have no triangles of the form $(r^i_{jk}, r^{i'}_{jk}, r^{i''}_{j'k'})$. This is a contradiction, and it completes the proof of proposition 5.4. ⊣

Theorem 1.1 now follows from propositions 4.2, 4.6, and 5.4.

6 Relation algebras

We briefly discuss the half of theorem 1.1 concerning *relation algebras.* We will not be detained long, as the arguments are nearly identical to those in the cylindric algebra case.

6.1 Definitions

We must now introduce relation algebras more formally.

The signature of relation algebras is $\{-, \cdot, 0, 1, 1', \smile, ;\}$, where $-, \cdot, 0, 1$ are as for cylindric algebras, $1'$ is a constant ('identity'), $\smile$ a unary function ('converse'), and ; a binary function ('composition'). A relation algebra is an algebra in this signature that satisfies certain equations which again we do not need to go into. We write RA for the class of relation algebras; it is a variety.

A *(simple) proper relation algebra* is an algebra in the signature of relation algebras of the form

$$\mathcal{A} = (A, -, \cap, \emptyset, {}^2U, 1'^{\mathcal{A}}, {}^{\smile\mathcal{A}}, ;^{\mathcal{A}}),$$

where U is some non-empty set, $(A, -, \cap, \emptyset, {}^2U)$ is a boolean subalgebra of the boolean algebra $(\wp({}^2U), -, \cap, \emptyset, {}^2U)$, $1'^{\mathcal{A}} = \{(a, a) : a \in U\}$, and for $X, Y \in A$, $X^{\smile\mathcal{A}} = \{(a, b) : (b, a) \in X\}$ and $X;^{\mathcal{A}} Y = \{(a, b) : \exists c((a, c) \in X \wedge (c, b) \in Y)\}$. Any proper relation algebra is an algebra of binary relations on a set.

A representation of an algebra $\mathcal{A}$ of the signature of relation algebras is an embedding h from $\mathcal{A}$ into a direct product of proper relation algebras. An algebra is said to be representable if it has a representation; any such algebra is necessarily a relation algebra. We write RRA for the class of all representable relation algebras; it is also a variety, contained in RA.

6.2 Results

Now take the dimension 'n' of sections 2–5 to be 3. The signature L (definition 3.7) of the graph M of proposition 2.6 now consists of binary relation symbols only.

Let $\mathcal{A}$ be the proper relation algebra of all binary relations on M that are definable by a formula of L^3 (the fragment of first-order logic over L consisting of all formulas using only the variables x, y, z) *with only two free variables.* Clearly, $\mathcal{A}$ is closed under the boolean operations, and also the relation algebra operations: $1'^{\mathcal{A}}$ is definable by $x = y$; the converse of the relation defined by $\varphi(x, y)$ is defined by $\varphi(y, x)$; and the composition of the relations defined by $\varphi(x, y)$ and $\psi(x, y)$ is defined by $\exists z(\varphi(x, z) \wedge \psi(z, y))$.

Lemma 6.1 *$\mathcal{A}$ is atomic.*

Proof. Define, for the relation algebra case, an MCA-formula to be one of the form $\alpha(x, y) \wedge \gamma_0(x, y) \wedge \gamma_1(y, x)$, where α is either $x = y$ or $R(x, y)$ for some $R \in L$, $\gamma_0(x, y)$ is either some $y_S(x, y)$ (if α is not equality or green) or $x = x$, otherwise, and similarly for $\gamma_1(y, x)$. Examples include $x = y$, $g_0^7(x, y)$, and $w_1(x, y) \wedge y_\omega(x, y) \wedge y_{\{1,5\}}(y, x)$, up to equivalence. Define Θ, the set of all partial isomorphisms of M of size at most three, as before. It is a 3-back-and-forth system on M. As in lemma 4.4, for any MCA-formula α and $\bar{a}, \bar{b} \in {}^2M$, if $M \models \alpha(\bar{a}) \wedge \alpha(\bar{b})$ then the map $(\bar{a} \mapsto \bar{b})$ is in Θ, so by fact 3.5, $\bar{a}$ and $\bar{b}$ are indistinguishable by any L^3-formula. It follows that each MCA-formula defines an atom of $\mathcal{A}$.

To show that any non-empty relation of $\mathcal{A}$ contains such an atom, let $\varphi(x, y) \in L^3$ define such a relation, and let W be as in definition 3.12 (for the 3-dimensional case). Then, in terms of definition 3.1, for arbitrary $\bar{a} \in {}^3M$ we have $M \models \exists xyz\varphi(\bar{a})$, so that by proposition 3.13, $M \models_W \exists xyz\varphi(\bar{a})$ also. So there exists $(a, b, c) \in \varphi^W$. Note that by definition of W, (a, b) lies in an atom of $\mathcal{A}$. Also, by proposition 3.13 again, $M \models \varphi(a, b, c)$, and as φ has only x, y free, we have $M \models \varphi(a, b)$. Thus, the relation defined by φ intersects an atom of $\mathcal{A}$, and thus contains one. $\dashv$

This shows that the atoms of $\mathcal{A}$ are the collections of all directed edges of M of a given isomorphism type, excepting those with type ρ.

For a relation symbol $R \in L$, write R^M for its interpretation $\{(a, a') \in {}^2M : M \models R(a, a')\}$ in M. Thus, $R^M \in \mathcal{A}$. But we may regard $\mathcal{A}$ as a subalgebra of $\mathcal{C}$, the complex algebra over $At\mathcal{A}$, via the map $a \mapsto \{x \in At\mathcal{A} : \mathcal{A} \models x \leq a\}$; this is easy to check. Therefore, we have $R^M \in \mathcal{C}$.

Lemma 6.2 *The complex algebra $\mathcal{C}$ over $At\mathcal{A}$ is not representable.*

Proof. Assume otherwise. As in lemma 5.5, it can be shown that $\mathcal{C}$ has a representation of the form $h : \mathcal{C} \to \mathcal{B}$, where $\mathcal{B}$ is a (simple) proper relation

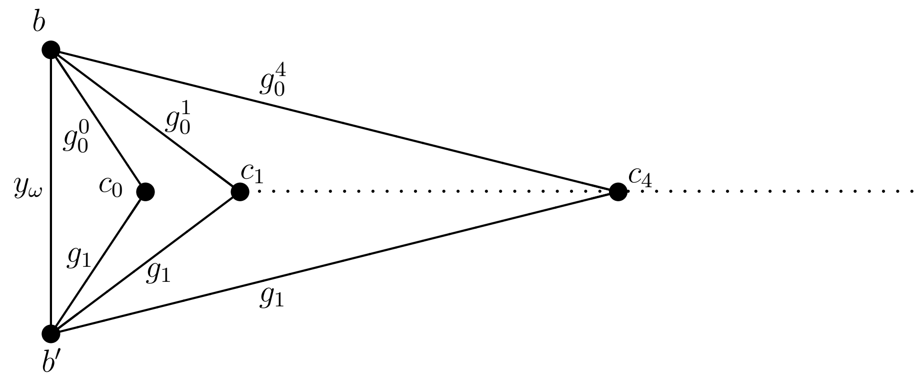

Figure 7: non-representability of the complex relation algebra

algebra of the form $(\wp(^2U), -, \cap, \ldots)$. Choose $(b, b') \in h(y_\omega^M)$ — as before (lemma 5.7), such a (b, b') exists. Now, by proposition 2.6, for each $i < \omega$ we evidently have $M \models \forall xy(y_\omega(x,y) \to \exists z(g_0^i(x,z) \wedge g_1(z,y)))$, so that $y_\omega^M \subseteq g_0^{i\,M}; g_1^M$. Hence, $(b, b') \in h(g_0^{i\,M}); h(g_1^M)$, so there is $c_i \in U$ with $(b, c_i) \in h(g_0^{i\,M})$ and $(c_i, b') \in h(g_1^M)$. See figure 7.

As in the cylindric algebra case, we can show that there is a red relation holding between any two distinct c_i. The six elements $1'$, $G = \bigcup_{i<\omega} g_0^{i\,M} \cup g_1^M$, $W = w_0^M \cup w_1^M$, and $R_{jk} = \bigcup_{i<\omega} r_{jk}^{i\,M}$ $(j < k < 3)$ of $\mathcal{C}$ form a finite partition of the atoms of $\mathcal{A}$. Now by definition of complex algebras, compositions of these elements (via ';') in $\mathcal{C}$ can be computed by taking the set of products of all atoms they contain. Using this, we see for example that $(G;G)\cdot G = 0$, because M has no green triangles. Since if $i \neq j$ then $(c_i, c_j) \in h(G;G)$, it follows that $(c_i, c_j) \notin h(G)$. It can be checked in a similar way that $(c_i, c_j) \notin h(1')$ and $(c_i, c_j) \notin h(W)$. So for each $i < j < \omega$ we must have $(c_i, c_j) \in h(R_{kl})$ for some $k < l < 3$.

By the pigeonhole principle, for some $i < j < 5$ and $k < l < 3$ we must have $(c_0, c_i), (c_0, c_j) \in h(R_{kl})$. Also, $(c_i, c_j) \in h(R_{k'l'})$ for some k', l'. But $R_{kl}; R_{kl} \cdot R_{k'l'} = 0$, since M has no triangles of the form $r_{kl}^i, r_{kl}^{i'}, r_{k'l'}^{i''}$ (with two suffixes the same). This leads to an impossibility, and completes the proof. ⊣

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