Available online at www.sciencedirect.com
ScienceDirect
International Journal of Solids and Structures 45 (2008) 2629–2650
INTERNATIONAL JOURNAL OF SOLIDS AND STRUCTURES
www.elsevier.com/locate/ijsolstr

Nonlinear analysis of structural elements under unilateral contact constraints by a Ritz type approach

Ricardo A.M. Silveira [a], Wellington L.A. Pereira [a], Paulo B. Gonçalves [b,*]

[a] Department of Civil Engineering, School of Mines, Federal University of Ouro Preto, Campus Universitário, Morro do Cruzeiro 35400-000 Ouro Preto, MG, Brazil
[b] Department of Civil Engineering, PUC-Rio, Rua Marquês de São Vicente, 225, Gávea 22453-900 Rio de Janeiro, RJ, Brazil

Received 5 September 2007; received in revised form 11 December 2007
Available online 28 December 2007

Abstract

A nonlinear modal solution methodology capable of solving equilibrium and stability problems of uni-dimensional structural elements (beams, columns and arches) with unilateral contact constraints is presented in this work. The contact constraints are imposed by an elastic foundation of the Winkler type, where special attention is given to the case in which the foundation reacts in compression only, characterizing the contact as unilateral. A Ritz type approach with moveable boundaries, where the coordinates defining the limits of the contact regions are considered as additional variables of the problem, is proposed to solve this class of unilateral contact problems. The methodology is illustrated by particular problems involving beams, beam-columns and arches, and the results are compared with available results obtained by finite element and mathematical programming techniques. It is concluded that the Ritz type approach proposed is particularly suited for the analysis of structural problems where the number, but not the length, of the contact regions between the bodies are known a priori. Therefore, it can substitute in these cases finite element applications and be used as a benchmark for more general and complex formulations as well.

Keywords: Unilateral contact; Stability; Ritz method; Tensionless foundation; Nonlinear structural analysis

1. Introduction

Contact interactions between a deformable structure and an elastic foundation are usually modeled by means of bilateral boundary conditions: displacements and/or forces prescribed in some (known) area of the structure. Such models are satisfactory for some engineering applications, but they stop being reliable when loss of contact occurs. Generally, in this case, it is necessary to establish unilateral boundary conditions as part of the solution, since the true contact area is unknown a priori. Structural elements used in foundation structures, pavements, flotation structures, beam-column joints, pipes on elastic foundation, piles partially

* Corresponding author. Tel.: +55 21 3527 1188; fax: +55 21 3527 1195.
E-mail addresses: ricardo@em.ufop.br (R.A.M. Silveira), paulo@civ.puc-rio.br (P.B. Gonçalves).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2007.12.012

embedded in soil, saddle-supported pipelines, composite laminate structures and protection shells in aggressive environments are examples of structures found in civil and mechanical engineering where unilateral contact may occur.

Even in the range of small deformations and under linear elastic behavior of the material, unilateral constraints introduce high nonlinearities, which cannot be dealt satisfactorily by usual nonlinear structural analysis methods. So, in order to study the equilibrium and stability of slender structural elements with unilateral constraints two types of nonlinearities, geometric and contact, must be taken into account and a reliable and efficient solution method is necessary.

In general, the first step to obtain the numerical solutions of contact problems consists in the discretization of the continuous system. For this, some numerical technique, such as Ritz method, finite element method (FEM) or boundary element method (BEM) is usually employed. After the discretization, attention is given to the selection of a proper methodology to treat adequately the contact constraints. Among several options found in the literature, one can mention:

(i) Transformation of the contact problem into a minimization problem without constraint through the adaptation of usual formulations of structural mechanics—differentiable functional and bilateral constraints—to the case of unilateral contact constraints. The convergence of these procedures, which are unavoidably of iterative nature, or incremental-iterative nature, is not guarantied. However, these procedures may or not introduce new concepts and existing computational codes for nonlinear analyses can be adapted to this particular case, leading to economy in computational time if no change in the contact region occurs between two load steps (Adan et al., 1994; Holmes et al., 1999; Silveira and Gonçalves, 2001; Li and Berger, 2003).

(ii) Use of mathematical programming techniques. This approach allows the solution of the contact problem with or without explicit elimination of unilateral constraints. Methods such as Lagrange's multipliers or penalties allow the elimination of the unilateral constraints (Simo et al., 1985; Barbosa, 1986; Wriggers and Imhof, 1993; Wriggers, 2002; Wriggers and Zavarise, 2004; Fisher and Wriggers, 2005). Usually these methods are based on the use of the special finite elements derived to simulate the impenetrability condition between two surfaces (Wriggers and Imhof, 1993; Wriggers, 2002). On the other hand, the unilateral constraints can be maintained in the formulation, retaining the original philosophy of the problem, and an alternative linear complementary problems (LCPs) can be obtained and solved by, for example, Lemke's or Dantig's algorithms (Lemke, 1968; Ascione and Grimaldi, 1984; Joo and Kwak, 1986; Barbosa, 1986; Silveira, 1995; Koo and Kwak, 1996; Silva, 1998; Silva et al., 2001; Silveira and Gonçalves, 2001; Wriggers, 2002; Pereira, 2003; Holanda and Gonçalves, 2003).

Numerical simulations dealing with the equilibrium and stability of structural elements under contact constraints, including here structure-foundation problems, can be easily found in the literature. For example, finite element applications including contact constraints in the stability of rods, limit-point behaviour of thick rubber spherical shells and large deformation post-buckling behavior of structures are presented, respectively, by Stein and Wriggers (1984), Endo et al. (1984) and Simo et al. (1986). Algorithms specifically designed to trace complex nonlinear equilibrium paths, such as arc-length procedures (Crisfield, 1991), have been used by Wriggers et al. (1987), Stein et al. (1990), Björkman (1992), Koo and Kwak (1996) and Silveira and Gonalves (2001), among others, to solve stability problems of structures with unilateral contact constraints. Tschöpe et al. (2003a,b) extended methods for the detection of critical points to problems with inequality constraints. Different formulations, algorithms and discretisation techniques for structural contact problems are described in depth on Wriggers' book (Wriggers, 2002). Recently Wriggers and Zavarise (2004) presented a state of art of the solution of contact problems within a computational mechanics approach.

The two general approaches defined above for the treatment of the contact constraints have also been applied to a wide range of structure-foundation problems. For example, Kadkhodayan (2006) studied the influence of deformable dies on the springback of circular plates and Hsu (2006) analyzed the behavior of non-uniform beams resting on a nonlinear media. Geotechnical applications, where the soil-structure interaction is highlighted, can be found in the works of Mezaini (2006), Küçükarslan and Banerjee (2005) and Maheshwari et al. (2004). Also recently, many papers were published concerning the nonlinear dynamic response of thin and moderately thick plates resting on a tensionless Winkler or Pasternak-type foundation (Yu et al., 2007; Güler and Celep, 2005; Celep and Güler, 2004; Celep and Gençoglu, 2003). Numerical approximations involving the stability, buckling and post-buckling behavior of plates under unilateral contact constraints imposed by elastic foundation appear in recent papers by Muradova and Stavroulakis (2006), Shen and Li (2004), Shen and Yu (2004), Shen and Teng (2004) and Holanda and Gonçalves (2003). Wang et al. (2005) provide an important review of the state of the art of beams and plates on elastic foundation; they include soil modeling as well as analytical and numerical possibilities for solving this class of contact problem.

This paper focuses on unilateral contact problems involving beams, columns and arches on a tensionless elastic foundation of the Winkler-type; it adopts the first alternative described above for the treatment of the contact constraints. Since 1990, the authors of this paper have analyzed several contact problems involving a deformable structure with contact constraints (Silveira, 1995; Silva et al., 2001; Silveira and Gonçalves, 2001; Pereira, 2003; Holanda and Gonçalves, 2003; Pereira and Silveira, 2006). These works attempted to establish reliable methodologies for the analysis of structures with unilateral boundary conditions. Therefore, this article can be considered as an extension of these previous ones, but adds a new contribution by providing an alternative methodology for the analysis of structures resting unilaterally on an elastic foundation.

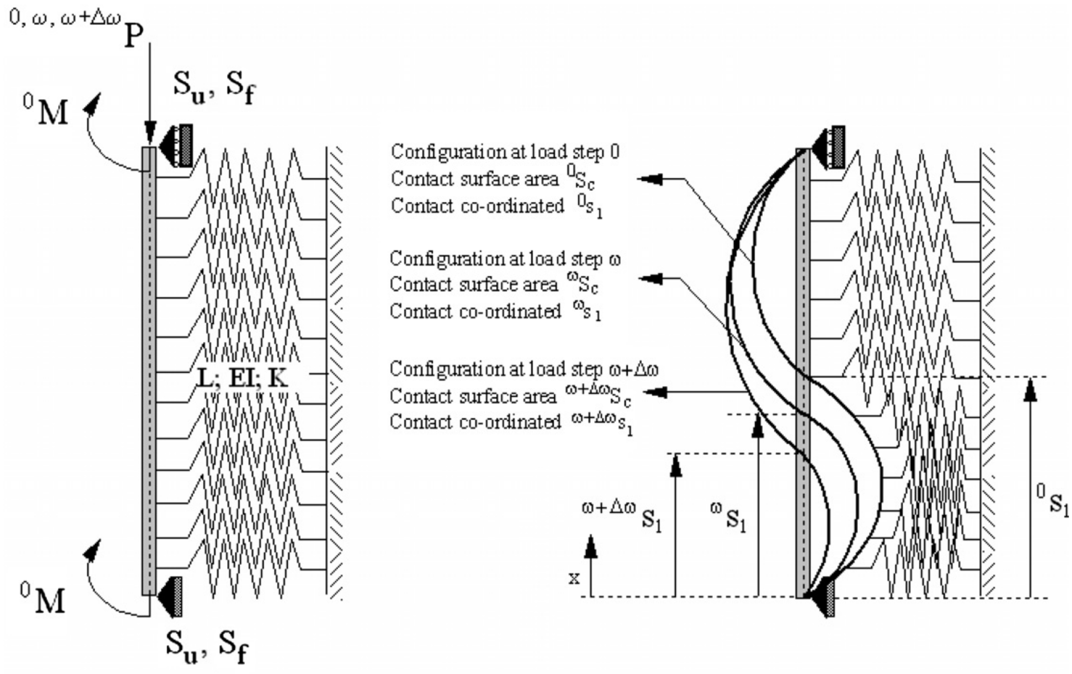
The aim of the present work is to develop a semi-analytical methodology, using a Ritz type approach, for the elastic equilibrium and instability analysis of beams, columns and arches resting on a tensionless Winkler-type elastic foundation. In the proposed Ritz approach the displacements and end points of the contact regions are taken as basic unknowns. This approach is particularly suited for the analysis of structural problems where the number, but not the location or length of the contact areas, is known a priori, leading to a fast convergence. When complicated loading cases are considered, one can perform initially an analysis considering bilateral contact and identify the number of regions where the foundation is under traction. This data can be considered as a starting point in the iterative procedure, leading usually to a small number of iterations. The number of contact regions can also be determined, but this leads to a more involved and numerically less stable algorithm. This leads to highly nonlinear equations. In order to solve the resulting algebraic nonlinear equations and obtain nonlinear equilibrium paths, the Newton–Raphson method is used together with an arc-length iteration procedure (Crisfield, 1991). This incremental-iterative strategy allows limit points to be passed and, consequently, snap buckling phenomena to be identified. Unilateral contact problems analyses and comparisons with existing results validate the proposed formulation.

Of course, the finite element method is the most appropriate tool to analyze complex nonlinear systems with unilateral constraints. However, it is usually expensive with respect to both storage and CPU costs, particularly in the analysis of two and three dimensional contact problems. As a result, it is difficult to deal with situations such as sensitivity analyses, optimization, feedback control problems and parametric analyses. Not surprisingly, in recent years a lot of attention has been paid to reducing the coast of nonlinear solutions by using reduced-order models (Rega and Troger, 2005). We believe the present methodology, although case specific and tested for uni-dimensional structural systems, is a semi-analytical reliable alternative for the construction of reduced-order approximations for several structure-foundation problems with unilateral constraints which can be effectively used in practical applications.

2. Formulation of the unilateral contact problem

Consider the structural system shown in Fig. 1a consisting of a bar and an elastic tensionless foundation, and assume that both bodies may undergo large deflections and rotations but small strains, within the elastic range of the material. In addition, the contact surface is assumed unbonded and frictionless. The column is defined as a solid elastic continuum which occupies a domain ${}^iV$ ($i = 0$, ω and $\omega + \Delta\omega$). Its boundary ${}^iS$ is composed of three regions: ${}^iS_u$, ${}^iS_f$ and ${}^iS_c$, where the surface forces are specified on ${}^iS_f$ and the displacements are specified on ${}^iS_u$. The remaining part, ${}^iS_c$, corresponds to the region where contact is likely to occur, which is not known a priori.

For the structural system, the equilibrium equations, the kinematic relations and the constitutive law are given, respectively, by:



(a) Engineering problem (b) Deformation pattern

Fig. 1. Structural system under unilateral contact constraints imposed by an elastic foundation.

$$\Delta S_{ij,j} + (\Delta u_{i,j}^{\omega+\Delta\omega} S_{jk,i})_{,k} = 0 \tag{1a}$$

$$\Delta \varepsilon_{ij} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + \Delta u_{k,i}\Delta u_{k,j}) \tag{1b}$$

$$\Delta S_{ij} = C_{ijkl}\Delta\varepsilon_{kl} \tag{1c}$$

where the customary summation convention is used. In Eq. (1a), ΔS_{ij} are the components of the 2nd Piola–Kirchhoff stress increment tensor, the unknowns of the problem, and ${}^{\omega+\Delta\omega}S_{ij}$ are the Cartesian components of this same tensor at state $\omega + \Delta\omega$, which are referred to the previous equilibrium configuration ω. Δu_i, $\Delta\varepsilon_{ij}$ and C_{ijkl} define the increment of displacement and strains and material properties, respectively.

As in many engineering applications, if the designer is interested only on the response of the foundation at the contact area, it is possible to construct simple mathematical models for describing the response of the foundation at the contact zone with a reasonable degree of accuracy. Using the well-known Winkler model (Hetényi, 1946; Kerr, 1964), the following constitutive equation can be written to describe the elastic foundation reaction:

$$\Delta r_b = C_b \Delta u_b \tag{2}$$

where Δr_b and Δu_b are the incremental compressive reaction and deflection of the foundation, respectively, and C_b is the foundation elastic modulus.

For the two bodies under investigation, the displacements and the surface forces are specified on ${}^iS_u$ and ${}^iS_f$, respectively, and the following conditions must be satisfied on S_c:

(i) The gap in the potential contact area, φ, after the increment of the displacements, must satisfy the following inequality constraint at configuration $\omega + \Delta\omega$:

$$\varphi \geqslant 0 \tag{3}$$

(ii) Under the assumption of a tensionless foundation model, contact pressure must be compressive, i.e.:

$$r_b \geqslant 0 \tag{4}$$

(iii) The complementary relation between φ and r_b is:

$$\int_{\omega+\Delta\omega S_c} r_b \varphi^{\omega+\Delta\omega} \mathrm{d}S_c = 0 \tag{5}$$

These three constraints define in a complete way the contact as being unilateral. Fig. 2 shows the domain of validity of these relations and the contact law.

For a given load increment, the solution of the unilateral contact problem can be obtained by solving Eq. (1a), together with Eqs. 1b, 1c and 2, and by satisfying the appropriate boundary conditions on S_u and S_f, as well as the restrictions (3)–(5) on S_c. However, the nonlinearity due to the unilateral constraints and the nonlinear strain–displacement relations make it difficult to solve this problem directly. For this reason, an equivalent minimization problem is formulated, which is particularly suitable for numerical analysis. According to Joo and Kwak (1986) and Silveira and Gonçalves (2001), the optimization's problem:

$$\text{Min } \Pi(\Delta u, \Delta u_b) \tag{6}$$

$$\text{Subject to } : -\varphi \leqslant 0, \text{ on } S_c \tag{7}$$

where,

$$\Pi = \int_{\omega_V} \left({}^{\omega}S_{ij} + \frac{1}{2}\Delta S_{ij}\right) \Delta\varepsilon_{ij}{}^{\omega} \, \mathrm{d}V + \int_{\omega+\Delta\omega S_c} \left({}^{\omega}r_b + \frac{1}{2}\Delta r_b\right) \Delta u_b^{\omega+\Delta\omega} \mathrm{d}S_c - \int_{\omega+\Delta\omega S_f} F_i \Delta u_i^{\omega+\Delta\omega} \mathrm{d}S_f \tag{8}$$

is equivalent to the unilateral contact problem described above by Eqs. (1) to (5).

3. Discretization procedure and modal solution

The solution of the minimization problem defined by Eqs. (6) and (7), using mathematical programming techniques, was presented by Silveira (1995), Silveira and Gonçalves (2001) and Holanda and Gonçalves (2003). Now, a different strategy of solution is proposed assuming that contact constraints (3)–(5) on S_c, and the elastic foundation displacements, can be introduced in the analysis by considering explicitly the coordinates defining the limits of the contact regions (s_k) as additional variables of the problem (see Fig. 1b). Hence, for a structural member in contact with a tensionless elastic foundation and subjected to conservative loads, the total potential energy functional can be rewritten as:

$$\overline{\Pi}_1(\mathbf{u}, \mathbf{Sc}, \lambda) = U(\mathbf{u}, \mathbf{Sc}) - \lambda \mathbf{F}_\mathbf{r}^\mathbf{T} \mathbf{u} \tag{9}$$

where U is the strain energy which is a function of the displacement vector $\mathbf{u}$ and of the vector $\mathbf{S_c}$, which contains the coordinates defining the limits of the contact regions (s_k). These coordinates are considered here as additional variables of the problem. Note that the length of each contact region is a function of the system parameters and is not known a priori. This characterizes the unilateral contact problem as nonlinear. $\mathbf{F_r}$ is a fixed load vector (reference vector) and λ is a scalar load multiplier.

If the Ritz method is applied, the following displacement field, written in matrix form, can be used to approximate $\overline{\Pi}_1$:

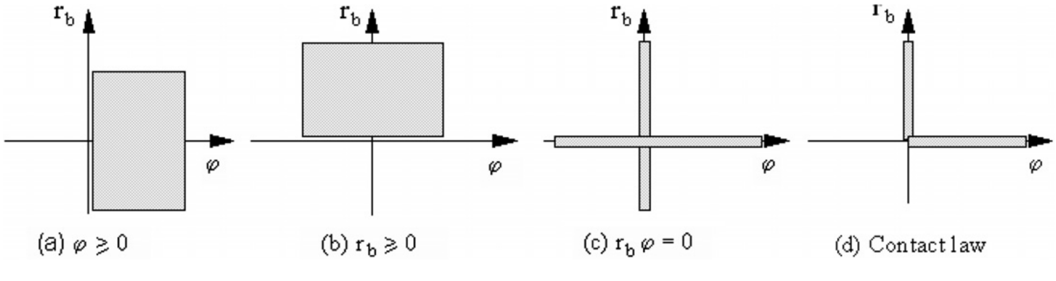


Fig. 2. Domain of validity of the contact constraints.

$$\mathbf{u} = \mathbf{\Psi A} \tag{10}$$

where the matrix $\mathbf{\Psi}$ contains the functions that satisfy the boundary conditions on S_u and the components of vector $\mathbf{A}$ are the unknown coefficients. Thus, substituting (10) into (9), considering a small variation in the total potential energy, and expanding Eq. (9) in a Taylor series, one obtains:

$$\delta\overline{\Pi}_1 = \frac{\partial\overline{\Pi}_1}{\partial\mathbf{A}}\delta\mathbf{A} + \frac{\partial\overline{\Pi}_1}{\partial\mathbf{Sc}}\delta\mathbf{Sc} + \frac{1}{2}\left[\delta\mathbf{A}^{\mathrm{T}}\frac{\partial^2\overline{\Pi}_1}{\partial\mathbf{A}^2}\delta\mathbf{A} + 2\delta\mathbf{A}^{\mathrm{T}}\frac{\partial^2\overline{\Pi}_1}{\partial\mathbf{A}\partial\mathbf{Sc}}\delta\mathbf{Sc} + \delta\mathbf{Sc}^{\mathrm{T}}\frac{\partial^2\overline{\Pi}_1}{\partial\mathbf{Sc}^2}\delta\mathbf{Sc}\right] + 0(\delta\mathbf{A}^3, \delta\mathbf{Sc}^3) \tag{11a}$$

or, using a more compact matrix notation:

$$\delta\overline{\Pi}_1 = \mathbf{g}^{\mathrm{T}}\delta\mathbf{U} + \frac{1}{2}(\delta\mathbf{U}^{\mathrm{T}}\mathbf{M}\delta\mathbf{U}) + 0(\delta\mathbf{U}^3) \tag{11b}$$

in which the vector $\mathbf{U}$ contains the unknown variables of the problem ($\mathbf{A}$ and $\mathbf{Sc}$), $\mathbf{g}$ is the gradient vector (out-of-balance load vector), and $\mathbf{M}$ is the Hessian matrix, which can be written as follow:

$$\mathbf{M} = \begin{bmatrix} \mathbf{K} & \mathbf{J} \\ \mathbf{J}^{\mathrm{T}} & \mathbf{S} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2\overline{\Pi}_1}{\partial\mathbf{A}^2} & \frac{\partial^2\overline{\Pi}_1}{\partial\mathbf{A}\partial\mathbf{Sc}} \\ \frac{\partial^2\overline{\Pi}_1}{\partial\mathbf{Sc}\partial\mathbf{A}} & \frac{\partial^2\overline{\Pi}_1}{\partial\mathbf{Sc}^2} \end{bmatrix} \tag{12}$$

where $\mathbf{K}$, $\mathbf{S}$ and $\mathbf{J}$ are the stiffness, contact and joining matrices, respectively.

For equilibrium, the change in Eq. (11b) should be stationary irrespective of $\delta\mathbf{U}$ and hence, the equilibrium equations are:

$$\frac{\partial\overline{\Pi}_1}{\partial\mathbf{U}} = \mathbf{g} = \begin{Bmatrix} \frac{\partial\overline{\Pi}_1}{\partial\mathbf{A}} \\ \frac{\partial\overline{\Pi}_1}{\partial\mathbf{Sc}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{13}$$

which represent a nonlinear algebraic equations system, involving polynomial or transcendental functions of s_k.

3.1. Nonlinear solution strategy

The incremental-iterative solution strategy for geometrically nonlinear elastic contact problems adopted in this work will be summarized in this section. In the Ritz method context, the equilibrium Eq. (13) can be rewritten as follow:

$$\mathbf{g} = \mathbf{F_i}\ (\mathbf{U}) - \lambda\mathbf{F_r} \quad \cong \mathbf{0} \tag{14}$$

where $\mathbf{F_i}$ defines a set of generalized internal forces in terms of the components $\mathbf{U} = \mathbf{U}(\mathbf{A}, \mathbf{Sc})$.

Fig. 3 shows the incremental-iterative solution strategy used to solve Eq. (13) or Eq. (14). Here a constant cylindrical arc-length constraint is adopted (Crisfield, 1991). In this solution scheme two distinct steps are required for each load increment: (1) a *predictor phase*, where approximations for $\Delta\lambda^0$, $\Delta\mathbf{A}^0$ and $\mathbf{Sc}^0$ are obtained; (2) a *corrector phase*, where these approximations are corrected to satisfy the equilibrium equations. The arc-length constraint equation is only applied on the unknown amplitudes $\mathbf{A}$.

4. Examples

The general semi-analytical approach proposed is now particularized for two structural systems under unilateral contact constraints.

4.1. Example 1: beam-column

Consider a beam-column of length L and bending stiffness EI, resting on a tensionless Winkler foundation of stiffness K subjected to two concentrated moments at the supports plus an axial load, as illustrated in Fig. 1a. For this structural system, the total potential energy is given by (Shames and Dym, 1995):

1. Load increment: nl = 1, 2,..., nlt
2. Compute: "tangent incremental solution" } *Predictor phase*
• Approximate: $\mathbf{S_c}$ and $\Delta\lambda^0$
• Compute: $\mathbf{M}$
• Compute: $\Delta\mathbf{U}^0$ ($\Delta\mathbf{U}^0 = \mathbf{M}^{-1}\,\Delta\lambda^0\mathbf{F_r}$)
3. Iterations - *equilibrium problem*: k = 1, 2,..., I_{max} } *Corrector phase*
• Compute: $\mathbf{g}^{(k-1)}$
• Check convergence:

$$\frac{\left\|\mathbf{g}^{(k-1)}\right\|}{\left\|\Delta\lambda^{(k-1)}\mathbf{F_r}\right\|} \leq \xi \ ?$$

Yes: Go to step 4
No: Compute the corrections: $\delta\mathbf{U}^k$ and $\delta\lambda^k$ using the constraint equation:

$$(\Delta\mathbf{A}^k)^T\Delta\mathbf{A}^k + (\Delta\lambda^k)^2\mathbf{F_r}^T\mathbf{F_r} = \Delta\ell^2$$

Updated variables: $\Delta\mathbf{U}^k$ and $\Delta\lambda^k$, go to step 3
4. Go to step 1 if nl < nlt; stop if nl = nlt.

Fig. 3. Nonlinear solution procedure based on the second order Newton–Raphson method and arc-length strategies.

$$\Pi_2 = \frac{\mathrm{EI}}{2}\int_0^L\left[1+\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]^2\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2\mathrm{d}x - \frac{P}{2}\int_0^L\left[1+\frac{1}{4}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\mathrm{d}x + \frac{K}{2}\int_0^{s_1}w^2\mathrm{d}x - M\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{(x=0)} - M\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{(x=L)} \tag{15}$$

where s_1 represents the length of the contact region. For this system, the expected behavior of the member is shown in Fig. 1b, where one contact region of length s_1 is expected.

For a simply supported bar, the following linear combination of harmonic functions can be used to approximate the displacement field

$$w = \sum_{i=1}^{n} W_i \sin\left(\frac{i\pi x}{L}\right) \tag{16}$$

where i is the number of half-waves, n is the total number of modes necessary to achieve convergence and W_i are the modal amplitudes. After substitution of (16) and its derivatives in Eq. (15), an approximated form for Π_2 is obtained. Hence:

$$\overline{\Pi}_2(W_i, s_1, \lambda) = U(W_i, s_1) - V(W_i, M, \lambda P) \tag{17}$$

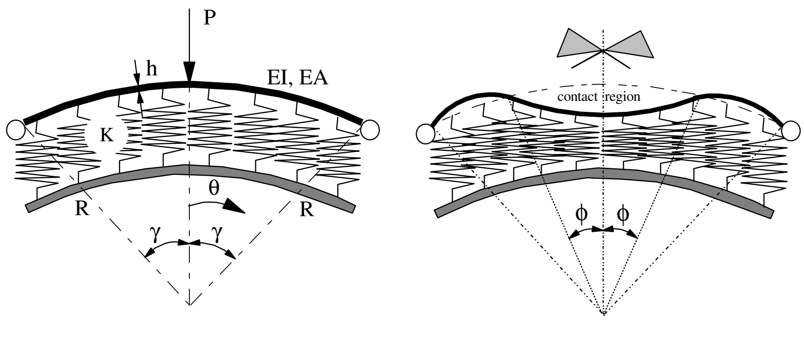
where U is the strain energy, which is a function of the modal amplitudes W_i and of the coordinate defining the limit of the contact region s_1 and V is the potential of the external load. The analytic expression for this approximation is shown in the Appendix A.

The nonlinear algebraic equations for this particular problem are, according to Eq. (13):

$$\frac{\partial\overline{\Pi}_2}{\partial\mathbf{U}} = \mathbf{g} = \begin{Bmatrix} \frac{\partial\overline{\Pi}_2}{\partial W_i}, \quad i = 1,\ldots,n \\ \frac{\partial\overline{\Pi}_2}{\partial s_1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ 0 \end{Bmatrix} \tag{18}$$

4.2. Example 2: shallow arch

Now, consider the arch shown in Fig. 4a. It is a circular pinned arch of radius R, length $2\gamma R$, bending stiffness EI and membrane stiffness EA in contact with a tensionless Winkler foundation of modulus K. For this system, the expected deformation pattern is shown in Fig. 4b, where a central contact region defined by the angles $\pm\phi$ is expected. The total potential energy of the arch may be written as (Brush and Almroth, 1975):



(a) Structural problem (b) Deformation pattern

Fig. 4. Arch under unilateral contact constraints imposed by an elastic foundation.

$$\Pi_3 = \frac{\mathrm{EI}}{2R^3}\int_{-\gamma}^{\gamma}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}-\frac{\mathrm{d}^2w}{\mathrm{d}\theta^2}\right)^2\mathrm{d}\theta + \frac{\mathrm{EA}}{2R}\int_{-\gamma}^{\gamma}\left[\frac{\mathrm{d}u}{\mathrm{d}\theta}+w+\frac{1}{2R}\left(u-\frac{\mathrm{d}w}{\mathrm{d}\theta}\right)^2\right]^2\mathrm{d}\theta + \frac{KR}{2}\int_{-\phi}^{\phi}w^2\mathrm{d}\theta - Pw(\theta=0) \tag{19}$$

where u and w are the tangential and transversal displacements, respectively, of a point along the centroidal-axis, and θ is the circumferential co-ordinate. These displacements for a simply supported arch may be approximated by:

$$u = \sum_{i_u}^{m} U_{i_u}\sin\left(\frac{(2i_u-1)\pi\theta}{2\gamma}\right) \tag{20a}$$

$$w = \sum_{i_w}^{n} W_{i_w}\cos\left(\frac{(2i_w-1)\pi\theta}{2\gamma}\right) \tag{20b}$$

where i_u and i_w are the number of half-waves, and U_{i_u} and W_{i_w} are the modal amplitudes.

The substitution of u and w and their derivatives in Eq. (19) leads to Π_3 as a function of the modal amplitudes U_{i_u} and W_{i_w}, contact angle ϕ, and load parameter λ, i.e.:

$$\overline{\Pi}_3(U_{i_u}, W_{i_w}, \phi, \lambda) = U(U_{i_u}, W_{i_w}, \phi) - V(W_{i_w}, \lambda P) \tag{21}$$

where the contact angle limit ϕ is the additional variables of the problem. The analytic expression for $\overline{\Pi}_3$ is shown in the Appendix B.

The equilibrium equations of the arch are then:

$$\frac{\partial\overline{\Pi}_3}{\partial\mathbf{U}} = \mathbf{g} = \begin{Bmatrix} \frac{\partial\overline{\Pi}_3}{\partial U_{i_u}}, & i_u = 1,\ldots,m \\ \frac{\partial\overline{\Pi}_3}{\partial W_{i_w}}, & i_w = 1,\ldots,n \\ \frac{\partial\overline{\Pi}_3}{\partial\phi} & \end{Bmatrix} = \begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\\ 0\end{Bmatrix} \tag{22}$$

5. Results and discussion

This section aims to validate the proposed semi-analytical methodology for the analysis of structural elements under unilateral constraints as well as the adopted nonlinear solution strategy. The results of this work are compared with available results obtained by finite element and mathematical programming techniques. The finite element methodology for the small displacement analysis where the contact problem is solved as a linear complementary problem, LCP, involving only the original variables subjected to inequality constraints, which is solved by Lemke's algorithm (Lemke, 1968), can be found in Silva et al. (2001), Pereira (2003) and Pereira and Silveira (2006). For the geometrically nonlinear analysis of slender structural elements under unilateral contact the relevant information is found in Silveira (1995), Silveira and Gonçalves (2001) and Holanda and Gonçalves (2003).

First the behavior of a beam subjected to three different loading cases is analyzed. They are illustrated in Fig. 5 together with the relevant parameters (given in consistent units) and the expected deformation pattern. Although small displacements and linear behavior of the material are considered, the unilateral contact must be solved by an iterative process since the contact lengths are unknown a priori. So, the proposed algorithm (see Fig. 3) is employed considering only one load step. In all the numerical analysis a convergence factor of $\xi = 10^{-3}$ is adopted. Consider first the beam illustrated in Fig. 5a. Ten terms are considered in Eq. (16). This is enough to achieve convergence in all examples presented here. Initially, an elastic foundation that reacts in compression and tension (bilateral contact problem) is considered. The results are presented in Fig. 6a where the anti-symmetrical behavior of the beam can be observed for different values of the foundation modulus $k = \mathrm{KL}^4/\mathrm{EI}$. Good agreement with the analytical and numerical results given by Hetényi (1946) and Pereira (2003) is observed. Now, a tensionless foundation is considered. The results for the beam deflection and elastic foundation reaction are presented for the same values of the foundation modulus k in Fig. 6b and c, respectively. The contact region (and the corresponding displacements) decreases steadily as the parameter k increases, while the beam displacements in the non-contact region increase. This is followed by a drastic increase in the foundation reaction in the contact region. The dependence of the contact area on the foundation stiffness is one of the main characteristics of tensionless foundation as compared with the conventional foundation. Again a good agreement is observed between the modal solution and the FE results.

The second contact problem is shown in Fig. 5c. The results are shown in Fig. 7a (bilateral contact constraints), and Fig. 7b and c (unilateral contact constraints). Again, the variation of the beam displacements and elastic foundation reaction for different values of the non-dimensional elastic foundation stiffness parameter k is presented. The good agreement with results obtained through finite element/LCP approach (Pereira, 2003) and analytical formulation (Hetényi, 1946) demonstrates the accuracy and efficiency of the proposed methodology. The influence of the foundation stiffness parameter k as well as of the type of contact constraints imposed is clearly noted for large values of k. There is for relatively stiff foundations a marked difference

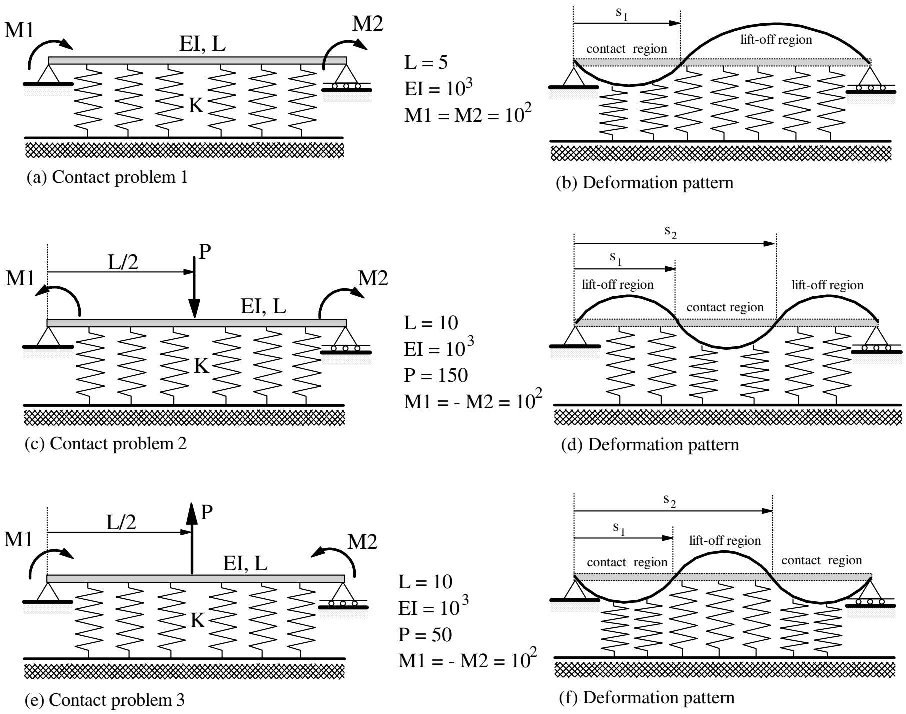
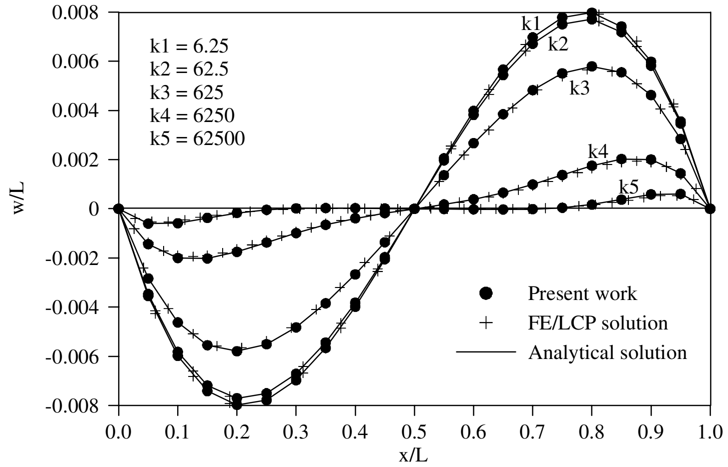
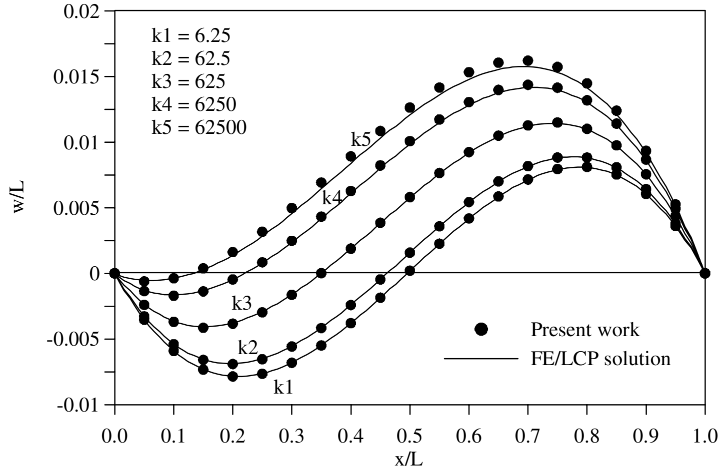


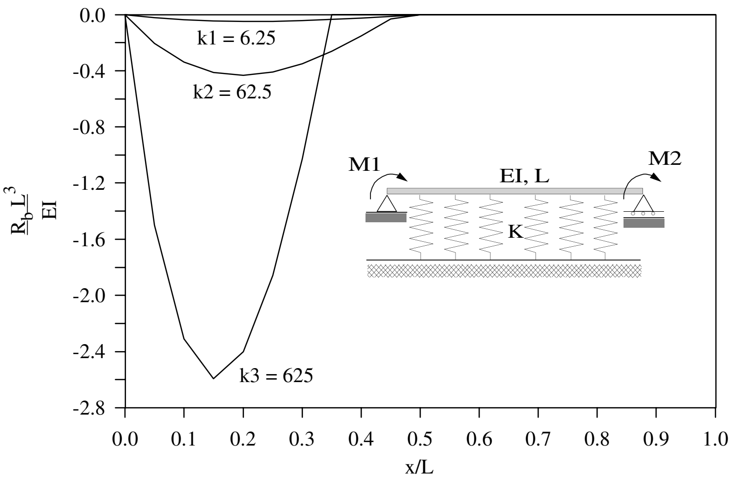
Fig. 5. Three examples of beams resting on a tensionless elastic foundation.



(a) Bilateral contact: beam displacement



(b) Unilateral contact: beam displacement



(c) Unilateral contact: elastic foundation reaction.

Fig. 6. Contact problem 1 (Fig. 5a): beam resting on an elastic foundation. (a) Bilateral contact: beam displacements for increasing values of the foundation stiffness, k. (b) Unilateral contact: beam displacements for increasing values of the foundation stiffness, k. (c) Unilateral contact: foundation reaction.

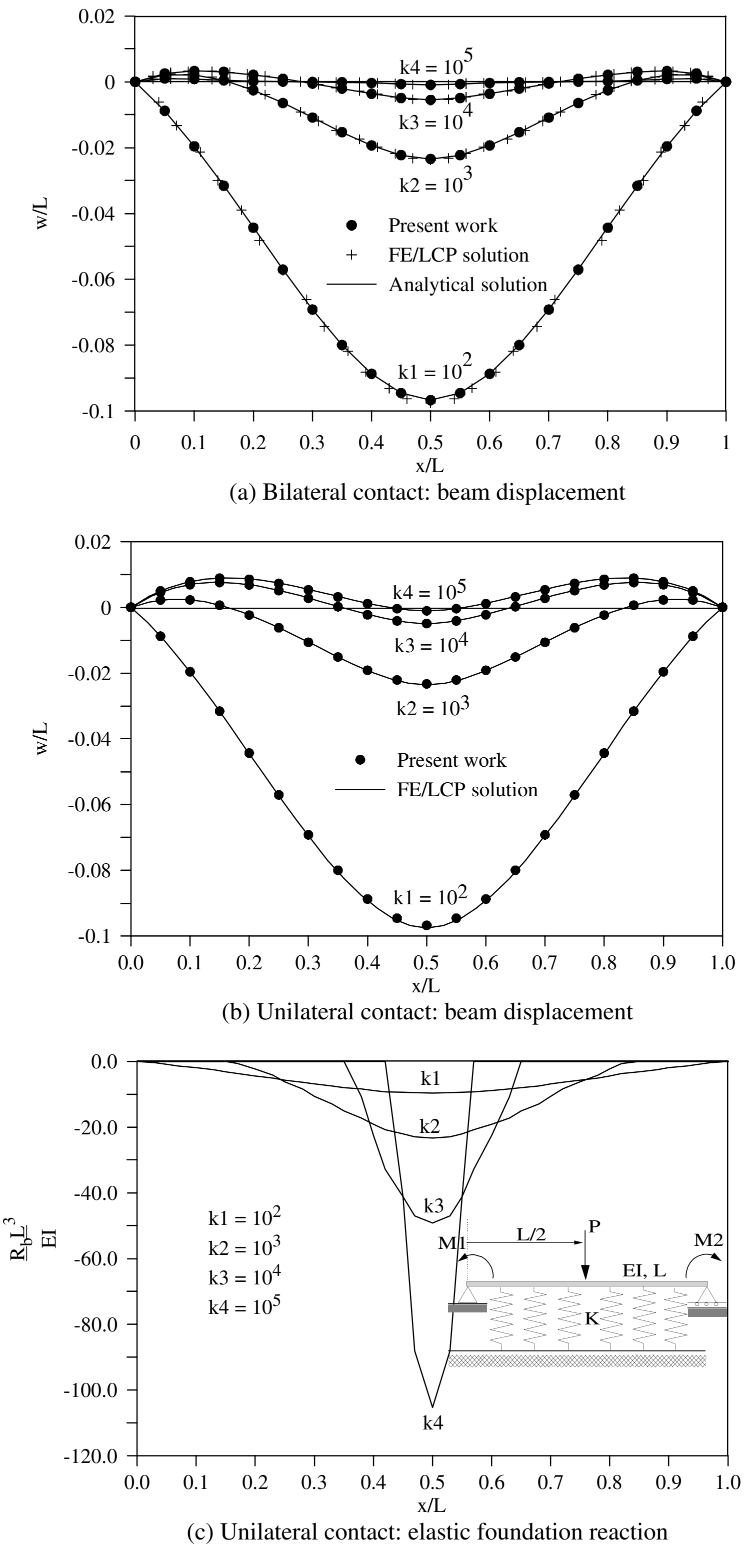
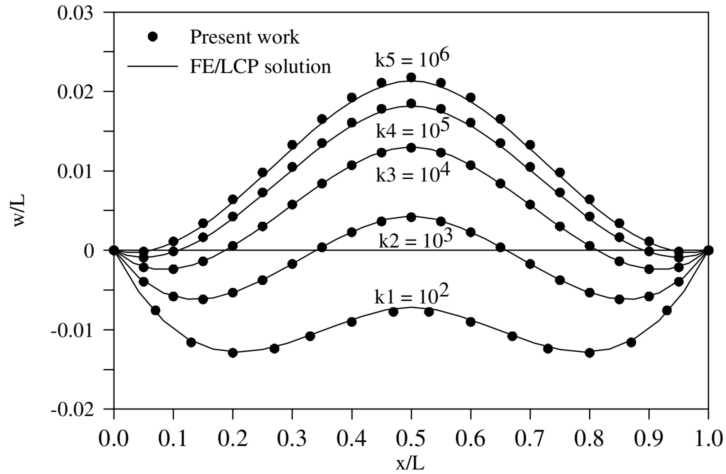
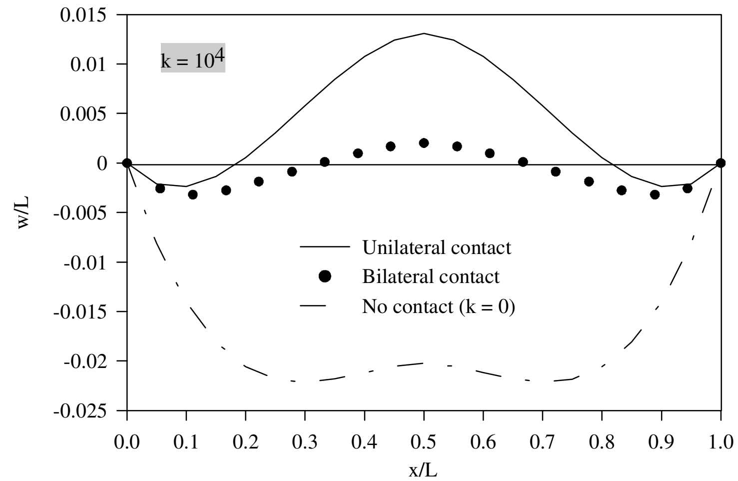


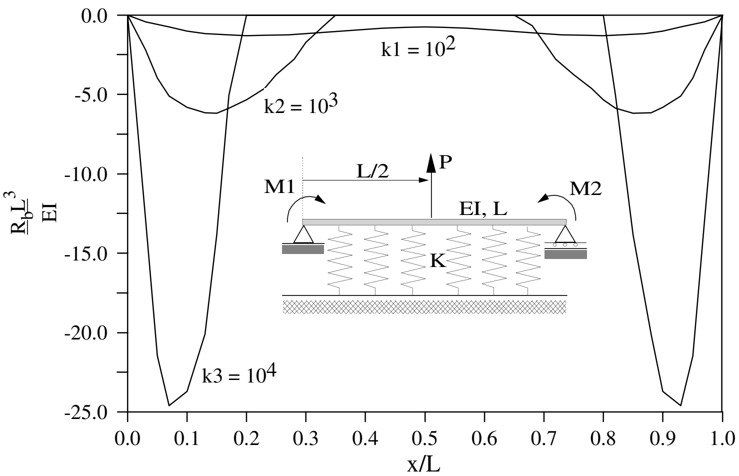
Fig. 7. Contact problem 2 (Fig. 5b): beam resting on an elastic foundation. (a) Bilateral contact: beam displacements for increasing values of the foundation stiffness, k. (b) Unilateral contact: beam displacements for increasing values of the foundation stiffness, k. (c) Unilateral contact: foundation reaction.



(a) Unilateral contact: beam displacement



(b) Comparison between bilateral and unilateral contact problems

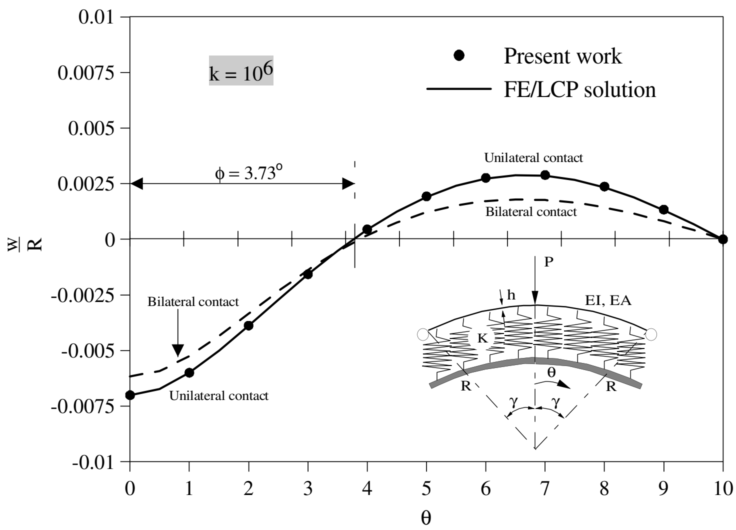


(c) Unilateral contact: elastic foundation reaction

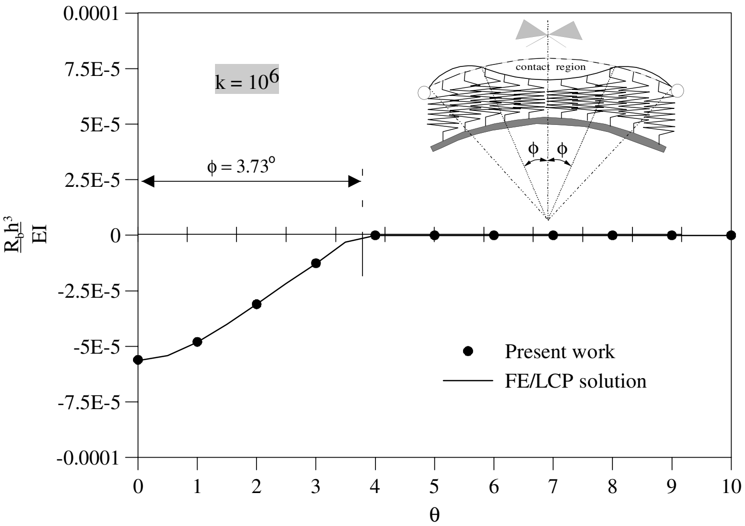
Fig. 8. Contact problem 3 (Fig. 5c): beam resting on an elastic foundation. (a) Bilateral contact: beam displacements for increasing values of the foundation stiffness, k. (b) Unilateral contact: beam displacements for increasing values of the foundation stiffness, k. (c) Unilateral contact: foundation reaction.

between the displacements of the tensionless and the conventional foundation models. Under unilateral constraints the beam displacement w increases on the non-contact region and decreases on the contact region. Consequently, considerable error may result if the unilateral character of the foundation is not taken into account in the analysis. As shown in Fig. 7, as k increases, we approach the limiting case of $k \to \infty$, the tensionless rigid foundation case. As in many numerical applications, the rigid foundation can be simulated by considering a very large value for k. Fig. 5e shows the third example. Fig. 8 shows the relevant results for different values of the k. Again a good agreement between the present results with those previously obtained using the FE/LCP approach (Pereira, 2003) is observed.

Now, consider the arch shown in Fig. 4a. It is a slender circular arch of radius R, length $2\gamma R$, bending stiffness EI and membrane stiffness EA in contact with a tensionless foundation of modulus K. For a concentrated load P applied at $\theta = 0$, the expected deformation pattern is shown in Fig. 4b, where a central contact region, defined by the angles $\pm\phi$, is expected. For the arch, five terms ($m = 5$) in Eq. (20a) and 10 terms ($n = 10$) in Eq. (20b) are considered to obtain convergence in all cases. The arch lateral displace-



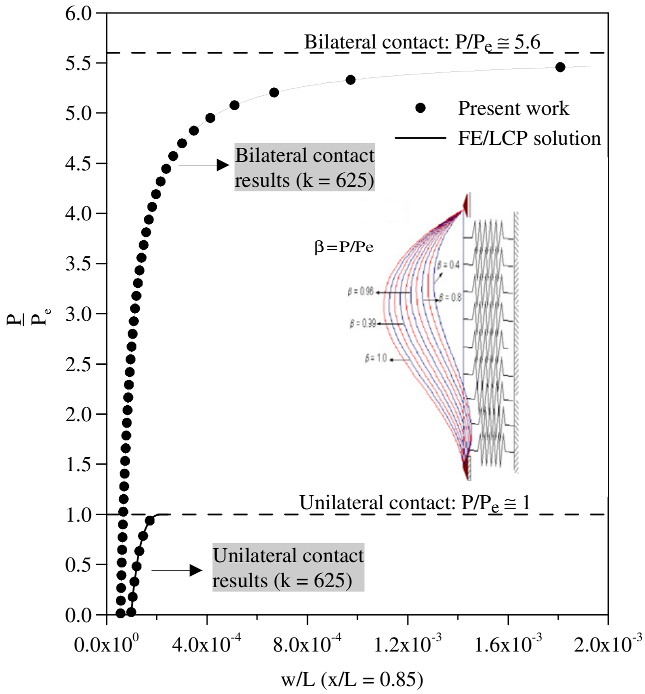
(a) Arch displacement



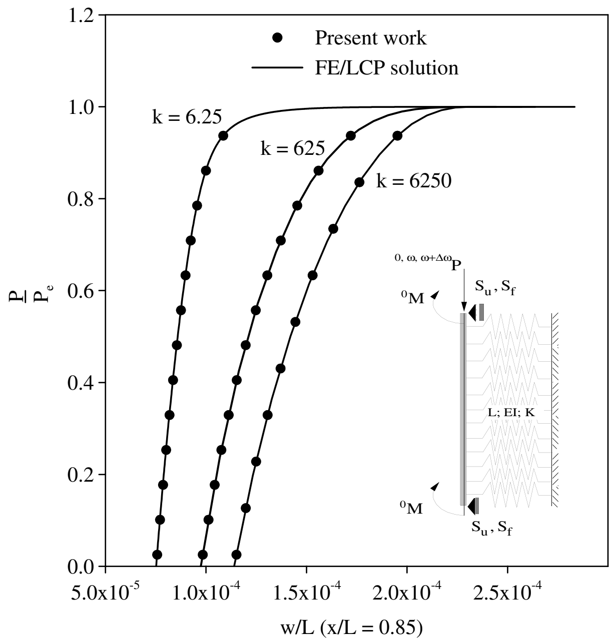
(b) Elastic foundation reaction

Fig. 9. Arch in contact with an elastic foundation.

ment w and elastic foundation reaction r_b are shown in Fig. 9a and b, respectively, considering the following data (Walker, 1969): $R/h = 500$ (h = thickness), $\gamma = 10°$, $EI = 1.4$, $EA = 420$, a non-dimensional elastic foundation stiffness parameter $k = KR^4/EI = 10^6$, and $P = 0.1$. Due to the problem symmetry, results are shown for half arch. The conventional foundation model (bilateral contact) was also considered



(a) Column equilibrium paths for the bilateral and unilateral contact problems

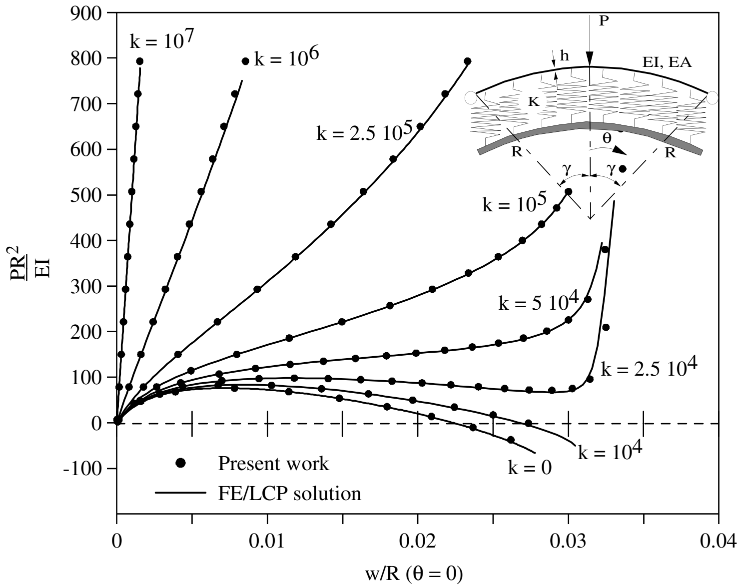


(b) Nonlinear equilibrium paths for various values of k (unilateral constraints)

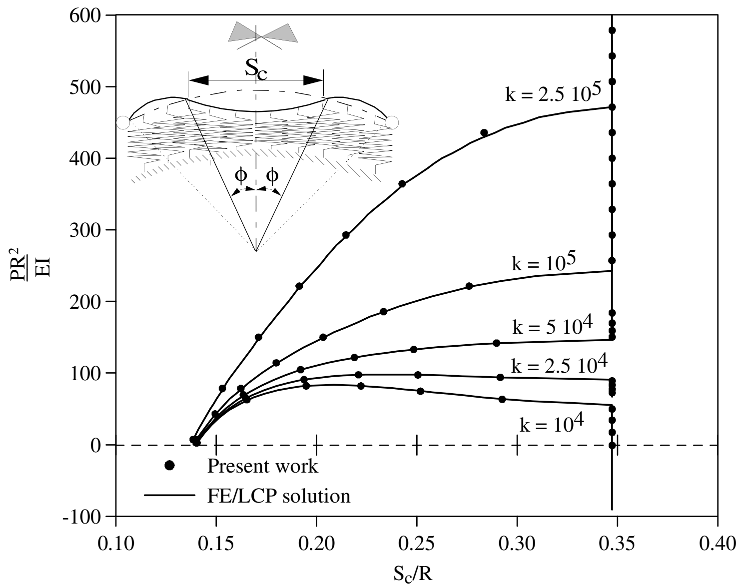
Fig. 10. Nonlinear response of a beam-column resting on an elastic foundation.

in Fig. 9a. One can observe that, under unilateral contact constraints, the arch displacement w increases in the non-contact region. The contact angle between the bodies $2\phi = 7.46^\circ$ coincide with that obtained using the FE/LCP approach (Silveira, 1995).

The first stability problem under contact constraints is illustred in Fig. 10a. It is a beam-column under an increasing compressive axial load P and constant bending moments at the supports (see also Fig. 1). The bending moments M at the supports have a constant small value and act as initial load imperfections, generating in the unilateral contact case a non-contact region. Thus, this analysis aims to verify the nonlinear response and changes on the contact and non-contact regions with the monotonic increment of the load P.



(a) Arch equilibrium paths for various values of k



(b) Variation of the contact region S_c between the bodies

Fig. 11. Nonlinear response and stability of a shallow arch under unilateral contact constraints.

The results are obtained through the proposed nonlinear solution strategy and by considering the following data: $L = 5$; $\mathrm{EI} = 10^3$; $\mathrm{EA} = 12\ 10^6$ and $M = 1$. Fig. 10a, where the non-dimensional load parameter P/P_e is plotted as a function of the non-dimensional displacement w/L, shows the nonlinear response of the column considering both bilateral and unilateral contact.

For the bilateral contact case (conventional Winkler model), the load-displacement response is asymptotic to the value of the theoretical critical load of the perfect column given by (Brush and Almroth, 1975):

$$P_{\mathrm{cr}} = \left(\frac{n\pi}{L}\right)^2 \mathrm{EI} + \left(\frac{L}{n\pi}\right)^2 K \tag{23}$$

For this geometry and a non-dimensional elastic foundation stiffness parameter $k = \mathrm{KL}^4/\mathrm{EI} = 625$, the column buckles with two half-waves ($n = 2$) and the following relationship is found $P_{\mathrm{cr}}/P_e \cong 5.6$, with $P_e = \pi^2\mathrm{EI}/\mathrm{L}^2$.

For the unilateral contact case, the equilibrium path is asymptotic to the first bifurcation load of the perfect column ($P_{\mathrm{cr}}/P_e \cong 1$). This response can be better understood observing the column deformation patterns also presented in Fig. 10a. Note the reduction of the contact region S_c with increasing P. After the separation of the bodies the bar deformation is similar to the first buckling mode ($n = 1$). This example highlights again the fact that considerable error may result if the appropriate foundation model is not taken into account in the analysis (Silva et al., 2001).

The influence of the foundation stiffness is illustrated in Fig. 10b, where again the non-dimensional load parameter P/P_e is plotted as a function of w/L ($x/L = 0.85$). The non-dimensional foundation stiffness parameter k assumes the values 6.25, 625 and 6250. Independent of the foundation stiffness, the equilibrium path is asymptotic to the first bifurcation load and reproduces the results obtained by Silveira (1995).

As a last example, consider the same slender circular arch analysed previously (Fig. 4a). Now, an incremental load P is applied at $\theta = 0$ and the nonlinear response is obtained, with the central contact region defined by the angles $\pm\phi$. The results are shown in Fig. 11a and b. In Fig. 11a, the variation of the lateral displacement w is plotted as a function of the load parameter $\mathrm{PR}^2/\mathrm{EI}$, for different values of the non-dimensional elastic foundation stiffness parameter $k = \mathrm{KR}^4/\mathrm{EI}$. The nonlinear equilibrium path of the system without foundation ($k = 0$) was originally presented by Walker (1969), with the arch exhibiting a snap-through behavior and a limit load $\mathrm{PR}^2/\mathrm{EI} = 76.3$. For a flexibly foundation ($k < 10^4$), no additional effects were observed in the pre and post-buckling behavior. With $k = 10^4$, a small increase is observed in the limit load ($\mathrm{PR}^2/\mathrm{EI} = 83.4$). For $k \geqslant 5 \times 10^4$, the snap-through behavior disappears, and for $k = 10^6$ and 10^7 the equilibrium paths are practically linear (very stiff foundation). Again, the results of Silveira (1995) agree well with those obtained in this work. Fig. 11b shows the variation of contact regions S_c ($2R\phi$) with the applied load.

6. Conclusions

A semi-analytical approach, based on the Ritz method, is proposed in this work for the linear and nonlinear analysis of structural elements in contact with a tensionless elastic foundation. In the Ritz approach, the modal amplitudes of the displacements as well as the co-ordinates of the lift-off points of the non-contact regions are taken as problem variables. This leads to an eminently nonlinear system of algebraic equations, even when the structure and foundation models are linear, which are solved by an incremental-iterative arch-length solution strategy. In order to study the applicability, efficiency and accuracy of the proposed methodoly, several examples involving beams, columns and shallow arches under unilateral contact constraints are analyzed and compared favorably with available results. The examples show that the dependence of the contact area on the foundation stiffness is one of the main characteristics of a tensionless foundation as compared with the conventional foundation. Confirming previous results obtained by the authors using a finite element formulation and mathematical programming techniques (Silva et al., 2001; Silveira and Gonçalves, 2001; Holanda and Gonçalves, 2003), the results show that in many contact problems considerable error may result if the elastic foundation is not adequately modelled and its unilateral constraint character is not taken into account.

This methodology is particularly suited for the analysis of structural problems where the number and location, but not the length, of the contact and non-contact regions are known a priori. In these cases, it can substitute large and time-consuming finite element packages usually used in this type of analysis. This simplified but accurate semi-analytical approach can be efficiently used to perform detailed parametric analyses and thus contribute to a better understanding of the effects of unilateral constraints in structural analysis. It can also be used as a benchmark for more general and complex formulations.

Acknowledgements

The authors are grateful for the financial support from the Brazilian National Council for Scientific and Technological Development (CNPq/MCT). They also acknowledge the support from FAPERJ-CNE, USIMINAS, FAPEMIG and Gorceix Foundation. Special thanks also to Professor Michael Engelhardt and his department at the University of Texas for their hospitality during the manuscript's preparation.

Appendix A

Expression for $\overline{\Pi}_2$ in terms of the coefficients obtained through the exact integration of the harmonic functions:

$$\begin{aligned}\overline{\Pi}_2 = &\sum_i^n\sum_j^n W_iW_j\alpha_{ij}^{2b} + \sum_i^n\sum_j^n\sum_k^n\sum_l^n W_iW_jW_kW_l\alpha_{ijkl}^{4b} + \sum_i^n\sum_j^n\sum_k^n\sum_l^n\sum_p^n\sum_q^n W_iW_jW_kW_lW_pW_q\alpha_{ijklpq}^{6b}\\ &+\sum_i^n\sum_j^n W_iW_j\alpha_{ij}^{2m} + \sum_i^n\sum_j^n\sum_k^n\sum_l^n W_iW_jW_kW_l\alpha_{ijkl}^{4m}\\ &+\sum_i^n\sum_j^n W_iW_j\alpha_{ij}^{2f} - \sum_i^n MW_i\alpha_i^M(x=0) - \sum_i^n MW_i\alpha_i^M(x=L)\end{aligned} \tag{24}$$

where:

$$\begin{aligned}\alpha_{ij}^{2b} &= \frac{\mathrm{EI}}{4}\frac{i^2j^2\pi^4}{L^4}\int_0^L\left\{\cos\left[(i-j)\frac{\pi x}{L}\right] - \cos\left[(i+j)\frac{\pi x}{L}\right]\right\}\,\mathrm{d}x\\
\alpha_{ijkl}^{4b} &= \frac{\mathrm{EI}}{16}\frac{i^2j^2kl\pi^6}{L^6}\int_0^L\left\{\cos\left[(i-j-k+l)\frac{\pi x}{L}\right] + \cos\left[(i-j+k-l)\frac{\pi x}{L}\right]\right.\\
&\quad - \cos\left[(i-j-k-l)\frac{\pi x}{L}\right] - \cos\left[(i-j+k+l)\frac{\pi x}{L}\right] + \cos\left[(i+j-k+l)\frac{\pi x}{L}\right]\\
&\quad + \cos\left[(i+j+k-l)\frac{\pi x}{L}\right]\\
&\quad \left.- \cos\left[(i+j-k-l)\frac{\pi x}{L}\right] - \cos\left[(i+j+k+l)\frac{\pi x}{L}\right]\right\}\,\mathrm{d}x\\
\alpha_{ijklpq}^{6b} &= \frac{\mathrm{EI}}{256}\frac{i^2j^2klpq\pi^8}{L^8}\int_0^L\left\{\cos\left[(i-j-k+l-p+q)\frac{\pi x}{L}\right] + \cos\left[(i-j-k+l+p-q)\frac{\pi x}{L}\right]\right.\\
&\quad - \cos\left[(i-j-k+l-p-q)\frac{\pi x}{L}\right] - \cos\left[(i-j-k+l+p+q)\frac{\pi x}{L}\right]\\
&\quad + \cos\left[(i-j+k-l-p+q)\frac{\pi x}{L}\right]\\
&\quad + \cos\left[(i-j+k-l+p-q)\frac{\pi x}{L}\right] - \cos\left[(i-j+k-l-p-q)\frac{\pi x}{L}\right]\\
&\quad - \cos\left[(i-j+k-l+p+q)\frac{\pi x}{L}\right]\end{aligned}$$

$$
\begin{aligned}
&+\cos\left[(i-j-k-l-p+q)\frac{\pi x}{L}\right]+\cos\left[(i-j-k-l+p-q)\frac{\pi x}{L}\right]\\
&-\cos\left[(i-j-k-l-p-q)\frac{\pi x}{L}\right]\\
&-\cos\left[(i-j-k-l+p+q)\frac{\pi x}{L}\right]+\cos\left[(i-j+k+l-p+q)\frac{\pi x}{L}\right]\\
&+\cos\left[(i-j+k+l+p-q)\frac{\pi x}{L}\right]\\
&-\cos\left[(i-j+k+l-p-q)\frac{\pi x}{L}\right]-\cos\left[(i-j+k+l+p+q)\frac{\pi x}{L}\right]\\
&+\cos\left[(i+j-k+l-p+q)\frac{\pi x}{L}\right]\\
&+\cos\left[(i+j-k+l+p-q)\frac{\pi x}{L}\right]-\cos\left[(i+j-k+l-p-q)\frac{\pi x}{L}\right]\\
&-\cos\left[(i+j-k+l+p+q)\frac{\pi x}{L}\right]\\
&+\cos\left[(i+j+k-l-p+q)\frac{\pi x}{L}\right]+\cos\left[(i+j+k-l+p-q)\frac{\pi x}{L}\right]\\
&-\cos\left[(i+j+k-l-p-q)\frac{\pi x}{L}\right]\\
&-\cos\left[(i+j+k-l+p+q)\frac{\pi x}{L}\right]+\cos\left[(i+j-k-l-p+q)\frac{\pi x}{L}\right]\\
&+\cos\left[(i+j-k-l+p-q)\frac{\pi x}{L}\right]\\
&-\cos\left[(i+j-k-l-p-q)\frac{\pi x}{L}\right]-\cos\left[(i+j-k-l+p+q)\frac{\pi x}{L}\right]\\
&+\cos\left[(i+j+k+l-p+q)\frac{\pi x}{L}\right]\\
&+\cos\left[(i+j+k+l+p-q)\frac{\pi x}{L}\right]-\cos\left[(i+j+k+l-p-q)\frac{\pi x}{L}\right]\\
&-\cos\left[(i+j+k+l+p+q)\frac{\pi x}{L}\right]\Bigg\}\mathrm{d}x
\end{aligned}
$$

$$
\alpha_{ij}^{2m}=\frac{P}{4}\frac{ij\pi^2}{L^2}\int_0^L\left\{\cos\left[(i-j)\frac{\pi x}{L}\right]+\cos\left[(i+j)\frac{\pi x}{L}\right]\right\}\mathrm{d}x
$$

$$
\begin{aligned}
\alpha_{ijkl}^{4m}=\frac{P}{64}\frac{ijkl\pi^4}{L^4}\int_0^L\Bigg\{&\cos\left[(i-j-k+l)\frac{\pi x}{L}\right]+\cos\left[(i-j+k-l)\frac{\pi x}{L}\right]+\cos\left[(i-j-k-l)\frac{\pi x}{L}\right]+\\
&+\cos\left[(i-j+k+l)\frac{\pi x}{L}\right]+\cos\left[(i+j-k+l)\frac{\pi x}{L}\right]+\cos\left[(i+j+k-l)\frac{\pi \mathrm{x}}{L}\right]\\
&+\cos\left[(i+j-k-l)\frac{\pi x}{L}\right]+\cos\left[(i+j+k+l)\frac{\pi x}{L}\right]\Bigg\}\mathrm{d}x
\end{aligned}
$$

$$
\alpha_{ij}^{2f}=\frac{K}{4}\int_0^{s_1}\left\{\cos\left[(i-j)\frac{\pi x}{L}\right]-\cos\left[(i+j)\frac{\pi x}{L}\right]\right\}\mathrm{d}x;\quad \alpha_i^M(x=0)=\frac{i\pi}{L};\quad \alpha_i^M(x=L)=\frac{i\pi}{L}\cos(i\pi)
$$

Appendix B

Expression for $\overline{\Pi}_3$ in terms of the coefficients obtained through the exact integration of the harmonics functions:

$$
\begin{aligned}
\overline{\Pi}_3 = & \sum_{i_u}^{m}\sum_{j_u}^{m} U_{i_u}U_{j_u}\alpha^{2b}_{i_u j_u} - \sum_{i_u}^{m}\sum_{i_w}^{n} U_{i_u}W_{i_w}\alpha^{2b}_{i_u i_w} + \sum_{i_w}^{n}\sum_{j_w}^{n} W_{i_w}W_{j_w}\alpha^{2b}_{i_w j_w} + \sum_{i_u}^{m}\sum_{j_u}^{m} U_{i_u}U_{j_u}\alpha^{2m}_{i_u j_u} \\
& + \sum_{i_u}^{m}\sum_{i_w}^{n} U_{i_u}W_{i_w}\alpha^{2m}_{i_u i_w} + \sum_{i_w}^{n}\sum_{j_w}^{n} W_{i_w}W_{j_w}\alpha^{2m}_{i_w j_w} + \sum_{i_u}^{m}\sum_{i_w}^{n}\sum_{j_w}^{n} U_{i_u}W_{i_w}W_{j_w}\alpha^{3m1}_{i_u i_w j_w} \\
& - \sum_{i_u}^{m}\sum_{j_u}^{m}\sum_{i_w}^{n} U_{i_u}U_{j_u}W_{i_w}\alpha^{3m1}_{i_u j_u i_w} + \sum_{i_u}^{m}\sum_{j_u}^{m}\sum_{k_u}^{m} U_{i_u}U_{j_u}U_{k_u}\alpha^{3m}_{i_u j_u k_u} + \sum_{i_w}^{n}\sum_{j_w}^{n}\sum_{k_w}^{n} W_{i_w}W_{j_w}W_{k_w}\alpha^{3m}_{i_w j_w k_w} \\
& - \sum_{i_u}^{m}\sum_{i_w}^{n}\sum_{j_w}^{n} U_{i_u}W_{i_w}W_{j_w}\alpha^{3m2}_{i_u i_w j_w} + \sum_{i_u}^{m}\sum_{j_u}^{m}\sum_{i_w}^{n} U_{i_u}U_{j_u}W_{i_w}\alpha^{3m2}_{i_u j_u i_w} \\
& + \sum_{i_u}^{m}\sum_{j_u}^{m}\sum_{k_u}^{m}\sum_{l_u}^{m} U_{i_u}U_{j_u}U_{k_u}U_{l_u}\alpha^{4m}_{i_u j_u k_u l_u} - \sum_{i_u}^{m}\sum_{i_w}^{n}\sum_{j_w}^{n}\sum_{k_w}^{n} U_{i_u}W_{i_w}W_{j_w}W_{k_w}\alpha^{4m}_{i_u i_w j_w k_w} \\
& + \sum_{i_u}^{m}\sum_{j_u}^{m}\sum_{i_w}^{n}\sum_{j_w}^{n} U_{i_u}U_{j_u}W_{i_w}W_{j_w}\alpha^{4m}_{i_u j_u i_w j_w} - \sum_{i_u}^{m}\sum_{j_u}^{m}\sum_{k_u}^{m}\sum_{i_w}^{n} U_{i_u}U_{j_u}U_{k_u}W_{i_w}\alpha^{4m}_{i_u j_u k_u i_w} \\
& + \sum_{i_w}^{n}\sum_{j_w}^{n}\sum_{k_w}^{n}\sum_{l_w}^{n} W_{i_w}W_{j_w}W_{k_w}W_{l_w}\alpha^{4m}_{i_w j_w k_w l_w} + \sum_{i_w}^{n}\sum_{j_w}^{n} W_{i_w}W_{j_w}\alpha^{2f}_{i_w j_w} - P\sum_{i_w}^{n} W_{i_w}\alpha^{P}_{i_w}(\theta=0)
\end{aligned}
\tag{25}
$$

where:

$$
\begin{aligned}
\alpha^{2b}_{i_u,j_u} &= \frac{\mathrm{EI}\pi^2}{16R^3\gamma^2}(2i_u-1)(2j_u-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[(i_u-j_u)\frac{\pi\theta}{\gamma}\right]+\cos\left[(i_u+j_u-1)\frac{\pi\theta}{\gamma}\right]\right\}\mathrm{d}\theta \\
\alpha^{2b}_{i_u,i_w} &= -\frac{\mathrm{EI}\pi^3}{16R^3\gamma^3}(2i_u-1)(2i_w-1)^2\int_{-\gamma}^{\gamma}\left\{\cos\left[(i_u-i_w)\frac{\pi\theta}{\gamma}\right]+\cos\left[(i_u+i_w-1)\frac{\pi\theta}{\gamma}\right]\right\}\mathrm{d}\theta \\
\alpha^{2b}_{i_w,j_w} &= \frac{\mathrm{EI}\pi^4}{64R^3\gamma^4}(2i_w-1)^2(2j_w-1)^2\int_{-\gamma}^{\gamma}\left\{\cos\left[(i_w-j_w)\frac{\pi\theta}{\gamma}\right]+\cos\left[(i_w+j_w-1)\frac{\pi\theta}{\gamma}\right]\right\}\mathrm{d}\theta \\
\alpha^{2m}_{i_u,j_u} &= \frac{\mathrm{EA}\pi^2}{16R\gamma^2}(2i_u-1)(2j_u-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[(i_u-j_u)\frac{\pi\theta}{\gamma}\right]+\cos\left[(i_u+j_u-1)\frac{\pi\theta}{\gamma}\right]\right\}\mathrm{d}\theta \\
\alpha^{2m}_{i_u,i_w} &= \frac{\mathrm{EA}\pi}{4R\gamma}(2i_u-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[(i_u-i_w)\frac{\pi\theta}{\gamma}\right]+\cos\left[(i_u+i_w-1)\frac{\pi\theta}{\gamma}\right]\right\}\mathrm{d}\theta \\
\alpha^{2m}_{i_w,j_w} &= \frac{\mathrm{EA}}{4R}\int_{-\gamma}^{\gamma}\left\{\cos\left[(i_w-j_w)\frac{\pi\theta}{\gamma}\right]+\cos\left[(i_w+j_w-1)\frac{\pi\theta}{\gamma}\right]\right\}\mathrm{d}\theta \\
\alpha^{3m1}_{i_u i_w j_w} &= \frac{\mathrm{EA}\pi^3}{64R^2\gamma^3}(2i_u-1)(2i_w-1)(2j_w-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(2i_u-2i_w+2j_w-1)\pi\theta}{2\gamma}\right]\right. \\
&\quad \left.+\cos\left[\frac{(2i_u+2i_w-2j_w-1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u-2i_w-2j_w+1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u+2i_w+2j_w-3)\pi\theta}{2\gamma}\right]\right\}\mathrm{d}\theta \\
\alpha^{3m1}_{i_u j_u i_w} &= -\frac{\mathrm{EA}\pi^2}{16R^2\gamma^2}(2j_u-1)(2i_w-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(2i_u+2j_u-2i_w-1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u-2j_u+2i_w-1)\pi\theta}{2\gamma}\right]\right. \\
&\quad \left.+\cos\left[\frac{(2i_u-2j_u-2i_w+1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u+2j_u+2i_w-3)\pi\theta}{2\gamma}\right]\right\}\mathrm{d}\theta \\
\alpha^{3m}_{i_u j_u k_u} &= \frac{\mathrm{EA}\pi}{16R^2\gamma}(2k_u-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(2i_u-2j_u+2k_u-1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u+2j_u-2k_u-1)\pi\theta}{2\gamma}\right]\right. \\
&\quad \left.+\cos\left[\frac{(2i_u-2j_u-2k_u+1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u+2j_u+2k_u-3)\pi\theta}{2\gamma}\right]\right\}\mathrm{d}\theta
\end{aligned}
$$

$$\alpha^{3m}_{i_w j_w k_w} = \frac{EA\pi^2}{32R^2\gamma^2}(2j_w-1)(2k_w-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(2i_w-2j_w+2k_w-1)\pi\theta}{2\gamma}\right]+\cos\left[\frac{(2i_w+2j_w-2k_w-1)\pi\theta}{2\gamma}\right]\right.$$
$$\left.-\cos\left[\frac{(2i_w-2j_w-2k_w+1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_w+2j_w+2k_w-3)\pi\theta}{2\gamma}\right]\right\}d\theta$$

$$\alpha^{3m2}_{i_u i_w j_w} = -\frac{EA\pi}{8R^2\gamma}(2j_w-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(2i_u+2i_w-2j_w-1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u-2i_w+2j_w-1)\pi\theta}{2\gamma}\right]\right.$$
$$\left.+\cos\left[\frac{(2i_u-2i_w-2j_w+1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u+2i_w+2j_w-3)\pi\theta}{2\gamma}\right]\right\}d\theta$$

$$\alpha^{3m2}_{i_u j_u i_w} = \frac{EA}{8R^2}\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(2i_u-2j_u-2i_w+1)\pi\theta}{2\gamma}\right]+\cos\left[\frac{(2i_u-2j_u+2i_w-1)\pi\theta}{2\gamma}\right]\right.$$
$$\left.-\cos\left[\frac{(2i_u+2j_u-2i_w-1)\pi\theta}{2\gamma}\right]-\cos\left[\frac{(2i_u+2j_u+2i_w-3)\pi\theta}{2\gamma}\right]\right\}d\theta$$

$$\alpha^{4m}_{i_u j_u k_u l_u} = \frac{EA}{64R^3}\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(i_u-j_u-k_u+l_u)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_u-j_u+k_u-l_u)\pi\theta}{\gamma}\right]\right.$$
$$-\cos\left[\frac{(i_u-j_u-k_u-l_u+1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u-j_u+k_u+l_u-1)\pi\theta}{\gamma}\right]$$
$$-\cos\left[\frac{(i_u+j_u-k_u+l_u-1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u+j_u+k_u-l_u-1)\pi\theta}{\gamma}\right]$$
$$\left.+\cos\left[\frac{(i_u+j_u-k_u-l_u)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_u+j_u+k_u+l_u-2)\pi\theta}{\gamma}\right]\right\}d\theta$$

$$\alpha^{4m}_{i_u i_w j_w k_w} = -\frac{EA\pi^3}{128R^3\gamma^3}(2i_w-1)(2j_w-1)(2k_w-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(i_u-i_w-j_w+k_w)\pi\theta}{\gamma}\right]\right.$$
$$+\cos\left[\frac{(i_u-i_w+j_w-k_w)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u-i_w-j_w-k_w+1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u-i_w+j_w+k_w-1)\pi\theta}{\gamma}\right]$$
$$-\cos\left[\frac{(i_u+i_w-j_w+k_w-1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u+i_w+j_w-k_w-1)\pi\theta}{\gamma}\right]$$
$$\left.+\cos\left[\frac{(i_u+i_w-j_w-k_w)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_u+i_w+j_w+k_w-2)\pi\theta}{\gamma}\right]\right\}d\theta$$

$$\alpha^{4m}_{i_u j_u i_w j_w} = \frac{3EA\pi^2}{128R^3\gamma^2}(2i_w-1)(2j_w-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(i_u-j_u-i_w+j_w)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_u-j_u+i_w-j_w)\pi\theta}{\gamma}\right]\right.$$
$$-\cos\left[\frac{(i_u-j_u-i_w-j_w+1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u-j_u+i_w+j_w-1)\pi\theta}{\gamma}\right]$$
$$-\cos\left[\frac{(i_u+j_u-i_w+j_w-1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u+j_u+i_w-j_w-1)\pi\theta}{\gamma}\right]$$
$$\left.+\cos\left[\frac{(i_u+j_u-i_w-j_w)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_u+j_u+i_w+j_w-2)\pi\theta}{\gamma}\right]\right\}d\theta$$

$$\alpha^{4m}_{i_u j_u k_u i_w} = -\frac{EA\pi}{32R^3\gamma}(2i_w-1)\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(i_u-j_u-k_u+i_w)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_u-j_u+k_u-i_w)\pi\theta}{\gamma}\right]\right.$$
$$-\cos\left[\frac{(i_u-j_u-k_u-i_w+1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u-j_u+k_u+i_w-1)\pi\theta}{\gamma}\right]$$
$$-\cos\left[\frac{(i_u+j_u-k_u+i_w-1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_u+j_u+k_u-i_w-1)\pi\theta}{\gamma}\right]$$
$$\left.+\cos\left[\frac{(i_u+j_u-k_u-i_w)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_u+j_u+k_u+i_w-2)\pi\theta}{\gamma}\right]\right\}d\theta$$

$$\alpha^{4m}_{i_w j_w k_w l_w} = \frac{\mathrm{EA}\pi^4}{1024R^3\gamma^4}(2i_w-1)(2j_w-1)(2k_w-1)(2l_w-1)$$

$$\int_{-\gamma}^{\gamma}\left\{\cos\left[\frac{(i_w-j_w-k_w+l_w)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_w-j_w+k_w-l_w)\pi\theta}{\gamma}\right]\right.$$

$$-\cos\left[\frac{(i_w-j_w-k_w-l_w+1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_w-j_w+k_w+l_w-1)\pi\theta}{\gamma}\right]$$

$$-\cos\left[\frac{(i_w+j_w-k_w+l_w-1)\pi\theta}{\gamma}\right]-\cos\left[\frac{(i_w+j_w+k_w-l_w-1)\pi\theta}{\gamma}\right]$$

$$\left.+\cos\left[\frac{(i_w+j_w-k_w-l_w)\pi\theta}{\gamma}\right]+\cos\left[\frac{(i_w+j_w+k_w+l_w-2)\pi\theta}{\gamma}\right]\right\}\mathrm{d}\theta$$

$$\alpha^{2f}_{i_w j_w} = \frac{\mathrm{KR}}{4}\int_{-\phi}^{\phi}\left\{\cos\left[(i_w-j_w)\frac{\pi\theta}{\gamma}\right]+\cos\left[(i_w+j_w-1)\frac{\pi\theta}{\gamma}\right]\right\}\mathrm{d}\theta;\quad \alpha^{P}_{i_w}(\theta=0)=1$$

References

Adan, N., Sheinman, I., Altus, E., 1994. Post-buckling behavior of beams under contact constraints. Transactions of the ASME Journal Applied Mechanics 61 (4), 764–772.

Ascione, L., Grimaldi, A., 1984. Unilateral contact between a plate and an elastic foundation. Meccanica 19, 223–233.

Barbosa, H.J.C., 1986. Numerical algorithms for contact problems in elasticity. D.Sc. Thesis, Federal University of Rio de Janeiro—COPPE, Rio de Janeiro, Brazil (in Portuguese).

Björkman, G., 1992. Path following and critical points for contact problems. Computational Mechanics 10 (3-4), 231–246.

Brush, D.O., Almroth, B.O., 1975. Buckling of Bars, Plates, and Shells. McGraw-Hill, New York.

Celep, Z., Gençoglu, M., 2003. Forced vibrations of a rigid circular plate on a tensionless Winkler edge support. Journal of Sound and Vibration 263 (4), 945–953.

Celep, Z., Güler, K., 2004. Static and dynamic responses of a rigid circular plate on a tensionless Winkler foundation. Journal of Sound and Vibration 276 (1-2), 449–458.

Crisfield, M.A., 1991. Non-linear Finite Element Analysis of Solids and Structures. John Wiley & Sons, West Sussex, England.

Endo, T., Oden, J.T., Becker, E.B., Miller, T., 1984. A numerical analysis of contact and limit-point behavior in a class of problems of finite elastic deformation. Computers and Structures 18 (5), 899–910.

Fisher, K.A., Wriggers, P., 2005. Frictionless 2D contact formulations for finite deformations based on the mortar method. Computational Mechanics 36, 226–244.

Güler, K., Celep, Z., 2005. Response of a rectangular plate-column system on a tensionless Winkler foundation subjected to static and dynamic loads. Structural Engineering and Mechanics—An International Journal 21 (6), 699–712.

Hetényi, M., 1946. Beams on Elastic Foundation. University of Michigan Press, Ann Arbor.

Holanda, A.S., Gonçalves, P.B., 2003. Post-buckling analysis of plates resting on a tensionless elastic foundation. Journal of Engineering Mechanics, ASCE 129 (4), 438–448.

Holmes, P., Domokos, G., Schmitt, J., Szeberényi, I., 1999. Constrained Euler buckling: an interplay of computation and analysis. Computer Methods in Applied Mechanics and Engineering 170 (3-4), 175–207.

Hsu, M.H., 2006. Mechanical analysis of non-uniform beams resting on nonlinear elastic foundation by the differential quadrature method. Structural Engineering and Mechanics—An International Journal 22 (3), 35–50.

Joo, J.W., Kwak, B.M., 1986. Analysis and applications of elasto-plastic contact problems considering large deformation. Computers and Structures 24 (6), 953–961.

Kadkhodayan, M., 2006. An investigation into the influence of deformable dies on the springback of circular plates. Scientia Iranica 13 (2), 201–205.

Kerr, A.D., 1964. Elastic and viscoelastic foundation models. Transactions of the ASME Journal Applied Mechanics 31, 491–498.

Koo, J.S., Kwak, B.M., 1996. Post-buckling analysis with frictional contact combining complementary relations and an arc-length method. International Journal for Numerical Methods in Engineering 39, 1161–1180.

Küçükarslan, S., Banerjee, P.K., 2005. Inelastic dynamic analysis of pile-soil-structure interaction. International Journal of Computational Engineering Science 5 (1), 245–258.

Lemke, C.E., 1968. On complementary pivot theory. In: Dantzig G.B., Yenott A.F. (Ed.), Mathematics of Decision Sciences.

Li, J., Berger, E.J., 2003. A semi-analytical approach to three-dimensional normal contact problems with friction. Computational Mechanics 30 (4), 310–322.

Maheshwari, P., Chandra, S., Basudhar, P.K., 2004. Response of beams on a tensionless extensible geosynthetic-reinforced earth bed subjected to moving loads. Computers and Geotechnics 31, 537–548.

Mezaini, N., 2006. Effects of soil-structures interactions on the analysis of cylindrical tanks. ASCE Practice Periodical on Structural Design and Construction 11 (1), 50–57.

Muradova, A.D., Stavroulakis, G.E., 2006. A unilateral contact model with buckling in von Kármán plates. Nonlinear Analysis: Real World Applications 8, 1261–1271.

Pereira, W.L.A., Silveira, R.A.M., 2006. Numerical analysis of long beams under unilateral contact imposed by elastic foundation. In: Proceedings of XXVII Ibero-Latin American Congress of Computational Methods in Engineering (XXVII CILAMCE), Pará, Brazil.

Pereira, W.L.A., 2003. Numerical formulation for analysis of beams in contact with elastic bases. M.Sc. Thesis, Federal University of Ouro Preto, Minas Gerais, Brazil (in Portuguese).

Rega, G., Troger, H., 2005. Dimension reduction of dynamical systems: methods, models, applications. Nonlinear Dynamics 41, 1–15.

Shames, I.H., Dym, C.L., 1995. Energy and Finite Element Methods in Structural Mechanics (SI Unit Edition). Taylor & Francis, Philadelphia.

Shen, H.-S., Li, Q.S., 2004. Post-buckling of shear deformable laminated plates resting on a tensionless elastic foundation subjected to mechanical or thermal loading. International Journal of Solids and Structures 41 (16–17), 4769–4785.

Shen, H.-S., Teng, J.G., 2004. Post-buckling analysis of unilaterally constrained laminated thin plates. Composite Structures 66, 571–578.

Shen, H.-S., Yu, L., 2004. Nonlinear bending behavior of Reissner–Mindlin plates with free edges resting on tensionless elastic foundations. International Journal of Solids and Structures 41 (16–17), 4809–4825.

Silva, A.R.D., Silveira, R.A.M., Gonçalves, P.B., 2001. Numerical methods for analysis of plates on tensionless elastic foundation. International Journal of Solids and Structures 38 (10), 2083–2100.

Silva, A.R.D., 1998. Analysis of plates under contact constraints. M.Sc. Thesis, Federal University of Ouro Preto, Minas Gerais, Brazil (in Portuguese).

Silveira, R.A.M., Gonçalves, P.B., 2001. A modal nonlinear solution for structural elements under contact constraints imposed by elastic foundations. In: Domínguez, J., Brebbia, C.A. (Eds.), Computational Methods in Contact Mechanics V. WIT Press, Southampton.

Silveira, R.A.M., Gonçalves, P.B., 2001. Analysis of slender structural elements under unilateral contact constraints. Structural Engineering and Mechanics—An International Journal 12 (1), 35–50.

Silveira, R.A.M., 1995. Analysis of slender structural elements under unilateral contact constraints. D.Sc. Thesis, Catholic University (PUC-Rio), Rio de Janeiro, Brazil (in Portuguese).

Simo, J.C., Wriggers, P., Schweizerhof, K.H., Taylor, R.L., 1986. Finite deformation post-buckling analysis involving inelasticity and contact constraints. International Journal for Numerical Methods in Engineering 23, 779–800.

Simo, J.C., Wriggers, P., Taylor, R.L., 1985. A perturbed Lagrangian formulation for the finite element solution of contact problems. Computer Methods in Applied Mechanics and Engineering 50, 163–180.

Stein, E., Wagner, W., Wriggers, P., 1990. Nonlinear stability—analysis of shell and contact-problems including branch-switching. Computational Mechanics 5 (6), 428–446.

Stein, E., Wriggers, P., 1984. Stability of rods with unilateral constraints, a finite element solution. Computers and Structures 19 (1-2), 205–211.

Tschöpe, H., Õnate, E., Wriggers, P., 2003a. Direct computation of instability points for contact problems. Computational Mechanics 31 (1-2), 173–178.

Tschöpe, H., Wriggers, P., Õnate, E., 2003b. Direct computation of instability points with inequality constraints using the FEM. Engineering Computations 20 (5-6), 611–628.

Walker, A.C., 1969. A non-linear finite element analysis of shallow circular arches. International Journal of Solids and Structures 5, 97–107.

Wang, Y.H., Tham, L.G., Cheung, Y.K., 2005. Beams and plates on elastic foundations: a review. Progress in Structural Engineering and Materials 7 (4), 174–182.

Wriggers, P., Imhof, M., 1993. On the treatment of nonlinear unilateral contact problems. Archive of Applied Mechanics 63, 116–129.

Wriggers, P., Wagner, W., Stein, E., 1987. Algorithms for non-linear contact constraints with application to stability problems of rods and shells. Computational Mechanics 2 (3), 215–230.

Wriggers, P., Zavarise, G., 2004. Computational contact mechanics. In: Stein, E., De Borst, R., Hughes, T.J.R. (Eds.), Encyclopedia of Computational Mechanics. John Wiley & Sons, West Sussex, England.

Wriggers, P., 2002. Computational Contact Mechanics. John Wiley & Sons, West Sussex, England.

Yu, L., Shen, H.-S., Huo, X.-P., 2007. Dynamic responses of Reissner–Mindlin plates with free edges resting on tensionless elastic foundations. Journal of Sound and Vibration 299, 212–228.