

# An alternative order parameter for the 4-state potts model

H.A. Fernandes<sup>a</sup>, E. Arashiro<sup>a</sup>, J.R. Drugowich de Felício<sup>a</sup>, A.A. Caparica<sup>b,\*</sup>

<sup>a</sup>*Departamento de Física e Matemática, Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto, Universidade de São Paulo, Avenida Bandeirantes, 3900 - CEP 14040-901, Ribeirão Preto, São Paulo, Brazil*

<sup>b</sup>*Instituto de Física, Universidade Federal de Goiás, C.P. 131, 74.001-970, Goiânia, Goiás, Brazil*

Received 21 December 2005; received in revised form 21 December 2005

Available online 20 March 2006

## Abstract

We have investigated the dynamic critical behavior of the two-dimensional 4-state Potts model using an alternative order parameter first used by Vanderzande [J. Phys. A 20 (1987) L549] in the study of the Z(5) model. We have estimated the global persistence exponent  $\theta_g$  by following the time evolution of the probability  $P(t)$  that the considered order parameter does not change its sign up to time  $t$ . We have also obtained the critical exponents  $\theta$ ,  $z$ ,  $\nu$ , and  $\beta$  using this alternative definition of the order parameter and our results are in complete agreement with available values found in literature.

© 2006 Elsevier B.V. All rights reserved.

*Keywords:* Short-time dynamics; Potts model; Critical phenomena; Monte Carlo methods

## 1. Introduction

In 1989, Janssen et al. [1] and Huse [2] pointed out, using renormalization group techniques and numerical calculations, respectively, that there is universality and scaling behavior even at an early stage of the time evolution of dynamic systems without conserved order parameter (model A in the terminology of Halperin et al. [3]).

Since then, a great deal of works on phase transitions and critical phenomena using Monte Carlo simulations in the short-time regime have been published and their results are in good agreement with theoretical predictions and numerical results found in equilibrium [4–13]. In addition, the new approach has proven to be useful in determining with good precision the dynamic exponent  $z$ , as well as the new exponent  $\theta$  which governs the so-called critical initial slip [14], the anomalous behavior of the magnetization when the system is quenched to the critical temperature  $T_c$ .

The dynamic scaling relation obtained by Janssen et al. [1] for the  $k$ th moment of the magnetization, extended to systems of finite size, is written as

$$M^{(k)}(t, \tau, L, m_0) = b^{k\beta/\nu} M^{(k)}(b^{-z}t, b^{1/\nu}\tau, b^{-1}L, b^{x_0}m_0), \quad (1)$$

\*Corresponding author.

*E-mail addresses:* [henrique@pg.fclrp.usp.br](mailto:henrique@pg.fclrp.usp.br) (H.A. Fernandes), [evearash@usp.br](mailto:evearash@usp.br) (E. Arashiro), [drugo@usp.br](mailto:drugo@usp.br) (J.R. Drugowich de Felício), [caparica@if.ufg.br](mailto:caparica@if.ufg.br) (A.A. Caparica).

where  $t$  is the time evolution,  $b$  is an arbitrary spatial scaling factor,  $\tau = (T - T_c)/T_c$  is the reduced temperature and  $L$  is the linear size of the lattice. The exponents  $\beta$  and  $\nu$  are as usual the equilibrium critical exponents associated respectively with the order parameter and the correlation length and  $x_0$  is related to the exponents  $z$ ,  $\theta$ ,  $\beta$  and  $\nu$  by the equation

$$x_0 = \theta z + \beta/\nu. \quad (2)$$

By setting the scaling factor  $b = t^{1/z}$  and  $\tau = 0$  in Eq. (1), we obtain the first moment of the magnetization

$$M(t) \sim m_0 t^\theta, \quad (3)$$

where  $m_0$  represents the initial magnetization of the system.

Far from equilibrium, another dynamic critical exponent was proposed by Majumdar et al. [15] studying the behavior of the global persistence probability  $P(t)$  that the order parameter has not changed its sign up to the time  $t$ . At criticality,  $P(t)$  is expected to decay algebraically as

$$P(t) \sim t^{-\theta_g}, \quad (4)$$

where  $\theta_g$  is the global persistence exponent. If the time evolution would be a Markovian process, then the exponent  $\theta_g$  should obey the equation [15]

$$\theta_g z = -\theta z + \frac{d}{z} - \frac{\beta}{\nu}. \quad (5)$$

However, as shown in several works [15–25] the exponent  $\theta_g$  is an independent critical index closely related to the non-Markovian character of the process.

In this paper, we estimate the global persistence exponent of the 4-state Potts model employing an alternative order parameter first used by Vanderzande [27] in the study of the Z(5) model. Our estimate is in complete agreement with the results obtained recently for the Ising model with three-spin interactions in one direction and for the 4-state Potts model [26] with its traditional order parameter (see Eq. (8)). In addition, comparing with estimates for the Ising and 3-state Potts models [17], there are consistently increasing values from  $q = 2$  to 4. In order to check the validity of that alternative order parameter, we also estimate the dynamic critical exponents  $\theta$  and  $z$ , along with the static critical exponents  $\nu$  and  $\beta$ . Our results [ $\theta = -0.046(9)$ ,  $z = 2.294(3)$ ,  $\nu = 0.669(6)$ , and  $\beta = 0.0830(6)$ ] are in complete agreement with previous results found in the literature. In Section 2 we describe the model and the order parameter. In Section 3 we show the short-time scaling relations and present our results. In Section 4 we summarize and conclude.

## 2. The model

The  $q$ -state Potts model [28,29] is a generalization of the Ising model that preserves the next-nearest neighbour interaction, works with only two energies (neighboring spins are in the same state or not) but permits to put at each site any number of states ( $0 \leq q < \infty$ ). This model encloses a quite number of other problems of statistical physics. It undergoes a first-order phase transition when  $q > 4$  and a continuous phase transition for  $q \leq 4$ . Thus along with the Ising model, the Potts model is an important laboratory to check new theories and algorithms in the study of critical phenomena. Although its exact solution is not known, several results were obtained during the last 50 years [29–32].

The Hamiltonian of the  $q$ -state Potts model is given by

$$\beta \mathcal{H} = -K \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}, \quad (6)$$

where  $\beta = 1/k_B T$  and  $k_B$  is the Boltzmann constant,  $\langle i,j \rangle$  represents nearest-neighbor pairs of lattice sites,  $K$  is the dimensionless ferromagnetic coupling constant and  $\sigma_i$  is the spin variable which takes the values  $\sigma_i = 0, \dots, q-1$  on the lattice site  $i$ . It is well known that the critical point of this model is given by [29]

$$K_c = \log(1 + \sqrt{q}). \quad (7)$$

Usually, the order parameter of this model is given by [4,33]

$$M_1(t) = \frac{1}{L^d(q-1)} \left\langle \sum_i (q\delta_{\sigma_i(t),1} - 1) \right\rangle, \tag{8}$$

where  $L$  is the linear size of the lattice and  $d$  is the dimension of the system.

In this paper, we use a different definition for the order parameter, first proposed by Vanderzande [27] studying the Z(5) model. It can be written as

$$M_2(t) = \frac{1}{L^d} \left\langle \sum_i (\delta_{\sigma_i(t),0} - \delta_{\sigma_i(t),1}) \right\rangle, \tag{9}$$

where the average  $\langle \dots \rangle$  is taken over independent initial configurations.

### 3. Results

We performed short-time Monte Carlo simulations to obtain the critical exponents for the 4-state Potts model.

Simulations were carried out for square lattices with periodic boundary conditions and dimensions  $L = 120, 180$  and  $240$ . We also used the lattice sizes  $L = 20, 30, 40, 50, 60$  and  $90$  just to estimate the exponent  $z$  through the scaling collapses for different lattice sizes. The estimates for each exponent were obtained from five independent bins in the critical temperature. For the exponents  $\theta_g, z, \beta$  and  $\nu$ , each bin consisted of 20 000 samples, whereas for the exponent  $\theta$  we have used 100 000 samples. When estimating the exponent  $z$  through the scaling collapses, we used 50 000 samples. The error bars are fluctuations of the averages obtained from those bins. The dynamic evolution of the spins is local and updated by the heatbath algorithm.

In the following sections we show the results for the dynamic and static exponents of the 4-state Potts model.

#### 3.1. The dynamic critical exponent $\theta_g$

First of all we are concerned with the global persistence probability  $P(t)$ . It is defined as the probability that the global order parameter has not changed its sign up to time  $t$ . For  $\tau = 0$ , the global persistence probability decays algebraically as [15]

$$P(t) \sim t^{-\theta_g}, \tag{10}$$

where  $\theta_g$  is the global persistence exponent.

In order to estimate the critical exponent  $\theta_g$ , a sharp preparation of the initial states is demanded in order to obtain a precise value for the initial magnetization  $m_0 \ll 1$ . After obtaining the critical exponent  $\theta_g$  for several values of the initial magnetization  $m_0$ , the final value is achieved from the limit  $m_0 \rightarrow 0$ .

In this moment, it is worth to explain how to obtain a small value of  $m_0$  in Eq. (9). First, each site on the lattice is occupied by a spin variable which takes the values  $\sigma = 0, 1, 2$  or  $3$  with equal probability. After, the magnetization is measured by using  $M_2(t)$  and then, the variables in the sites are randomly chosen up to obtain a null value for the magnetization. The last procedure is to change  $\delta$  sites on the lattice in order to obtain the desired initial magnetization. It is given simply by

$$m_0 = \frac{\delta}{L^2} \tag{11}$$

and a value of  $m_0$  is obtained changing  $\delta$  sites occupied by  $\sigma = 2$  or  $3$  and substituting them by  $\sigma = 0$ .

In Fig. 1 we show the behavior of the global persistence probability for  $L = 240$  and  $m_0 = 0.000625$  in double-log scales, together with the behavior of the exponent  $\theta_g$  for  $m_0 = 0.005, 0.0025, 0.00125$  and  $0.000625$ . In order to obtain these initial magnetizations for this lattice,  $\delta$  should correspond to  $\delta = 288, 144, 72$ , and  $36$ .

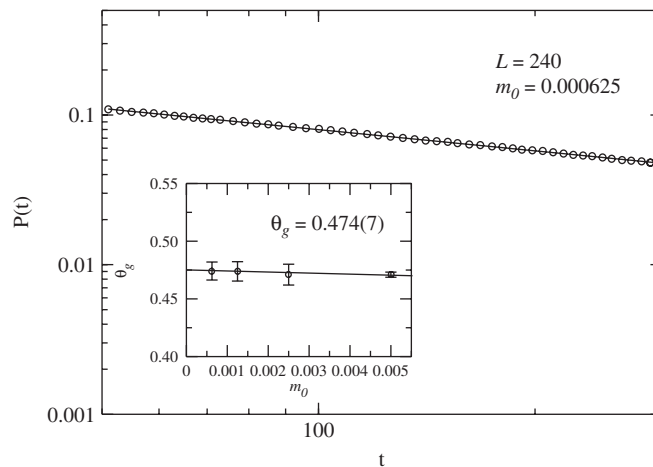


Fig. 1. The time evolution of the global persistence probability  $P(t)$  for a lattice size  $L = 240$  and  $m_0 = 0.000625$ . The error bars calculated over five sets of 20 000 samples are smaller than the symbols. The inset displays the exponent  $\theta_g$  for four different initial magnetizations, as well as its extrapolated values.

The extrapolated value of  $\theta_g$ , when  $m_0 \rightarrow 0$  and  $L = 240$  is

$$\theta_g = 0.474(7). \quad (12)$$

The extrapolated values of  $\theta_g$  for  $L = 120$  and  $180$  are shown in Table 1. We emphasize that sometimes the value of the initial magnetizations for the lattice size  $L = 180$  are slightly different from those of the  $L = 240$ . This is necessary in order to obtain an integer value of  $\delta$ .

The estimates obtained with the three lattice sizes show that the finite size effects are less than the statistical errors. Therefore, we conclude that the values for an infinite lattice are within the error bars of our results for  $L = 240$ .

### 3.2. The dynamic critical exponent $\theta$

Another critical exponent found only in the nonequilibrium state is the exponent  $\theta$  that characterizes the anomalous behavior of the order parameter in the short-time regime. Formerly, a positive value was always associated to this exponent [4,8,34–38] and the phenomenon was known as critical initial slip. However, some models can exhibit negative values for the exponent  $\theta$ . This is the case, for instance, of the Baxter–Wu model [5] and the tricritical Ising model, anticipated by Janssen et al. [39] and numerically confirmed by da Silva et al. [7].

In this paper we reobtain the dynamic critical exponent  $\theta$  for the 4-state Potts model using the order parameter described in Section 2 (Eq. (9)).

Usually the exponent  $\theta$  has been calculated using Eq. (3) or through the autocorrelation

$$A(t) \sim t^{\theta-d/z}, \quad (13)$$

where  $d$  is the dimension of the system. In the present work however we estimated the exponent  $\theta$  using the time correlation of the magnetization [37]

$$C(t) = \langle M(0)M(t) \rangle, \quad (14)$$

which behaves as  $t^\theta$  when  $\langle M(t=0) \rangle = 0$ . The average is taken over a set of random initial configurations. Initially, this approach had shown to be valid only for models which exhibit up-down symmetry [37]. Nevertheless, it has been demonstrated recently that this approach is more general and can include models with other symmetries [40]. This approach can thus be used for the  $q \neq 2$  Potts models.

Table 1  
Global persistence exponent  $\theta_g$

$L$	$\theta_g$
120	0.470(5)
180	0.473(6)
240	0.474(7)

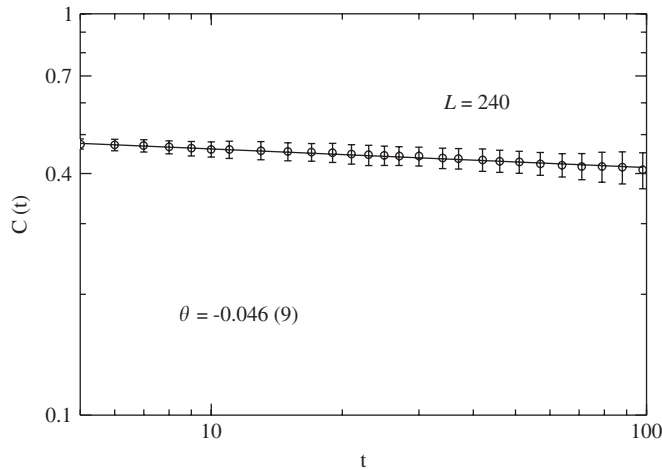


Fig. 2. Time correlation of the total magnetization for samples with  $\langle M(t=0) \rangle = 0$ . The error bars were calculated over five sets of 100 000 samples.

When compared to the other two techniques (Eq. (3) and Eq.(13)), this method has at least two advantages. It does not demand a careful preparation of the initial configurations ( $m_0 \ll 1$ ) neither a delicate limit  $m_0 \rightarrow 0$ , as well as the knowledge in advance of the exponent  $z$  (see Eq. (13)), which is an order of magnitude greater than  $\theta$ . In this case, a small relative error in  $z$  induces a large error in  $\theta$ .

In Fig. 2 we show the time dependence of the time correlation  $C(t)$  in double-log scales for the system with  $L = 240$ . The linear fit of this curve leads to the value  $\theta = -0.046(9)$ .

For the lattice sizes  $L = 120$  and  $180$  we obtained respectively  $\theta = -0.045(8)$  and  $-0.046(8)$ . These results are in good agreement with those found for the same model using the order parameter of the Eq. (8) [26,33].

### 3.3. The dynamic critical exponent $z$

The critical exponent  $z$  was estimated independently by means of two techniques. We began using mixed initial conditions, in order to obtain the function  $F_2(t)$ , given by [41]

$$F_2(t) = \frac{\langle M^2(t) \rangle_{m_0=0}}{\langle M(t) \rangle_{m_0=1}^2} \sim t^{d/z}, \tag{15}$$

where  $d$  is the dimension of the system. This approach proved to be very efficient in estimating the exponent  $z$  for a great number of models [5,7,8,19,38,42]. In this technique, for different lattice sizes, the double-log curves of  $F_2$  versus  $t$  fall on the same straight line, without any rescaling of time, resulting in more precise estimates for  $z$ .

The time evolution of  $F_2$  is shown on log-scales in Fig. 3 for  $L = 240$ .

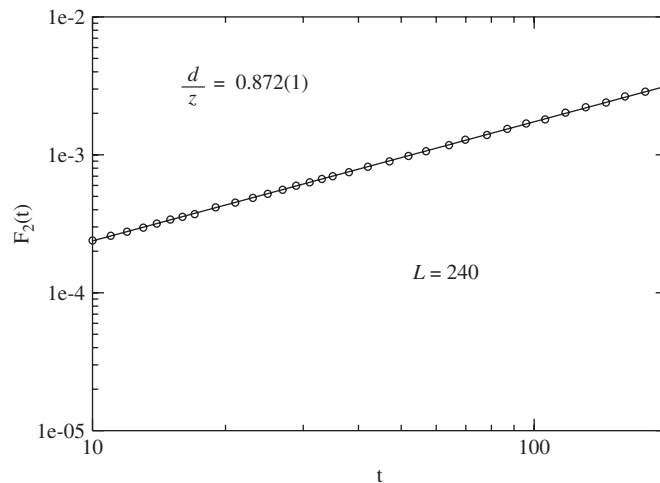


Fig. 3. The time evolution of  $F_2(t)$ . The error bars are smaller than the symbols. Each point represents an average over five sets of 20 000 samples.

The slope of the straight line gives

$$\frac{d}{z} = 0.872(1), \quad (16)$$

yielding

$$z = 2.294(3). \quad (17)$$

This result is in good agreement with the values  $z = 2.294(6)$  recently obtained for the Baxter–Wu model [5],  $z = 2.3(1)$  for the Ising model with multispin interactions [43], and  $z = 2.290(3)$  for the 4-state Potts model [41].

For  $L = 120$  we have obtained  $z = 2.296(5)$  and for  $L = 180$  we have obtained  $z = 2.295(5)$  indicating that the finite size effects are less than the statistical errors.

The second technique consists of studying the parameter

$$R(T, t, L) = \left\langle \left( \text{sign} \frac{1}{L} \sum_{top} (\delta_{\sigma_i(t),0} - \delta_{\sigma_i(t),1}) \right) \left( \text{sign} \frac{1}{L} \sum_{bottom} (\delta_{\sigma_i(t),0} - \delta_{\sigma_i(t),1}) \right) \right\rangle \quad (18)$$

introduced by de Oliveira [44]. In this case, the scaling relation for  $T = T_c$  is given by [45]

$$R(T = T_c, t, L_1) = R(T = T_c, b^{-z}t, bL_1), \quad (19)$$

with  $b = L_2/L_1$ . This equation shows that the dynamical exponent  $z$  can be easily estimated by adjusting the time rescaling factor  $b^{-z}$  in order to obtain the best scaling collapse of the curves for two different lattice sizes.

Fig. 4 shows the parameter  $R$  as a function of the time (full lines), as well as the scaling collapse (open circles) between different pairs of lattice for samples with ordered initial configurations ( $m_0 = 1$ ).

The best values of  $z$ , obtained through the  $\chi^2$  test [46] for different scaling collapses are shown in Table 2.

Our results obtained for the collapse of  $R(t)$  are in good agreement with our results arising from  $F_2(t)$ , as well as the results for the 4-state Potts model [41], and the Baxter–Wu model which belongs to the same universality class [5].

### 3.4. The static critical exponents $\nu$ and $\beta$

With the value of the exponent  $z$  in hand, we can estimate the static exponent  $\nu$  taking the derivative of the logarithm of the order parameter

$$M(t, \tau) = t^{-\beta/\nu z} M(1, t^{1/\nu z} \tau) \quad (20)$$

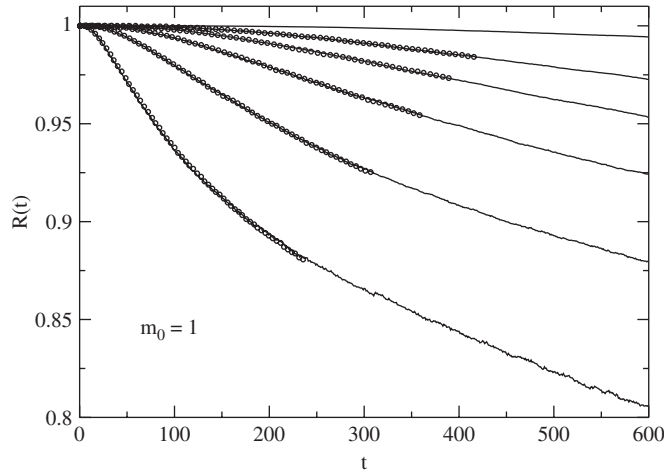


Fig. 4.  $R(t)$  vs  $t$  with ordered initial configurations  $m_0 = 1$ . The full lines show the behavior of  $R(t)$  for lattices  $L = 20, 30, 40, 50, 60$  and  $90$  (from the bottom to the top) and the corresponding time rescaled curves for lattices  $L = 30, 40, 50, 60,$  and  $90$  (open circles). The exponent  $z$  obtained for each collapse is shown in Table 2. The error bars, calculated over five sets of 50 000 samples, are smaller than the symbols.

Table 2  
Estimates of the dynamical exponent  $z$  for the best scaling collapse of  $R(t)$

$L_2 \mapsto L_1$	$z$
$30 \mapsto 20$	2.27(5)
$40 \mapsto 30$	2.28(4)
$50 \mapsto 40$	2.28(5)
$60 \mapsto 50$	2.29(3)
$90 \mapsto 60$	2.28(3)

with respect to  $\tau$  in the critical point

$$\partial_\tau \ln M(t, \tau)|_{\tau=0} = t^{1/\nu z} \partial_{\tau'} \ln F(\tau')|_{\tau'=0}. \tag{21}$$

This equation follows a power law that does not depend on  $L$  and the function  $F(\tau')$  is a scaling function which modifies the power law at  $\tau' \neq 0$  but still in the critical domain. In numerical simulations we approximate the derivative by a finite difference. Our results were obtained using finite differences of  $K_c \pm \delta$  with  $\delta = 0.001$ . In Fig. 5 the power law increase of Eq. (21) is plotted in double-log scales for  $L = 240$ .

From the slope of the curve we estimate the exponent  $1/\nu z$  for the three lattice sizes. Using the exponent  $z$  calculated previously, we obtain  $\nu = 0.670(9)$  for  $L = 120$ ,  $\nu = 0.668(6)$  for  $L = 180$ , and  $\nu = 0.669(6)$  for  $L = 240$ .

Finally, we evaluate the static exponent  $\beta$  following the decay of the order parameter in initially ordered samples ( $m_0 = 1$ ). At the critical temperature  $\tau = 0$ , the scaling law of Eq. (20) allows one to obtain  $\beta/\nu z$  which in turn leads to the exponent  $\beta$ , using the previous result obtained for the product  $\nu z$ . In Fig. 6 we show the time evolution of the magnetization in double-log scale for  $L = 240$ .

A linear fit of this straight line gives the value  $\beta/\nu z = 0.0541(1)$  leading to  $\beta = 0.0830(6)$ . For  $L = 120$  we obtained  $\beta = 0.0834(7)$  and for  $L = 180$  we obtained  $\beta = 0.0835(4)$ .

Our results for  $\nu$  and  $\beta$  are in good agreement with the exact results  $\nu = \frac{2}{3}$  and  $\beta = \frac{1}{12}$  [31,32].

### 3.5. Anomalous dimension $x_0$

Finally, we calculate the value of the anomalous dimension  $x_0$  of the magnetization which is introduced to describe the dependence of the scaling behavior of the initial conditions. It is related to the exponents  $\theta, z,$

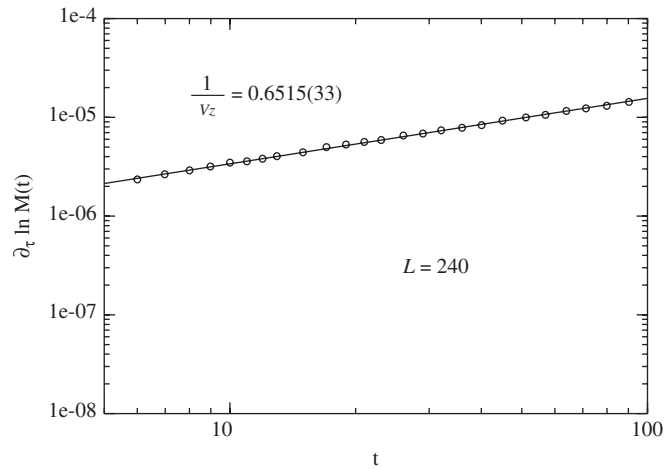


Fig. 5. The time evolution of the derivative  $\partial_\tau \ln M(t, \tau)|_{\tau=0}$  on log–log scales in a dynamic process starting from an ordered state ( $m_0 = 1$ ). The error bars are smaller than the symbols. Each point represents an average over five sets of 20000 samples.

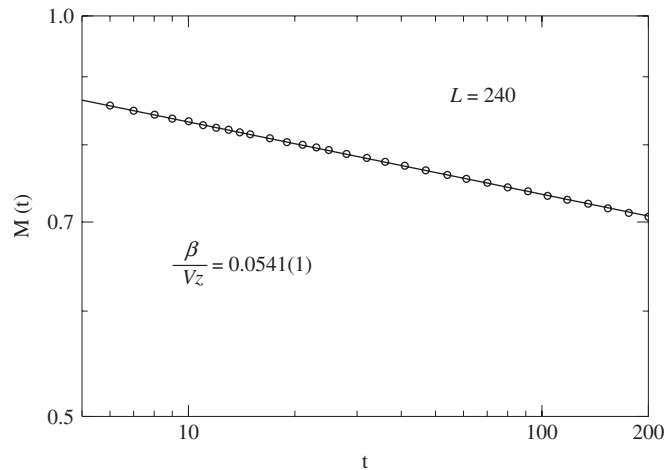


Fig. 6. The time evolution of the magnetization for initially ordered samples ( $m_0 = 1$ ). The error bars calculated over five sets of 20000 samples are smaller than the symbols.

Table 3  
The exponent  $x_0$  for the 4-state Potts model

$L$	$x_0$
120	0.021(21)
180	0.019(20)
240	0.019(23)

and  $\beta/v$  by

$$x_0 = \theta z + \beta/v. \tag{22}$$

Table 3 shows the values of  $x_0$  obtained with the exponents estimated all along this paper.

Our results show that the anomalous dimension of the 4-state Potts model has a null value whose meaning is the presence of the marginal operator, i.e., the operator which has the scaling dimension equal to dimensionality of the system and whose effect is not modified under renormalization-group operations. An



alike value was found recently by Arashiro et al. [26] for this model and for the Ising model with three-spin interactions in one direction.

#### 4. Discussion and conclusions

In this paper we revisited the 4-state Potts model in order to obtain the global persistence exponent  $\theta_g$  using an order parameter first proposed by Vanderzande [27] in the study of the  $Z(5)$  model. The results are in good agreement with each other. By using this alternative order parameter, we have also estimated the dynamic critical exponents  $\theta$  and  $z$ , as well as the well-known static exponents  $\nu$  and  $\beta$ . The exponent  $\theta$  was estimated using the time correlation of the magnetization, whereas to obtain the exponent  $z$  we used the function  $F_2(t)$  which combines simulations performed with different initial conditions and scaling collapse for the parameter  $R$  introduced by de Oliveira. The static exponents were obtained through the scaling relations for the magnetization and its derivative with respect to the temperature at  $T_c$ . Our results, when compared with available values in literature support the reliability of this new order parameter.

#### Acknowledgment

This work was supported by the Brazilian agencies CAPES, FAPESP and FUNAPE-UFG.

#### References

- [1] H.K. Janssen, B. Schaub, B. Schmittmann, *Z. Phys. B: Condens. Matter* 73 (1989) 539.
- [2] D.A. Huse, *Phys. Rev. B* 40 (1989) 304.
- [3] B.I. Halperin, P.C. Hohenberg, S.-K. Ma, *Phys. Rev. B* 10 (1974) 139.
- [4] B. Zheng, *Int. J. Mod. Phys. B* 12 (1998) 1419.
- [5] E. Arashiro, J.R. Drugowich de Felício, *Phys. Rev. E* 67 (2003) 046123.
- [6] K. Okano, L. Schulke, K. Yamagishi, B. Zheng, *Nucl. Phys. B* 485 (1997) 727.
- [7] R. da Silva, N.A. Alves, J.R. Drugowich de Felício, *Phys. Rev. E* 66 (2002) 026130.
- [8] H.A. Fernandes, J.R. Drugowich de Felício, A.A. Caparica, *Phys. Rev. B* 72 (2005) 054434.
- [9] M. Santos, W. Figueiredo, *Phys. Rev. E* 63 (2001) 042101.
- [10] B.C.S. Grandi, W. Figueiredo, *Phys. Rev. E* 70 (2004) 056109.
- [11] M. Pleimling, F. Iglói, *Phys. Rev. Lett.* 92 (2004) 145701.
- [12] M. Pleimling, F. Iglói, *Phys. Rev. B* 71 (2005) 094424.
- [13] A. Malakis, S.S. Martinos, I.A. Hadjiagapiou, *Physica A* 327 (2003) 349.
- [14] Z.B. Li, L. Schulke, B. Zheng, *Phys. Rev. Lett.* 74 (1995) 3396.
- [15] S.N. Majumdar, A.J. Bray, S. Cornell, C. Sire, *Phys. Rev. Lett.* 77 (1996) 3704.
- [16] S.N. Majumdar, A.J. Bray, *Phys. Rev. Lett.* 91 (2003) 030602.
- [17] L. Schulke, B. Zheng, *Phys. Lett. A* 233 (1997) 93.
- [18] R. da Silva, N.A. Alves, J.R. Drugowich de Felício, *Phys. Rev. E* 67 (2003) 057102.
- [19] R. da Silva, N. Alves, *Physica A* 350 (2005) 263.
- [20] F. Ren, B. Zheng, *Phys. Lett. A* 313 (2003) 312.
- [21] E.V. Albano, M.A. Muñoz, *Phys. Rev. E* 63 (2001) 031104.
- [22] M. Saharay, P. Sen, *Physica A* 318 (2003) 243.
- [23] H. Hinrichsen, H.M. Koduvely, *Eur. Phys. J. B* 5 (1998) 257.
- [24] P. Sen, S. Dasgupta, *J. Phys. A* 37 (2004) 11949.
- [25] B. Zheng, *Mod. Phys. Lett. B* 16 (2002) 775.
- [26] E. Arashiro, H. A. Fernandes, J.R. Drugowich de Felício, e-print cond-mat/0603436 (unpublished).
- [27] C. Vanderzande, *J. Phys. A* 20 (1987) L549.
- [28] R.B. Potts, *Proc. Cambridge Phil. Soc.* 48 (1952) 106.
- [29] F.Y. Wu, *Rev. Mod. Phys.* 54 (1992) 235.
- [30] M.P.M. den Nijs, *J. Phys. A* 12 (1979) 1857.
- [31] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, New York, 1982.
- [32] E. Domany, E.K. Riedel, *J. Appl. Phys.* 49 (1978) 1315.
- [33] R. da Silva, J.R. Drugowich de Felício, *Phys. Lett. A* 333 (2004) 277.
- [34] A. Jaster, E. Manville, L. Schulke, B. Zheng, *J. Phys. A* 32 (1999) 1395.
- [35] L. Schulke, B. Zheng, *Phys. Lett. A* 204 (1995) 295.
- [36] T. Tomé, J.R. Drugowich de Felício, *Phys. Rev. E* 53 (1996) 108.

- [37] T. Tomé, M.J. de Oliveira, *Phys. Rev. E* 58 (1998) 4242.
- [38] N. Alves, J.R. Drugowich de Felício, *Mod. Phys. Lett. B* 17 (2003) 209.
- [39] H.K. Janssen, K. Oerding, *J. Phys. A* 27 (1994) 715.
- [40] T. Tomé, *J. Phys. A* 36 (2003) 6683.
- [41] R. da Silva, N.A. Alves, J.R. Drugowich de Felício, *Phys. Lett. A* 298 (2002) 325.
- [42] R. da Silva, R. Dickman, J.R. Drugowich de Felício, *Phys. Rev. E* 70 (2001) 067701.
- [43] C.S. Simões, J.R. Drugowich de Felício, *Mod. Phys. Lett. B* 15 (2001) 487.
- [44] P.M.C. de Oliveira, *Europhys. Lett.* 20 (1992) 621.
- [45] M. Silvério Soares, J. Kamphorst Leal da Silva, F.C. Sá Barreto, *Phys. Rev. B* 55 (1997) 1021.
- [46] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, *Numerical Recipes*, Cambridge University Press, London, 1986.