Digital Filter Design Using Root Moments for Sum-of-All-Pass Structures From Complete and Partial Specifications

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***Abstract*—This paper is concerned with the development of digital filter design procedures for transfer functions in the form of sum-of-all-pass in which the requirements may be partially specified. Specifically, the requirements for a digital filter or equalizer in amplitude $A(\theta)$, or phase $\phi(\theta)$, or possibly group-delay response $\tau(\theta)$, may be specified from measurements over a limited a set of frequencies $\{\theta_1, \theta_2\}$. The problem is to develop techniques for the design a transfer function $H(z)$ satisfying these specifications and constrained to be in the form of a sum-of-two-all-pass functions. The proposed solution is based on the use of *root moments*. The companion problem concerned with the estimation of the orders of the required all-pass filters is also examined and a solution proposed based on the same context of root moments.**

***Index Terms*—All-pass filters, fundamental relationships, Newton identities, root moments.**

I. Introduction

THE purpose of this paper is to develop a design method for digital filters and equalizers that have transfer functions expressed as the sum of two all-pass functions, under the constraint that the requirements are only partially specified over a frequency interval that does not cover the entire range. The design of sum-of-all-pass digital filters from specifications that cover the entire frequency range has received considerable attention over the years. The sum-of-all-pass realization structure has its conceptual basis in the theory of classical lattice filters, and its interpretation through the scattering parameters first proposed by Fettweis in 1971 for wave digital filters in [1]. An earlier contribution by Gold and Rader in 1969 can be traced as a precursor of such sum-of-all-pass structures, where the problem of expressing elliptic filter transfer functions in this form for compact transfer function representation was proposed [2]. In the same year, these structures are proposed by Constantinides for notch filter design [3].

The sum-of-all-pass configuration is known to have many desirable attributes such as robustness to quantization errors, computationally efficiency, compactness with respect to realizations, and others [4]–[14]. We have counted more than 100 publications in the area over the past 20 years, making it a fertile and seemingly an inexhaustible area of research. The optimal design of piecewise constant magnitude filters and the efficient computations associated with sum-of-all-pass structures made an impact in the theory of transmultiplexers [15]–[17]. The problem, in one form or another, is examined periodically over the years [18]–[24]. Renewed interest in the subject is shown recently particularly from the perspective of designing such structures to satisfy requirements that while they do not have the standard piecewise constant gain specifications, they are, nevertheless, defined over the complete frequency range [25]–[28]. It is noted that in [28], the sum-of-all-pass structures produced are referred to as having minimum phase. It is shown in the present publication that such structures cannot have minimum phase but must be of mixed phase. Indeed, this specific attribute imposes limitations on the range of infinite-impulse response (IIR) transfer functions realizable by such means. The nonminimum-phase nature of the required transfer function complicates the problem considerably particularly for the general case when nonstandard gain/magnitude requirements are encountered. The standard piecewise constant case has been addressed in recent publications by several authors as in [26], [29]–[32]. The use of such structures for notch filters has also been taken up more recently in [33] and [34] and in an adaptive filter form in [35]–[37].

An important aspect of such design structures is related to the orders of the all-pass filters to be employed. In [28], in line with standard practice [23], [25]–[27], the orders are assumed to be known *a priori*. However, it is shown in the present paper that the required all-pass orders need not be assumed *a priori* and that they can be determined from the phase responses of the component all-pass functions, and hence from the specifications of the initial filter design problem.

II. The Problem and Approach Specified

The approach taken in the present paper is based on the use of root moments. Root moments are symmetric functions of the roots of a polynomial and include amongst them the *higher order root moments* and the *Wronski moments* [38], [39]. The first-order root moments, or simply the root moments, are parameters that are related to the differential cepstrum as indicated elsewhere [39]–[41].

Let a real digital filter or equalizer transfer function be written as

$$H(z) = \frac{1}{2}\left[A_1(z) + A_2(z)\right] \tag{1}$$

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where $A_i(z)$, $i = 1, 2$ are real stable all-pass functions. In the notch filter design, one of the all-pass functions is normally taken to be unity, while the other is designed to be a stable transfer function [3], [33], [34], [37], [42], [43]. We assume that the required transfer function $H(z)$ is to be as given in (1). For the purposes of the present paper, we identify three basic problems in the filter design domain as follows:

Problem 1: given an magnitude specification limited over the frequency range (θ_1, θ_2) to derive the transfer function;

Problem 2: given a phase specification limited over the frequency range (θ_1, θ_2) to derive the transfer function $H(z)$;

Problem 3: given a magnitude and phase specification limited over the frequency ranges (θ_1, θ_2), and (θ_3, θ_4), which may be noncontiguous and nonoverlapping, to derive the transfer function $H(z)$.

In the above problem definitions, we have two unknown stable transfer functions, the all-pass filters $A_1(z)$ and $A_2(z)$, and hence we need two independent conditions for their determination. Problem 1 has been solved many years ago for the standard piecewise constant low-pass, high-pass, bandpass, and bandstop filters as a classical equiripple filter design problem [2], [44]. In these approaches, however, only the magnitude response is taken into consideration. It is implicitly assumed that the phase characteristic is unimportant. Since then, there have many developments on the fundamental ideas by incorporating phase or group delay specifications, but these essentially have not deviated from the original standard definitions in which the specifications cover the entire frequency range. The present paper outlines the principles and develops procedures to effect a solution to Problem 1. The interrelated Problems 2 and 3 are examined in a separate paper.

III. Structural Properties

The theoretical basis for the development of the methods is obtained in a straightforward manner as shown below. Some analytical development is allowed only with respect to those aspects of the problem that are not well known, or else results are given compactly. We assume that the required transfer function $H(z)$ is real. Let $A_1(z)$ be stable and of order n_1 and $A_2(z)$ stable and of order n_2. At this juncture, both orders are assumed to be known, but we shall develop techniques to estimate these at a later stage of the paper. We can write therefore [45]

$$A_1(z) = \frac{z^{-n_1} D_1(z^{-1})}{D_1(z)} \tag{2}$$

and

$$A_2(z) = \frac{z^{-n_2} D_2(z^{-1})}{D_2(z)} \tag{3}$$

where $D_i(z)$, $i = 1, 2$ are the respective denominators of the all-pass functions. The required transfer function then takes the form

$$H(z) = \frac{1}{2}\left[\frac{z^{-n_1} D_1(z^{-1})}{D_1(z)} + \frac{z^{-n_2} D_2(z^{-1})}{D_2(z)}\right] = \frac{N(z)}{D(z)} \tag{4}$$

where the numerator $N(z)$ and denominator $D(z)$ are given by

$$N(z) = \frac{1}{2}\cdot\left[z^{-n_1} D_1(z^{-1}) D_2(z) + z^{-n_2} D_2(z^{-1}) D_1(z)\right] \tag{5}$$

and

$$D(z) = D_1(z) D_2(z). \tag{6}$$

It is clear from the above that both numerator and denominator are of the same order $n = n_1 + n_2$ when there are no cancellation of factors. We note the following properties.

A. The Palindromic Polynomial Numerator

The numerator $N(z)$ of the required transfer function is a palindromic real polynomial.

This is evident by constructing $N(z^{-1})$ from (5) as

$$N(z^{-1}) = \frac{1}{2}\cdot\left[z^{n_1} D_1(z) D_2(z^{-1}) + z^{n_2} D_2(z) D_1(z^{-1})\right] \tag{7}$$

and hence we obtain the palindrome

$$z^{-(n_1+n_2)} N(z^{-1}) = N(z). \tag{8}$$

Let us assume at this juncture that there are no zeros on the circumference of the unit circle. This can be ensured by preprocessing the specifications as indicated in [46]. In addition, in view of (8), let

$$N(z) = N_1(z)\tilde{N}_1(z) \tag{9}$$

where $N_1(z)$ contains all zeros inside the unit circle and $\tilde{N}_1(z)$ contains all zeros outside the unit circle. Since $N(z)$ is palindromic, it has its zeros in reciprocal pairs, and hence apart from a constant factor, we can write [47]

$$|N_1(z)| = \left|\tilde{N}_1(z)\right|. \tag{10}$$

B. The Perfect Square Numerator

The amplitude response of the required transfer function $H(z)$ of (1) *has special form as below.*

$$|H(z)|_{z=e^{j\theta}} = \frac{A^2(\theta)}{B(\theta)} \tag{11}$$

where $A(\theta)$ and $B(\theta)$ are real functions.

This is clearly evident from (4) and (8), from which we have

$$|H(z)| = \left|\frac{N_1(z)\cdot\tilde{N}_1(z)}{D(z)}\right| \tag{12}$$

where $A(\theta) = |N_1(z)| = |\tilde{N}_1(z)|$, $B(\theta) = |D(z)|$, and hence in the above form in view of (10). Thus, the above sum-of-all-pass structure imposes an additional constraint on the required transfer function. Namely, it is required to have a magnitude response with a numerator that is a perfect square, while the denominator, apart from it being stable, it is unconstrained. This point makes a fundamental deviation from standard design techniques and complicates the design problem considerably. Indeed, if the denominator were likewise a perfect square, then the overall problem could be reduced to the standard design methods.

There is one further and significant complication that is concerned with the phase response of a sum-of-all-pass transfer function as indicated below.

C. Nonminimum Phase

The sum-of-all-pass transfer function of $H(z)$ has nonminimum-phase response.

This is a consequence of where the numerator $N(z)$ of the transfer function, apart from any zeros on the unit circle, if it has zeros internal to the unit circle, then it must also have zeros outside the unit circle. Indeed, from the same considerations it is seen that it has as many zeros outside the unit circle as there are zeros inside the unit circle, notwithstanding their being in reciprocal form. Clearly, in the main, $H(z)$ cannot be minimum phase as stated in [28], but it must have mixed phase.

We shall employ root moments in our design approach as they enable us, among other benefits, to unravel the constraint imposed on the form of the numerator of $H(z)$ to be a perfect square. Root moments have already been used effectively in a range of digital filter design problems [48], [49]. In our present case, we set

$$H(z) = K \cdot \frac{N_1(z) \cdot \tilde{N}_1(z)}{D(z)} \tag{13}$$

$$N_1(z) = \prod_{i=1}^{\frac{n}{2}} (1 - \alpha_i z^{-1}) \tag{14}$$

and hence

$$\tilde{N}_1(z) = \prod_{i=1}^{\frac{n}{2}} \left(1 - \frac{1}{\alpha_i} z^{-1}\right) \tag{15}$$

where the roots $|\alpha_i| < 1$. The order n is taken to be even for simplicity of analysis.

The root moments of $N_1(z)$ and $\tilde{N}_1(z)$ are then given by

$$S_m^{N_1} = \sum_{i=1}^{\frac{n}{2}} \alpha_i^m \tag{16}$$

and

$$S_m^{\tilde{N}_1} = \sum_{i=1}^{\frac{n}{2}} \left(\frac{1}{\alpha_i}\right)^m \tag{17}$$

respectively [50]. Therefore, it is clear that

$$S_{-m}^{\tilde{N}_1} = S_m^{N_1}. \tag{18}$$

Let the denominator $D(z)$ be of the form

$$D(z) = \prod_{i=1}^{n} (1 - \beta_i z^{-1}) \tag{19}$$

where $|\beta_i| < 1$. Therefore, the form of the overall transfer function of (1) is

$$H(z) = K \cdot \frac{\prod_{i=1}^{\frac{n}{2}} (1 - \alpha_i z^{-1}) \cdot \prod_{i=1}^{\frac{n}{2}} \left(1 - \frac{1}{\alpha_i} z^{-1}\right)}{\prod_{i=1}^{n} (1 - \beta_i z^{-1})} \tag{20}$$

which can be further written in the form

$$H(z) = \tilde{K} \cdot \frac{\prod_{i=1}^{\frac{n}{2}} (1 - \alpha_i z^{-1}) \cdot \prod_{i=1}^{\frac{n}{2}} (1 - \alpha_i z)}{\prod_{i=1}^{n} (1 - \beta_i z^{-1})} \cdot z^{-\frac{n}{2}} \tag{21}$$

where $\tilde{K}$ includes the original gain K and all constant factors arising from the rewriting of $\tilde{N}_1(z)$.

IV. Fundamental Relationships

We can express the magnitude and phase functions of a transfer function in a form that makes them amenable to separate treatment as follows [11], [40], [41], [46], [50]. On taking the logarithm of (21) and expanding the terms into an infinite series, we obtain the following:

$$\ln \frac{H(z)}{\tilde{K}} = -\frac{n}{2} \ln z - \sum_{m=1}^{\infty} \frac{S_m^{N_1}(z^{-m} + z^m) - S_m^D z^{-m}}{m} \tag{22}$$

where

$$S_m^{N_1} = \sum_{i=1}^{\frac{n}{2}} \alpha_i^m \tag{23}$$

$$S_m^D = \sum_{i=1}^{n} \beta_i^m. \tag{24}$$

It is observed that $S_m^{N_1}$ and S_m^D tend to zero for large m. The implication of this is that the polynomials associated with these root moments have their roots located inside the unit circle and hence stability is assured.

The parameters $S_m^{N_1}$ and S_m^D can also be seen as the cepstrum coefficients in the manner of [28], [51], and [52]. An alternative interpretation based on the properties of polynomials falls within the area of the theory of equations in which these are known as the *root moments* of an equation. This is the interpretation adopted in the present paper [41], [53]. Now for $z = e^{j\theta}$, we let $H(z) = C(\theta) \cdot e^{j\phi(\theta)}$ so that

$$\ln \frac{C(\theta)}{\tilde{K}} + j\phi(\theta) = -j\frac{n}{2}\theta - \sum_{m=1}^{\infty} \frac{2S_m^{N_1}}{m} \cos m\theta + \sum_{m=1}^{\infty} \frac{S_m^D}{m} e^{-jm\theta}. \tag{25}$$

Thus

$$\ln C(\theta) = \ln \tilde{K} - \sum_{m=1}^{\infty} \frac{\left(2S_m^{N_1} - S_m^D\right)}{m} \cdot \cos m\theta \tag{26}$$

$$\phi(\theta) = -\frac{n}{2}\theta - \sum_{m=1}^{\infty} \frac{S_m^D}{m} \cdot \sin m\theta. \tag{27}$$

The above (26) and (27) are referred to, in this context, as the *fundamental relationships*. For real transfer functions, (26) is an even Fourier series while (27), after an adjustment to take care of the linear term, is an odd Fourier series.

A. The Newton Identities

The coefficients of a given polynomial and its root moments are connected via a recursion relationships, known as the Newton Identities, as below [40], [50], [52], [54]–[56].

$$S_m + p_1 S_{m-1} + p_2 S_{m-2} + \ldots + m p_m = 0, \qquad m \leq n \tag{28}$$

$$S_m + p_1 S_{m-1} + p_2 S_{m-2} + \ldots + p_n S_{m-n} = 0, \qquad m > n \tag{29}$$

with $S_0 = n$. These equations are known in signal processing circles in the context of cepstra [41], [51], [52]. However, they are considerably older than that. They are also known in the mathematical theory of equations as the *Newton Identities*. They essentially date back to the 17th century Scottish mathematician Gregory who first developed such relationships for up to fifth-degree polynomials. The generalization of these relationships for any degree is due to the genius of Newton [56]. There is an implication of sufficiency in the Newton Identities in that given the set $\{p_i\}$, one can determine the set $\{S_i\}$, and conversely, given $\{S_i\}$, one can determine $\{p_i\}$.

B. Fourier and Fundamental Relationships

The root moments can be obtained by inverse Fourier analysis of the fundamental relationships.

Thus, from (26) and (27), we obtain

$$2S_m^{N_1} - S_m^D = -\frac{m}{\pi}\int_{-\pi}^{\pi} \ln\left[\frac{C(e^{j\theta})}{\tilde{K}}\right]\cos m\theta d\theta \tag{30}$$

$$S_m^D = -\frac{m}{\pi}\int_{-\pi}^{\pi}\left[\phi(\theta) + \frac{n}{2}\theta\right]\sin m\theta d\theta. \tag{31}$$

The above relationships in practice will have to be computed from the specifications given, perhaps, as a list of numbers. Effectively the magnitude and phase responses are sampled and hence the integrals need to be evaluated as sums. If the sampling is regular, then we can use the fast Fourier transform (FFT) as an efficient means for their computation [39], [53]. With irregular sampling, other numerical, and not necessarily as efficient, techniques need to be used. It is important to note that by taking a finite number of terms in the fundamental relationships in view of their Fourier form, we are essentially performing a least-squares approximation. More refined forms of approximation involving different norms may be implemented, but this is an area open for further development.

C. Fundamental Relationships for the All-Pass Filter

The above fundamental relationships take a specific form for an all-pass filter $H_a(\theta) = A_a(\theta)\exp(j\phi_a(\theta))$ as follows [53].

Since the magnitude is unity, we have

$$\ln(A_a(\theta)) = 0 \tag{32}$$

while for the phase response, we obtain

$$\phi_a(\theta) = -p\theta + \sum_{m=1}^{\infty}\frac{2S_m^F}{m}\sin m\theta \tag{33}$$

where p is the order of the all-pass with a numerator $F(z)$.

It is clear that once the root moments of an all-pass filter have been determined its transfer function is directly obtainable from the Newton Identities. Thus, the design procedure is almost completed when the root moments have been estimated. With the few additional points below, we are in a position to describe the various steps of the solution.

D. The Monotonic Phase of the All-Pass Filter

The phase response of a stable all-pass filter is a monotonically decreasing function of frequency.

This is a pivotal property of stable all-pass filters. The property is shown in [57] by employing the *separation property* of reactance functions [58]–[60].

E. The Sum-of-All-Pass Filter

The magnitude response of $H(z)$ in (1) *is given by*

$$C(\theta) = \cos\frac{\phi_1(\theta) - \phi_2(\theta)}{2} \tag{34}$$

and the corresponding phase response is given as

$$\phi(\theta) = \frac{\phi_1(\theta) + \phi_2(\theta)}{2} \tag{35}$$

where two all-pass filters are

$$A_1 = e^{j\phi_1(\theta)} \tag{36}$$

$$A_2 = e^{j\phi_2(\theta)}. \tag{37}$$

This is a well-known set of equations, easily derivable as demonstrated in [4], [5], [45], and [61]–[63]. They are used at different stages of the paper en route to the estimation of the all-pass phase functions.

There is evident temptation to view (34) and (35) as furnishing the necessary conditions for the all-pass phase response estimation by making, as in [28], the identification

$$\phi_1(\theta) = \phi_d(\theta) + \cos^{-1}|H_d(\theta)| \tag{38}$$

$$\phi_2(\theta) = \phi_d(\theta) - \cos^{-1}|H_d(\theta)|. \tag{39}$$

However, these conditions, as they stand, are inadequate, as their differentials cannot be both jointly negative, monotonically decreasing for all frequencies, as required in Section IV-D. Moreover, the inverse cosine term above, in a strict sense, needs to be in an unwrapped form. Therefore, some adjustments are needed to ensure that these constraints are met. In [28], these appear to be implemented with additive constant and linear terms to the phase equations appropriately. The process needs to identify the positions at which the phase changes take place and add multiples of π as necessary. Despite such steps when the gain is equal to 1, the slope of the phase functions is infinite, and this situation cannot be taken into account by such means.

F. The Companion Filter

A filter $G(z)$, companion to the required $H(z)$, may be defined such that $G(z)$ has a magnitude response identical to $H(z)$ but has minimum phase.

This is a very useful concept in the ensuing design process, as it forms the initial point for further design considerations. From its definition, the companion filter has a transfer function of the form

$$G(z) = K \cdot \frac{\left[\prod_{i=1}^{\frac{n}{2}} (1-\alpha_i z^{-1})\right]^2}{\prod_{i=1}^{n} (1-\beta_i z^{-1})}. \tag{40}$$

The fundamental relationships for the companion filter are

$$\ln F(\theta) = \ln K - \sum_{m=1}^{\infty} \frac{2S_m^{N_1} - S_m^D}{m} \cos m\theta \tag{41}$$

$$\phi_G(\theta) = \sum_{m=1}^{\infty} \frac{2S_m^{N_1} - S_m^D}{m} \sin m\theta. \tag{42}$$

The transfer functions $H(z)$ and $G(z)$ have the same amplitude form while their phase forms are substantially different. The important observation to note is that the minimum-phase transfer function $G(z)$ is now constructible from the given magnitude specifications alone, as described below.

A general observation can be made here. It is clear from the above that the sum-of-all-pass structure imposes a range of conditions that limit the design procedures. Specifically, it is observed that if there are joint gain and phase requirements, then the phase specifications must be monotonically decreasing functions of frequency. This follows from (35) and from the results in Section IV-D.

V. The Design Procedure

In order to obtain the root moments $S_m^{N_1}$ and S_m^D in Section IV, we need two independent equations. Thus, we need to use both the amplitude and phase forms of the fundamental relationships, or some other independent information. However, in the definition of Problem 1, only the magnitude response is given, and this may be so, over a limited range of frequencies. Use of the magnitude form alone would imply a minimum-phase transfer function as given in [39], which is clearly not the case in this problem. However, the companion minimum-phase filter is constructible as given in the following subsection.

A. Construction of Companion Transfer Function $G(z)$

From (41) assuming $K = 1$, we obtain

$$2S_m^{N_1} - S_m^D = -\frac{m}{\pi} \int_{-\pi}^{\pi} \ln\left[C(\theta)\right] \cos m\theta d\theta \tag{43}$$

where $C(\theta)$ is the given magnitude specification.

From (30) with $S_m = 2S_m^{N_1} - S_m^D$, we can determine a large degree polynomial with coefficients $\{h_r\} h_0 \neq 0$. Its root moments S_m will tend to zero exponentially as they are the sum of the root moments of a minimum-phase function, as can be seen from (23) and (24). Hence, in practice, the degree of the polynomial $\{h_r\}$ need not to be very large. From Section IV-B, the evaluation of the root moments may be put into effect by using an FFT, in which case the polynomial degree implicitly will be allowed to be the same as the length of the FFT. This length can be as large as required, thereby obviating the need to have *a priori* the degree of the polynomial.

We note that the polynomial $\sum_{r=0}^{N} h_r z^{-1}$, $h_0 \neq 0$ is a Taylor series expansion to $G(z)$ in (40). This is evident from the root moments S_m as given in (41), and they clearly correspond to a ratio of two polynomials [50]. We can now write the companion transfer function as

$$G(z) = K \cdot \frac{\left[\prod_{i=1}^{\frac{n}{2}} (1-\alpha_i z^{-1})\right]^2}{\prod_{i=1}^{n} (1-\beta_i z^{-1})} = \frac{N_{\min}(z)}{D(z)} \tag{44}$$

or effectively

$$\frac{N_{\min}(z)}{D(z)} \cong h_0 + h_1 z^{-1} + \ldots + h_N z^{-N} \tag{45}$$

where N is the FFT length chosen to be sufficiently large. Now, we set

$$N_{\min}(z) = a_0 + a_1 z^{-1} + \ldots + a_n z^{-n} \tag{46}$$

$$D(z) = 1 + b_1 z^{-1} + \ldots + b_n z^{-n} \tag{47}$$

and from (45), we have equivalently

$$N_{\min}(z) = D(z)(h_0 + h_1 z^{-1} + \ldots + h_N z^{-N}). \tag{48}$$

By equating terms in (48), we can obtain both $N_{\min}(z)$ and $D(z)$. Indeed since the left-hand side is of degree n, all coefficients on the right-hand side of degree higher than n must be identically zero. This condition yields $D(z)$. Moreover, by equating the remaining coefficients of lower powers we obtain, in conjunction with $D(z)$, the numerator $N_{\min}(z)$. The overall order n is assumed to be known at this juncture, but a lower bound on its value can be estimated from the specifications as indicated in Section VI.

B. Construction of the Required Transfer Function

At this juncture, we have constructed the rational transfer function of the companion minimum-phase filter $G(z)$. It is now a straightforward matter to construct the required transfer function $H(z)$ in finite-impulse response (FIR) form [39]. However, the requirement is to have a rational form, and this requires further consideration. We note that the denominators of $H(z)$ and $G(z)$ are identical, and hence the denominator of the required $H(z)$ is directly obtainable from the preceding considerations. The numerator of $H(z)$ requires more effort and can be obtained from the numerator of $G(z)$ as follows. We determine the root moments $S_m^{N_{\min}}$ of $N_{\min}(z)$. Because of the squared form of $N_{\min}(z)$, as it can be seen from (44), these will be equal to twice the root moments of $S_m^{N_1}$ of the minimum-phase part of $N(z)$, i.e.,

$$S_m^{N_{\min}} = 2S_m^{N_1}. \tag{49}$$

Hence, we can determine the root moments of the factor $N_1(z)$, and from the Newton Identities, we can construct $N_1(z)$. The maximum phase part $\tilde{N}_1(z)$ of $N(z)$ is simply the mirror image version of $N_1(z)$, which can be constructed by reversing the order of the coefficients of $N_1(z)$.

So far, we have managed to construct the transfer function $H(z)$ in a form given as in (20). This form now needs to be expressed as the sum of two all-pass functions as given by (1). In general, not all rational forms $N(z)/D(z)$ can be expressed as the sum of two all-pass functions as can be seen from (44). Nevertheless, the specific development in this paper so far has been so geared as to ensure that this form is attainable. To achieve this, we note from (6) that

$$D_2(z) = \frac{D(z)}{D_1(z)} \tag{50}$$

and from (5), we have

$$\begin{aligned} N(z) &= \frac{1}{2}\left[z^{-n_1}D_1(z^{-1})\frac{D(z)}{D_1(z)} + z^{-n_2}\frac{D(z^{-1})}{D_1(z^{-1})}D_1(z)\right] \\ &= \frac{1}{2}\left[D(z)\cdot A_1(z) + z^{-n}D(z^{-1})\cdot A_1(z^{-1})\right] \end{aligned} \tag{51}$$

where $n = n_1 + n_2$. On the unit circle $z = e^{j\theta}$, we can write this equation as

$$\left|N(e^{j\theta})\right|\cdot e^{j\phi_N(\theta)} = \left|D(e^{j\theta})\right|\cdot e^{-j\frac{n}{2}\theta}\cos\left[\phi_D(\theta)+\phi_1(\theta)+\frac{n}{2}\theta\right]. \tag{52}$$

In this equation, the unknown quantity is the phase response $\phi_1(\theta)$ of the all-pass transfer function $A_1(\theta)$, the order having been estimated prior to this development as indicated in VI. Hence, we determine the phase response of one of the all-pass functions needed in the realization.

The subproblem now is to determine the all-pass transfer function from its phase response. To do this, in our root moments context, we note that the fundamental relationships for an all-pass transfer function can be obtained in a manner similar to the one followed for (26) and (27), so that

$$\ln A(\theta) = 0 \tag{53}$$

$$\phi_A(\theta) = -M\theta - 2\sum_{m=1}^{\infty}\frac{S_m^A \sin m\theta}{m} \tag{54}$$

where M is the order of the all-pass, and $A(\theta) = 1$, S_m^A are the root moments of the denominator. Therefore, an all-pass function can be constructed from its phase response through its corresponding root moments and by using the Newton Identities. The order M can be determined as indicated in VI. Thus, from (54), we obtain $\phi_1(\theta)$ and hence $A_1(\theta)$.

The all-pass function $A_2(\theta)$ can be determined in a similar fashion. However, we note that the product of the denominators of $A_1(\theta)$ and $A_2(\theta)$ is already known, as given in (47), and since $D_1(z)$ has been determined, $D_2(z)$ is obtained immediately. It is not advisable, however, to obtain $D_2(z)$ by long division as this process is susceptible to accumulated errors. A better approach is to use the root moments algebra to obtain the root moments of $D_2(z)$ from those of $D(z)$ and then determine $D_1(z)$, and hence through the Newton Identities, we determine $D_2(z)$ [50].

C. Design From Partial Response Specifications

We now consider the case when the magnitude response is partially specified over a range of frequencies (θ_1, θ_2). This range may be the union of subranges over which measurements may have been carried out. The specifications of the overall problem allow two alternative routes to be taken, in that either it is required to make strict use of the measurements alone, or freedom is allowed to insert some fictitious response in the unspecified regions. In both cases, the conditions pertaining to the magnitude behavior at the fixed points $\theta = 0$ and $\theta = \pi$ must be observed. Specifically, for real transfer functions, we require that any consistent magnitude $A(\theta)$ function should be such that at $\theta = 0$ and π either $A(\theta) = 0$ or $dA(\theta)/d\theta = 0$ or both of these conditions are jointly satisfied [64]. In the unspecified regions, we have the freedom to insert any response consistent with *a priori* requirements on it being a magnitude function. The first stage of the design objective is to estimate the appropriate root moments and then proceed along the same lines as above.

1) No Insertions Allowed: In this case, only the measured data and the fixed-point conditions are allowed to be used. We interpret, therefore, the fundamental relationship of (26) as an interpolating formula that can be evaluated at the fixed and measured points. The unknown parameters are the root moments. Since the companion filter is minimum phase, its root moments decay exponentially and hence a finite number of terms N in the summation need be taken. The interpolation formula (26) evaluated at the overall frequency specifications yields a set of linear equations in the unknown first N root moments. Matrix inversion yields these and, hence, the FIR form of the transfer function as in (13). From this point, we proceed along the same path as above. Such an approach is likely to produce substantial deviations due to the poor sampling process involved particularly in relation to the unspecified regions.

2) Insertions Allowed: The introduction of the missing parts of the response is essentially a degree of freedom that can be used to advantage in the design. It can be done in a variety of ways. Indeed, an optimization framework can be adopted in which we seek to determine the optimum insertions so as to minimize the order of the transfer function. This approach is taken in Example 1 in which a simple constraint on the gain outside the regions of measurements should be below a certain value. This is a case of importance in many applications that are gain limited. The more general case of optimal *functional* insertion is beyond the scope of the present paper and is the subject of further study. It is clear from Fig. 1 that even a simple nonoptimal insertion in the unspecified regions improves matters considerably. The insertion need only conform to the magnitude requirements, but a more reasoned approach would also ensure that it be continuous in frequency.

There remains the question regarding the way that the orders of the transfer function and of the required all-pass filters are selected. These points are examined below.

VI. Order Selection

In this section, we deal with the following problems:

1) given a minimum-phase FIR filter transfer function to estimate a lower bound on the order of a minimum-phase IIR transfer function that has both its numerator and denominator of the same order;
2) given the phase response of a stable all-pass filter to determine its order.

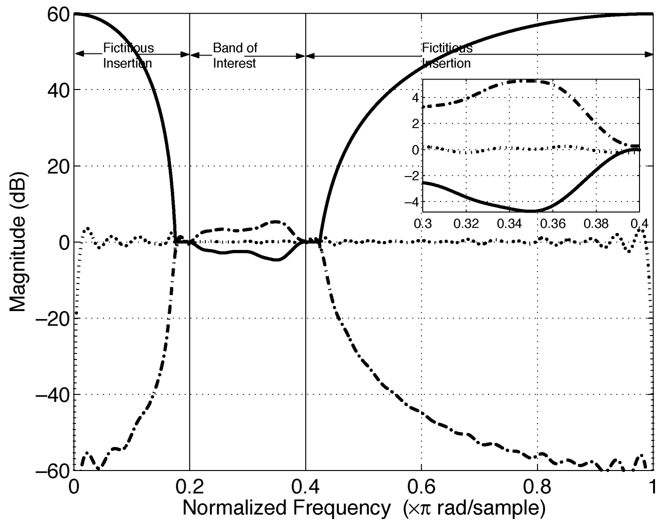

Fig. 1. Optical equalizer example. Dotted line: Desired magnitude response. Response in $[0, 0.2\pi]$ and $[0.4\pi, \pi]$ is fictitious. Dashed–dotted line: Achieved sum of two all-pass magnitude response. Solid line: Equalized magnitude response.

The first problem is approached as follows.

A. A Bound on Order

We note that the root moments $S_m^{\rm FIR}$ of the given FIR transfer function and of the IIR transfer function will be related as $S_m^{\rm FIR} = S_m^{\rm num} - S_m^{\rm den}$. Thus, we can write

$$\left|S_m^{\rm FIR}\right| \leq \left|S_m^{\rm num}\right| + \left|S_m^{\rm den}\right|. \tag{55}$$

However, since the IIR filter is minimum phase and has the same order for numerator and denominator, the above inequality becomes

$$\left|S_m^{\rm FIR}\right| \leq n\rho_{\rm num}^m + n\rho_{\rm den}^m \tag{56}$$

where $\rho_{\rm num}$ and $\rho_{\rm den}$ are the largest magnitudes of the roots of the numerator and denominator respectively. On taking these equal to unity, we have

$$n > \max\frac{\left|S_m^{\rm FIR}\right|}{2}. \tag{57}$$

The above gives a means of placing a lower bound on the order of the companion filter. In normal filter responses with significant undulations in the magnitude, it is found that the bound is rather conservative and a more representative value for the general IIR case is about twice the value given in inequality (57). For the specific case under consideration, this bound also tends to be rather conservative. However, a local search can be put into effect, if desired, to improve the quality of approximation and is an area for further exploration.

B. Order of All-Pass From Phase Response

The second problem, concerned with the order estimation of an all-pass given its phase response, is more easily and accurately tractable. Clearly, the procedures based on magnitude response are inapplicable [45], [47], [52], [65], [66]. The unwrapped phase response in the fundamental relationships of (54), evaluated at $\theta = \pi$, yields immediately the all-pass order as

$$M = -\frac{\phi(\pi)}{\pi}. \tag{58}$$

This is also a consequence of the argument principle in complex variable theory, and it essentially corresponds to the winding number of the FIR transfer function [67], [68].

The phase response of an all-pass transfer function also furnishes appropriate bounds [68] as indicated below.

C. Bound on Order From Root of Maximum Modulus

The root with the largest modulus can furnish a bound on the order as follows [68]. We assume that we are examining a minimum-phase polynomial transfer function for which have its root moments. These moments may have been derived from either the gain or the phase specifications and can be used in the manner indicated in [40]. In this context, a restricted form of second-order root moments is employed to determine the sum and product of the roots that have the largest modulus. An iterative procedure leads to the modulus of the complex root with the largest modulus. Let this modulus be $\rho_{\max}$, and for the problem under examination, let us examine the phase fundamental relationship for minimum-phase systems. We can write, therefore, the following:

$$|\phi(\theta)| \leq \sum_{m=1}^{\infty} \frac{\left|S_m^{\rm FIR}\right|}{m} |\sin m\theta|. \tag{59}$$

If we set $|S_m^{\rm FIR}| \leq n\rho_{max}^m$, where n is the estimate of the FIR order, then we have

$$n \geq \frac{\phi_{\max}}{\ln(1+\rho_{\max})} \tag{60}$$

where $\phi_{\max}$ is the maximum value of the minimum-phase function. It is interesting also to note that a bound on this maximum value can be placed from (59), under the assumption that the root moments are given, as

$$\phi_{\max} \leq \sum_{m=1}^{\infty} \frac{\left|S_m^{\rm FIR}\right|}{m}. \tag{61}$$

This expression may be easily evaluated over a finite number of terms since the root moments decay exponentially with the summation index.

There is an alternative, and more general route, to estimating the order, which relies on the estimation of the order of a polynomial from its root moments [68]. The root moments of the denominator of an all-pass filter are directly derived from (54), and the problem can now be posed as indicated below.

D. Order From Root Moments

The problem can be specified as follows: Given the root moments of a minimum-phase polynomial in z^{-1} estimate the order of the polynomial. The various ramifications and solutions to it are dealt with in a forthcoming paper [68]. The pivotal observation is made that if the order is n, then the Newton Identities alone can yield the root moment S_{n+1}, obtainable from all the previous root moments $[S_1, S_2, S_3, \ldots, S_n]$ and the estimated polynomial coefficients $[1, p_1, p_2, p_3, \ldots, p_n]$. The polynomial coefficients up to index n are also evaluated recursively from the Newton Identities. Thus, a complete enumeration procedure can be set up starting with $n = 1$ and then checking the estimated next root moment against the value obtained from the specifications. If there is correspondence between these two values, then the order has been determined. In practice, it is found that we need to test the correspondence for more than one value between the estimated and evaluated root moments [68].

VII. Examples

There are many problems in signal processing where approximation of magnitude response is of paramount importance, typically in the equalization of a communication channel, which is to approximate to the inverse of the channel magnitude response. We consider three design examples. The first one arises in optical signal processing. The specific case we have taken is from optical signal processing for the design of Mach–Zehnder equalizers [71]–[73].

Two further design examples are given below. The first example is to approximate a complete magnitude response given as a set of points. The points are sampled from a thirtieth-order FIR bandpass filter frequency response. A portion of the magnitude response is also taken for the partial case which serves as the second example.

A. Example 1

This is an example that has arisen from a real problem in optical system equalization. Measurements on the performance of an optical system produce a variation in the gain and phase over a broad range of wavelengths. The design requirement is to produce an equalizer of the Mach–Zehnder form to correct the performance. The Mach–Zehnder structure is the optical equivalent of the sum-of-all-pass structure we consider in this paper. The measurements are normalized and shown in Fig. 1. The measurements, and hence the design specifications, are limited over a portion of the frequency axis. We introduce in the solution artificial behavior in the unspecified regions. However, this is done in such a way as to ensure that the system would not require further optical amplification, this being a design requirement. The two all-pass structures are found to be of order 24 and the equalized responses are also shown in Fig. 1.

B. Example 2

In this example, the magnitude response to be approximated is bandpass with its passband located in the range 0.325π–0.525π and its stopbands are $0 - 0.2\pi$ and $0.65\pi - \pi$. The passband attenuation is less than 0.3 dB and the stopband attenuation better than 60 dB. The result of the design using the proposed method is shown in Fig. 2. Both all-pass filters are found to be of order 18. It is noted that since the frequency response of a bandpass filter at $\theta = 0$ and $\theta = \pi$ must have high attenuation, it is the difference of two all-pass filters that must be considered rather than their sum. Another point to be observed is that as a preamble to the sum-of-all-pass procedure, one can design another structure, say an FIR of large order, which is then reduced to the required form by the proposed method. This is an area worth further investigation.

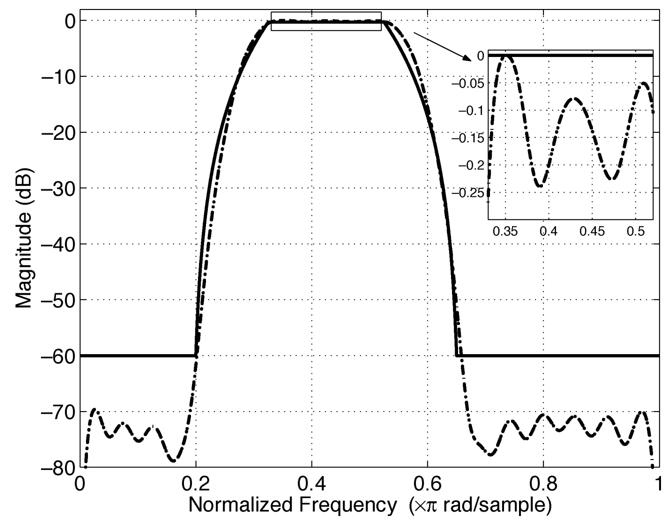

Fig. 2. Complete magnitude approximation. Solid line: Desired magnitude requirements. Dashed–dotted line: Achieved sum of two all-pass magnitude response. Both all-pass filters are of order 18.

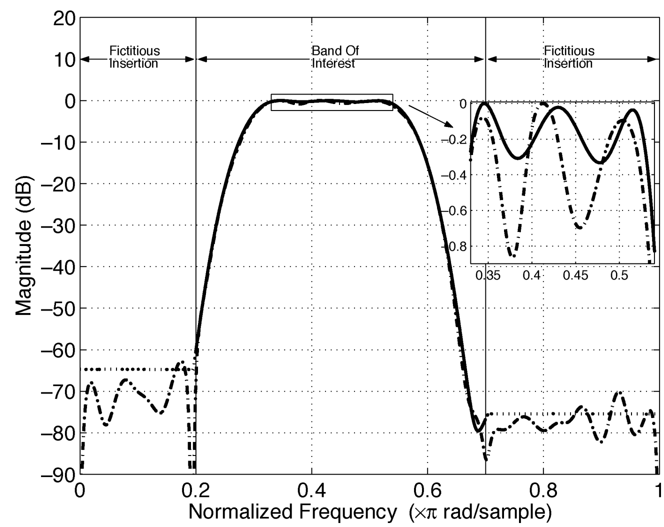

Fig. 3. Partial magnitude approximation using matrix inversion. Solid line: Desired partial magnitude response in the frequency range $[0.2\pi, 0.7\pi]$. Dashed–dotted line: Achieved sum of two all-pass magnitude response. Both all-pass filters are of order 20. Dotted line: Complete magnitude response from matrix inversion.

C. Example 3

The magnitude response to be approximated in this example is a central fragment of the desired magnitude in Fig. 2 located in the frequency range $[0.2\pi, 0.7\pi]$. Two alternate approaches given above are employed. The first approach is to use the matrix inversion to obtain S_m^{FIR}. The result is shown in Fig. 3. The second approach is to add some fictitious magnitude response in the "don't care regions." The results are shown in Fig. 4.

VIII. Conclusion

A framework is presented to the design of digital filters that have transfer functions expressed as the sum of two all-pass

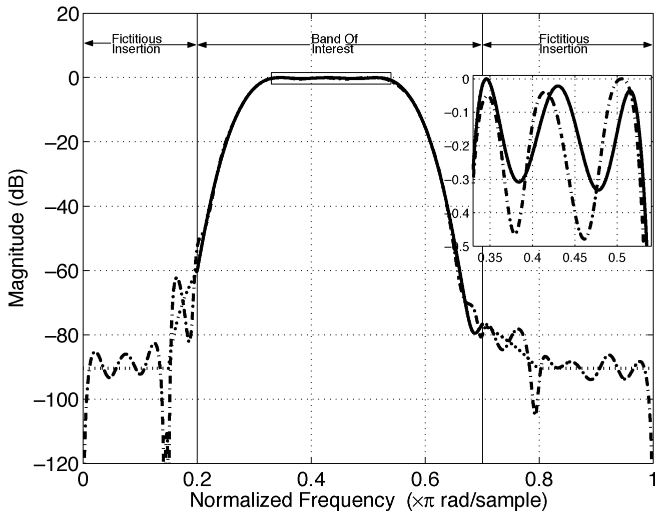

Fig. 4. Partial magnitude approximation by adding fictitious magnitude. Solid line: Desired partial magnitude response in the frequency range $[0.2\pi, 0.7\pi]$. Dashed–dotted line: Achieved sum of two all-pass magnitude response. Both all-pass filters are of order 20. Dotted line: Complete magnitude recovered by adding fictitious magnitude.

functions. The problem has been examined over the years from various perspectives and the one adopted in this paper is based upon the concept of root moments. By suitable use of such parameters, we have been able to determine the phase responses of the all-pass filters and their required orders and to put into effect designs from partial gain specifications. In the process of developing the techniques, we have derived some significant properties of the sum-of-all-pass structures and have indicated corrections or improvements to earlier publications that appeared in the open literature. Some examples are given to illustrate the design procedure. The software developed for this purpose has been written in Matlab and will be made downloadable from the website http://www.commsp.ee.ic.ac.uk/people/agc.html.

REFERENCES

[1] A. Fettweis, "Digital filter structure related to classical filter networks," *Arch. Elektr. Uebertrag.*, vol. 25, pp. 78–89, 1971.

[2] B. Gold and C. Rader, *Digital Processing of Signals*. London, U.K.: McGraw-Hill, 1969.

[3] A. G. Constantinides, "Digital notch filters," *Electron. Lett.*, vol. 5, pp. 198–199, 1969.

[4] R. A. Valenzuela, "Techniques for transmultiplexer design," Ph.D. dissertation, Electrical Engineering Dept., Imperial College London, London, U.K., Jun. 1982.

[5] R. A. Valenzuela and A. G. Constantinides, "Digital signal processing schemes for efficient interpolation and decimation," in *Proc. Inst. Elect. Eng.—Electronic Circuits Systems*, vol. 130, 1983, pp. 225–235.

[6] P. Vaidyanathan, P. Regalia, and S. Mitra, "Design of doubly-complementary IIR digital filters using a single complex allpass filter, with multirate applications," *IEEE Trans. Circuits Syst.*, vol. 34, no. 4, pp. 378–389, Apr. 1987.

[7] D. Harris, "Design of stable all-pass filters," in *IEEE Int. Conf. Acoustics, Speech, Signal Processing*, vol. 4, Apr. 1979, pp. 813–817.

[8] F. J. Harris and A. G. Constantinides, "On the requantization of data to implement digital filters with reduced resolution arithmetic," in *Proc. Int. Symp. Signal Processing Applications (ISSPA)*, 1989, pp. 25–28.

[9] T. Saramaki and M. Renfors, "A Remez-type algorithm for designing digital filters composed of all-pass sections based on phase approximations," in *Proc. 38th Midwest Symp. Circuits Systems*, Aug. 13–16, 1995, pp. 571–575.

[10] T. Saramaki and M. Bhattacharya, "Multiplierless realization of recursive digital filters using allpass structures," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing*, vol. 2, Apr. 6–10, 2003, p. II-8.

[11] K. Rajamani and Y. S. Lai, "A novel method for designing allpass digital filters," *IEEE Signal Process. Lett.*, vol. 6, pp. 207–209, Aug. 1999.

[12] M. Lang, "Allpass filter design and applications," *IEEE Trans. Signal Process.*, vol. 46, no. 9, pp. 2505–2514, Sep. 1998.

[13] K. C. Haddad, Y. Y. Yang, N. P. Galatsanos, and H. Stark, "Allpass filter design using projection-based method under group delay constraints," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing*, vol. 6, May 7–11, 2001, pp. 3833–3836.

[14] G. Stancic and B. Djuric, "Design of IIR digital filters using allpass networks," in *Proc. 4th Int. Conf. Telecommunications Modern Satellite, Cable Broadcasting Services*, vol. 1, Oct. 13–15, 1999, pp. 145–148.

[15] A. G. Constantinides and R. A. Valenzuela, "Transmultiplexing: Factors influencing the design of TDM to FDM conversion," in *Proc. Int. Conf. Information Control (INFO II 1979)*, 1980, pp. 260–263.

[16] ——, "A new recursive interpolator with applications in transmultiplexer design," presented at the Int. Conf. Signal Processing, Florence, Italy, Sep. 2–5, 1981.

[17] ——, "A new low complexity modular transmultiplexer design," in *Proc. IEEE Int. Conf. Communications*, vol. 18, 1981, pp. 1–5.

[18] H. K. Kwan, "Improved systolic allpass digital filters for very high-speed applications," *Electron. Lett.*, vol. 28, no. 22, pp. 2061–2063, 1992.

[19] M. Renfors, Y. Neuvo, and S. Mitra, "A new class of uniform filter banks based on recursive Nth-band filters," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing*, vol. 11, Apr. 1986, pp. 2555–2558.

[20] P. A. Regalia, S. K. Mitra, and P. P. Vaidyanathan, "The digital all-pass filter: A versatile signal processing building block," *Proc. IEEE*, vol. 76, pp. 19–37, Jan. 1988.

[21] T. I. Laakso, T. Q. Nguyen, and R. D. Koilpillai, "Designing allpass filters using the eigenfilter method," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing*, vol. 3, Apr. 27–30, 1993, pp. 77–80.

[22] A. Kanellakis, P. Agathoklis, and A. Tawfik, "A note on the orthogonal realization of allpass digital filters and their finite wordlength performance," *IEEE Trans. Signal Process.*, vol. 44, no. 10, pp. 2616–2619, Oct. 1996.

[23] B. Nowrouzian, "A novel approach to the exact design of the follow-the-leader LDI digital allpass networks," in *Proc. 35th Midwest Symp. Circuits Systems*, vol. 2, Aug. 9–12, 1992, pp. 788–791.

[24] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[25] C. K. Lu, S. Summerfield, and M. Anderson, "A new pipelined architecture for allpass digital filters based on the 3-port adaptor," in *Proc. IEEE Workshop Signal Processing Systems Design Implementation*, Nov. 3–5, 1997, pp. 431–437.

[26] P. Lowenborg, H. Johansson, and L. Wanhammar, "A class of two-channel hybrid analog/digital filter banks," in *Proc. 42nd Midwest Symp. Circuits Systems*, vol. 1, Aug. 8–11, 1999, pp. 14–17.

[27] S. C. Chan, J. S. Mao, and K. L. Ho, "A new design method for two-channel perfect reconstruction IIR filter banks," *IEEE Signal Process. Lett.*, vol. 7, no. 8, pp. 221–223, Aug. 2000.

[28] H. H. Dam, S. Nordebo, and L. Svensson, "Design of minimum-phase digital filters as the sum of two allpass functions using the cepstrum technique," *IEEE Trans. Signal Process.*, vol. 51, no. 3, pp. 726–731, Mar. 2003.

[29] B. Jaworski and T. Saramaki, "Linear phase IIR filters composed of two parallel allpass sections," in *Proc. IEEE Int. Symp. Circuits Systems*, vol. 2, May 30–Jun. 2 1994, pp. 537–540.

[30] S. Lawson, T. Wicks, and A. Jones, "Design and implementation of cascaded allpass digital filter structures with magnitude and delay constraints," in *Proc. 6th Int. Conf. Digital Processing Signals in Communications*, Sep. 2–6, 1991, pp. 31–35.

[31] M. M. Doss, "Linear phase bandpass sampled-data filters," in *Proc. 1995 20th Int. Conf. Microelectronics*, vol. 2, Sep. 12–14, 1999, pp. 736–739.

[32] I. Tabus, C. Popeea, and J. Astola, "Optimizing the compaction gain in a class of IIR filters," in *Proc. IEEE Int. Symp. Circuits Systems*, vol. 3, May 30–Jun. 2 1999, pp. 528–531.

[33] C. C. Tseng and S. C. Pei, "Complex notch filter design using allpass filter," *Electron. Lett.*, vol. 34, no. 10, pp. 966–967, 1998.

[34] S. C. Pei and C. C. Tseng, "IIR multiple notch filter design based on allpass filter," *IEEE Trans. Circuits Systems II, Analog Digit. Signal Process.*, vol. 44, no. 2, pp. 133–136, Feb. 1997.

[35] K. Martin and M. Padmanabhan, "Lerner-based nonuniform filter-banks and some of their applications," in *Proc. IEEE Int. Symp. Circuits Systems*, vol. 2, May 3–6, 1992, pp. 1003–1006.

[36] T. J. Lim and M. D. Macleod, "Adaptive allpass filtering for nonminimum-phase system identification," in *Proc. Inst. Elect. Eng. Vision Image Signal Processing*, vol. 141, Dec. 1994, pp. 373–379.

[37] J. A. Chambers and A. G. Constantinides, "Frequency tracking using constrained adaptive notch filters synthesised from allpass sections," in *Proc. Inst. Elect. Eng. Radar Signal Processing*, vol. 137, Dec. 1990, pp. 475–481.

[38] E. T. Whittaker, *The Calculus of Observations*, 2nd ed. Glasgow, U.K.: Blackie, 1929.
[39] A. G. Constantinides and W. M. Li, "A framework for the design of digital filters and equalizers from partial specifications based on root moments," *IEEE Trans. Signal Process.*, 2004, submitted for publication.
[40] T. Stathaki, "Cumulant-based and algebraic techniques for signal modeling," Ph.D. dissertation, Electrical Engineering Dept., Imperial College London, London, U.K., 1994.
[41] T. Stathaki and A. G. Constantinides, "Root moments: An alternative interpretation of cepstra for signal feature extraction and modeling," in *1995 Conf. Rec. 29th Asilomar Conf. Signals, Systems, Computers*, vol. 2, Oct. 30–Nov. 1 1995, pp. 1477–1481.
[42] D. S. Maric, S. Hiti, C. C. Stancu, D. B. Nagashima, and J. M. Rutledge, "An application of a constrained adaptive lattice-structure allpass-based notch filter for advanced control of surface-mounted permanent-magnet synchronous drives," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 46, no. 12, pp. 1513–1516, Dec. 1999.
[43] A. Fabre, F. Martin, and M. Hanafi, "Current mode allpass/notch and bandpass filters with reduced sensitivities," *Electron. Lett.*, vol. 26, no. 18, pp. 1495–1496, 1990.
[44] A. G. Constantinides and R. A. Valenzuela, "A new class of efficient interpolators and decimators with applications in transmultiplexing," in *Proc. Int. Symp. Circuits Systems*, Rome, Italy, May 10–12, 1981, pp. 260–263.
[45] S. K. Mitra, *Digital Signal Processing: A Computer-Based Approach*. New York: McGraw-Hill, 1998.
[46] W. M. Li, "Root moments and digital filter design," Ph.D. dissertation, Electrical and Electronic Engineering Dept., Imperial College London, London, U.K., 2005.
[47] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*. London, U.K.: Clarendon Press, Sep. 2002.
[48] I. Fotinopoulos, T. Stathaki, and A. G. Constantinides, "A method for FIR filter design from joint amplitude and group delay characteristics," in *Conf. Rec. 35th Asilomar Conf. Signals, Systems Computers*, vol. 1, 2001, pp. 621–625.
[49] I. Fotinopoulos and T. Stathaki, "Equiripple minimum phase FIR filter design from linear phase systems using a novel technique for polynomial factorization," in *IEEE Global Telecommunications Conf. 2000 (IEEE GLOBECOM)* , vol. 3, Dec. 2000, pp. 1659–1663.
[50] T. Stathaki, "Root moments: a digital signal-processing perspective," *Proc. Inst. Elect. Eng. Vision, Image, Signal Processing*, vol. 145, pp. 293–302, Aug. 1998.
[51] A. Polydoros and A. T. Fam, "The differential cepstrum definition and properties," in *Proc. IEEE Int. Symp. Circuits Systems*, Apr. 1981, pp. 77–80.
[52] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1999.
[53] A. G. Constantinides and W. M. Li, "Root moments and minimum phase FIR filter design," *IEEE Trans. Signal Process.*, 2004, submitted for publication.
[54] T. Stathaki, A. G. Constantinides, and G. Stathakis, "Minimum phase FIR filter design from linear phase systems using root moments," in *Proc. 1998 IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, vol. 3, May 12–15, 1998, pp. 1317–1320.
[55] T. Stathaki and I. Fotinopoulos, "Equiripple minimum phase FIR filter design from linear phase systems using root moments," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 48, no. 6, pp. 580–587, Jun. 2001.
[56] I. Todhunter, *Theory of Equations*. London, U.K.: McMillan, 1882.
[57] A. G. Constantinides and H. G. Dimopoulos, "Active *RC* filters derivable from *LC* ladder filters via linear transformations," *Electron. Circuits Syst.*, vol. 1, pp. 17–21, 1976.
[58] N. Balabanian and T. A. Bickart, *Electrical Network Theory*. New York: Wiley, 1969.
[59] V. Valkenburg, *Introduction to Modern Network Synthesis*. New York: Wiley, 1960.
[60] V. Belevitch, *Classical Network Theory*. San Francisco, CA: Holden-Day, 1968.
[61] A. N. Willson and H. J. Orchard, "Insights into digital filters made as the sum of two allpass functions," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 42, no. 3, pp. 129–137, Mar. 1995.
[62] S. R. Pillai, "Design of three channel complementary filters as a weighted sum of allpass sections," *Electron. Lett.*, vol. 29, pp. 719–721, Apr. 1993.
[63] I. W. Selesnick, "Lowpass filters realizable as coupled allpasses: Design via a new flat delay filter," in *Proc. IEEE Symp. Advances Digital Filtering Signal Processing*, Jun. 5–6, 1998, pp. 135–138.
[64] A. G. Constantinides, Digital signal processing and digital filters, in M.Sc. Course Lecture Notes, Electrical Electronics Engineering Dept., Imperial College London, London, U.K., 2004.
[65] F. Brglez, "Digital filter design with short word-length coefficients," *IEEE Trans. Circuits Syst.*, vol. 25, no. 12, pp. 1044–1050, Dec. 1978.
[66] H. K. Kwan, "Tunable bandpass passive digital filter design," in *Proc. 2002 45th Midwest Symp. Circuits Systems*, vol. 3, Aug. 2002, pp. III-481–III-484.
[67] Z. Nehari, *Conformal Mapping*. New York: Dover, 1975.
[68] A. G. Constantinides and W. M. Li, "An algebraic approach to the estimation of the order of FIR filters from complete and partial magnitude and phase specifications," *IEEE Trans. Signal Process.*, 2004, submitted for publication.
[69] A. G. Constantinides, "Frequency functions—A class of real functions lowpass digital filters having equiripple passbands," in *Proc. IEEE Int. Symp. Circuit Theory*, 1970, pp. 134–135.
[70] S. U. Pillai and T. I. Shim, *Spectrum Estimation and System Identification*. New York: Springer, 1993.
[71] C. K. Madsen and J. H. Zhao, *Optical Filter Design and Analysis: A Signal Processing Approach*, ser. Wiley Microwave and Optical Engineering. New York: Wiley, Jul. 1999.
[72] C. K. Madsen, E. J. Laskowski, J. Bailey, M. A. Cappuzzo, S. Chandrasekhar, L. T. Gomez, A. Griffin, P. Oswald, and L. W. Stulz, "The application of integrated ring resonators to dynamic dispersion compensation," in *Proc. IEEE/LEOS All-Optical Networking*, Jul. 15–17, 2002, p. 2–29.
[73] C. K. Madsen, "Integrated waveguide allpass filter tunable dispersion compensators," in *Proc. Optical Fiber Communication Conf. Exhibit*, Mar. 17–22, 2002, pp. 131–132.

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