Paradox as Basis of Science

Wasaburo Unno
RIST, Kinki University, Higashi-Osaka-shi, Osaka-577, Japan

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Abstract

Olbers' paradox is known to have profound relation with the cosmological model, viz., the expanding universe and the cosmological large-scale fractal structure. A well-known mathematical paradox that the length of two sides of a triangle is equal to that of the remaining side is considered to be an good introduction of the fractal geometry. A line segment of an infinite length can be constructed in the same way as the triangle paradox is constructed.

Key Words: Paradox; fractal geometry.

1 Introduction

Achilles cannot overtake a turtle because the turtle goes ahead of Achilles by some distance when Achilles reaches the previous turtle's position. This is a well-known paradox which can be resolved in terms of the concept of uniform convergence, because successive intervals of time and position become less and less exponentially so that the total time lapse under consideration is set to be finite from the beginning. It is, therefore, recognized as an error in logic by means of insufficient assumptions. In any case, the paradox is a useful tool to introduce the concept of differentiation in high school education. Likewise, any paradox may provide a productive medium for a corresponding science to grow. We will consider the case of Olbers' paradox, and then discuss the paradox of a line segment of infinite length of which the application is not yet known.

2 Olbers' Paradox

A star has a finite size. Angular size of a star is therefore finite and is inversely proportional to square of the distance of the star from an observer. Since number of stars increases proportionally to square of the distance, the product is finite, and the total area covered by stars become infinite, when integrated over infinite distance. This means that the earth is surrounded by the wall of furnace composed of stellar surfaces in all directions. Radiations emitted from stellar surface will be absorbed but reemitted in the local black body temperature again by another stellar surface. The result is then that the temperature of the interstellar space as well as the stellar surface temperature becomes the same and all lives on the earth will die out (evaporated) with heat.

There are two ways to resolve this Olbers' paradox. One is the expanding universe and the other is the fractal structure of the universe. Radiations emitted from a star in a distant galaxy is red-shifted to lower energies when received on the earth so that night sky can be dark. The application of this interpretation, however, is realistic for the cosmic 3K black body radiation which was emitted by 3000K plasma at the epoch of matter-radiation decoupling at $z(=\delta\lambda/\lambda) = 1000$. The darkness of optical night sky is more properly interpreted by the large scale fractal structure at a distance scale of $z = 1$. This is the concept of the Charliér universe which is composed of hierarchical clusterings of stars and star-clusters up to super- super-clusters of galaxies. The star-filling factor of space becomes less and less as larger and larger volume is concerned. The result is that only a very small portion of the sky is covered by all

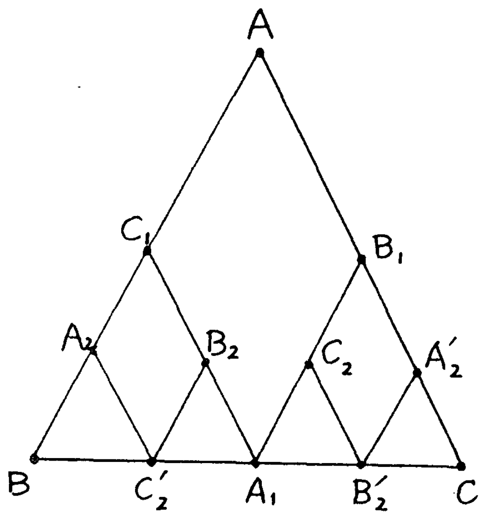

Figure 1: Paradox: $2 = 1$

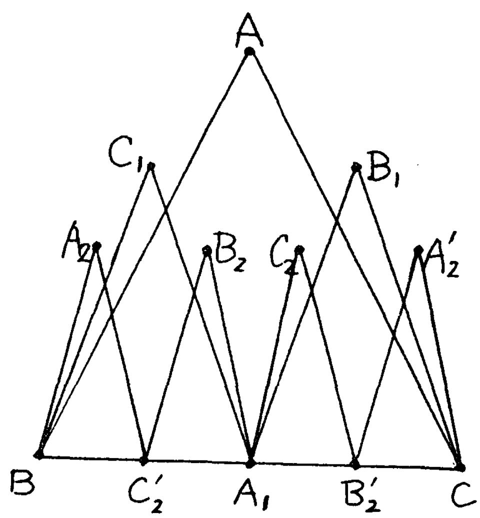

Figure 2: Paradox: $\infty = 1$

the stellar images.

Olbers' paradox is thus considered to be closely related to the structure of the universe. The roll of a paradox for the basic scientific concept can be appreciated also in this context.

3 The Paradox of a Triangle and its Extension

The total length of two sides of a triangle is equal to the length of the remaining side. The proof of this paradox goes as follows. Let ABC denote a triangle in Figure 1. Let A_1, B_1, and C_1 be the mid-points on the sides opposite to A, B, and C. Likewise, A_2, A_2', B_2, B_2', C_2, and C_2' are defined in Figure 1. Evidently, the length of line $BC_1A_1B_1C$ is equal to that of line BAC, and so is the length of line $BA_2C_2'B_2A_1C_2B_2'A_2'C$. Repeating this procedure infinite times, we find the total length of the zig-zag line is the same as the original length which is 2 in case of a right triangle. At the same time, height of peak points of a zig-zag line comes down to half of that of the previous zig-zag line so that all the peak points converge to the straight line BC. We have, therefore, a straight line segment of length 2 coinciding with a straight line segment of length 1, in contradiction with the axiom of Euclidean geometry which we have employed. This is a well-known paradox (see, e.g. Yamaguchi,1986) but I don't know who invented it. Extending this idea, we can easily prove a more extreme paradox saying that infinity is equal to unity. Let the height of C_1 and B_1 be somewhat larger than half the height of A, let the height of A_2, B_2, C_2, and A_2' be somewhat larger than half the height of C_1 and B_1, and so on, as shown in Figure 2. Then, the limit of the lengths of the zig-zag lines in this case will be infinite while the limiting zig-zag line will again coincide with the straight line segment BC. Apparently, the interpretation of this paradox is the same as that of the paradox of infinite length of British coast line discussed by Mandelbroe(1982). To cover the n-th zig- zag line by a sphere of radius of $(1/2)^n$, we need roughly $[2(1+f)]^n$ spheres, where $(1+f)/2$ is the ratio of the heights of peaks of successive zig-zag lines. The limiting zig-zag straight line has a fractal dimension of $1 + \log(1+f)/\log 2$ for $0 < f < 1$, and therefore it is not an ordinary line. If the height of peaks is less than half of that of the previous line ($f < 0$), the limiting line will be simply BC, a line of dimension 1. The case of $f = 0$ is intermediate but is peculiar in the sense that the length is 2 of dimension 1. The discontinuity of the length with variable f-value at $f = 0$ is noteworthy.

4 Wave of Infinitesimal Wavelength

Line BAC in Figure 2 can be made smooth and fit with a sine- curve of half wavelength $\lambda/2$. Then, take the mirror image of B_1 with respect to BC and draw a sine curve of wavelength $\lambda/2$ through $BC_1A_1B_1C$. The amplitude of the second sine curve is taken to be larger than half of that of the

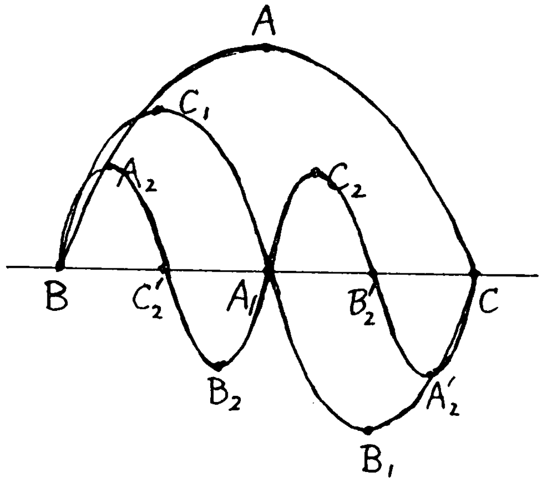

Figure 3: Paradox: $\infty = 1$

original one. If we continue this procedure indefinitely, we will have a sine curve of which both the wave length and the amplitude tend to be zero but the total length becomes infinite, (Figure 3). Thus, we have again an apparent straight line segment whose length is infinite. The de Broglie wave length, λ, of a particle of mass m moving with speed c is given by

$$\lambda = h/mc \tag{1}$$

The energy, $E = h\nu = hc/\lambda = mc^2$, of the wave motion transverse to the direction of propargation is Ka^2, where ν denotes the frequency, a the wave amplitude and K the constant. Consider the n-th particle or wave of which wave length $\lambda^{(n)}$ is half of the wave length $\lambda^{(n-1)}$ of the (n-1)-th wave and the amplitude $a^{(n)}$ is $(2)^{-1/2}$ times the amplitude $a^{(n-1)}$. In this case, since $E^{(n)} = 2E^{(n-1)}$, the oscillation velocity $V^{(n)}$ of the medium is $(2)^{1/2}$ of $V^{(n-1)}$, and the frequency, $\nu = V/a$, satisfies the relation: $\nu^{(n)} = 2\nu^{(n-1)}$, in agreement with the relation $E = h\nu$.

Therefore, a particle can be considered to be an oscillation of vacuum of which the wave form is shown as the limiting curve in Figure 3. If the Planck constant h is taken to be zero, then the wave length λ of a particle becomes zero and the length of the sine curve goes to infinity. To avoid the infinity, the Planck constant must be finite. The advantage of the above picture of a particle lies in the fact that we do not need to invoke on the complementarity of wave and particle, although the latter picture are proved to be very useful.

At present, the paradox of pseudo line having infinite length on finite line segment has no scientific meaning. However, it is a pleasant thought to consider that such a paradox may help the birth of a new science in future.

References

[1] Mandelbrot, B.B., 1982, *The Fractal Geometry of Nature*, W.H. Freeman Co., p.25.

[2] Yamaguchi, Y., 1986, *Chaos and Fractal* (in Japanese), Blue Backs, Kodansha, p.144.