均衡型 C_4-Bowtie デザイン

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Balanced C_4-Bowtie Designs

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In graph theory, the decomposition problems of graphs are very important topics. Various types of decompositions of many graphs can be seen in the literature of gaph theory.
We show that the necessary and sufficient condition for the existence of a balanced C_4-bowtie decomposition of the complete multi-graph λK_n is $\lambda(n-1) \equiv 0 \pmod{16}$ and $n \geq 7$.
This decomposition is called a balanced C_4-bowtie design.

Key words: Balanced C_4-bowtie decomposition, Complete multi-graph, Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. *The complete multi-graph* λK_n is the complete graph K_n in which every edge is taken λ times. Let C_4 be *the cycle* on 4 vertices. *The C_4-bowtie* is a graph of 2 edge-disjoint C_4's with a common vertex and the common vertex is called *the center of the C_4-bowtie.*
When λK_n is decomposed into edge-disjoint sum of C_4-bowties, we say that *λK_n has a C_4-bowtie decomposition.* Moreover, when every vertex of λK_n appears in the same number of C_4-bowties, we say that *λK_n has a balanced C_4-bowtie decomposition* and this number is called *the replication number.* This balanced C_4-bowtie decomposition of λK_n is called *a balanced C_4-bowtie design.*

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_4-bowtie decomposition of λK_n is $\lambda(n-1) \equiv 0 \pmod{16}$ and $n \geq 7$.

2. Balanced C_4-bowtie decomposition of λK_n

We use the following notation for a C_4-bowtie.

Notation. We denote a C_4-bowtie passing through $v_1 - v_2 - v_3 - v_4 - v_1 - v_5 - v_6 - v_7 - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7)\}$.

We have the following theorem.

Theorem 1. If λK_n has a balanced C_4-bowtie decomposition, then
$\lambda(n-1) \equiv 0 \pmod{16}$ and $n \geq 7$.

Proof. Suppose that λK_n has a balanced C_4-bowtie decomposition. Let b be the number of C_4-bowties and r be the replication number. Then $b = \lambda n(n-1)/16$ and $r = 7\lambda(n-1)/16$. Among r C_4-bowties having a vertex v of λK_n, let r_1 and r_2 be the numbers of C_4-bowties in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4r_1 + 2r_2 = \lambda(n-1)$. From these relations, $r_1 = \lambda(n-1)/16$ and $r_2 = 3\lambda(n-1)/8$. Thus, $\lambda(n-1) \equiv 0$ (mod 16). Since a bowtie is a subgraph of λK_n, $n \geq 7$.

Note. The condition $\lambda(n-1) \equiv 0$ (mod 16) and $n \geq 7$ in Theorem 1 can be classified as follows: (i) $n \equiv 1$ (mod 16), $n \geq 17$ for $\lambda \equiv 1, 3, 5, 7, 9, 11, 13, 15$ (mod 16), (ii) $n \equiv 1$ (mod 8), $n \geq 9$ for $\lambda \equiv 2, 6, 10, 14$ (mod 16), (iii) $n \equiv 1$ (mod 4), $n \geq 9$ for $\lambda \equiv 4, 12$ (mod 16), (iv) $n \equiv 1$ (mod 2), $n \geq 7$ for $\lambda \equiv 8$ (mod 16), and (v) $n \geq 7$

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for $\lambda \equiv 0 \pmod{16}$.

We have the following theorem.

Theorem 2. If λK_n has a balanced C_4-bowtie decomposition, then $s\lambda K_n$ has a balanced C_4-bowtie decomposition for every s.

Proof. Obvious. Repeat s times the balanced C_4-bowtie decomposition of λK_n.

We have the following theorems.

Theorem 3. When $n \equiv 1 \pmod{16}$ and $n \geq 17$, λK_n has a balanced C_4-bowtie decomposition for every λ.

Proof. Put $n = 16t + 1$. Construct tn C_4-bowties as follows:
$\{(i, i+1, i+10t+2, i+8t+1), (i, i+2, i+10t+4, i+8t+2)\}$,
$\{(i, i+3, i+10t+6, i+8t+3), (i, i+4, i+10t+8, i+8t+4)\}$,
...
$\{(i, i+2t-1, i+14t-2, i+10t-1), (i, i+2t, i+14t, i+10t)\}$ $(i = 1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of K_n. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Note. We consider the vertex set V of λK_n as $V = \{1, 2, ..., n\}$. The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Theorem 4. When $\lambda \equiv 0 \pmod 2$, $n \equiv 1 \pmod 8$ and $n \geq 9$, λK_n has a balanced C_4-bowtie decomposition.

Proof. We consider 2 cases.
Case 1. $n = 9$. Construct 9 C_4-bowties as follows:
$\{(i, i+1, i+4, i+7), (i, i+2, i+6, i+5)\}$ $(i = 1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of $2K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.
Case 2. $n \equiv 1 \pmod 8$ **and** $n \geq 17$. Put $n = 8t + 1$ $(t \geq 2)$. Construct tn C_4-bowties as follows:
$\{(i, i+1, i+5t+2, i+4t+1), (i, i+2, i+5t+4, i+4t+2)\}$,
$\{(i, i+3, i+5t+6, i+4t+3), (i, i+4, i+5t+8, i+4t+4)\}$,
...
$\{(i, i+t, i+7t, i+5t), (i, i+1, i+5t+2, i+4t+1)\}$,
$\{(i, i+2, i+5t+4, i+4t+2), (i, i+3, i+5t+6, i+4t+3)\}$,
$\{(i, i+4, i+5t+8, i+4t+4), (i, i+5, i+5t+10, i+4t+5)\}$,
...
$\{(i, i+t-1, i+7t-2, i+5t-1), (i, i+t, i+7t, i+5t)\}$ $(i = 1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of $2K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Theorem 5. When $\lambda \equiv 0 \pmod 4$, $n \equiv 1 \pmod 4$ and $n \geq 9$, λK_n has a balanced C_4-bowtie decomposition.

Proof. Put $n = 4t+1$. Construct tn C_4-bowties as follows:
$\{(i, i+1, i+2t+2, i+2t+1), (i, i+2t, i+6t, i+4t)\}$,
$\{(i, i+2, i+2t+4, i+2t+2), (i, i+2t-1, i+6t-1, i+4t-1)\}$,
$\{(i, i+3, i+2t+6, i+2t+3), (i, i+4, i+2t+8, i+2t+4)\}$,
$\{(i, i+5, i+2t+10, i+2t+5), (i, i+6, i+2t+12, i+2t+6)\}$,
...
$\{(i, i+2t-3, i+6t-3, i+4t-3), (i, i+2t-2, i+6t-2, i+4t-2)\}$ $(i = 1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of $4K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Theorem 6. When $\lambda \equiv 0 \pmod 8$, $n \equiv 1 \pmod 2$ and $n \geq 7$, λK_n has a balanced C_4-bowtie decomposition.

Proof. We consider 5 cases.
Case 1. $n \equiv 1 \pmod 4$ **and** $n \geq 9$. By Theorem 5 and Theorem 2, λK_n has a balanced C_4-bowtie decomposition.
Case 2.1. $n = 7$. Construct 21 C_4-bowties as follows:
$\{(i, i+1, i+2, i+4), (i, i+6, i+5, i+3)\}$,
$\{(i, i+2, i+4, i+1), (i, i+5, i+3, i+6)\}$,
$\{(i, i+3, i+6, i+5), (i, i+4, i+1, i+2)\}$ $(i = 1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of $8K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.
Case 2.2. $n = 11$. Construct 55 C_4-bowties as

follows:
$\{(i,i+1,i+2,i+4),(i,i+10,i+9,i+7)\}$,
$\{(i,i+2,i+4,i+8),(i,i+9,i+7,i+3)\}$,
$\{(i,i+3,i+6,i+1),(i,i+8,i+5,i+10)\}$,
$\{(i,i+4,i+8,i+5),(i,i+7,i+3,i+6)\}$,
$\{(i,i+5,i+1,i+2),(i,i+6,i+1,i+2)\}$
$(i=1,2,...,n)$.
Then they comprise a balanced C_4-bowtie decomposition of $8K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Case 2.3. $n = 15$. Construct 105 C_4-bowties as follows:
$\{(i,i+1,i+8,i+7),(i,i+3,i+5,i+2)\}$,
$\{(i,i+2,i+3,i+1),(i,i+5,i+9,i+4)\}$,
$\{(i,i+3,i+5,i+2),(i,i+1,i+8,i+7)\}$,
$\{(i,i+4,i+7,i+3),(i,i+6,i+11,i+5)\}$,
$\{(i,i+5,i+9,i+4),(i,i+7,i+13,i+6)\}$,
$\{(i,i+6,i+11,i+5),(i,i+4,i+7,i+3)\}$,
$\{(i,i+7,i+13,i+6),(i,i+2,i+3,i+1)\}$
$(i=1,2,...,n)$.
Then they comprise a balanced C_4-bowtie decomposition of $8K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Case 2.4. $n \equiv 3$ **(mod** 4) **and** $n \geq 19$. Put $n = 4t+3$ $(t \geq 4)$. Construct $(2t+1)n$ C_4-bowties as follows:
$\{(i,i+1,i+2t+2,i+2t+1),(i,i+t+2,i+2t+3,i+t+1)\}$,
$\{(i,i+j,i+2j-1,i+j-1),(i,i+j+t+1,i+2j+2t+1,i+j+t)\}$ $(j=2,3,...,t)$,
$\{(i,i+t+1,i+2t+1,i+t),(i,i+2,i+3,i+1)\}$,
$\{(i,i+t+2,i+2t+3,i+t+1),(i,i+1,i+2t+2,i+2t+1)\}$,
$\{(i,i+j,i+2j-1,i+j-1),(i,i+j-t,i+2j-2t-1,i+j-t-1)\}$ $(j=t+3,t+4,...,2t-1)$,
$\{(i,i+2t,i+4t-1,i+2t-1),(i,i+t+1,i+2t+1,i+t)\}$,
$\{(i,i+2t+1,i+4t+1,i+2t),(i,i+t,i+2t-1,i+t-1)\}$
$(i=1,2,...,n)$.
Then they comprise a balanced C_4-bowtie decomposition of $8K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Theorem 7. When $\lambda \equiv 0 \pmod{16}$ and $n \geq 7$, λK_n has a balanced C_4-bowtie decomposition.

Proof. We consider 7 cases.
Case 1. $n \equiv 1$ **(mod** 2) **and** $n \geq 7$. By Theorem 6 and Theorem 2, λK_n has a balanced C_4-bowtie decomposition.
Case 2.1. $n = 8$. Construct 56 C_4-bowties as follows:
$\{(i,i+1,i+5,i+6),(i,i+4,i+7,i+2)\}$,
$\{(i,i+2,i+3,i+1),(i,i+5,i+6,i+4)\}$,
$\{(i,i+3,i+1,i+5),(i,i+6,i+4,i+7)\}$,
$\{(i,i+4,i+7,i+2),(i,i+3,i+1,i+5)\}$,
$\{(i,i+5,i+6,i+4),(i,i+7,i+2,i+3)\}$,
$\{(i,i+6,i+4,i+7),(i,i+2,i+3,i+1)\}$,
$\{(i,i+7,i+2,i+3),(i,i+1,i+5,i+6)\}$
$(i=1,2,...,n)$.
Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Case 2.2. $n = 10$. Construct 90 C_4-bowties as follows:
$\{(i,i+1,i+6,i+2),(i,i+5,i+4,i+7)\}$,
$\{(i,i+2,i+8,i+1),(i,i+6,i+7,i+4)\}$,
$\{(i,i+3,i+5,i+9),(i,i+8,i+1,i+6)\}$,
$\{(i,i+4,i+2,i+3),(i,i+7,i+9,i+8)\}$,
$\{(i,i+5,i+4,i+7),(i,i+1,i+6,i+2)\}$,
$\{(i,i+6,i+7,i+4),(i,i+9,i+3,i+5)\}$,
$\{(i,i+7,i+9,i+8),(i,i+4,i+2,i+3)\}$,
$\{(i,i+8,i+1,i+6),(i,i+3,i+5,i+9)\}$,
$\{(i,i+9,i+3,i+5),(i,i+2,i+8,i+1)\}$
$(i=1,2,...,n)$.
Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Case 2.3. $n = 12$. Construct 132 C_4-bowties as follows:
$\{(i,i+1,i+7,i+2),(i,i+6,i+11,i+10)\}$,
$\{(i,i+2,i+3,i+5),(i,i+11,i+10,i+8)\}$,
$\{(i,i+3,i+5,i+9),(i,i+8,i+4,i+1)\}$,
$\{(i,i+4,i+1,i+7),(i,i+9,i+6,i+11)\}$,
$\{(i,i+5,i+9,i+6),(i,i+10,i+8,i+4)\}$,
$\{(i,i+6,i+11,i+10),(i,i+1,i+7,i+2)\}$,
$\{(i,i+7,i+2,i+3),(i,i+5,i+9,i+6)\}$,
$\{(i,i+8,i+4,i+1),(i,i+7,i+2,i+3)\}$,
$\{(i,i+9,i+6,i+11),(i,i+2,i+3,i+5)\}$,
$\{(i,i+10,i+8,i+4),(i,i+3,i+5,i+9)\}$,
$\{(i,i+11,i+10,i+8),(i,i+4,i+1,i+7)\}$
$(i=1,2,...,n)$.
Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Case 2.4. $n = 14$. Construct 182 C_4-bowties as follows:
$\{(i,i+1,i+8,i+2),(i,i+9,i+4,i+7)\}$,
$\{(i,i+2,i+3,i+5),(i,i+10,i+6,i+11)\}$,
$\{(i,i+3,i+5,i+9),(i,i+11,i+8,i+1)\}$,
$\{(i,i+4,i+7,i+13),(i,i+12,i+10,i+6)\}$,
$\{(i,i+5,i+9,i+14),(i,i+13,i+12,i+10)\}$,
$\{(i,i+6,i+11,i+8),(i,i+7,i+13,i+12)\}$,

$\{(i, i+7, i+13, i+12), (i, i+8, i+2, i+3)\}$,
$\{(i, i+8, i+2, i+3), (i, i+5, i+9, i+4)\}$,
$\{(i, i+9, i+4, i+7), (i, i+1, i+8, i+2)\}$,
$\{(i, i+10, i+6, i+11), (i, i+2, i+3, i+5)\}$,
$\{(i, i+11, i+8, i+1), (i, i+3, i+5, i+9)\}$,
$\{(i, i+12, i+10, i+6), (i, i+4, i+7, i+13)\}$,
$\{(i, i+13, i+12, i+10), (i, i+6, i+11, i+8)\}$
$(i = 1, 2, ..., n)$.

Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Case 2.5. $n \equiv 0, 2 \pmod 6$, $n \geq 18$. We consider 3 subcases.

Subcase 2.5.1. $n \equiv 2, 6, 14, 18 \pmod{24}$, $n \geq 18$. Put $n = 24t + 4a + 2$ $(t \geq 0)$. Then $a = 4, 6, 7, 9$.

Construct $(24t + 4a + 1)n$ C_4-bowties as follows:
$\{(i, i+1, i+12t+2a+2, i+2), (i, i+12t+2a+3, i+4, i+7)\}$,
$\{(i, i+2, i+3, i+5), (i, i+12t+2a+4, i+6, i+11)\}$,
$\{(i, i+3, i+5, i+9), (i, i+12t+2a+5, i+8, i+15)\}$,
$\{(i, i+4, i+7, i+13), (i, i+12t+2a+6, i+10, i+19)\}$,
...
$\{(i, i+6t+a-2, i+12t+2a-5, i+24t+4a-11), (i, i+18t+3a, i+12t+2a-2, i+24t+4a-5)\}$,
$\{(i, i+6t+a-1, i+12t+2a-3, i+24t+4a-7), (i, i+18t+3a+1, i+12t+2a, i+24t+4a-1)\}$,
$\{(i, i+6t+a, i+12t+2a-1, i+24t+4a-3), (i, i+18t+3a+2, i+12t+2a+2, i+1)\}$,
$\{(i, i+6t+a+1, i+12t+2a+1, i+24t+4a+1), (i, i+18t+3a+3, i+12t+2a+4, i+6)\}$,
$\{(i, i+6t+a+2, i+12t+2a+3, i+4), (i, i+18t+3a+4, i+12t+2a+6, i+10)\}$,
$\{(i, i+6t+a+3, i+12t+2a+5, i+8), (i, i+18t+3a+5, i+12t+2a+8, i+14)\}$,
$\{(i, i+6t+a+4, i+12t+2a+7, i+12), (i, i+18t+3a+6, i+12t+2a+10, i+18)\}$,
...
$\{(i, i+12t+2a-2, i+24t+4a-5, i+24t+4a-12), (i, i+24t+4a, i+24t+4a-2, i+24t+4a-6)\}$,
$\{(i, i+12t+2a-1, i+24t+4a-3, i+24t+4a-8), (i, i+24t+4a+1, i+24t+4a, i+24t+4a-2)\}$,
$\{(i, i+12t+2a, i+24t+4a-1, i+24t+4a-4), (i, i+12t+2a+1, i+24t+4a+1, i+24t+4a)\}$,
$\{(i, i+12t+2a+1, i+24t+4a+1, i+24t+4a), (i, i+12t+2a+2, i+2, i+3)\}$,
$\{(i, i+12t+2a+2, i+2, i+3), (i, i+12t+2a, i+24t+4a-1, i+24t+4a-4)\}$,
$\{(i, i+12t+2a+3, i+4, i+7), (i, i+1, i+12t+2a+2, i+2)\}$,
$\{(i, i+12t+2a+4, i+6, i+11), (i, i+2, i+3, i+5)\}$,
$\{(i, i+12t+2a+5, i+8, i+15), (i, i+3, i+5, i+9)\}$,
$\{(i, i+12t+2a+6, i+10, i+19), (i, i+4, i+7, i+13)\}$,
...
$\{(i, i+18t+3a, i+12t+2a-2, i+24t+4a-5), (i, i+6t+a-2, i+12t+2a-5, i+24t+4a-11)\}$,
$\{(i, i+18t+3a+1, i+12t+2a, i+24t+4a-1), (i, i+6t+a-1, i+12t+2a-3, i+24t+4a-7)\}$,
$\{(i, i+18t+3a+2, i+12t+2a+2, i+1), (i, i+6t+a, i+12t+2a-1, i+24t+4a-3)\}$,
$\{(i, i+18t+3a+3, i+12t+2a+4, i+6), (i, i+6t+a+1, i+12t+2a+1, i+24t+4a+1)\}$,
$\{(i, i+18t+3a+4, i+12t+2a+6, i+10), (i, i+6t+a+2, i+12t+2a+3, i+4)\}$,
$\{(i, i+18t+3a+5, i+12t+2a+8, i+14), (i, i+6t+a+3, i+12t+2a+5, i+8)\}$,
$\{(i, i+18t+3a+6, i+12t+2a+10, i+18), (i, i+6t+a+4, i+12t+2a+7, i+12)\}$,
...
$\{(i, i+24t+4a, i+24t+4a-2, i+24t+4a-6), (i, i+12t+2a-2, i+24t+4a-5, i+24t+4a-12)\}$,
$\{(i, i+24t+4a+1, i+24t+4a, i+24t+4a-2), (i, i+12t+2a-1, i+24t+4a-3, i+24t+4a-8)\}$
$(i = 1, 2, ..., n)$.

Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Subcase 2.5.2. $n \equiv 0, 8 \pmod{24}$, $n \geq 24$. Put $n = 24t + 8a$ $(t \geq 0)$. Then $a = 3, 4$.

Construct $(24t + 8a - 1)n$ C_4-bowties as follows:
$\{(i, i+1, i+12t+4a+1, i+2), (i, i+12t+4a+2, i+4, i+7)\}$,
$\{(i, i+2, i+3, i+5), (i, i+12t+4a+3, i+6, i+11)\}$,
$\{(i, i+3, i+5, i+9), (i, i+12t+4a+4, i+8, i+15)\}$,
...
$\{(i, i+3t+a, i+6t+2a-1, i+12t+4a-3), (i, i+15t+5a+1, i+6t+2a+2, i+12t+4a+3)\}$,
$\{(i, i+3t+a+1, i+6t+2a+1, i+1), (i, i+15t+5a+2, i+6t+2a+4, i+12t+4a+7)\}$,
$\{(i, i+3t+a+2, i+6t+2a+3, i+12t+4a+5), (i, i+15t+5a+3, i+6t+2a+6, i+12t+4a+11)\}$,
$\{(i, i+3t+a+3, i+6t+2a+5, i+12t+4a+9), (i, i+15t+5a+4, i+6t+2a+8, i+12t+4a+15)\}$,
...
$\{(i, i+6t+2a-1, i+12t+4a-3, i+24t+8a-7), (i, i+18t+6a, i+12t+4a, i+24t+8a-1)\}$,
$\{(i, i+6t+2a, i+12t+4a-1, i+24t+8a-3), (i, i+18t+6a+1, i+12t+4a+2, i+4)\}$,
$\{(i, i+6t+2a+1, i+1, i+12t+4a+1), (i, i+18t+6a+2, i+12t+4a+4, i+8)\}$,

$\{(i, i+6t+2a+2, i+12t+4a+3, i+6), (i, i+18t+6a+3, i+12t+4a+6, i+12)\}$,
$\{(i, i+6t+2a+3, i+12t+4a+5, i+10), (i, i+18t+6a+4, i+12t+4a+8, i+16)\}$,
...
$\{(i, i+12t+4a-2, i+24t+8a-5, i+24t+8a-10), (i, i+24t+8a-1, i+24t+8a-2, i+24t+8a-4)\}$,
$\{(i, i+12t+4a-1, i+24t+8a-3, i+24t+8a-6), (i, i+12t+4a, i+24t+8a-1, i+24t+8a-2)\}$,
$\{(i, i+12t+4a, i+24t+8a-1, i+24t+8a-2), (i, i+12t+4a+1, i+2, i+3)\}$,
$\{(i, i+12t+4a+1, i+2, i+3), (i, i+12t+4a-1, i+24t+8a-3, i+24t+8a-6)\}$,
$\{(i, i+12t+4a+2, i+4, i+7), (i, i+1, i+12t+4a+1, i+2)\}$,
$\{(i, i+12t+4a+3, i+6, i+11), (i, i+2, i+3, i+5)\}$,
$\{(i, i+12t+4a+4, i+8, i+15), (i, i+3, i+5, i+9)\}$,
...
$\{(i, i+15t+5a, i+6t+2a, i+12t+4a-1), (i, i+3t+a-1, i+6t+2a-3, i+12t+4a-7)\}$,
$\{(i, i+15t+5a+1, i+6t+2a+2, i+12t+4a+3), (i, i+3t+a, i+6t+2a-1, i+12t+4a-3)\}$,
$\{(i, i+15t+5a+2, i+6t+2a+4, i+12t+4a+7), (i, i+3t+a+1, i+6t+2a+1, i+1)\}$,
$\{(i, i+15t+5a+3, i+6t+2a+6, i+12t+4a+11), (i, i+3t+a+2, i+6t+2a+3, i+12t+4a+5)\}$,
$\{(i, i+15t+5a+4, i+6t+2a+8, i+12t+4a+15), (i, i+3t+a+3, i+6t+2a+5, i+12t+4a+9)\}$,
...
$\{(i, i+18t+6a, i+12t+4a, i+24t+8a-1), (i, i+6t+2a-1, i+12t+4a-3, i+24t+8a-7)\}$,
$\{(i, i+18t+6a+1, i+12t+4a+2, i+4), (i, i+6t+2a, i+12t+4a-1, i+24t+8a-3)\}$,
$\{(i, i+18t+6a+2, i+12t+4a+4, i+8), (i, i+6t+2a+1, i+1, i+12t+4a+1)\}$,
$\{(i, i+18t+6a+3, i+12t+4a+6, i+12), (i, i+6t+2a+2, i+12t+4a+3, i+6)\}$,
$\{(i, i+18t+6a+4, i+12t+4a+8, i+16), (i, i+6t+2a+3, i+12t+4a+5, i+10)\}$,
...
$\{(i, i+24t+8a-1, i+24t+8a-2, i+24t+8a-4), (i, i+12t+4a-2, i+24t+8a-5, i+24t+8a-10)\}$
$(i = 1, 2, ..., n)$.

Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Subcase 2.5.3. $n \equiv 12, 20$ **(mod 24)**, $n \geq 20$. Put $n = 24t+8a+4$ $(t \geq 0)$. Then $a = 2, 4$. Construct $(24t+8a+3)n$ C_4-bowties as follows:

$\{(i, i+1, i+12t+4a+3, i+2), (i, i+12t+4a+4, i+4, i+7)\}$,
$\{(i, i+2, i+3, i+5), (i, i+12t+4a+5, i+6, i+11)\}$,
$\{(i, i+3, i+5, i+9), (i, i+12t+4a+6, i+8, i+15)\}$,
...
$\{(i, i+3t+a-1, i+6t+2a-3, i+12t+4a-7), (i, i+15t+5a+2, i+6t+2a, i+12t+4a-1)\}$,
$\{(i, i+3t+a, i+6t+2a-1, i+12t+4a-3), (i, i+15t+5a+3, i+6t+2a+2, i+1)\}$,
$\{(i, i+3t+a+1, i+6t+2a+1, i+12t+2a+1), (i, i+15t+5a+4, i+6t+2a+4, i+12t+4a+7)\}$,
$\{(i, i+3t+a+2, i+6t+2a+3, i+12t+2a+5), (i, i+15t+5a+5, i+6t+2a+6, i+12t+4a+11)\}$,
...
$\{(i, i+6t+2a, i+12t+4a-1, i+24t+8a-3), (i, i+18t+6a+3, i+12t+4a+2, i+24t+8a+3)\}$,
$\{(i, i+6t+2a+1, i+12t+4a+1, i+24t+8a+1), (i, i+18t+6a+4, i+12t+4a+4, i+4)\}$,
$\{(i, i+6t+2a+2, i+1, i+12t+4a+3), (i, i+18t+6a+5, i+12t+4a+6, i+8)\}$,
$\{(i, i+6t+2a+3, i+12t+4a+5, i+6), (i, i+18t+6a+6, i+12t+4a+8, i+12)\}$,
$\{(i, i+6t+2a+4, i+12t+4a+7, i+10), (i, i+18t+6a+7, i+12t+4a+10, i+16)\}$,
...
$\{(i, i+12t+4a, i+24t+8a-1, i+24t+8a-6), (i, i+24t+8a+3, i+24t+8a+2, i+24t+8a)\}$,
$\{(i, i+12t+4a+1, i+24t+8a+1, i+24t+8a-2), (i, i+12t+4a+2, i+24t+8a+3, i+24t+8a+2)\}$,
$\{(i, i+12t+4a+2, i+24t+8a+3, i+24t+8a+2), (i, i+12t+4a+3, i+2, i+3)\}$,
$\{(i, i+12t+4a+3, i+2, i+3), (i, i+12t+4a+1, i+24t+8a+1, i+24t+8a-2)\}$,
$\{(i, i+12t+4a+4, i+4, i+7), (i, i+1, i+12t+4a+3, i+2)\}$,
$\{(i, i+12t+4a+5, i+6, i+11), (i, i+2, i+3, i+5)\}$,
$\{(i, i+12t+4a+6, i+8, i+15), (i, i+3, i+5, i+9)\}$,
...
$\{(i, i+15t+5a+2, i+6t+2a, i+12t+4a-1), (i, i+3t+a-1, i+6t+2a-3, i+12t+4a-7)\}$,
$\{(i, i+15t+5a+3, i+6t+2a+2, i+1), (i, i+3t+a, i+6t+2a-1, i+12t+4a-3)\}$,
$\{(i, i+15t+5a+4, i+6t+2a+4, i+12t+4a+7), (i, i+3t+a+1, i+6t+2a+1, i+12t+4a+1)\}$,
$\{(i, i+15t+5a+5, i+6t+2a+6, i+12t+4a+11), (i, i+3t+a+2, i+6t+2a+3, i+12t+4a+5)\}$,
$\{(i, i+15t+5a+6, i+6t+2a+8, i+12t+4a+15), (i, i+3t+a+3, i+6t+2a+5, i+12t+4a+9)\}$,
...
$\{(i, i+18t+6a+3, i+12t+4a+2, i+24t+8a+3), (i, i+6t+2a, i+12t+4a-1, i+24t+8a-3)\}$,
$\{(i, i+18t+6a+4, i+12t+4a+4, i+4), (i, i+$

$6t+2a+1, i+12t+4a+1, i+24t+8a+1)\}$,
$\{(i, i+18t+6a+5, i+12t+4a+6, i+8), (i, i+6t+2a+2, i+1, i+12t+4a+3)\}$,
$\{(i, i+18t+6a+6, i+12t+4a+8, i+12), (i, i+6t+2a+3, i+12t+4a+5, i+6)\}$,
$\{(i, i+18t+6a+7, i+12t+4a+10, i+16), (i, i+6t+2a+4, i+12t+4a+7, i+10)\}$,
...
$\{(i, i+24t+8a+3, i+24t+8a+2, i+24t+8a), (i, i+12t+4a, i+24t+8a-1, i+24t+8a-6)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Case 2.6. $n \equiv 4 \pmod 6$, $n \geq 16$. We consider 4 subcases.

Subcase 2.6.1. $n \equiv 16 \pmod{24}$, $n \geq 16$. Put $n = 24t + 16$ $(t \geq 0)$.
Construct $(24t+15)n$ C_4-bowties as follows:
$\{(i, i+1, i+12t+9, i+12t+10), (i, i+24t+15, i+12t+6, i+12t+3)\}$,
$\{(i, i+2, i+12t+11, i+12t+14), (i, i+12t+8, i+12t+7, i+12t+5)\}$,
$\{(i, i+3, i+12t+13, i+12t+18), (i, i+12t+9, i+12t+10, i+12t+12)\}$,
$\{(i, i+4, i+12t+15, i+12t+22), (i, i+12t+10, i+12t+12, i+12t+16)\}$,
$\{(i, i+5, i+12t+17, i+12t+26), (i, i+12t+11, i+12t+14, i+12t+20)\}$,
...
$\{(i, i+3t+2, i+18t+11, i+24t+14), (i, i+15t+8, i+18t+8, i+24t+8)\}$,
$\{(i, i+3t+3, i+18t+13, i+2), (i, i+15t+9, i+18t+10, i+24t+12)\}$,
$\{(i, i+3t+4, i+18t+15, i+6), (i, i+15t+10, i+18t+12, i+12t+8)\}$,
$\{(i, i+3t+5, i+18t+17, i+10), (i, i+15t+11, i+18t+14, i+4)\}$,
$\{(i, i+3t+6, i+18t+19, i+14), (i, i+15t+12, i+18t+16, i+8)\}$,
$\{(i, i+3t+7, i+18t+21, i+18), (i, i+15t+13, i+18t+18, i+12)\}$,
...
$\{(i, i+6t+4, i+24t+15, i+12t+6), (i, i+18t+10, i+24t+12, i+12t)\}$,
$\{(i, i+6t+5, i+1, i+12t+9), (i, i+18t+11, i+24t+14, i+12t+4)\}$,
$\{(i, i+6t+6, i+3, i+12t+13), (i, i+18t+12, i+12t+8, i+12t+7)\}$,
$\{(i, i+6t+7, i+5, i+12t+17), (i, i+18t+13, i+2, i+12t+11)\}$,
$\{(i, i+6t+8, i+7, i+12t+21), (i, i+18t+14, i+4, i+12t+15)\}$,
$\{(i, i+6t+9, i+9, i+12t+25), (i, i+18t+15, i+6, i+12t+19)\}$,
...
$\{(i, i+9t+6, i+6t+3, i+24t+13), (i, i+21t+12, i+6t, i+24t+7)\}$,
$\{(i, i+9t+7, i+6t+5, i+1), (i, i+21t+13, i+6t+2, i+24t+11)\}$,
$\{(i, i+9t+8, i+6t+7, i+5), (i, i+21t+14, i+6t+4, i+24t+15)\}$,
$\{(i, i+9t+9, i+6t+9, i+9), (i, i+21t+15, i+6t+6, i+3)\}$,
$\{(i, i+9t+10, i+6t+11, i+13), (i, i+21t+16, i+6t+8, i+7)\}$,
$\{(i, i+9t+11, i+6t+13, i+17), (i, i+21t+17, i+6t+10, i+11)\}$,
...
$\{(i, i+12t+8, i+12t+7, i+12t+5), (i, i+24t+14, i+12t+4, i+12t-1)\}$,
$\{(i, i+12t+9, i+12t+10, i+12t+12), (i, i+3, i+12t+13, i+12t+18)\}$,
$\{(i, i+12t+10, i+12t+12, i+12t+16), (i, i+4, i+12t+15, i+12t+22)\}$,
$\{(i, i+12t+11, i+12t+14, i+12t+20), (i, i+5, i+12t+17, i+12t+26)\}$,
...
$\{(i, i+15t+8, i+18t+8, i+24t+8), (i, i+3t+2, i+18t+11, i+24t+14)\}$,
$\{(i, i+15t+9, i+18t+10, i+24t+12), (i, i+3t+3, i+18t+13, i+2)\}$,
$\{(i, i+15t+10, i+18t+12, i+12t+8), (i, i+3t+4, i+18t+15, i+6)\}$,
$\{(i, i+15t+11, i+18t+14, i+4), (i, i+3t+5, i+18t+17, i+10)\}$,
$\{(i, i+15t+12, i+18t+16, i+8), (i, i+3t+6, i+18t+19, i+14)\}$,
$\{(i, i+15t+13, i+18t+18, i+12), (i, i+3t+7, i+18t+21, i+18)\}$,
...
$\{(i, i+18t+10, i+24t+12, i+12t), (i, i+6t+4, i+24t+15, i+12t+6)\}$,
$\{(i, i+18t+11, i+24t+14, i+12t+4), (i, i+6t+5, i+1, i+12t+9)\}$,
$\{(i, i+18t+12, i+12t+8, i+12t+7), (i, i+6t+6, i+3, i+12t+13)\}$,
$\{(i, i+18t+13, i+2, i+12t+11), (i, i+6t+7, i+5, i+12t+17)\}$,
$\{(i, i+18t+14, i+4, i+12t+15), (i, i+6t+8, i+7, i+12t+21)\}$,
$\{(i, i+18t+15, i+6, i+12t+19), (i, i+6t+9, i+9, i+12t+25)\}$,
...

$\{(i, i+21t+12, i+6t, i+24t+7), (i, i+9t+6, i+6t+3, i+24t+13)\}$,
$\{(i, i+21t+13, i+6t+2, i+24t+11), (i, i+9t+7, i+6t+5, i+1)\}$,
$\{(i, i+21t+14, i+6t+4, i+24t+15), (i, i+9t+8, i+6t+7, i+5)\}$,
$\{(i, i+21t+15, i+6t+6, i+3), (i, i+9t+9, i+6t+9, i+9)\}$,
$\{(i, i+21t+16, i+6t+8, i+7), (i, i+9t+10, i+6t+11, i+13)\}$,
$\{(i, i+21t+17, i+6t+10, i+11), (i, i+9t+11, i+6t+13, i+17)\}$,
...
$\{(i, i+24t+13, i+12t+2, i+12t-5), (i, i+12t+7, i+12t+5, i+12t+1)\}$,
$\{(i, i+24t+14, i+12t+4, i+12t-1), (i, i+1, i+12t+9, i+12t+10)\}$,
$\{(i, i+24t+15, i+12t+6, i+12t+3), (i, i+2, i+12t+11, i+12t+14)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Subcase 2.6.2. $n \equiv 4$ **(mod** 24), $n \geq 28$. Put $n = 24t+28$ $(t \geq 0)$.
Construct $(24t+27)n$ C_4-bowties as follows:
$\{(i, i+1, i+12t+15, i+12t+16), (i, i+24t+27, i+12t+12, i+12t+9)\}$,
$\{(i, i+2, i+12t+17, i+12t+20), (i, i+12t+14, i+12t+13, i+12t+11)\}$,
$\{(i, i+3, i+12t+19, i+12t+24), (i, i+12t+15, i+12t+16, i+12t+18)\}$,
$\{(i, i+4, i+12t+21, i+12t+28), (i, i+12t+16, i+12t+18, i+12t+22)\}$,
$\{(i, i+5, i+12t+23, i+12t+32), (i, i+12t+17, i+12t+20, i+12t+26)\}$,
...
$\{(i, i+3t+4, i+18t+21, i+12t+14), (i, i+15t+16, i+18t+18, i+24t+22)\}$,
$\{(i, i+3t+5, i+18t+23, i+4), (i, i+15t+17, i+18t+20, i+24t+26)\}$,
$\{(i, i+3t+6, i+18t+25, i+8), (i, i+15t+18, i+18t+22, i+2)\}$,
$\{(i, i+3t+7, i+18t+27, i+12), (i, i+15t+19, i+18t+24, i+6)\}$,
$\{(i, i+3t+8, i+18t+29, i+16), (i, i+15t+20, i+18t+26, i+10)\}$,
...
$\{(i, i+6t+7, i+24t+27, i+12t+12), (i, i+18t+19, i+24t+24, i+12t+6)\}$,
$\{(i, i+6t+8, i+1, i+12t+15), (i, i+18t+20, i+24t+26, i+12t+10)\}$,
$\{(i, i+6t+9, i+3, i+12t+19), (i, i+18t+21, i+12t+14, i+12t+13)\}$,
$\{(i, i+6t+10, i+5, i+12t+23), (i, i+18t+22, i+2, i+12t+17)\}$,
$\{(i, i+6t+11, i+7, i+12t+27), (i, i+18t+23, i+4, i+12t+21)\}$,
$\{(i, i+6t+12, i+9, i+12t+31), (i, i+18t+24, i+6, i+12t+25)\}$,
...
$\{(i, i+9t+11, i+6t+7, i+24t+27), (i, i+21t+23, i+6t+4, i+24t+21)\}$,
$\{(i, i+9t+12, i+6t+9, i+3), (i, i+21t+24, i+6t+6, i+24t+25)\}$,
$\{(i, i+9t+13, i+6t+11, i+7), (i, i+21t+25, i+6t+8, i+1)\}$,
$\{(i, i+9t+14, i+6t+13, i+11), (i, i+21t+26, i+6t+10, i+5)\}$,
$\{(i, i+9t+15, i+6t+15, i+15), (i, i+21t+27, i+6t+12, i+9)\}$,
...
$\{(i, i+12t+14, i+12t+13, i+12t+11), (i, i+24t+26, i+12t+10, i+12t+5)\}$,
$\{(i, i+12t+15, i+12t+16, i+12t+18), (i, i+3, i+12t+19, i+12t+24)\}$,
$\{(i, i+12t+16, i+12t+18, i+12t+22), (i, i+4, i+12t+21, i+12t+28)\}$,
$\{(i, i+12t+17, i+12t+20, i+12t+26), (i, i+5, i+12t+23, i+12t+32)\}$,
...
$\{(i, i+15t+16, i+18t+18, i+24t+22), (i, i+3t+4, i+18t+21, i+12t+14)\}$,
$\{(i, i+15t+17, i+18t+20, i+24t+26), (i, i+3t+5, i+18t+23, i+4)\}$,
$\{(i, i+15t+18, i+18t+22, i+2), (i, i+3t+6, i+18t+25, i+8)\}$,
$\{(i, i+15t+19, i+18t+24, i+6), (i, i+3t+7, i+18t+27, i+12)\}$,
$\{(i, i+15t+20, i+18t+26, i+10), (i, i+3t+8, i+18t+29, i+16)\}$,
...
$\{(i, i+18t+19, i+24t+24, i+12t+6), (i, i+6t+7, i+24t+27, i+12t+12)\}$,
$\{(i, i+18t+20, i+24t+26, i+12t+10), (i, i+6t+8, i+1, i+12t+15)\}$,
$\{(i, i+18t+21, i+12t+14, i+12t+13), (i, i+6t+9, i+3, i+12t+19)\}$,
$\{(i, i+18t+22, i+2, i+12t+17), (i, i+6t+10, i+5, i+12t+23)\}$,
$\{(i, i+18t+23, i+4, i+12t+21), (i, i+6t+11, i+7, i+12t+27)\}$,
$\{(i, i+18t+24, i+6, i+12t+25), (i, i+6t+12, i+9, i+12t+31)\}$,

...
$\{(i, i+21t+23, i+6t+4, i+24t+21), (i, i+9t+11, i+6t+7, i+24t+27)\}$,
$\{(i, i+21t+24, i+6t+6, i+24t+25), (i, i+9t+12, i+6t+9, i+3)\}$,
$\{(i, i+21t+25, i+6t+8, i+1), (i, i+9t+13, i+6t+11, i+7)\}$,
$\{(i, i+21t+26, i+6t+10, i+5), (i, i+9t+14, i+6t+13, i+11)\}$,
$\{(i, i+21t+27, i+6t+12, i+9), (i, i+9t+15, i+6t+15, i+15)\}$,
...
$\{(i, i+24t+25, i+12t+8, i+12t+1), (i, i+12t+13, i+12t+11, i+12t+7)\}$,
$\{(i, i+24t+26, i+12t+10, i+12t+5), (i, i+1, i+12t+15, i+12t+16)\}$,
$\{(i, i+24t+27, i+12t+12, i+12t+9), (i, i+2, i+12t+17, i+12t+20)\}$
$(i = 1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Subcase 2.6.3. $n \equiv 22 \pmod{24}$, $n \geq 22$. Put $n = 24t+22$ $(t \geq 0)$.
Construct $(24t+21)n$ C_4-bowties as follows:
$\{(i, i+1, i+12t+12, i+12t+13), (i, i+24t+21, i+12t+9, i+24t+17)\}$,
$\{(i, i+2, i+3, i+12t+16), (i, i+12t+11, i+12t+10, i+24t+19)\}$,
$\{(i, i+3, i+5, i+12t+20), (i, i+12t+12, i+12t+13, i+4)\}$,
$\{(i, i+4, i+7, i+12t+24), (i, i+12t+13, i+12t+15, i+8)\}$,
$\{(i, i+5, i+9, i+12t+28), (i, i+12t+14, i+12t+17, i+12)\}$,
...
$\{(i, i+3t+3, i+6t+5, i+24t+20), (i, i+15t+12, i+18t+13, i+12t+4)\}$,
$\{(i, i+3t+4, i+6t+7, i+2), (i, i+15t+13, i+18t+15, i+12t+8)\}$,
$\{(i, i+3t+5, i+6t+9, i+6), (i, i+15t+14, i+18t+17, i+1)\}$,
$\{(i, i+3t+6, i+6t+11, i+10), (i, i+15t+15, i+18t+19, i+12t+16)\}$,
$\{(i, i+3t+7, i+6t+13, i+14), (i, i+15t+16, i+18t+21, i+12t+20)\}$,
$\{(i, i+3t+8, i+6t+15, i+18), (i, i+15t+17, i+18t+23, i+12t+24)\}$,
...
$\{(i, i+6t+6, i+12t+11, i+12t+10), (i, i+18t+15, i+24t+19, i+24t+16)\}$,
$\{(i, i+6t+7, i+12t+13, i+12t+15), (i, i+18t+16, i+24t+21, i+24t+20)\}$,
$\{(i, i+6t+8, i+12t+15, i+12t+19), (i, i+18t+17, i+1, i+12t+12)\}$,
$\{(i, i+6t+9, i+12t+17, i+12t+23), (i, i+18t+18, i+3, i+5)\}$,
$\{(i, i+6t+10, i+12t+19, i+12t+27), (i, i+18t+19, i+5, i+9)\}$,
$\{(i, i+6t+11, i+12t+21, i+12t+31), (i, i+18t+20, i+7, i+13)\}$,
...
$\{(i, i+9t+8, i+18t+15, i+24t+19), (i, i+21t+17, i+6t+1, i+12t+1)\}$,
$\{(i, i+9t+9, i+18t+17, i+1), (i, i+21t+18, i+6t+3, i+12t+5)\}$,
$\{(i, i+9t+10, i+18t+19, i+5), (i, i+21t+19, i+6t+5, i+12t+9)\}$,
$\{(i, i+9t+11, i+18t+21, i+9), (i, i+21t+20, i+6t+7, i+12t+13)\}$,
$\{(i, i+9t+12, i+18t+23, i+13), (i, i+21t+21, i+6t+9, i+12t+17)\}$,
$\{(i, i+9t+13, i+18t+25, i+17), (i, i+21t+22, i+6t+11, i+12t+21)\}$,
...
$\{(i, i+12t+9, i+24t+17, i+12t+1), (i, i+24t+18, i+12t+3, i+24t+5)\}$,
$\{(i, i+12t+10, i+24t+19, i+12t+5), (i, i+2, i+12t+14, i+6)\}$,
$\{(i, i+12t+11, i+24t+21, i+12t+9), (i, i+24t+20, i+12t+7, i+24t+13)\}$,
$\{(i, i+12t+12, i+2, i+12t+14), (i, i+12t+10, i+12t+8, i+24t+15)\}$,
$\{(i, i+12t+13, i+4, i+12t+18), (i, i+1, i+12t+12, i+2)\}$,
$\{(i, i+12t+14, i+6, i+12t+22), (i, i+24t+19, i+12t+5, i+24t+9)\}$,
$\{(i, i+12t+15, i+8, i+12t+26), (i, i+3, i+12t+16, i+10)\}$,
...
$\{(i, i+15t+13, i+6t+4, i+24t+18), (i, i+3t+1, i+18t+12, i+12t+2)\}$,
$\{(i, i+15t+14, i+6t+6, i+12t+11), (i, i+3t+2, i+18t+14, i+12t+6)\}$,
$\{(i, i+15t+15, i+6t+8, i+4), (i, i+3t+3, i+18t+16, i+12t+10)\}$,
$\{(i, i+15t+16, i+6t+10, i+8), (i, i+3t+4, i+18t+18, i+12t+14)\}$,
$\{(i, i+15t+17, i+6t+12, i+12), (i, i+3t+5, i+18t+20, i+12t+18)\}$,
$\{(i, i+15t+18, i+6t+14, i+16), (i, i+3t+6, i+18t+22, i+12t+22)\}$,
...
$\{(i, i+18t+16, i+12t+10, i+12t+8), (i, i+$

$6t+4, i+24t+18, i+24t+14)\}$,
$\{(i, i+18t+17, i+1, i+12t+12), (i, i+6t+5, i+24t+20, i+24t+18)\}$,
$\{(i, i+18t+18, i+12t+14, i+12t+17), (i, i+6t+6, i+12t+11, i+24t+21)\}$,
$\{(i, i+18t+19, i+12t+16, i+12t+21), (i, i+6t+7, i+2, i+3)\}$,
$\{(i, i+18t+20, i+12t+18, i+12t+25), (i, i+6t+8, i+4, i+7)\}$,
$\{(i, i+18t+21, i+12t+20, i+12t+29), (i, i+6t+9, i+6, i+11)\}$,
...
$\{(i, i+21t+19, i+18t+16, i+24t+21), (i, i+9t+7, i+6t+2, i+12t+3)\}$,
$\{(i, i+21t+20, i+18t+18, i+3), (i, i+9t+8, i+6t+4, i+12t+7)\}$,
$\{(i, i+21t+21, i+18t+20, i+7), (i, i+9t+9, i+6t+6, i+12t+11)\}$,
$\{(i, i+21t+22, i+18t+22, i+11), (i, i+9t+10, i+6t+8, i+12t+15)\}$,
$\{(i, i+21t+23, i+18t+24, i+15), (i, i+9t+11, i+6t+10, i+12t+19)\}$,
...
$\{(i, i+24t+21, i+24t+20, i+12t+7), (i, i+12t+9, i+12t+6, i+24t+11)\}$
$(i=1, 2, ..., n)$.
Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.

Subcase 2.6.4. $n \equiv 10 \pmod{24}$, $n \geq 34$. Put $n = 24t+34$ $(t \geq 0)$.
Construct $(24t+33)n$ C_4-bowties as follows:
$\{(i, i+1, i+12t+18, i+12t+19), (i, i+24t+33, i+12t+15, i+24t+29)\}$,
$\{(i, i+2, i+3, i+12t+22), (i, i+12t+17, i+12t+16, i+24t+31)\}$,
$\{(i, i+3, i+5, i+12t+26), (i, i+12t+18, i+12t+19, i+4)\}$,
$\{(i, i+4, i+7, i+12t+30), (i, i+12t+19, i+12t+21, i+8)\}$,
$\{(i, i+5, i+9, i+12t+34), (i, i+12t+20, i+12t+23, i+12)\}$,
...
$\{(i, i+3t+4, i+6t+7, i+24t+30), (i, i+15t+19, i+18t+21, i+12t+8)\}$,
$\{(i, i+3t+5, i+6t+9, i+12t+17), (i, i+15t+20, i+18t+23, i+12t+12)\}$,
$\{(i, i+3t+6, i+6t+11, i+4), (i, i+15t+21, i+18t+25, i+12t+16)\}$,
$\{(i, i+3t+7, i+6t+13, i+8), (i, i+15t+22, i+18t+27, i+12t+20)\}$,
$\{(i, i+3t+8, i+6t+15, i+12), (i, i+15t+23, i+18t+29, i+12t+24)\}$,
$\{(i, i+3t+9, i+6t+17, i+16), (i, i+15t+24, i+18t+31, i+12t+28)\}$,
...
$\{(i, i+6t+9, i+12t+17, i+12t+16), (i, i+18t+24, i+24t+31, i+24t+28)\}$,
$\{(i, i+6t+10, i+12t+19, i+12t+21), (i, i+18t+25, i+24t+33, i+24t+32)\}$,
$\{(i, i+6t+11, i+12t+21, i+12t+25), (i, i+18t+26, i+1, i+12t+18)\}$,
$\{(i, i+6t+12, i+12t+23, i+12t+29), (i, i+18t+27, i+3, i+5)\}$,
$\{(i, i+6t+13, i+12t+25, i+12t+33), (i, i+18t+28, i+5, i+9)\}$,
$\{(i, i+6t+14, i+12t+27, i+12t+37), (i, i+18t+29, i+7, i+13)\}$,
...
$\{(i, i+9t+13, i+18t+25, i+24t+33), (i, i+21t+28, i+6t+5, i+12t+9)\}$,
$\{(i, i+9t+14, i+18t+27, i+3), (i, i+21t+29, i+6t+7, i+12t+13)\}$,
$\{(i, i+9t+15, i+18t+29, i+7), (i, i+21t+30, i+6t+9, i+12t+17)\}$,
$\{(i, i+9t+16, i+18t+31, i+11), (i, i+21t+31, i+6t+11, i+12t+21)\}$,
$\{(i, i+9t+17, i+18t+33, i+15), (i, i+21t+32, i+6t+13, i+12t+25)\}$,
$\{(i, i+9t+18, i+18t+35, i+19), (i, i+21t+33, i+6t+15, i+12t+29)\}$,
...
$\{(i, i+12t+15, i+24t+29, i+12t+7), (i, i+24t+30, i+12t+9, i+24t+17)\}$,
$\{(i, i+12t+16, i+24t+31, i+12t+11), (i, i+2, i+12t+20, i+6)\}$,
$\{(i, i+12t+17, i+24t+33, i+12t+15), (i, i+24t+32, i+12t+13, i+24t+25)\}$,
$\{(i, i+12t+18, i+2, i+12t+20), (i, i+12t+16, i+12t+14, i+24t+27)\}$,
$\{(i, i+12t+19, i+4, i+12t+24), (i, i+1, i+12t+18, i+2)\}$,
$\{(i, i+12t+20, i+6, i+12t+28), (i, i+24t+31, i+12t+11, i+24t+21)\}$,
$\{(i, i+12t+21, i+8, i+12t+32), (i, i+3, i+12t+22, i+10)\}$,
...
$\{(i, i+15t+21, i+6t+8, i+24t+32), (i, i+3t+3, i+18t+22, i+12t+10)\}$,
$\{(i, i+15t+22, i+6t+10, i+2), (i, i+3t+4, i+18t+24, i+12t+14)\}$,
$\{(i, i+15t+23, i+6t+12, i+6), (i, i+3t+5, i+18t+26, i+1)\}$,
$\{(i, i+15t+24, i+6t+14, i+10), (i, i+3t+$

$6, i+18t+28, i+12t+22)\}$,
$\{(i, i+15t+25, i+6t+16, i+14), (i, i+3t+7, i+18t+30, i+12t+26)\}$,
$\{(i, i+15t+26, i+6t+18, i+18), (i, i+3t+8, i+18t+32, i+12t+30)\}$,
...
$\{(i, i+18t+25, i+12t+16, i+12t+14), (i, i+6t+7, i+24t+30, i+24t+26)\}$,
$\{(i, i+18t+26, i+1, i+12t+18), (i, i+6t+8, i+24t+32, i+24t+30)\}$,
$\{(i, i+18t+27, i+12t+20, i+12t+23), (i, i+6t+9, i+12t+17, i+24t+33)\}$,
$\{(i, i+18t+28, i+12t+22, i+12t+27), (i, i+6t+10, i+2, i+3)\}$,
$\{(i, i+18t+29, i+12t+24, i+12t+31), (i, i+6t+11, i+4, i+7)\}$,
$\{(i, i+18t+30, i+12t+26, i+12t+35), (i, i+6t+12, i+6, i+11)\}$,
...
$\{(i, i+21t+29, i+18t+24, i+24t+31), (i, i+9t+11, i+6t+4, i+12t+7)\}$,
$\{(i, i+21t+30, i+18t+26, i+1), (i, i+9t+12, i+6t+6, i+12t+11)\}$,
$\{(i, i+21t+31, i+18t+28, i+5), (i, i+9t+13, i+6t+8, i+12t+15)\}$,
$\{(i, i+21t+32, i+18t+30, i+9), (i, i+9t+14, i+6t+10, i+12t+19)\}$,
$\{(i, i+21t+33, i+18t+32, i+13), (i, i+9t+15, i+6t+12, i+12t+23)\}$,
$\{(i, i+21t+34, i+18t+34, i+17), (i, i+9t+16, i+6t+14, i+12t+27)\}$,
...
$\{(i, i+24t+33, i+24t+32, i+12t+13), (i, i+12t+15, i+12t+12, i+24t+23)\}$
$(i=1,2,...,n)$.
Then they comprise a balanced C_4-bowtie decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-bowtie decomposition.
This completes the proof of Theorem 7.

Therefore, we have the following main theorem and its corollary.

Main Theorem. λK_n has a balanced C_4-bowtie decomposition if and only if $\lambda(n-1) \equiv 0 \pmod{16}$ and $n \geq 7$.

Corollary. K_n has a balanced C_4-bowtie decomposition if and only if $n \equiv 1 \pmod{16}$, $n \geq 17$.

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