Generalisation – Graphs and Colourings

Christina Zarb
christina.zarb@um.edu.mt

Abstract

The interaction between practice and theory in mathematics is a central theme. Many mathematical structures and theories result from the formalisation of a real problem. Graph Theory is rich with such examples. The graph structure itself was formalised by Leonard Euler in the quest to solve the problem of the Bridges of Königsberg.

Once a structure is formalised, and results are proven, the mathematician seeks to generalise. This can be considered as one of the main praxis in mathematics.

The idea of generalisation will be illustrated through graph colouring. This idea also results from a classic problem, in which it was well known by topographers that four colours suffice to colour any map such that no countries sharing a border receive the same colour. The proof of this theorem eluded mathematicians for centuries and was proven in 1976. Generalisation of graphs to hypergraphs, and variations on the colouring theme will be discussed, as well as applications in other disciplines.

Keywords: *graph theory, vertex colouring, generalization.*

Introduction

Graph Theory is an extremely vast and rich area of mathematics, falling under the broader topic of Combinatorics. Its origins trace back to the eighteenth century, the Swiss mathematician Leonard Euler and the city of Königsberg (now Kaliningrad).

In the early 18th century, the citizens of Königsberg spent their days walking on the intricate arrangement of bridges across the waters of the Pregel River, which surrounded two central landmasses connected by a bridge. Additionally, the first landmass (an island) was connected by two bridges to the lower bank of the Pregel and also by two bridges to the upper bank, while the other landmass (which split the Pregel into two branches) was connected to the lower bank by one bridge and to the upper bank by one bridge, for a total of seven bridges, as shown in Figure 1 (Reid, 2010). According to folklore, the question arose of whether a citizen could take a walk through the town in such a way that each bridge would be crossed exactly once (Carlson, 2006).

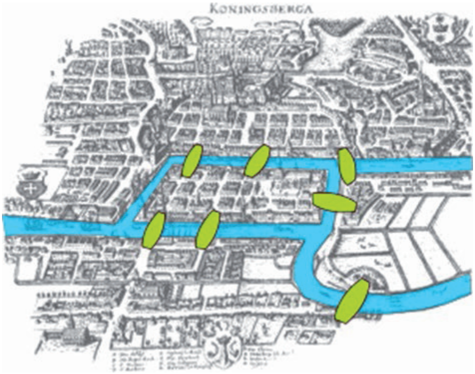

Figure 1

In 1735 the Swiss mathematician Leonhard Euler presented a solution to this problem, concluding that such a walk was impossible. He argued, that for this to be possible, for each bridge entering a land mass, there must be another bridge to allow an exit, meaning that each landmass, with the possible exception of the initial and terminal ones if they are not identical, must serve as an endpoint of an even number of bridges.

It would be several years before mathematicians would picture the Königsberg bridge problem as a graph consisting of vertices representing the landmasses and edges representing the bridges, as shown in Figure 2 (Carlson, 2006). The degree of a vertex of a graph specifies the number of edges incident to it. In modern graph theory, an Eulerian trail traverses each edge of a graph once and only once. Thus, Euler’s assertion that a graph possessing such a path has at most two vertices of odd degree was the first theorem in graph theory (Euler, 1736).

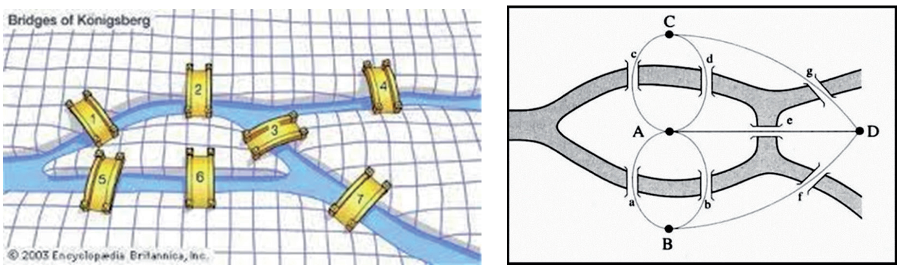

Figure 2

The aim of this paper is to explore the concept of generalisation in mathematics, which is one of the most important properties one considers when dealing with mathematical structures, through the theory of graphs, as well as the interaction between theory and application to real world problems. To further explore and discuss this notion, we consider the topic of graph colouring. We first give some basic definitions and results, and we then develop the various generalisations, both of the graph structure, as well as the concept of colouring.

We first give the formal definition of a graph. A graph is an ordered pair $G = (V, E)$ such that V is a set, called the vertex set, and E is a family of 2-element subsets of V, called the edge set, that is $E \subseteq \{\{u, v\}: u, v \in V\}$. In Figure 3, the vertices are the points A, B, C, D, E, F, G which are marked as nodes, and the edges are

$$\{\{A, B\}, \{A, E\}, \{B, E\}, \{C, D\}, \{C, E\}, \{C, F\}, \{D, E\}, \{D, F\}, \{F, G\}\}.$$

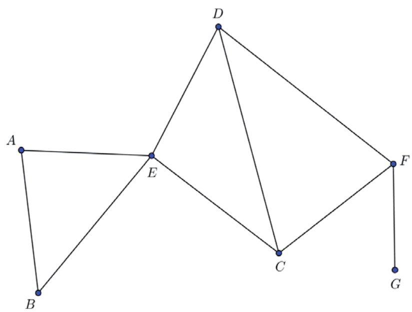

Figure 3

The structure and definition of a graph led to further studies, and several results and types of graphs were studied and constructed, as well as properties of such structures, some of which we give here, and others later on. As mentioned, the degree of a vertex is the number of edges incident to it. A graph is said to be simple if it has no loops or multiedges. A complete graph K_n is a graph on n vertices such that every pair of vertices are joined by an edge, that is the number of edges is $\binom{n}{2}$. A bipartite graph G is one in which the vertex set can be partitioned into two sets X and Y that is $V = V(G) = X \cup Y$ and $X \cap Y = \varnothing$, and such that any edge joins a vertex in X to a vertex in Y. A complete bipartite graph $K_{m,n}$ is one in which $|X| = m$, $|Y| = n$ and $E = \{xy: x \in X, y \in Y\}$, that is all possible edges joining a vertex in X to a vertex in Y.

When a graph is represented graphically, we consider the notion of a plane graph, which is a graph for which, in the graphical representation, no two edges cross each other. A graph is said to be planar if there exists a plane representation of the graph. Planar graphs are characterized by the famous Theorem by Kuratowski (Nishizeki & Chiba, 1988), which states that a graph is planar if and only if it does not contain any subdivision of K_5 or $K_{3,3}$. A subdivision of an edge is the operation where the edge is replaced by a path of length 2, the internal vertex added to the original graph. A subdivision of a graph G is a graph achieved by a sequence of edge-subdivisions on G.

Arguably, one of the most famous theorems in graph theory is the Four Colour Theorem (for a history of this see (Mitchem, 1981)). This states that, given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colours are required to colour the regions of the map so that no two adjacent regions have the same colour. The set of regions of a map can be represented more abstractly as a graph that has a vertex for each region and an edge for every pair of regions that share a boundary segment. The resulting graph is always planar. Hence, in graph-theoretic terminology, the Four Colour Theorem states that the vertices of every planar graph can be coloured with at most four colours so that no two adjacent vertices receive the same colour. It is generally accepted that this is what initiated the study of vertex colourings in graphs. The proof of this "well-known fact" eluded mathematicians for over 100 years and was finally proved by Appel and Haken in 1976 (Appel & Haken, 1976), the first major theorem to be proved using a computer. Through this problem, the notion of vertex colouring is introduced.

The paper is organised as follows: we first discuss generalisations of the graph structure in different ways, and give some ideas of possible applications. We then consider colouring, and look at various generalisations of this theme, leading to a unifying definition of the concept of vertex colouring of graphs and their generalisations, which encompasses all the possible variations discussed. Finally, we look at some interesting applications of graphs and colourings, particularly in computer science and telecommunication technology.

Generalisation of the Graph structure

If we look back at the formal definition of a graph, a graph is made up of a set of elements which we call vertices, and a family of subsets of this set, all of order two. This of course suggests that the structure can be generalised in more than one way. One such way is to give the edges direction, that is the edges are now ordered pairs of vertices, and are often referred to as arcs. We call this a directed graph, or digraph (Weisstein, 2000). Graphically we use an arrow on the edge to indicate its direction. A graph can be considered to be a special case of a digraph in which an edge can be considered to be two arcs, one in each direction. We will not dwell on this concept but rather look at the next idea of generalisation in more detail, particularly in the area of colouring.

Another way of generalising the graph structure is to remove the restriction of the size of the subsets, that is, rather than taking only subsets of size two, we can take larger subsets of vertices. This gives rise to the concept of a hypergraph. A hypergraph is made up of a set of elements called vertices, and a family of non-empty subsets called hyperedges or edges, which can now be of any size. An *r*-uniform hypergraph is a hypergraph in which all edges are of size *r*. So essentially a graph is a 2-uniform hypergraph.

The idea of looking at a family of sets as a similar structure to a graph started around 1960, by Berge and Lovasz amongst others (Berge, 1997). In regarding each set as a "generalised edge" and in calling the family itself a "hypergraph", their initial idea was to try to extend certain classical results of Graph Theory. However, it was noticed that this generalisation often led to simplification; moreover, one single statement, sometimes remarkably simple, could unify several theorems on graphs. In addition, the theory of hypergraphs is seen to be a very useful tool for the solution of integer optimization problems when the matrix has certain special properties which involve scheduling and location problems.

Several graph operations and results were thus studied in this new context, amongst which was the concept of colouring. In the next section we shall look at the area of colouring and several variations on the original theme which result when we consider hypergraphs. For standard graph theoretical notation we refer to (West, 2000).

Colouring Graphs and Hypergraphs

As previously mentioned, the Four Colour Theorem is considered the first graph colouring problem to be posed. It led to the definition of a proper vertex colouring of a graph, that is an assignment of colours to the vertices such that no two adjacent vertices receive the same colour, as well as the chromatic number of a graph $G, \chi(G)$, which is the minimum number of colours required for a proper colouring of G. Several families of graphs where studied with the aim of determining their chromatic number. Clearly, one can see that the chromatic number of a bipartite graph is 2, and this is another characterisation of bipartite graphs, and the chromatic number of a complete graph K_n is n. The proof of the Four Colour Theorem finally showed that the chromatic number of planar graphs is 4.

An obvious extension to consider was the colouring of the edges of a graph. An edge-colouring of a graph is an assignment of colours to the edges of the graph so that no two incident edges have the same colour. In 1964, Vizing (Vizing, 1964) proved that the minimum number of colours needed to edge-colour a simple graph, termed the chromatic index χ', is either its maximum degree Δ, or Δ+1. What followed was a study to classify all known graphs as Class 1 if $\chi' = \Delta$, or Class 2, when $\chi' = \Delta + 1$. This classification is not yet complete.

The colouring of hypergraphs started in 1966 when Erdös and Hajnal (Erdös & Hajnal, 1966) introduced the notions of colouring and the chromatic number of a hypergraph, and obtained the first important results. From the definition, in a proper colouring no edge is allowed to be monochromatic. In the literature these colourings are sometimes called classic colourings. This generalization of graph colourings initiated a wide area of further research. First, some old problems in set systems were formulated as colouring problems and many results in graph colouring were extended to hypergraphs. For example, using a natural generalization of the degree of a vertex, the classic theorem of Brooks', which for classic graph colouring states that the chromatic number is at most Δ, except for complete graphs and odd cycles (Brooks, 1941), was shown to hold for hypergraphs (Berge, 1989).

The fact that hypergraphs can contain more than two vertices in an edge allowed the exploration of different and more restricted colouring notions. The classic concept of hypergraph colouring is asymmetric: with the chromatic number as its central notion, this theory focuses on the minimum number of colours, while the maximum number of colours has no mathematical interest since a totally multicoloured vertex-set is always feasible. However, one can now consider the maximum number of colours when excluding polychromatic edges.

Voloshin (1993) introduced the concept of a mixed hypergraph colouring, which eliminated the above asymmetry and opened up an entirely new direction of research. Instead of H = (V, E), the basic idea is to consider a structure H = (V, C, D), termed a mixed hypergraph, with two families of subsets called C-edges and D-edges. By definition, a proper λ-colouring of a mixed hypergraph H = (V, C, D) is a mapping c : V → {1, 2, . . . , λ} for which two conditions hold:

- every C ∈ C has at least two vertices of a **C**ommon colour;
- every D ∈ D has at least two vertices of **D**ifferent colours.

A mixed hypergraph in which each edge is both a C-edge and a D-edge is called a bihypergraph (Voloshin, 1993). Here we require that each edge is non-monochromatic and non-rainbow (polychromatic), or NMNR for short, as termed in (Caro, et al., 2015). Note that, for classic graph colouring, a graph may be considered to be a 2-uniform mixed hypergraph in which all edges are D-edges.

This concept led to the discovery of new principal properties of colourings that do not exist in classical graph and hypergraph colourings: first of all, we now look at a lower chromatic number $\underline{\chi}$ and an upper chromatic number $\overline{\chi}$. The chromatic spectrum of a hypergraph H is the set of values k such that $\underline{\chi} \leq k \leq \overline{\chi}$ and H is *k*-colourable. We now have hypergraphs which are uncolourable, and perhaps most counterintuitively, which have gaps in the chromatic spectrum, that is there may exist integers *a, b* where *a*<*b*, such that the hypergraph H is *a*-colourable and *b*-colourable, but not *t*-colourable for some *a* < *t* < *b*.

So with the more general structure of a hypergraph, new concepts in vertex colourings emerged, such as mixed hypergraphs, and this led to further vertex-colouring variations. Bujtás and Tuza (2009) define colour-bounded hypergraphs as follows. We are given two integers, s_i and t_i associated with each edge e_i – a proper colouring requires that for each edge e_i, the number of colours used lies between s_i and t_i (inclusive). We can have a colouring such that $s_i = \alpha$, and $t_i = \beta$ are the same for each edge – we call this an (α,β)-colouring (Caro, et al., 2015). Given a hypergraph *H*, we have a proper (α,β)-colouring if the number of colours for every edge lies between α and β. So, for an r-uniform hypergraph, an NMNR-colouring is equivalent to a (2,r-1)-colouring, a classic colouring is equivalent to a (2,r)-colouring. This again illustrates the idea of generalisation, where a new definition encompasses previous definitions as special cases. Some properties of NMNR-colourings, such as uncolourability and chromatic spectrum gaps, can be extended to $(\alpha, \beta) -$ colourings.

We now describe *L*-colourings and *Q*-colourings of *r*-uniform hypergraphs, which can further encompass several different types of hypergraph colourings described in the literature. Such ideas, which originated in Voloshin's seminal work in Jiang et al. (2002), were more explicitly studied in Milici et al. (2001) , Griggs et al. (2008) and Quatrocchi (2001), and studied in much more generality for oriented graphs in Dvörak et al. (2010). We define *L*- and *Q*-colourings as follows (Caro, et al., 2016).

Let *H* be an *r*-uniform hypergraph, $r \geq 2$ and consider $E \in E(H)$. Then a colouring of the vertices of *E* induces a partition π of *r* whose parts are the numbers of vertices of each colour appearing in *E*. This partition is called the colour pattern of *E* and is written as $pat(E) = (n_1, n_2, \dots, n_k)$, where $n_1 \geq n_2 \geq \dots \geq n_k \geq 1$ and $\sum_{i=1}^{k} n_i = r$.

For any edge $E \in E(H)$, we assign $Q(E) \subseteq P(r)$, where *P(r)* denotes the set of all possible partitions of *r*. A colouring of the vertices of *H* is said to be an *L*-colouring, where $L = \{Q(E_i): i = 1, \dots, |E(H)|\}$, if $\forall E_i \in E(H),\ pat(E_i) \in Q(E_i)$. In the case when all the edges are assigned the same family of partitions *Q*, i.e. $Q(E_i) = Q, \forall E_i \in E(H)$, we call this a *Q*-colouring.

The main types which have been studied, some of which have already been mentioned, are now described in terms of *L*- and *Q*-colourings.

Two particularly important partitions of *r* will be used several times - the monochromatic partition (*r*) and the rainbow partition $(1,1,\dots,1)$. *M* and *R* are used to represent these partitions respectively.

- For classical graph colourings, $Q = \{\pi \in P(r): \pi = R\}$.
- For classical colourings of hypergraphs, $Q = \{\pi \in P(r): \pi \neq M\}$.
- In Voloshin colourings of hypergraphs or mixed hypergraphs (Voloshin, 1993), there exist two types of edges, *D*-edges and *C*-edges. A *D*-edge cannot be monochromatic, that is all vertices of the edge having the same colour, while a *C*-edge cannot be polychromatic(rainbow), that is all vertices having a different colour. Hence for all *D*-edges, $Q_D = Q(E) =$

$\{\pi \in P(r): \pi \neq M\}$, while for all *C*-edges, $Q_C = Q(E) = \{\pi \in P(r): \pi \neq R\}$, and $L = \{Q_C \cup Q_D\}$.

- A special case of this type of colouring is a non-monochromatic non-rainbow (NMNR) colouring, as discussed in (Caro, et al., 2015), which is a *Q*-colouring where $Q = \{\pi \in P(r): \pi \notin \{M, R\}\}$. Such hypergraphs are often referred to as bi-hypergraphs.
- An (α, β)-colouring of a hypergraph *H*, as described in (Caro, et al., 2015), is the case where $Q = \{\pi \in P(r): \pi = (n_1, n_2, \dots, n_k), \alpha \leq k \leq \beta\}$. This is based on the concept of colour-bounded hypergraphs first defined by Bujtás and Tuza in (Bujtás & Tuza, 2009). Observe that classical hypergraph colourings are (*2,r*)-colourings, while NMNR-colourings are (*2,r-1*)-colourings.
- Bujtás and Tuza defined another type of hypergraph colouring with further restrictions in (Bujtás & Tuza, 2009): a stably-bounded hypergraph is a hypergraph together with four colour-bound functions which express restrictions on vertex colourings. Formally, an *r*-uniform stably bounded hypergraph is a six-tuple *H* = (*V(H),E(H), s, t, a, b*), where *s,t,a* and *b* are integer-valued functions defined on *E(H)*, called colour-bound functions. We assume throughout that $1 \leq s \leq t \leq r$ and $1 \leq a \leq b \leq r$ hold. A proper vertex colouring of *H*= (*V(H), E(H), s, t, a, b*) satisfies the following three conditions for every edge $E \in E(H)$:.
    - The number of different colours assigned to the vertices of *E* is at least *s* and at most *t*.
    - There exists a colour assigned to at least *a* vertices of *E*.
    - Each colour is assigned to at most *b* vertices of *E*.

  Hence, such a colouring is a *L*-colouring where *L* is a family of Q_i such that for each Q_i, $Q_i = \{\pi \in P(r): \pi = (n_1, n_2, \dots, n_k), n_1 \geq \cdots \geq n_k, s \leq k \leq t, a \leq n_1 \leq b\}$.
- Another type of colouring which has been defined is conflict-free colouring, in which each edge contains a vertex with a “unique colour” that does not appear on any other vertex in *E*. The surveys (Pach & Tardos, 2009) and (Smorodinsky, 2013) include interesting results about, and applications of this type of colouring. In this case, $Q = \{\pi \in P(r): \pi = (n_1, n_2, \dots, n_k), n_1 \geq n_2 \geq \cdots \geq n_k = 1\}$.

This illustrates the power of generalisation, one of the main praxis in mathematics. What started as a single structure (the graph) and a defined concept for it (vertex colouring), led to variations of both structure and defined concept, leading to one unifying definition such that every variation can be described as a restricted case of this definition.

Applications

Mathematical structures, concepts and results are interesting in themselves, but more so when they prove to have important applications in solving problems from other disciplines. As we have seen, structures themselves are often the results of trying to solve an existing problem. We now look at a few applications of some of the results and methods which we have discussed.

We have seen that the main inspiration for the definition and study of vertex colourings in graphs was the Four Colour Theorem. The Groupe Spécial Mobile (GSM) was created in 1982 to provide a standard for a mobile telephone system. The first GSM network was launched in 1991 by Radiolinja in Finland with joint technical infrastructure maintenance from Ericsson. Today, GSM is the most popular standard for mobile phones in the world, used by over 2 billion people across more than 212 countries. GSM is a cellular network with its entire geographical range divided into hexagonal cells. Each cell has a communication tower which connects with mobile phones within the cell. All mobile phones connect to the GSM network by searching for cells in the immediate vicinity. GSM networks operate in only four different frequency ranges. The reason why only four different frequencies suffice is clear: the map of the cellular regions can be properly coloured by using only four different colours! So, the vertex colouring algorithm may be used for assigning at most four different frequencies for any GSM mobile phone network.

In fact, several applications of hypergraphs and hypergraph colouring in the world of telecommunications, particularly in the solution of resource allocation. One such example involves wireless communication, which is used in many different situations such as mobile telephony, radio and TV broadcasting, satellite communication, etc. In each of these situations a frequency assignment problem arises with application-specific characteristics. Several different modelling approaches have been developed for each of the features of the problem, such as the handling of interference among radio signals, the availability of frequencies, and the optimization criterion. The work described in Even et al. (2003) and Smorodinsky (2003) proposed to model frequency assignment to cellular antennas as conflict-free colouring. In the model developed in this research, one can use a very “small” number of distinct frequencies in total, to assign to a large number of antennas in a wireless network.

Another interesting application of mixed hypergraph colouring is the Byzantine agreement problem (Jaffe, et al., 2012). A set of n processors, any f of whom may be arbitrarily faulty, must reach agreement on a value proposed by one of the correct processors. The system is said to be f-tolerant if there is a protocol which ensures that after a finite number of steps, all correct processors receive the right value. It is a celebrated result that unless $n > 3f$, Byzantine agreement is impossible, due to the fact that faulty processors can equivocate, that is, say different things to different processors. It has been found that if all processors are grouped into all possible triplets to form a complete 3-uniform hypergraph such that within triplets, equivocation is not possible, then equivocation can be avoided across all processors and the system can be made Byzantine f-fault tolerant even when $n > 2f$. However, using all possible triplets as communication problems is too expensive in terms of algorithmic complexity. Using vertex-colouring theory, conditions for which a system is f-tolerant can be found without considering all possible triplets, this reducing processing time significantly.

References

Appel, K. & Haken, W. (1976) Every planar map is four colorable.. *Bulletin of the American mathematical Society,,* 82(5), 711-712.

Berge, C. (1989) *Hypergraphs: Combinatorics of Finite Sets.* s.l.:North-Holland.

Berge, C. (1997) Motivations and history of some of my conjectures. *Discrete Mathematics,* Volume 165/166, 61-70.

Brooks, R. L. (1941) On colouring the nodes of a network. *Mathematical Proceedings of the Cambridge Philosophical Society,* 37(194-197).

Bujtás, C. and Tuza, Z. (2009) Color-bounded hypergraphs, I: General results. *Discrete Mathematics,* Volume 309, 4890-4092.

Carlson, S. (2006) *Encyclopaedia Brittanica.* [Online] Available at: https://www.britannica.com/science/Konigsberg-bridge-problem [Accessed 26 August 2019].

Caro, Y., Lauri, J. and Zarb, C. (2015) Constrained colouring and σ-hypergraphs. *Discussiones Mathematicae Graph Theory,* 35(1), 171-189.

Caro, Y., Lauri, J. and Zarb, C. (2016) Selective hypergraph colourings. *Discrete Mathematics,* 339(4), 1232-1241.

Dvörak, J., Kara, K., Kral, O. and Pangrac, D. (2010) Pattern hypergraphs. *The Electronic Journal of Combinatorics,* 17(1), R15.

Erdös, P. and Hajnal, A. (1966) On chromatic number of graphs and set-systems. *Acta Math. Acad. Sci. Hungar.,* Volume 17, 61-99.

Euler, L. (1736) "Solutio problematis ad geometriam situs pertinentis". *Acad. Sci. U. Petrop,* Volume 8, 128-140.

Even, G., Lotker, Z., Ron, D. and Smorodinsky, S. (2003) Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular network. *Siam. J. Comput.,* Volume 33, 94-136.

Griggs, T., Lo Faro, G. and Quattrocchi, G. (2008) On some colouring of 4-cycle systems with specified block colour patterns. *Discrete Mathematics,* 308(2), 465-478.

Jaffe, A., Moscibroda, T. and Sen (2012) On the price of equivocation in byzantine agreement. In: *Proceedings of the 2012 ACM symposium on Principles of distributed computing.* s.l.:ACM, 309-318.

Jiang, T. et al. (2002) The Chromatic Spectrum of Mixed Hypergraphs. *Graphs and Combinatorics,* 18(2), 309-318.

Milici, S., Rosa, A. and Voloshin, V. (2001) Colouring steiner systems with specified block colour patterns. *Discrete Mathematics,* 240(13), 145-160.

Mitchem, J. (1981) On the history and solution of the four-color map problem. *The Two-Year College Mathematics Journal,* 12(2), 108-116.

Nishizeki, T. and Chiba, N. (1988) *Planar graphs: Theory and algorithms (Vol. 32).* s.l.:Elsevier.

Pach, J. and Tardos, G. (2009) Conflict-free colourings of graphs and hypergraphs. *Combinatorics, Probability and Computing,* 18(5), 819-834.

Quattrocchi, G. (2001) Colouring 4-cycle systems with specified block colour patterns: the case of embedding P3-designs. *Journal of Combinatorics,* 8(1), R24.

Reid, J. (2010) *Wikipedia (digital image).* [Online] Available at: https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg [Accessed August 2019].

Smorodinsky, S. P. t. S. o. C. (2003) *Combinatorial Problems in Computational Geometry.* PhD thesis, School of Computer Science, Tel-Aviv University,: s.n.

Smorodinsky, S. (2013) Conflict-free coloring and its applications. In: *Geometry-Intuitive, Discrete, and Convex.* s.l.:Springer, 331-389.

Vizing, V. G. (1964) On an estimate of the chromatic class of a p-graph. *Discret Analiz,* Volume 3, 25-30.

Voloshin, V. (1993) The mixed hypergraphs. *Computer Sci. J. Moldova,* Volume 1, 45-52.

Weisstein, E. (2000) *"Directed Graph." From MathWorld--A Wolfram Web Resource..* [Online] Available at: http://mathworld.wolfram.com/DirectedGraph.html [Accessed August 2019].

West, D. (2000) *Introduction to Graph Theory.* 2nd ed. s.l.:Prentice Hall.

Bio-note

Dr Christina Zarb is a Mathematics lecturer at the Junior College, University of Malta, where she has been teaching for seven years. She previously taught in other Higher Education Institutions including MCAST and St. Martins Institute. She is also a casual lecturer at the University of Malta. Christina has a Bachelor's degree in Mathematics and Computer Science, as well as a Master's degree and Ph.D. in Mathematics, all conferred by the University of Malta. Her area of specialisation is Graph Theory and Combinatorics. She has taught mathematics at various levels, starting from secondary school up to post-graduate level and is very interested in the transition process between the different levels of education, particularly in mathematics.