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Malik Rachmanov
The University of Texas Rio Grande Valley

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Malik Rakhmanov
California Institute of Technology, Pasadena, California 91125
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New static spherically symmetric solutions of the Einstein-Maxwell gravity with the dilaton field are found. The solutions correspond to black holes and naked singularities. In addition to mass and electric charge these solutions are labeled by a new parameter, the dilaton charge of the black hole. Depending on the values of electric and dilaton charges there are different types of solutions. The solutions exhibit a new type of a symmetry. Namely one solution transforms into another when the mass and the dilaton charge are interchanged. We also found that there is a finite interval of values of electric charge for which no black hole can exist. This gap separates two different types of solutions. Inside the gap the solution can exist only if the dilaton charge is exactly equal to the negative of the mass. The behavior of the solutions in the extremal regime is also analyzed.

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I. INTRODUCTION

The low-energy limit of string theory includes a scalar dilaton field, which is massless in all finite orders of perturbation theory [1]. However, in order not to conflict with classical tests of the tensor character of gravity, the physical dilaton should have mass. This mass is assumed to arise due to nonperturbative features of the quantum theory. At the classical level, and at distance scales small compared to the dilaton Compton wavelength, we can neglect the mass and study the effect of the dilaton on low-energy physics. In particular, the dilaton modifies Maxwell's equations and it affects the geometry of the space-time. For example, solutions that correspond to electrically charged black holes are modified by the presence of the dilaton. Such solutions were studied in [4]–[10], where it was shown that the dilaton changes the causal structure of the black hole and leads to curvature singularities at finite radii. The black hole solution obtained in [9] was also extensively studied in connection with extremal dilaton black holes. It was argued that such a black hole behaves like an elementary particle in the sense that its excitation spectrum has an energy gap [11]–[13].

Although the dilaton field naturally arises in string theory its existence from the point of view of general relativity is quite problematic. A generic scalar field can violate the equivalence principle; see the discussion in [2]. On the other hand, in black hole physics the inclusion of a scalar field leads to the appearance of a "baryon number" associated with the field. In the case of the dilaton field this is the dilaton charge. It is generally believed that no parameters other than mass, electric charge, and angular momentum can be associated with a black hole (see [3]). This conjecture essentially rules out the existence of the scalar field in the exterior of a static black hole. Indeed, if we assume that the black hole has a regular horizon, then following the arguments in [3] we arrive at the conclusion that the scalar field must be constant. On the other hand, the inclusion of the scalar field immediately results in singularities at the horizon. Strictly speaking, there is no horizon anymore. The surface of metric discontinuity becomes the surface where the scalar curvature diverges. In this case the arguments of [3] do not apply because the surface integral diverges. This leaves the possibility for the existence of the scalar field in the exterior of a black hole if one admits singularities at finite radii. However, on very general grounds it is unlikely that such a surface of singularities can ever appear.

In this paper we will not discuss all these issues. Here we simply assume that there is a scalar field in the exterior of the electrically charged black hole and study possible consequences of that. We will see that in most cases the dilaton field destroys the horizons of the black hole and leads to appearance of the curvature singularities. It will be shown that there is a one-parameter family of spherically symmetric asymptotically flat solutions with different dilaton charges. For a particular choice of the dilaton charge the solution reduces to the one obtained in [9], from now on referred to as the Garfinkle-Horowitz-Strominger (GHS) solution. This is the only true black hole solution of the entire family. The other solutions correspond to naked singularities of different types.

We will use geometrical units $c = G = 1$ throughout the paper. The line interval for a static spherically symmetric space-time can be written as

$$ds^2 = -\alpha^2 dt^2 + \beta^2 dr^2 + \gamma^2 (d\theta^2 + \sin^2\theta \, d\varphi^2), \qquad (1)$$

where α, β, and γ are functions of the radial coordinate r only. The solution that corresponds to the black hole with mass M and electric charge Q is given by the Reissner-Nordström metric with $\gamma^2(r) = r^2$ and

$$\alpha^2(r) = \beta^{-2}(r) = \left(1 - \frac{r_1}{r}\right)\left(1 - \frac{r_2}{r}\right),$$

where

$$r_{1,2} = M \pm \sqrt{M^2 - Q^2}.$$

This solution has two horizons $r = r_{1,2}$ when $Q^2 < M^2$. If $Q^2 = M^2$ the two horizons coincide and the black hole is said to be extremal. In the case when $Q^2 > M^2$ there are no horizons.

The metric is divergent when r approaches r_1 or r_2. However, invariant quantities made out of components of the Riemann curvature tensor are regular at the horizons. In particular, the scalar curvature is zero everywhere except the origin, where the true singularity resides.

The geometry of space-time is very different in the presence of the dilaton. Even in the case of pure dilaton gravity, when there is no electromagnetic field, a singularity appears at a finite radius.

Inclusion of the dilaton ϕ leads to the appearance of a conserved charge, the dilaton charge. In static space-time it is defined by

$$D = \frac{1}{4\pi}\oint \nabla_k \phi \, dS^k .$$

The integration is taken over a spacelike surface enclosing the origin. The conservation means that the value of the dilaton charge does not depend on the choice of the surface. This is a simple consequence of the equation of motion for the scalar field: $\nabla_\mu \nabla^\mu \phi = 0$.

Static spherically symmetric solutions of the Einstein equations and the dilaton equation are completely defined by the dilaton charge D and the mass M. The metric components are given by

$$\alpha^2(r) = \beta^{-2}(r) = \left(1 - \frac{r_s}{r}\right)^{\frac{2M}{r_s}} , \tag{2}$$

$$\gamma^2(r) = r^2 \left(1 - \frac{r_s}{r}\right)^{1-\frac{2M}{r_s}} . \tag{3}$$

The dilaton field is defined up to an arbitrary constant, its value at infinity:

$$\phi(r) = \phi_\infty + \frac{D}{r_s} \ln\left(1 - \frac{r_s}{r}\right) . \tag{4}$$

The constant is irrelevant and can be set to zero. The quantity that plays the role of the Schwarzschild radius is defined by

$$r_s = 2\sqrt{M^2 + D^2}. \tag{5}$$

Thus the metric shows a singularity at $r = r_s$. Unlike the Schwarzschild and the Reissner-Nordström solutions this is a true singularity. This can be seen from the formula for the scalar curvature:

$$R(r) = \frac{2D^2}{r^{2+\frac{2M}{r_s}} (r - r_s)^{2-\frac{2M}{r_s}}} .$$

Therefore, the scalar curvature becomes infinite as r approaches r_s for any nonzero value of the dilaton charge.

In the following sections we describe solutions that include both a dilaton and an electromagnetic field. The solutions share the main features of the Reissner-Nordström geometry and pure dilaton gravity. These are the appearance of two horizons and the presence of a curvature singularity at the horizons.

II. ACTION AND SYMMETRIES

The form of the action in four dimensions is suggested by the low-energy limit of string theory

$$S = -\frac{1}{16\pi}\int \sqrt{|g|}\,(R - 2\nabla_\mu\phi\nabla^\mu\phi - e^{-2\phi}F_{\mu\nu}F^{\mu\nu})\, d^4x.$$

The equations of motion for the metric $g_{\mu\nu}$, the vector potential A_μ and the dilaton field ϕ are

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{6}$$

$$\nabla_\mu (e^{-2\phi} F^{\mu\nu}) = 0, \tag{7}$$

$$\nabla_\mu \nabla^\mu \phi + \frac{1}{2} e^{-2\phi} F_{\mu\nu} F^{\mu\nu} = 0. \tag{8}$$

The components of the Einstein tensor and the energy-momentum tensor are given in Appendix.

The symmetries of the action are general covariance and gauge symmetry. In addition, the action is invariant under the global scale transformations

$$\tilde{\phi}(x) = \phi(x) + \Lambda,$$

$$\tilde{A}_\mu(x) = e^\Lambda A_\mu(x).$$

This freedom can be eliminated by specifying ϕ_∞, the value of the dilaton at infinity. If nonzero this value will result in a screening of the electric charge:

$$E(r \to \infty) \simeq \frac{Q e^{2\phi_\infty}}{r^2}.$$

In what follows we assume that $\phi_\infty = 0$. The equations for nonzero ϕ_∞ can be obtained by suitable redefinitions.

The Nöther current corresponding to the global scale transformations is

$$J_\mu = \nabla_\mu \phi + e^{-2\phi} F_{\mu\nu} A^\nu .$$

Note that this current is not gauge invariant. However the conserved charge, associated with the current, is gauge invariant provided that only those gauge transformations that vanish at infinity are allowed.

III. EQUATIONS OF MOTION

The metric for a static spherically symmetric space-time is given by Eq. (1). This form of the metric remains unchanged under the following transformation, which is a remnant of general coordinate invariance:

$$r \to \tilde{r} \quad \text{and} \quad \beta^2 \to \beta^2 \left(\frac{dr}{d\tilde{r}}\right)^2 . \tag{9}$$

We will use this freedom to choose β^2 in such a way that the Einstein equations of motion are simplified.

A spherically symmetric electric field is everywhere radial:

$$F_{rt} = f(r).$$

The Maxwell equations (7) can then be integrated and give a generalization of the Gauss law for curved spacetime with the dilaton:

$$\frac{\gamma^2}{\alpha\beta}e^{-2\phi}f = Q, \tag{10}$$

where Q is electric charge. The Einstein equations (6) are not all independent. This can easily be seen in the orthonormal frame, where $G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}$. The corresponding components of the energy-momentum tensor also coincide (see Appendix). We take the following linear combinations of them as independent equations:

$$G_{\hat{r}\hat{r}} + G_{\hat{\theta}\hat{\theta}} = 0, \tag{11}$$

$$G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = 2\frac{\phi'^2}{\beta^2}, \tag{12}$$

$$G_{\hat{t}\hat{t}} - G_{\hat{r}\hat{r}} = 2e^{-2\phi}\frac{f^2}{\alpha^2\beta^2}. \tag{13}$$

The prime denotes differentiation with respect to the radial coordinate r. Note that Eq. (11) contains only metric fields. It can be written as

$$\left[\frac{(\alpha\gamma)'\gamma}{\beta}\right]' = \alpha\beta. \tag{14}$$

This equation suggests the gauge choice

$$\alpha\beta = 1. \tag{15}$$

By making this choice we fix the freedom in Eq. (9). Then Eq. (14) can be integrated and gives

$$(\alpha\gamma)^2 = (r - r_1)(r - r_2), \tag{16}$$

where r_1 and r_2 are arbitrary constants. The situation when both r_1 and r_2 are real and positive corresponds to a black hole with two horizons located at $r = r_{1,2}$. An extremal black hole arises when $r_1 = r_2$. In the case when both r_1 and r_2 are complex numbers, with r_2 the complex conjugate of r_1, the solution gives a naked singularity. All this is in close analogy with the Reissner-Nordström solution. We will often use the linear combinations of the parameters:

$$\bar{r} = \frac{r_1 + r_2}{2} \quad \text{and} \quad \Delta = r_1 - r_2.$$

In the gauge, Eq. (15), the Einstein equations (12) and (13) take the form

$$\gamma'' + \gamma\phi'^2 = 0, \tag{17}$$

$$(\alpha^2\gamma\gamma')' = 1 - e^{-2\phi}\gamma^2 f^2. \tag{18}$$

The dilaton equation (8) also simplifies and can be written as

$$(\alpha^2\gamma^2\phi')' = e^{-2\phi}\gamma^2 f^2. \tag{19}$$

The differential equations (17)–(19) together with the constraints (15),(16) constitute a complete set of equations of motion.

To solve the equations of motion we reduce them to an equation with only one field; this will be the dilaton field. First combining Eqs. (18) and (19), we obtain

$$(\alpha^2\gamma^2\phi')' + (\alpha^2\gamma\gamma')' = 1.$$

Then integrating it and taking into account Eq. (16) we obtain

$$\phi' + \frac{\gamma'}{\gamma} = \frac{r - C}{(r - r_1)(r - r_2)}. \tag{20}$$

Here C is an arbitrary constant of integration with the dimensions of length. This parameter determines the type of solution. By making the particular choice $C = r_1$(or r_2) one can remove the singularity at $r = r_1$(or r_2). However, it is not possible to remove both singularities at the same time. We will show that the constant C is essentially given by the dilaton charge of the solution.

Eliminating γ between Eqs. (20) and (17), we obtain the equation with the dilaton field only:

$$\phi'' - 2\,\phi'^2 + 2\,\frac{r - C}{(r - r_1)(r - r_2)}\,\phi' = \frac{(C - r_1)(C - r_2)}{(r - r_1)^2(r - r_2)^2}. \tag{21}$$

This is an ordinary differential equation with singular coefficients. The choice $C = r_1$ makes the coefficient in front of ϕ' regular at $r = r_1$. It also makes the right-hand side of the equation vanish. In this case the exact solution was found in [9]. A very similar solution may be found if one chooses $C = r_2$. Here we allow C to be arbitrary. With arbitrary C the dilaton equation (21) also can be integrated exactly.

Before we describe the integration let us introduce some notation that will prove to be useful. Instead of C we will use σ defined by the equation

$$C = \bar{r} + \sigma.$$

It will be also convenient to introduce μ and ν defined by

$$\mu = \frac{\sigma}{\Delta}, \qquad \nu^2 = \frac{1}{2} - \frac{\sigma^2}{\Delta^2}.$$

Note that μ and ν are not independent variables.

IV. INTEGRATION OF THE DILATON EQUATION

To integrate the dilaton equation (21) we first introduce a new radial coordinate ρ by the formula

$$\rho = \frac{r - r_1}{r - r_2}.$$

Here we assume that r_1 and r_2 are real and unequal with $r_1 > r_2$. Let us also introduce a new function $\psi(\rho)$:

$$\phi'(r) = \frac{(1-\rho)^2}{2\rho\Delta}\,\psi(\rho).$$

These choices of the new function and variable greatly simplify the dilaton equation (21). It can now be written as

$$\rho \frac{d\psi}{d\rho} = (\psi + \mu)^2 - \nu^2. \tag{22}$$

Then one can see that there are three different types of solutions. They correspond to the following possibilities:

type I $\nu^2 > 0$,

type II $\nu^2 < 0$,

type III $\nu^2 = 0$.

To complete the integration of the dilaton equation let us write the integral of Eq. (22) for the type-I solution

$$\psi + \mu - \nu = \frac{2\nu\rho^{2\nu}}{\text{const} - \rho^{2\nu}}. \tag{23}$$

All three types of solutions will be discussed in the following sections.

V. SOLUTIONS OF TYPE I

The condition $\nu^2 > 0$ implies that a solution of type I has values of the parameter μ that are restricted to the interval $|\mu| < 1/\sqrt{2}$. In this range ν is a real number. We take ν positive. Integration of Eq. (23) gives a solution for the dilaton field:

$$e^{2\phi} = \frac{\rho^{-\mu}\Delta}{r_1\rho^{-\nu} - r_2\rho^{\nu}}. \tag{24}$$

Knowing ϕ, one can find a solution for the metric components. For the first component α^2 we obtain

$$\alpha^2 = \frac{\rho^{\mu}\Delta}{r_1\rho^{-\nu} - r_2\rho^{\nu}}. \tag{25}$$

Then the solution for the metric component γ^2 can be found from Eq. (16). We rewrite this equation in terms of the new radial coordinate

$$\alpha^2\gamma^2 = \frac{\rho\Delta^2}{(1-\rho)^2}. \tag{26}$$

The solution for the electric field is a simple consequence of the Gauss law, Eq. (10):

$$f = \frac{(1-\rho)^2}{\rho} \frac{Q}{(r_1\rho^{-\nu} - r_2\rho^{\nu})^2}. \tag{27}$$

These equations are written for the exterior of the black hole, $r > r_1$. General equations describing both the exterior and the interior of the black hole would include absolute values. In order not to complicate the equations, we assume that $r > r_1$ and omit the absolute value signs whenever possible.

The arbitrary constants r_1 and r_2, which correspond to the horizons, can be found from the asymptotic behavior at spatial infinity of the g_{tt} component of the metric:

$$\alpha^2(r) \simeq 1 - \frac{2M}{r}.$$

One more condition is needed. The asymptotic behavior of the electric field does not provide one, because it is satisfied automatically. We take instead Eq. (19). So the result is

$$r_1(\nu + \mu) + r_2(\nu - \mu) = 2M, \tag{28}$$

$$2\nu^2 r_1 r_2 = Q^2. \tag{29}$$

These equations allow us, in principle, to find r_1 and r_2. However, one should be careful when using these equations because μ and ν themselves depend on r_1 and r_2 through Δ. Apart from M and Q, only σ is a completely arbitrary parameter.

VI. DILATON CHARGE

The solution for the pure dilaton black hole suggests that the free parameter σ must somehow be related to the dilaton charge. To calculate the dilaton charge one needs to know the scalar potential A_t. For the static electric field, Eq. (27), the scalar potential is defined, up to an arbitrary constant, as its value at infinity:

$$A_t(r) = A_t(\infty) - \frac{Q}{2\nu}\left(\frac{\rho^{-\nu} - \rho^{\nu}}{r_1\rho^{-\nu} - r_2\rho^{\nu}}\right).$$

Knowing the potential, one can find the components of the dilaton current and calculate the flux of the dilaton field through a closed spacelike surface. The result is

$$\text{flux} = \oint J_k\, dS^k = 4\pi[M - \sigma - QA_t(\infty)].$$

Note that the flux does not depend on the choice of the surface. The dilaton charge is defined as the flux per unit solid angle. Assuming that the electric potential vanishes at infinity, we obtain the relation between the free parameter σ and the dilaton charge:

$$D = M - \sigma.$$

Thus we see that the arbitrary constant C is linearly related to the dilaton charge. The formula for the dilaton charge, although quite simple, has many consequences. One of them is the possibility of having a black-hole solution with two horizons even when the mass is zero.

VII. THE HORIZONS AND THE GAP

Let us analyze the horizons of the black hole with dilaton charge. To find the locations of the horizons we need to solve Eqs. (28) and (29), which we rewrite now as

$$\nu(r_1 + r_2) = M + D, \tag{30}$$

$$2\nu^2 r_1 r_2 = Q^2. \tag{31}$$

Note that ν depends on r_1 and r_2 through Δ. One way to solve these equations is to find Δ first. Simple calculations lead to

$$\Delta^2 = 4(M^2 + D^2 - Q^2). \tag{32}$$

Knowing Δ, we can now solve Eqs. (30), (31) for r_1 and r_2. The result is

$$r_{1,2} = \sqrt{M^2 + D^2 - Q^2}\left[\pm 1 + \frac{M+D}{\sqrt{(M+D)^2 - 2Q^2}}\right]. \tag{33}$$

The plus sign corresponds to r_1. We see that both $r_{1,2}$ are real and positive provided that

$$Q^2 < \frac{1}{2}(M+D)^2.$$

This is the region of validity of the type-I solution we described earlier. Thus we see that the space-time around the type-I black hole has two horizons.

If we increase the electric charge so that it approaches the critical value given by

$$Q^2 = \frac{1}{2}(M+D)^2, \tag{34}$$

both horizons extend to infinity and $\nu^2 \to +0$. Therefore in the limit this particular case corresponds to the solution of type III.

If we further increase the electric charge so that

$$\frac{1}{2}(M+D)^2 < Q^2 < M^2 + D^2, \tag{35}$$

we find that both $r_{1,2}$ become complex numbers with the same imaginary part. That means r_1 and r_2 are not complex conjugate to each other. Now one can see that $\nu^2 < 0$, and we are in the region of the type II solution.

Finally, the electric charge can be so large that

$$Q^2 > M^2 + D^2.$$

Here again $r_{1,2}$ are complex numbers, but now r_2 is the complex conjugate of r_1. Of course, Δ becomes pure imaginary. As a consequence of this $\nu^2 > 0$ and we again have a solution of type I, but now without horizons.

An important observation comes from the condition in Eq. (35). Namely, the upper limit is always greater than or equal to the lower limit. This leads to the existence of a gap that separates the two regions of the type-I solution. In either of these regions the solution with given mass and electric charge is a one-parameter family with arbitrary values of the dilaton charge. We will see that inside the gap there are no solutions unless the dilaton charge takes a special value.

When the upper limit coincides with the lower limit, the solution of type I with two horizons can be continuously transformed into the solution of type I with no horizons. This is only possible when the gap shrinks to a single point. This happens if the dilaton charge takes the value

$$D = M.$$

For all other values of the dilaton charge there is a finite gap, the region where real solutions are forbidden.

The different regions and the boundaries between them are shown in Fig. 1. It is convenient to take the pair (D, Q^2) as Cartesian coordinates on the plane. Then the curves corresponding to the boundaries in Eq. (35) are two parabolas, one being above the other. The gap is the region in between the curves (the shaded area in Fig. 1). Above and below the gap there are solutions of type I. The gap shrinks to a point when the two curves touch each other. This is the point with coordinates $(M, 2M^2)$. At this point the curves have a common tangent line defined by

$$Q^2 = 2MD.$$

Inside the gap real solutions do not exist, except when $D = -M$. This segment of a vertical line corresponds to the solution of type II.

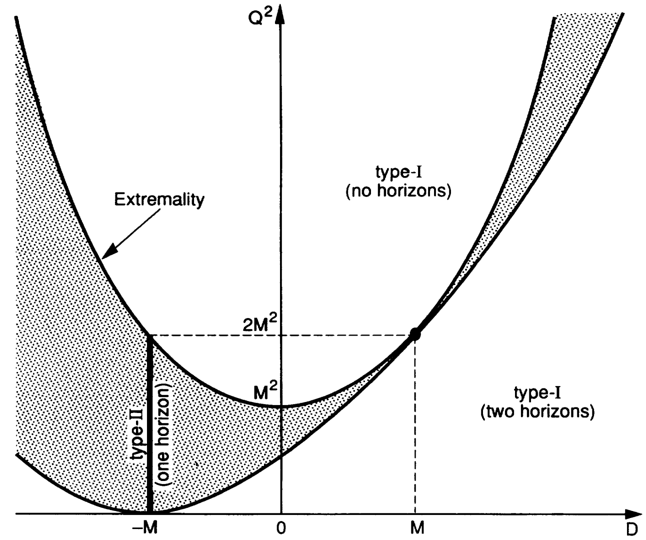

FIG. 1. The gap and the regions for different types of solutions.

VIII. DUALITY OF TYPE I SOLUTIONS

There is one feature of a solution of type I which is peculiar to dilaton black holes only. This is a symmetry between the mass and the dilaton charge of the solution. Let us take a solution of type I which is characterized by mass M and dilaton charge D. If we interchange M and D, we obtain another solution of type I. The equations for the metric and dilaton fields of this solution can be obtained from the corresponding equations of the former solution by the substitution $\mu \to -\mu$. The new solution is quite different from the old one, though both solutions have horizons at the same locations. The two solutions are dual to each other in the sense that the role of the mass in one solution is played by the dilaton charge in the other. The relation between the two solutions leads us to think of them as parts of one solution. Namely, suppose the solution of the type I above has $D < M$. By gradually increasing the dilaton charge to values $D > M$, we continuously transform the solution into its dual. Of course, at the moment when $D = M$ the solution and its dual coincide. Examples of dual solutions are given in the next section.

IX. SPECIAL CASES OF TYPE I SOLUTIONS

It is interesting to consider various special cases of type-I solutions with two horizons. First let us set one of the parameters to zero, while the other two remain unrestricted.

(i) $Q = 0$. When the electric charge vanishes the inner horizon shrinks to a point. For the outer horizon we find that $r_1 = r_s$ [see Eq. (5)], and the solution reduces to the one of pure dilaton gravity, Eqs. (2)–(4). This is not surprising because setting $Q = 0$ is equivalent to dropping the Maxwell term in the action.

(ii) $D = 0$. When the dilaton charge vanishes, the solution is specified by mass and electric charge, as in the case of the Reissner-Nordström black hole. However, it is very different from the latter. One of the differences is that values of the electric charge in the interval $M^2/2 < Q^2 < M^2$ are not allowed. We will also see that the curvature becomes singular at the horizons.

(iii) $M = 0$. Unlike other classical black holes, the dilaton black hole can exist without mass. The role of the mass in the formation of the horizons is now played by the dilaton charge. This solution is dual to the previous one.

(iv) $Q^2 = 2MD$. Let us now consider a special case of the solution of type I that corresponds to the tangent line. In this case the horizons are given by particularly simple expressions. For $D < M$ they are

$$r_1 = 2M \quad \text{and} \quad r_2 = 2D. \tag{36}$$

The metric also becomes very simple:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(1 - \frac{2D}{r}\right) (d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{37}$$

The dilaton is formally independent of mass:

$$\phi(r) = \frac{1}{2} \ln \left(1 - \frac{2D}{r}\right). \tag{38}$$

This is the GHS solution. It has the remarkable property of being regular at the outer horizon. As we noted earlier, this solution corresponds to the choice $C = r_1$ in the dilaton equation.

Quite naturally, for $D > M$ we find that the horizons are switched,

$$r_1 = 2D \quad \text{and} \quad r_2 = 2M, \tag{39}$$

while the metric and the dilaton are given by the same expressions as before. This is the dual solution. It corresponds to the choice $C = r_2$, and it is regular at the inner horizon.

X. CURVATURE SINGULARITIES

As we already saw in the case of pure dilaton gravity, the scalar curvature is divergent at a finite value of the radial coordinate. This is also true for the general solution of type I. One way to study the behavior of the curvature is to use the equations of motion. Note that when the equations of motion are satisfied we have a simple formula for the scalar curvature, $R = 2\alpha^2\phi'^2$ (see Appendix). Using this formula, one can see that when $r \to r_1{+}0$, or equivalently $\rho \to +0$, the curvature diverges as

$$R \propto (\mu - \nu)^2 \rho^{-n}, \text{ where } n = 2 - \mu - \nu \geq 1.$$

At the other horizon $r \to r_2{-}0$, or equivalently $\rho \to +\infty$, the scalar curvature is also divergent:

$$R \propto (\mu + \nu)^2 \rho^{m}, \quad \text{where} \quad m = 2 + \mu - \nu \geq 1.$$

These equations show that generically the scalar curvature is divergent at both horizons. One can make the curvature regular at $r = r_1$ by setting $\mu = \nu$. This condition leads to the GHS solution. Therefore the only solution of type I that has regular curvature at $r = r_1$ is the GHS solution. Similarly one can set $\mu = -\nu$ and obtain the solution with regular curvature at $r = r_2$. This is the dual solution. It is also clear that there is no solution for which the scalar curvature is regular at both horizons.

XI. SOLUTIONS OF TYPE II AND TYPE III

We know that when Q^2 takes the critical value given by Eq. (34) the parameter ν vanishes and we should obtain a solution of type III. In principle, we should be able to find the equations for the metric components and the dilaton field of the type III solution from the corresponding equations of the type I solution by taking the limit $\nu^2 \to +0$. However, as Q^2 approaches the critical value both horizons extend to infinity while Δ remains finite. The immediate consequence is that in the limit $\rho = 1$ for all values of r. One can see then that some equations, for example Eq. (26), become undefined. Thus the limit $\nu^2 = 0$ is divergent and there are no solutions of type III.

Now let us turn to the solution of type II, i.e., $\nu^2 < 0$. This solution appears only when the condition in Eq. (35) is satisfied. This is the region inside the gap. Here r_1 and r_2 are complex but not complex conjugate to each other. Rather, they are related by $r_2 = -r_1^*$. This means that the sum $r_1 + r_2$ is pure imaginary. This situation is very undesirable, because the quadratic form in Eq. (16) must be real. Remember that the quadratic form is a product of the metric components $(-g_{tt})g_{\theta\theta}$. Therefore the only possibility for a real solution to exist in this region is for $r_2 = -r_1$. This can happen if the dilaton charge takes the special value

$$D = -M.$$

Knowing this, it is easy to construct the solution of type II from the known solution of type I. The result is

$$e^{2\phi} = \frac{\rho^{-\mu}}{\cos(\lambda \ln \rho)}, \quad \alpha^2 = \frac{\rho^{\mu}}{\cos(\lambda \ln \rho)},$$

where λ is a real root of $\lambda^2 = -\nu^2$. Again, γ^2 can be found from Eq. (26). The solution above is written in terms of the radial coordinate ρ now defined by

$$\rho = \frac{r - r_1}{r + r_1}.$$

The black hole described by this solution has only one horizon for all allowed values of the electric charge. It is located at

$$r_1 = \sqrt{2M^2 - Q^2}.$$

Thus we see that inside the gap the solution exists only if the dilaton charge "compensates" the mass. For all other values of the dilaton charge there are no real solutions in this region.

XII. SOLUTION OF TYPE I ABOVE THE GAP

The last possibility to consider is the region above the gap:

$$Q^2 > M^2 + D^2.$$

In this case both r_1 and r_2 are complex and one is complex conjugate to the other. Let us denote $r_1 = u + iv$, where u and v can be found from Eq. (33). Then the radial coordinate ρ becomes a phase on the complex plane:

$$\rho = \frac{r - r_1}{r - r_1^*} = e^{-2i\eta}, \quad \text{where} \quad \tan\eta = \frac{v}{r - u}.$$

Knowing this, one can readily obtain a solution for the dilaton field;

$$e^{2\phi} = \frac{e^{\frac{\sigma}{v}\eta}}{\cos 2\nu\eta + \frac{u}{v}\sin 2\nu\eta},$$

and the first metric component

$$\alpha^2 = \frac{e^{-\frac{\sigma}{v}\eta}}{\cos 2\nu\eta + \frac{u}{v}\sin 2\nu\eta}.$$

The other metric component γ^2, again, can be found from Eq. (16). This equation written in terms of η takes the form

$$\alpha^2\gamma^2 = \frac{v^2}{\sin^2\eta}.$$

The solution has no horizon at all. Since $\nu^2 > 0$, it is type I, though quite different from the type I solution we described above.

XIII. EXTREMAL BLACK HOLES

In Sec. VII we obtained the expression for Δ in terms of the mass and the electric and dilaton charges. Extremal black holes appear when Δ vanishes. One can see that no matter where the horizon is located, extremality occurs when

$$Q^2 = M^2 + D^2. \tag{40}$$

If we approach extremality from above with some arbitrary value of the dilaton charge we inevitably obtain that $r_1 = r_2 = 0$. If we try to approach extremality from below by increasing the electric charge in the solution of type I, then we encounter the gap well before we meet extremality (see Fig. 1). By approaching extremality along the line of the solution of type II we again obtain that $r_1 = r_2 = 0$. The only possibility that remains is to approach the extremal curve, Eq. (40), at the point where the gap disappears. This is the only point where the solution of type I with two horizons can reach extremality. Thus we see that an extremal black hole can have a nonvanishing horizon only if the extremality condition Eq. (40) is supplemented by

$$D = M.$$

As we know, the point is located at $(M, 2M^2)$. If we approach this point from anywhere below the tangent line, $Q^2 = 2MD$, we obtain the solution

$$\alpha^2 = e^{2\phi} = \frac{r - r_1}{r + r_0}, \tag{41}$$

$$\gamma^2 = (r + r_0)(r - r_1). \tag{42}$$

Here the horizon r_1 and the shift r_0 are given by

$$r_1 = \sqrt{2}M \quad \text{and} \quad r_0 = (2 - \sqrt{2})M.$$

We can also approach the point by moving along the tangent line; see Eqs. (36)–(39). This way we obtain the same solution, namely Eqs. (41) and (42), but with zero shift

$$r_1 = 2M \quad \text{and} \quad r_0 = 0.$$

It is clear that we cannot redefine the radial coordinate in the solution of type I so that both procedures would lead to the solution with zero shift. The ambiguity in the shift comes from the fact that the orders of singularity of the coefficients in the dilaton equation (21) change abruptly when $r_1 = r_2$. The value of the shift is unphysical due to the symmetry $\tilde{r} = r + \text{const}$ and can be set to zero. Thus we see that at the point $(M, 2M^2)$ the extremal solution indeed has a finite horizon. The horizon, however, is singular. Since the scalar curvature diverges at the horizon (see Sec. X) it is a singularity rather than an event horizon.

XIV. ARBITRARY COUPLING CONSTANT

To study the dependence of the dilaton black hole solution on the strength of interaction between the dilaton and the electromagnetic field, an arbitrary coupling constant a was introduced in [9]. This parameter modifies the Maxwell term in the action in the following way:

$$e^{-2a\phi}F_{\mu\nu}F^{\mu\nu}.$$

The introduction of this coupling constant makes it possible to have both weak ($a \ll 1$) and strong ($a \gg 1$) coupling regimes. Then the solution should reduce to the Reissner-Nordström solution in the limit $a=0$. The choice $a=1$ emerges from string theory. We already described all the possible black hole solutions in the case $a=1$. The overall picture described above changes very little when the constant a is allowed to take arbitrary values. Again there is a finite gap that separates two regions of the solution of type I. The boundaries of the gap are given by

$$\frac{(M+aD)^2}{a^2+1} < Q^2 < M^2+D^2.$$

Inside the gap a solution exists only if the dilaton charge takes the special value $D=-M/a$. The solution in this case will be type II. The boundaries of the gap touch each other at the point

$$D=aM \quad \text{and} \quad Q^2=(a^2+1)M^2.$$

Again, this is the only point where an extremal solution can have a nonvanishing horizon.

Explicit formulas show that the solution of type I indeed reduces to the Reissner-Nordström solution in the limit $a=0$. The solution of type II exists only for $a>0$.

XV. CONCLUDING REMARKS

We have seen that the dilaton field drastically affects the space-time geometry of electrically charged black holes. The inclusion of the dilaton almost inevitably destroys the horizons of the Reissner-Nordström solution. The only exception is the GHS solution, which has a regular outer horizon but whose inner horizon is singular. In this paper we described all possible solutions assuming that the dilaton charge can take arbitrary values. It is interesting that we found the gap, the region where no real solution can exist. We also have seen that the dilaton charge plays a role quite similar to that of the mass. Moreover, we found that the simple interchange of the mass and the dilaton charge in one of the solutions gives us another solution of the same set of equations of motion. This fact we called duality. These are all effects of the dilaton field. Although the existence of the dilaton black holes is problematic due to the curvature singularities, the exact solutions we described above provide a simple framework for studying different effects of the dilaton in general relativity.

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APPENDIX

It is convenient to write the components of various tensors in the orthonormal frame defined by the tetrad

$$e^{\hat{\mu}}_{\nu} = \mathrm{diag}\{\alpha,\, \beta,\, \gamma,\, \gamma\sin\theta\}.$$

In this frame the nonzero components of the Riemann curvature tensor for the metric of Eq. (1) are

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \frac{1}{\alpha\beta}\left(\frac{\alpha'}{\beta}\right)',$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \frac{1}{\beta^2}\frac{\alpha'\gamma'}{\alpha\gamma},$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{1}{\beta\gamma}\left(\frac{\gamma'}{\beta}\right)',$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{\gamma^2} - \frac{\gamma'^2}{\beta^2\gamma^2}.$$

The components of the Ricci tensor and the scalar curvature can be found from these formulas by contraction. The Einstein tensor has the nonzero components

$$G_{\hat{t}\hat{t}} = \frac{2}{\beta^2}\left(\frac{\beta'\gamma'}{\beta\gamma} - \frac{\gamma''}{\gamma}\right) + \frac{1}{\gamma^2} - \frac{\gamma'^2}{\beta^2\gamma^2},$$

$$G_{\hat{r}\hat{r}} = \frac{2}{\beta^2}\frac{\alpha'\gamma'}{\alpha\gamma} - \frac{1}{\gamma^2} + \frac{\gamma'^2}{\beta^2\gamma^2},$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{1}{\beta^2}\left(\frac{\alpha''}{\alpha} - \frac{\alpha'\beta'}{\alpha\beta} + \frac{\alpha'\gamma'}{\alpha\gamma} - \frac{\beta'\gamma'}{\beta\gamma} + \frac{\gamma''}{\gamma}\right).$$

In the orthonormal frame the components of the energy-momentum tensor are especially simple:

$$8\pi T_{\hat{\mu}\hat{\nu}} = \frac{\phi'^2}{\beta^2}\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix} + e^{-2\phi}\frac{f^2}{\alpha^2\beta^2}\begin{pmatrix}1&&&\\&-1&&\\&&1&\\&&&1\end{pmatrix}.$$

The electromagnetic part of the energy-momentum tensor is traceless. Thus the only contribution to the scalar curvature comes from the dilaton field. By taking the trace of the Einstein equation one finds that $R = 2\beta^{-2}\phi'^2$. This equation provides a simple way to obtain the scalar curvature for a metric that is a solution of Einstein equations.

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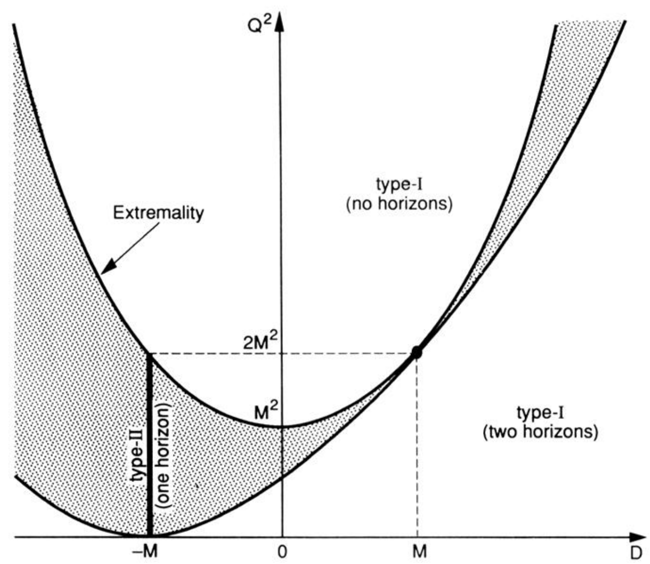

FIG. 1. The gap and the regions for different types of solutions.