1996

Hierarchical search strategy for the detection of gravitational waves from coalescing binaries

Soumya Mohanty
The University of Texas Rio Grande Valley

S. Dhurandhar

Hierarchical search strategy for the detection of gravitational waves from coalescing binaries

S. D. Mohanty and S. V. Dhurandhar
Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind Pune 411 007, India
(Received 18 March 1996)

The detection of gravitational waves from coalescing compact binaries would be a computationally intensive process if a single bank of template wave forms (i.e., a one-step search) is used. We present, in this paper, a method which leads to a large reduction in the computational power required as compared to a one-step search. This method is a hierarchical search strategy involving two template banks. We show that the computational power required by such a two-step search, for an on-line detection of the one-parameter family of Newtonian signals, is 1/8 of that required when an on-line one-step search is used. This reduction is achieved when signals having a strength of ~ 8.8 are required to be detected with a probability of ~ 0.95 and an average of one false event per year. We present approximate formulas for the detection probability of a signal and the false alarm probability. We investigate the effect of statistical correlations on these probabilities and incorporate these effects wherever possible. Our numerical results are specific to the noise power spectral density expected for the initial LIGO. [S0556-2821(96)00224-X]

PACS number(s): 04.80.Nn, 95.55.Ym, 95.75.Pq, 97.80.Af

I. INTRODUCTION

The inspiral, because of gravitational radiation reaction, of a binary composed of compact massive objects (neutron stars or black holes) will produce a gravitational wave signal [1] which, during the last few minutes before merger, will lie within the bandwidths of upcoming laser interferometric detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) [2], VIRGO [3], and GEO600. The wave form of this signal can be computed with enough accuracy to allow pattern-matching techniques, such as matched filtering, to considerably enhance the signal to noise ratio [1,4]. Therefore, it should be possible to detect such events up to a large distance and hence observe a significant event rate.

The technique of matched filtering is equivalent to correlating the detector output with a wave form, called the *template*, which is constructed from the expected signal itself. This technique is derived from the more general method of maximum likelihood detection [5]. When applied to the detection of a signal with a fixed wave form, this method involves the computation of a functional, called the likelihood ratio (LR), of the given data. A detection is announced if the LR exceeds a preset threshold. For coalescing binaries, however, a wide range of signal wave forms are possible corresponding to different values of the parameters which characterize a binary, such as the masses and spins of the compact bodies among others. The LR then becomes dependent on these parameters. In such a case, it is required that the LR be maximized over the whole parameter space and the maximum, called the test statistic, be compared with a threshold. Depending on the particular realization of noise present in the data, it is possible that the test statistic stays below the threshold even though a signal is present. It is also possible that it exceeds the threshold in the absence of a signal because of noise alone. Thus, a given signal can be detected only with a certain probability called the detection probability. The probability with which noise alone can masquerade as a signal is called the false alarm probability.

Computationally, efficient methods for the maximization of the LR are known for only a subset of the parameters involved. It is for the maximization over one such parameter that the correlation of the detector output with a template wave form is computed. For the remaining parameters, however, there is no straightforward method. The strategy which has been considered most often is to carry out the maximization over a discrete set of values for these parameters [6–9]. Therefore, a set of templates would be required corresponding to the set of values used for these parameters. Such a set is called a *bank* of templates. The use of a bank of templates implies, however, that not all of the possible signals can be detected equally well. This is because the signal to noise ratio, hence the detection probability of a signal, is reduced if its corresponding template is not used in obtaining the correlations [6,7]. Such a mismatch is bound to arise in a bank of templates for most of the signals. An arbitrarily fine spacing of the parameter values is not possible because of practical limitations imposed by the time required for the maximization and the available computing power. One criterion which can be used to space the parameter values is that all signals with a given minimum energy be detected with a given minimum detection probability. This is one of the criteria we adopt in this paper.

A method to set up a bank of templates satisfying a similar requirement was presented in [6,7]. It has been termed as the Sathyaprakash-Dhurandhar (SD) formalism for a *one-step search*. However, detection probability was not used explicitly by the authors in their computations. It was found only after the set of templates was obtained. Also, statistical dependencies among the correlation outputs were neglected. The SD formalism can be used, nevertheless, to obtain estimates of the computational power required for implementing the detection strategy outlined above. Such estimates [8,9], for the on-line detection of coalescing binary wave forms incorporating post-Newtonian corrections, appear to be excessive. Therefore, it would be desirable to reduce the computational load if possible. One such way could be to use a hierarchy of template banks in which information provided by a lower level is used to choose a subset of templates from the next level. The first level in such a strategy would consist of a template bank with a coarse spacing in parameter space but a lower threshold. In the next level a more finely spaced set of templates and a higher threshold would be used but only around those templates, in the previous level, which produce a crossing of the lower threshold. An appropriate choice of thresholds and spacings would reduce the number of templates employed compared to that required by a one-step search and hence, save on the computational cost.

The use of such a strategy, known as a *hierarchical search*, has been proposed a number of times [8–10]. However, a detailed formalism for the same has not been given so far in the context of the detection of gravitational wave signals. We present in this paper a rigorous formalism to describe a two-step hierarchical search and a first estimate of the numbers involved. We use the family of Newtonian wave forms for our computations and the noise power spectral density expected for the initial LIGO. We have not used more accurate wave forms so as to avoid the extra complications that would come in because of a larger number of parameters. They would obscure an important objective of this paper which is to highlight some subtle issues regarding hierarchical search in general. However, this formalism can be easily extended to accommodate a larger number of parameters as well as a larger number of intermediate stages.

The main result of this paper is that the computational power required for an on-line two-step search can be up to a factor of 8 smaller than that for an on-line one-step search. This happens when a detection probability of $\simeq 0.95$ is sought for signals having a signal to noise ratio (SNR) of 8.8σ and the false alarm is kept so that there is, on the average, not more than one false event per year. An SNR of 8.8σ corresponds to a distance of 38.5 Mpc for a 1.4 – 1.4 $M_\odot$ binary for the noise power spectral density used here. This factor of 8 is, however, not the last word since although our formalism can yield higher factors (~ 13) , the assumption of statistical independence made in some of the formulas breaks down. The correlations among the outputs of the templates become important when the templates are placed finely and this happens when the factor of 8 is exceeded.

A convenient parameter for characterizing the family of Newtonian signals is the chirp time which roughly equals the time taken by a binary to coalesce starting from a fiducial orbital frequency. This parameter depends on the masses of the components of the binary. The maximum value used for the chirp time, in this paper, is 138.0 sec which corresponds to a 0.5–0.5 $M_\odot$ binary. The lowest value used for the chirp time is 2.0 sec. We find that the minimum computational power required, for an on-line two-step search, for this range of chirp time is 167 MFlops. We also present our results using a lower value of 32.0 sec for the maximum chirp time which corresponds to a 1.2–1.2 $M_\odot$ binary. For this case, the computational power required is 32 MFlops.

Another objective of this paper is to study the effect of statistical correlations on the distributions of the test statistic. It is found that, in the presence of correlations, at least one of these distributions can be fit quite accurately by a distribution with a smaller set of statistically independent samples. The effective number of samples required does not seem to have any simple connection with the autocorrelation function of the noise. The issue of distribution functions of the test statistic was partially addressed in [11]. The authors found the distribution function of a subsidiary test statistic which was maximized over a single parameter only, namely, the initial phase of the signal. We use the complete test statistic in this paper.

We also present a refined version of the SD formalism that explicitly takes detection probability into account. We emphasize that nothing radically new has been incorporated in this formalism. Our main purpose, however, is to use it as a guide in setting up the two-step search and to provide a more concrete context for a discussion of the distribution functions of the test statistic. In this connection, we present a semianalytic method for the computation of the drop in signal to noise ratio as a function of the difference between the chirp times of a signal and a template.

The rest of the paper is organized as follows. In Sec. II we discuss the distribution functions which govern the test statistic in the presence and absence of a signal. These will be required to obtain the detection probability of a signal and the false alarm probability. We start in Sec. II A with an outline of the signal detection problem and the method of maximum likelihood detection. The family of signals and the noise to be used in this paper will be described in Sec. II B. We obtain the test statistic relevant to this choice of signals and noise in Sec. II C. We discuss the distribution functions of this test statistic in Sec. II D. An approximate expression for detection probability is obtained which will also be used in the case of a two-step search. The false alarm probability will turn out to be dependent on the method used for the computation of the test statistic. Hence, it will be discussed in Sec. II E along with a rigorous reformulation of the SD formalism for a one-step search.

Section III is devoted to the two-step hierarchical search. The distribution functions obtained in the previous section will be required here also. In Sec. III A, a general formalism, and associated set of notations, is introduced to describe a two-step hierarchical search. In Sec. III B it is shown that there exists an optimum set of spacings and thresholds which minimizes the computational requirement. An algorithm to obtain this optimum solution is presented. In Sec. III C the computing power required for an on-line two-step search is estimated and numerical results are presented. We conclude with Sec. IV.

II. FALSE ALARM AND DETECTION PROBABILITIES

Some of the results in the following will be obtained from or compared with Monte Carlo simulations. For these simulations, we have mainly used the Gaussian random number generator, G05FDF, provided in the NAg library of numerical routines. Wherever possible, the results have been checked for consistency with those obtained using the routine GASDEV provided in Ref. [12].

A. The signal detection problem

We denote the parameters characterizing a family of wave forms by $\overline{\mu}$ and the family itself by $s(t;\overline{\mu})$. A specific member of the family is identified by a subscript on $\overline{\mu}$. We denote a segment of the output of a detector as $x(t)$, where $t=0$ and $t=T$ will denote the beginning and the end of the segment, respectively. If the signals have finite durations then T should be larger than the longest signal. A particular realization of the noise contaminating $x(t)$ will be denoted as $n(t)$. For a given $x(t)$, the problem of detection consists of discriminating between the following cases:

$$x(t)=\begin{cases} n(t) & \text{when there is no signal,} \\ n(t)+s(t;\overline{\mu_s}) & \text{when there is a signal with parameters } \overline{\mu_s}\,. \end{cases}$$

The range of every parameter will, in general, be finite. In the absence of any prior information, it must be assumed that all the parameter values, within their respective ranges, are equally likely to occur. In such a case, the method of *maximum likelihood detection* [5] can be used. When the noise is a stationary Gaussian random process, maximum likelihood detection (MLD) reduces to (a) the computation of a *test statistic*, defined below, and (b) its comparison with a threshold η. The test statistic Λ is obtained as

$$\Lambda=\max_{\overline{\mu}}(\langle x(t),s(t;\overline{\mu})\rangle-\tfrac{1}{2}\langle s(t;\overline{\mu}),s(t;\overline{\mu})\rangle), \qquad (1)$$

where the maximization is performed over all the values of the parameters. The angular brackets enclosing two functions denote an inner product which can be written in the Fourier domain as ('' $\widetilde{\ }$ '' denotes a Fourier transform)

$$\langle x(t),g(t)\rangle=\int_0^\infty df\,\frac{1}{S_n(f)}[\widetilde{x}(f)\widetilde{g}^*(f)+\widetilde{x}^*(f)\widetilde{g}(f)], \qquad (2)$$

where the Hermitian property of the Fourier transform of a real function has been used to restrict the domain of integration to positive frequencies. $S_n(f)$ is the power spectral density of the noise which is defined by

$$E[\widetilde{n}^*(f)\widetilde{n}(f')]=S_n(f)\,\delta(f-f'), \qquad (3)$$

where $E[z]$ stands for the ensemble average of a random variable z.

Consider, the detection problem mentioned above. Because of the presence of noise, Λ becomes a random variable. Its probability density function will depend on the presence or absence of the signal. We denote the two distribution functions as $P_1(\Lambda;\overline{\mu_s})$ when the signal is present, and $P_0(\Lambda)$ when there is no signal. The different distributions arise because the presence of the signal changes the mean values of the noise samples. It turns out that when $\Lambda>\eta$, it is more probable that the density function of which Λ is a sample is $P_1(\Lambda;\overline{\mu_s})$ rather than $P_0(\Lambda)$. This allows a discrimination to be made between the two cases above. Because of the random nature of Λ though, there is no guarantee that it will cross the threshold whenever the signal is present. In general, in the presence of any signal, one can only associate a probability with the event that $\Lambda>\eta$. This is known as the detection probability of that signal. For the case being considered here, the detection probability, $Q_d(\eta;\overline{\mu_s})$, is

$$Q_d(\eta;\overline{\mu_s})=1-\int_0^\eta P_1(\Lambda;\overline{\mu_s})d\Lambda. \qquad (4)$$

The event $\Lambda>\eta$ can also occur in the absence of any signal because of noise alone. This then leads to an error in detection. The probability of such an event is called the false alarm probability (shortened to ''false alarm'' henceforth) which we denote by $Q_0(\eta)$:

$$Q_0(\eta)=1-\int_0^\eta P_0(\Lambda)d\Lambda. \qquad (5)$$

It is the allowed false alarm, along with $S_n(f)$, that fixes η. From Eqs. (5) and (4), we see that the determination of detection and false alarm probabilities requires a knowledge of the cumulative distribution functions of Λ.

B. The noise power spectral density and the Newtonian wave form

The noise in ground-based laser interferometric detectors will have, in general, both a Gaussian and a non-Gaussian component. The main sources for the Gaussian component [13] are the shot noise because of photon counting, the thermal noise in the mirror suspensions along with the mirror itself and seismic noise. The non-Gaussian component can be contributed by numerous sources such as sudden strain releases in the mirror suspension wires or even lightning strikes [2,14]. It should be possible to remove most of the non-Gaussian component by using environmental monitors and looking for coincidence among detectors located at widely separated sites. It is, therefore, assumed usually that the detector noise will be a Gaussian random process. Over a time scale of hours, it can also be assumed to be stationary. Thus, the method of maximum likelihood detection, as outlined in the previous section, can be used in this case.

The power spectral density of the Gaussian noise component rises very steeply towards the low frequency end because of seismic effects. At the high frequency end, it is dominated by photon shot noise which leads to a rise towards higher frequencies. Thus, the data will have to be bandpassed with a low frequency seismic cutoff f_a and a high frequency cutoff f_c. We use the power spectral density expected for the initial LIGO as given in [13]. Accordingly, we choose $f_a=40$ Hz and $f_c=1$ kHz.

The lowest-order approximation to the wave form of the gravitational wave emitted by a coalescing compact binary is provided by the quadrupole formalism [15]. The response of an interferometric detector to such a wave form can be written as [7],

$$h(t;\mathcal{A},t_a,\xi,\Phi)=\mathcal{A}a(t-t_a,\xi)\cos[\phi(t-t_a,\xi)+\Phi]. \quad (6)$$

We choose this wave form as the signal that is to be detected. The parameter $\mathcal{A}$ takes into account the distance to the binary as well as various geometrical factors connected with the orientation of the orbital plane of the binary relative to the plane of the sky and the orientation of the detector antenna pattern [1,16,17,18]. When the detector is optimally oriented and the plane of the binary coincides with that of the sky,

$$\mathcal{A}=1.92\times10^{-23}\left[\frac{\xi}{25.0}\right]^{-1}\left[\frac{r}{100\text{ Mpc}}\right]^{-1}\left[\frac{f_a}{40\text{ Hz}}\right]^{-2}. \quad (7)$$

Because of the Earth's rotational and orbital motions, the orientation of the detector antenna pattern will change with time, making $\mathcal{A}$ time dependent. For observations lasting a few minutes, however, it is effectively a constant. The other, time-dependent, part of the amplitude is

$$a(t,\xi)=\left[1-\frac{t}{\xi}\right]^{-1/4}. \quad (8)$$

The phase of the wave form $\phi(t,\xi)$ can be expressed as

$$\phi(t,\xi)=2\pi\int_0^t f(t',\xi)dt'. \quad (9)$$

The integrand $f(t,\xi)$ is the instantaneous frequency of the signal which is given by

$$f(t,\xi)=f_a a(t,\xi)^{3/2}. \quad (10)$$

Thus, the wave form is a *chirp* whose amplitude and instantaneous frequency increase with time. The rate at which the instantaneous frequency increases is governed by the parameter ξ, called the *chirp time*,

$$\xi=34.54\left[\frac{\mathcal{M}}{M_\odot}\right]^{-5/3}\left[\frac{f_a}{40\ \text{Hz}}\right]^{-8/3}\text{ sec}, \quad (11)$$

where $\mathcal{M}$, the *chirp mass*, is the following combination of the reduced mass μ and the total mass M of the binary:

$$\mathcal{M}=(\mu^3M^2)^{1/5}. \quad (12)$$

Because of the seismic cutoff, the amplitude of the signal becomes negligible when its instantaneous frequency lies below f_a. The time at which the instantaneous frequency of the wave form reaches f_a is denoted by t_a. The high frequency cutoff f_c will also force the amplitude to a negligible value for instantaneous frequencies beyond f_c. In addition, this nature of the wave form will change when the compact bodies plunge towards one another once the last stable orbit is reached. This would happen when $f(t)\sim10^3$ Hz. Thus, the infinite instantaneous frequency implied by Eq. (10) at $t=\xi$ will not be reached in reality. The wave form has, therefore, an effectively finite duration given, to a very good approximation, by ξ. For a binary consisting of neutron stars with $m_1=m_2=1.4M_\odot$ the value of $\xi=24.8$ sec. For binaries with less massive components, ξ would be larger. In this paper we present our results for two different ranges of chirp times. The minimum value $\xi_{\min}$ is the same for both the ranges, $\xi_{\min}=2.0$ sec. The maximum values of the chirp time are taken as $\xi_{\max}=32.0$ sec (for $m_1=m_2\simeq1.2M_\odot$) and $\xi_{\max}=138.0$ sec (for $m_1=m_2\simeq0.5M_\odot$). The parameter t_a can be taken as the *time of arrival* of the signal. The phase of the signal at t_a is denoted by Φ.

The test statistic [Eq. (1)] was expressed in terms of Fourier domain representations. It has not been possible so far to obtain the Fourier transform of $h(t;\mathcal{A},t_a,\xi,\Phi)$ exactly, but it can be calculated approximately by using the method of stationary phase [7]. The approximate form for positive frequencies is

$$\tilde{h}(f;\mathcal{A},t_a,\xi,\Phi)=\mathcal{A}\sqrt{\xi}\left[\frac{2}{3f_a}\right]^{1/2}\left[\frac{f}{f_a}\right]^{-7/6}\exp[i\psi(f)], \quad (13)$$

where

$$\psi(f)=-2\pi ft_a+2\pi f_a\xi\alpha(f)+\Phi+\frac{\pi}{4},$$

$$\alpha(f)=\frac{1}{5}\left(8-3\left[\frac{f}{f_a}\right]^{-5/3}-5\frac{f}{f_a}\right). \quad (14)$$

We can also write the wave form in Eq. (6) as

$$h(t;\mathcal{A},t_a,\xi,\Phi)=\mathcal{A}h_0(t-t_a;\xi)\cos(\Phi)+\mathcal{A}h_{\pi/2}(t-t_a;\xi)\sin(\Phi), \quad (15)$$

where

$$h_0(t;\xi)=a(t,\xi)\cos[\phi(t,\xi)], \quad (16)$$

$$h_{\pi/2}(t;\xi)=a(t,\xi)\cos[\phi(t,\xi)+\pi/2]. \quad (17)$$

This representation will be helpful in what follows. The "quadrature" components, h_0 and $h_{\pi/2}$, have the properties

$$\langle h_0(t;\xi),h_0(t;\xi)\rangle\approx\langle h_{\pi/2}(t;\xi),h_{\pi/2}(t;\xi)\rangle, \quad (18)$$

$$|\langle h_0(t;\xi),h_{\pi/2}(t;\xi)\rangle| \ll\sqrt{\langle h_0(t;\xi),h_0(t;\xi)\rangle\langle h_{\pi/2}(t;\xi),h_{\pi/2}(t;\xi)\rangle}. \quad (19)$$

It should be noted that if the Fourier transforms of h_0 and $h_{\pi/2}$ obtained in the stationary phase approximation were to be used, then Eq. (18) would be an equality while the left-hand side (LHS) of Eq. (19) would vanish. In practice the ratio of LHS to the right-hand side (RHS) in Eq. (19) is typically $\sim10^{-3}$. In the following, this ratio will turn out to be the statistical correlation between two Gaussian variables and it will not make any significant difference if it is taken to be zero. Similarly, the two sides in Eq. (18) can be taken to be equal in the following. Actually, these approximations are not a limitation because we can always orthonormalize h_0 and $h_{\pi/2}$ using Schmidt's orthogonalization. For our analysis, however, such small effects will not make much difference and in the following, we treat Eq. (18) as an equality while the LHS of Eq. (19) is taken to be zero.

C. The test statistic and its computation

We now apply the strategy of maximum likelihood detection to the noise and family of signals described above. The test statistic in this case can be written as

$$\Lambda = \max_{(\mathcal{A},t_a,\xi,\Phi)} [\langle x(t), h(t;\mathcal{A},t_a,\xi,\Phi)\rangle - \tfrac{1}{2}\langle h(t;\mathcal{A},t_a,\xi,\Phi), h(t;\mathcal{A},t_a,\xi,\Phi)\rangle]. \quad (20)$$

The maximization over $\mathcal{A}$ and Φ can be carried out, in a straightforward manner [19], using Eqs. (15), (18), and (19) (see the comments below the last two equations). This yields an equivalent test statistic which we continue to denote as Λ:

$$\Lambda = \max_{(t_a,\xi)} \sqrt{C_0^2(t_a,\xi) + C_{\pi/2}^2(t_a,\xi)}, \quad (21)$$

where

$$C_0(t_a,\xi) = \langle x(t), \mathcal{N}_h h_0(t-t_a;\xi)\rangle = \mathcal{N}_h \int_{f_a}^{f_c} df \frac{1}{S_n(f)} [\tilde{x}(f)\tilde{h}_0^*(f;\xi)e^{2\pi i f t_a} + \tilde{x}^*(f)\tilde{h}_0(f;\xi)e^{-2\pi i f t_a}], \quad (22)$$

$$C_{\pi/2}(t_a,\xi) = \langle x(t), \mathcal{N}_h h_{\pi/2}(t-t_a;\xi)\rangle = \mathcal{N}_h \int_{f_a}^{f_c} df \frac{1}{S_n(f)} [\tilde{x}(f)\tilde{h}_{\pi/2}^*(f;\xi)e^{2\pi i f t_a} + \tilde{x}^*(f)\tilde{h}_{\pi/2}(f;\xi)e^{-2\pi i f t_a}]. \quad (23)$$

The functions $\mathcal{N}_h \tilde{h}_0(f;\xi)/S_n(f)$ and $\mathcal{N}_h \tilde{h}_{\pi/2}(f;\xi)/S_n(f)$ are the Fourier domain representations of the *templates* $h_0(t;\xi)$ and $h_{\pi/2}(t;\xi)$, respectively. The presence of noise turns $C_0(t_a,\xi)$ and $C_{\pi/2}(t_a,\xi)$ into random variables. Demanding that their variances be unity fixes the normalization constant $\mathcal{N}_h$. We express $\mathcal{N}_h$ explicitly in Sec. II D.

A linear correlation $c(\tau)$ [20,21], between two wave forms $r(t)$ and $g(t)$ is given by

$$c(\tau) = \int_{-\infty}^{\infty} r(t-\tau)g(t)dt = \int_0^{\infty} df[\tilde{r}^*(f)\tilde{g}(f)e^{2\pi i f\tau} + \tilde{r}(f)\tilde{g}^*(f)e^{-2\pi i f\tau}].$$

A comparison with Eqs. (22) and (23) shows that, for a fixed ξ, C_0 (or $C_{\pi/2}$) can be computed, as functions of t_a, by simply obtaining a linear correlation between $x(t)$ and the template $h_0(t;\xi)$ [or $h_{\pi/2}(t;\xi)$]. The role of the lag τ in a correlation is played here by t_a. Since a correlation can be efficiently computed using a fast Fourier transform (FFT), maximization over t_a becomes a straightforward operation of computing the correlation of $x(t)$ with two templates, squaring and summing these correlations for each value of τ, and finding the maximum of the result over τ. However, as mentioned in Sec. I, the remaining parameter ξ is not amenable to such a simple treatment. The method most commonly discussed, in this connection, is to perform the maximization over a discrete set $\{\xi_j\}$. This would yield an approximation to the test statistic.

Thus, the overall form of the detection strategy which results, given $\{\xi_j\}$ and a threshold η, is the following. For each $\xi_m \in \{\xi_j\}$, the detector output is separately correlated with two templates $h_0(t;\xi_m)$ and $h_{\pi/2}(t;\xi_m)$ to obtain $C_0(\tau,\xi_m)$ and $C_{\pi/2}(\tau,\xi_m)$, respectively. Then, the quantity $X(\tau;\xi_m) = \sqrt{C_0^2(\tau,\xi_m) + C_{\pi/2}^2(\tau,\xi_m)}$ is computed. We call $X(\tau;\xi_m)$ the *rectified* output of a template with chirp time ξ_m. We denote the maximum, over τ, of a rectified output $X(\tau;\xi_m)$ by λ_m. These operations lead, therefore, to the construction of a set $\{\lambda_j\}$. An approximation to the test statistic is then obtained which we again denote by Λ:

$$\Lambda = \max_j \{\lambda_j\}. \quad (24)$$

Finally, Λ would be compared with η. This method for the calculation of the test statistic has been termed as a *one-step search.*

The set $\{\xi_j\}$, as well as the corresponding set of templates, will be termed as a *bank of templates*. It is convenient to refer to the set of two templates, $h_0(t;\xi)$ and $h_{\pi/2}(t;\xi)$, having the same chirp time as a single "template." This has already been done tacitly in the case of a rectified output. This "template" can then be labeled with its chirp time. Thus, a term such as "template ξ_m" is understood in the following to mean a set of two templates, both having the same chirp time ξ_m. Also, templates (in the above sense) "lying" close or far from each other would, actually, imply the same for their chirp times. A helpful picture to adopt is to treat the set $h_0(t;\xi_m)$ and $h_{\pi/2}(t;\xi_m)$ as a black box which has a single input, namely, $x(t)$, and a single output, namely, $X(\tau;\xi_m)$. Each box is labeled by a chirp time and is called a template. This terminology should ensure clarity in the following which would otherwise get masked by a lot of redun-

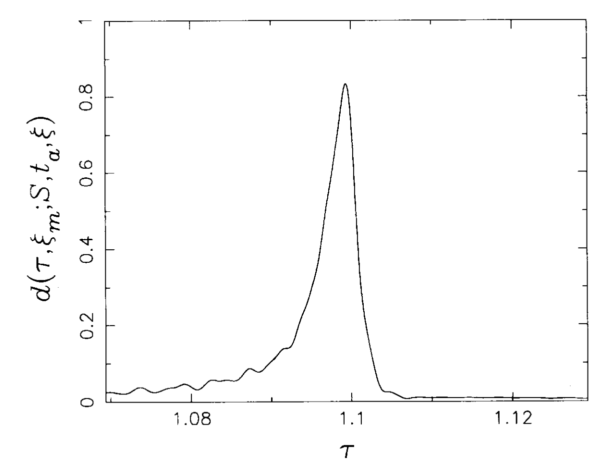

FIG. 1. The processed form of a signal. The template chirp time was chosen as $\xi_m = 4.0$ sec while the signal chirp time $\xi = 4.10$ sec. The time of arrival of the signal was chosen to be $t_a = 1.0$ sec and the strength [see Eq. (26)], $S = 1.0$.

dant text. However, on the few occasions when the correct use of the term ''template'' would be required, we will explicitly use h_0 and $h_{\pi/2}$.

It is important to realize that Λ, obtained as above, is only an approximation to the true test statistic. No claim can be made as to the uniqueness of the procedure used to obtain this approximation. Also, the cumulative distribution functions which characterize Λ will now depend on the method used to obtain it. In the present case, this dependence will be in terms of a set of parameters characterizing the bank of templates. If the template chirp times were chosen arbitrarily, all of them would have to be treated as parameters. In the case of a two-step search, the dependence would be on two banks of templates and two thresholds. It turns out that the family of Newtonian wave forms has a nice feature which allows the bank of templates to be characterized by a single parameter. However, this need not be true in general. In fact, this simplicity will not be obtained when a parameter such as the chirp mass is used instead of the chirp time. We will denote the distribution functions of Λ as $\mathcal{F}_0(z)$ when no signal is present, and $\mathcal{F}_1(z;\overline{\mu_s})$ when a signal with parameters $\overline{\mu_s}$ is present. The dependence of these functions on the template bank used is implicit but we will make it explicit wherever required.

We now define some quantities which will be of use later on. Consider a signal $h(t;\overline{\mu_s})$ where $\overline{\mu_s}$ stands for the parameters of the signal. We denote by $d(\tau,\xi_m;\overline{\mu_s})$, for $\xi_m \in \{\xi_j\}$, the quantity

$$d(\tau,\xi_m;\overline{\mu_s})=\mathcal{N}_h\sqrt{\langle h(t;\overline{\mu_s}),h_0(t-\tau;\xi_m)\rangle^2+\langle h(t;\overline{\mu_s}),h_{\pi/2}(t-\tau;\xi_m)\rangle^2}. \tag{25}$$

This is the rectified output of a template when $x(t)=h(t;\overline{\mu_s})$ and noise is absent. We call such a rectified output as the *processed form* of the signal produced by the template with chirp time ξ_m. One such processed form is shown in Fig. 1. The processed form of a signal has the property that a change in the signal's time of arrival simply translates it along the rectified output. It is easy to show, using the stationary phase approximation, that the processed form is independent of the signal parameter Φ. However, the exact processed form is not strictly independent of Φ but the variation is typically $\sim 0.1\%$ around the peak of the processed form. This is an entirely negligible effect in our analysis. Therefore, we will henceforth suppress Φ whenever the signal parameters in $d(\tau,\xi_m;\overline{\mu_s})$ are required to be expressed explicitly.

The maximum value that any processed form of a signal can have is called the *strength S* of the signal [7]. This value is attained only for a template with the same chirp time as that of the signal. For a signal with amplitude $\mathcal{A}$,

$$S=\frac{\mathcal{A}}{\mathcal{N}_h}, \tag{26}$$

where $\mathcal{N}_h$ is the normalization constant for a template with the same chirp time as that of the signal.

Our definition of the strength of a signal is identical, within the orthonormality of h_0 and $h_{\pi/2}$, to the *signal to noise ratio* $(S/N)[h]$, defined in [16]. The inner product $(f|g)$ defined in [16] is identical to our inner product $\langle f,g\rangle$. The signal to noise ratio in [16] is given by $(h|h)^{1/2}$ and hence, is equal to $\langle h,h\rangle^{1/2}$. Thus, we get $\langle h,h\rangle^{1/2} =\mathcal{A}\langle h_0,h_0\rangle^{1/2}=(\mathcal{A}/\mathcal{N}_h)[\mathcal{N}_h^2\langle h_0,h_0\rangle]^{1/2}=(\mathcal{A}/\mathcal{N}_h)$ (by the definition of $\mathcal{N}_h$) which is nothing but the strength S defined as above. Henceforth, we use S instead of $\mathcal{A}$ to parametrize a signal. In this paper, the typical value for the strength is $S\sim 9.0$. For the canonical binary consisting of neutron stars with an identical mass of 1.4 $M_\odot$ for both the components, this value of the strength corresponds to a distance of 38.0 Mpc. A signal of strength 10.0 from such a binary would correspond to a lower distance of 34.0 Mpc. Note that the strength of a signal also depends on the noise power spectral density. For a detector with less noise, the same strength would correspond to a larger distance.

When the template chirp time is not the same as that of the signal, the maximum of the processed form will be reduced. We call the maximum of the processed form in such a case as the *observed strength* $S_{\rm obs}$ of the signal in that template. The observed strength should depend only on the chirp times of the signal and the template involved (apart from $\mathcal{A}$ of course) because of the properties of the processed form mentioned above. If the observed strength of a signal with $S=1$ is determined as a function of the signal and template chirp times [7], a determination of the observed strength when $S\neq 1$ becomes trivial. We denote this function by $\mathcal{H}(\xi_t,\xi_s)$, where ξ_t is the template chirp time and ξ_s denotes the chirp time of the signal. We will see later that $\mathcal{H}(\xi_t,\xi_s)$ plays an important role in the determination of the bank of templates.

Actually, this function is the result of a maximization of the *ambiguity* function [5] over a subset of the parameters involved. The parameters that are maximized over are the initial phase and the time of arrival. They have been termed as *extrinsic* parameters [8], in contrast with the remaining parameters, such as the chirp time, which have been called the *intrinsic* parameters. We, therefore, call $\mathcal{H}(\xi_t,\xi_s)$ the *intrinsic ambiguity* function. Given a signal with $S=1$ and a chirp time ξ_s, it is formally defined as

$$\mathcal{H}(\xi_t,\xi_s)=\max_\tau d(\tau,\xi_t;\overline{\mu_s}). \tag{27}$$

In terms of the intrinsic ambiguity function, the observed strength can be obtained as

$$S_{\rm obs}=S\mathcal{H}(\xi_t,\xi_s). \tag{28}$$

It is possible to compute $\mathcal{H}(\xi_t,\xi_s)$ approximately using the stationary phase approximation, to the Fourier transform of the Newtonian wave form, given in Eq. (13). First, the processed form $d(\tau,\xi_t;\overline{\mu_s})$ can be expressed, in this approximation, as

$$d(\tau,\xi_t;\overline{\mu_s})=\frac{1}{\beta}\left[\left(\int_{f_a}^{f_c}df\frac{1}{f^{7/3}S_n(f)}\times\cos[2\pi\Delta\xi f_a\alpha(f)-2\pi\Delta t_a f]\right)^2+\left(\int_{f_a}^{f_c}df\frac{1}{f^{7/3}S_n(f)}\times\sin[2\pi\Delta\xi f_a\alpha(f)-2\pi\Delta t_a f]\right)^2\right]^{1/2}, \quad (29)$$

$$\beta=\int_{f_a}^{f_c}df\frac{1}{S_n(f)}f^{-7/3}, \quad (30)$$

where $\Delta t_a=\tau-t_a$ (t_a is the signal arrival time now) and $\Delta\xi=\xi_t-\xi_s$. Note that only the difference between the template and signal chirp times occurs here. We then choose that value of Δt_a for which the instantaneous ''frequency'' of both the integrands vanishes at $f=f_0$ (where f_0 is kept as a parameter). A simple maximization yields

$$\Delta t_a=\Delta\xi\left(\left[\frac{f_a}{f_0}\right]^{8/3}-1\right). \quad (31)$$

The intrinsic ambiguity function can now be expressed as

$$\mathcal{H}(\xi_t,\xi_s)=\mathcal{H}(\Delta\xi)=\frac{1}{\beta}\left[\left(\int_{f_a}^{f_c}****\right)^2+\left(\int_{f_a}^{f_c}****\right)^2\right]^{1/2}, \quad (32)$$

where the rows of asterisks stand for the same integrands as in Eq. (29) but with Δt_a replaced by Eq. (31). The integrals can be evaluated numerically for a given f_0. To fix f_0, we obtain the exact intrinsic ambiguity function using correlations and find, empirically, the value of f_0 which produces the best agreement with it. We find $f_0=310$ Hz.

Our method is actually akin to a second stationary phase approximation but we do not have a formal proof for our procedure. Nonetheless, we find the approximation to be quite good and use Eq. (32) in our analysis. In Fig. 2 the approximate form is compared with one obtained numerically without any approximations. The discrepancy between the two curves is seen to be significant only for large chirp times. We do not expect a significant change in our final results because of the use of this approximation.

Note that $\mathcal{H}(\xi_t,\xi_s)$ in Eq. (32) depends only on $|\Delta\xi|$ and not on ξ_t and ξ_s separately. This behavior is replicated by the exact intrinsic ambiguity function also. This simplifies matters a lot when it comes to setting up a bank of templates. It is important to realize, however, that this behavior of $\mathcal{H}$ is dependent on the choice of intrinsic parameters. For instance, the use of $\mathcal{M}$, the chirp mass, instead of ξ, will make the intrinsic ambiguity function dependent on both the signal and template chirp masses.

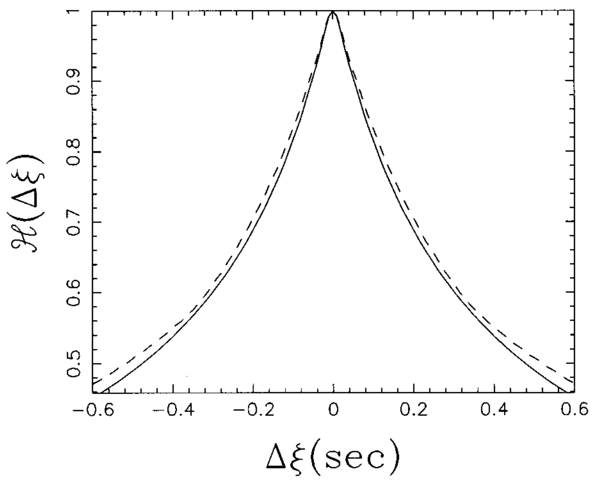

FIG. 2. A plot of the intrinsic ambiguity function $\mathcal{H}(\Delta\xi)$. The solid curve was obtained semianalytically [Eq. (32)], while the dashed curve was obtained numerically using correlations. The value of f_0 chosen was 310 Hz.

In [7], the template chirp times chosen were such that $\mathcal{H}$ dropped to ~ 0.9 in the middle of two consecutive templates. Such a drop in $\mathcal{H}$ corresponds, here, to $\Delta\xi=0.040$ sec. The difference between consecutive template chirp times, in a one-step search, would typically be of this order. Thus, the number of template chirp times that will be required would be ~ 1000.

The practical implementation of a one-step search would be the following. The detector output will be sampled with a sufficiently small sampling interval Δ to give the time series $[x_i=x(i\Delta);i=0,\dots,N-1]$. Since the upper frequency cutoff of the bandpass filter, in Sec. II B, was chosen to be $f_c=1$ kHz, the Nyquist sampling rate would be 2 kHz. We take the sampling frequency to be 2048 Hz, the nearest power of 2, which implies $\Delta=1.0/2048.0$ sec. The time series should, of course, be longer than the duration of the longest template or equivalently, the largest chirp time $\xi_{\max}$. Such time series will be required for the templates, h_0 and $h_{\pi/2}$, also. In the time series of a template with chirp time ξ_m, samples for $i>\xi_m/\Delta$ will be zero since the template has a finite duration of ξ_m. Therefore, when a correlation between the template and the detector output is taken using an FFT, the preferable method, only the first $N-\xi_m/\Delta$ samples will be the result of a linear correlation [4]. It is desirable to have equal lengths of correlations for every template. Hence, only the first $N_p=N-\xi_{\max}/\Delta$ samples will be retained in each correlation and the rest discarded. We call N_p, the *padding* for the template bank. A useful figure for N_p is $\sim 5\times 10^5$ corresponding to $N=256\times 2048$ and $\xi_{\max}=32.0$ sec. A time series corresponding to $X(\tau;\xi_m)$ will then be obtained whose ith sample we denote as $X_i(\xi_m)$ ($i=0,\dots,N_p$ now):

$$X_i(\xi_m)=\sqrt{C_0^2(i\Delta,\xi_m)+C_{\pi/2}^2(i\Delta,\xi_m)}. \quad (33)$$

As before, λ_m will then be found as

$$\lambda_m = \max_i \{X_i(\xi_m)\}. \tag{34}$$

Henceforth, we use this discrete form of the detection strategy.

The description of a one-step search, presented above, is not complete. The bank of templates and the threshold are yet to be fixed. In other words, the parameters characterizing a template bank have to be determined so that some desirable conditions are satisfied. These conditions are framed in terms of false alarm and detection probabilities which depend on the distribution functions, $\mathcal{F}_1(z;\overline{\mu_s})$ and $\mathcal{F}_0(z)$, of Λ. We will now discuss these distribution functions.

D. Distributions of the test statistic

The distribution functions of Λ can be constructed from the distributions of its building blocks. We start at the lowest level, i.e., the distributions of C_0 and $C_{\pi/2}$. In the following, the parameters of a signal will be denoted by $\overline{\mu_s} = (\mathcal{A}, t_a, \xi, \Phi)$. The context in which these symbols will be used in the following will be clear enough so that no confusion should arise because of the use of the same, in the previous section, as parameters for maximization. A typical member of the bank of templates will be denoted by ξ_m, unless specified otherwise.

The random variables $C_0(i\Delta,\xi_m)$ and $C_{\pi/2}(i\Delta,\xi_m)$ are obtained by correlating the detector output $x(t)$ with two templates $h_0(t;\xi_m)$ and $h_{\pi/2}(t;\xi_m)$, respectively. Thus, C_0 and $C_{\pi/2}$ are linear combinations of the time samples of $x(t)$. Since the noise in $x(t)$ was assumed to be Gaussian random process, $C_0(i\Delta,\xi_m)$ and $C_{\pi/2}(i\Delta,\xi_m)$ will, therefore, be Gaussian random variables. In the absence of a signal, their mean values would be zero but in the presence of a signal $h(t;\overline{\mu_s})$, their mean values would be

$$E[C_a(i\Delta,\xi_m)] = \langle h(t;\overline{\mu_s}), \mathcal{N}_h h_a(t-i\Delta;\xi_m)\rangle, \tag{35}$$

for $a=0,\pi/2$. Their covariance matrix can be calculated as follows ($a=0,\pi/2$ and $b=0,\pi/2$) where we have suppressed arguments and parameters at some places for clarity:

$$\begin{aligned}
&E[C_a(i,\xi_m)C_b(i,\xi_m)] - E[C_a(i,\xi_m)]E[C_b(i,\xi_m)]\\
&\quad = \mathcal{N}_h^2 E\{[\langle h_a, n(t)\rangle + \langle h_a, h(t)\rangle][\langle h_b, n(t)\rangle\\
&\qquad + \langle h_b, h(t)\rangle]\} - \mathcal{N}_h^2 \langle h_a, h(t)\rangle\langle h_b, h(t)\rangle\\
&\quad = \mathcal{N}_h^2 E[\langle h_a(t-i\Delta;\xi_m), n(t)\rangle\langle h_b(t-i\Delta;\xi_m), n(t)\rangle]\\
&\quad = \mathcal{N}_h^2 \langle h_a(t-i\Delta;\xi_m), h_b(t-i\Delta;\xi_m)\rangle,
\end{aligned} \tag{36}$$

where Eq. (3) has been used in the last step.

The variance of C_0 or $C_{\pi/2}$ can be computed from the above equation by putting $a=b$:

$$\begin{aligned}
&E[C_a^2(i\Delta,\xi_m)] - E[C_a(i\Delta,\xi_m)]^2\\
&\quad = \mathcal{N}_h^2 \langle h_a(t-i\Delta;\xi_m), h_a(t-i\Delta;\xi_m)\rangle\\
&\quad = 2\mathcal{N}_h^2 \int_{f_a}^{f_c} \frac{|\widetilde{h}(f;\xi_m)|^2}{S_n(f)}\, df.
\end{aligned} \tag{37}$$

The normalization $\mathcal{N}_h$ can now be chosen to make the variance unity. It follows from Eq. (19) and the above that $C_0(i\Delta,\xi_m)$ and $C_{\pi/2}(i\Delta,\xi_m)$ are very nearly statistically independent. We take them to be exactly so. We will now move on to the random variable $X_i(\xi_m)$ defined in Eq. (33).

In general, if x_1 and x_2 are two uncorrelated, zero-mean Gaussian random variables with unit variances then the distribution of $u=\sqrt{x_1^2+x_2^2}$ is given by the Rayleigh distribution function $R(u)$ [22]:

$$R(u) = u e^{-u^2/2}. \tag{38}$$

On the other hand, if x_1 and x_2 have an expectation value of μ_1 and μ_2, respectively, other moments being the same as before, then the probability density function of u is given by the Rician distribution $\mathrm{Ri}(u)$ [22]:

$$\mathrm{Ri}(u) = u \exp[-(u^2+d^2)/2] I_0(du), \tag{39}$$

where $d^2 = \mu_1^2 + \mu_2^2$, and $I_0(x)$ is the modified Bessel function of the first kind, of order zero. For $du \gg 1$,

$$\mathrm{Ri}(u) \sim \sqrt{\frac{u}{2\pi d}} \exp[-(u-d)^2/2], \tag{40}$$

while for $d=0$, it goes into a Rayleigh distribution. The asymptotic form given above shows that a Rician density behaves such as a Gaussian when $u \sim d$.

From Eq. (33) and what has been said about the moments of $C_0(i\Delta,\xi_m)$ and $C_{\pi/2}(i\Delta,\xi_m)$, it follows that $X_i(\xi_m)$ has a Rayleigh density when a signal is absent and a Rician density when a signal is present. The quantity corresponding to d, in the case of $X_i(\xi_m)$, will depend on both the signal and the template parameters. We denote it, therefore, by $d_i(\xi_m;\overline{\mu_s})$:

$$d = d_i(\xi_m;\overline{\mu_s}) = \mathcal{N}_h \sqrt{\langle h(t;\overline{\mu_s}), h_0(t-i\Delta;\xi_m)\rangle^2 + \langle h(t;\overline{\mu_s}), h_{\pi/2}(t-i\Delta;\xi_m)\rangle^2}. \tag{41}$$

The time series $(d_i(\xi_m;\overline{\mu_s}); i=0,\ldots,N_p-1)$ is nothing but a sampled version of the processed form, $d(\tau,\xi_m;\overline{\mu_s})$, of the signal which was introduced in the previous section. The samples are taken at $\tau=i\Delta$.

The next step in the determination of the distributions of the test statistic would be to obtain the distribution function of λ_m, the maximum over a single rectified output. We denote it by $F_{1,m}(\eta;\overline{\mu_s})$, in the presence of a signal with parameters $\overline{\mu_s}$ and $F_{0,m}(\eta)$, in the absence of a signal (note the use of ''F'' in these symbols as opposed to the ''$\mathcal{F}$'' used in the distribution functions of the test statistic Λ).

Let the joint probability density function of all the samples in a given rectified output be $P_{(b)}[X_0(\xi_m),X_1(\xi_m),\ldots,X_{N_p-1}(\xi_m)]$ where $b=1$ corresponds to the presence of a signal (parameters suppressed) and $b=0$ corresponds to the absence of a signal. The cumulative distribution function of λ_m is then

$$F_{b,m}(z)=\Pr\{\lambda_m\leqslant z\}=\{X_i(\xi_m)\leqslant z \text{ for all i}\}$$
$$=\int_0^z\cdots\int_0^z P_{(b)}(X_0,X_1,\ldots,X_{N_p-1})$$
$$\times dX_0,\ldots,dX_{N_p-1}, \tag{42}$$

where we have again suppressed the signal parameters for clarity. We find it difficult to proceed further because solving such an integral, assuming that $P_{(b)}$ itself could be calculated first, appears to be an intractable task. This is because of the large values that N_p can take ($\sim 10^5$) and the fact that it is difficult to separate the interdependence of the variables.

The distribution of the maximum over a set of statistically dependent random variables appears to be a difficult problem in general. Even the case of the maximum over a set of more than a few correlated Gaussian random variables, does not appear to have been solved exactly. There are ways of approximately calculating this distribution [23] but it is impractical to apply them to a case of more than four or five variables. On the other hand, if there were no statistical dependence among the X_i, it would be trivial to solve the integral because $P_{(b)}$ would then be just a product of the density functions of the variables. The density functions of $X_i(\xi_m)$ have already been obtained above. Therefore, in this case,

$$F_{0,m}(z)=\left(\int_0^z R(x)dx\right)^{N_p} \tag{43}$$

when a signal is absent, and

$$F_{1,m}(z;\overline{\mu_s})=\prod_{j=0}^{N_p-1}\int_0^z \mathrm{Ri}(x,d_j(\xi_m;\overline{\mu_s}))dx \tag{44}$$

when a signal with parameters $\overline{\mu_s}$ is present.

An obvious way to estimate the true distribution of λ_m is to perform a Monte Carlo simulation. For the case when there is no signal, we found a surprisingly simple result: The estimated distribution can be fit, almost exactly, by the distribution of Eq. (43) but with a reduced number of samples $N_{\rm eff}<N_p$. This behavior was also noted in [24]. It was found that the effective number of samples required depends on the

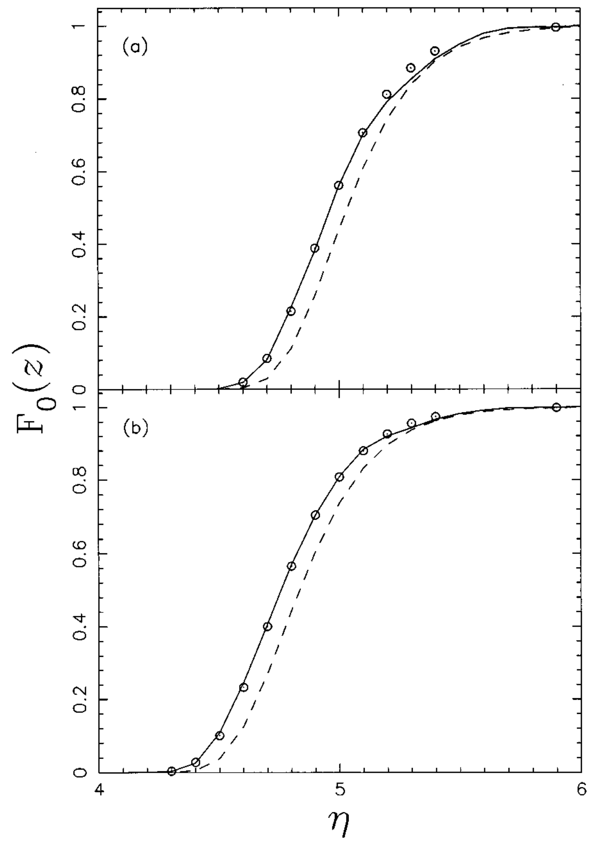

FIG. 3. Monte Carlo estimates of $F_0(z)$, the distribution function of the maximum over a single rectified output in the absence of a signal. The padding used in (a) was $N_p\times\Delta=108.0$ sec. For (b) $N_p\times\Delta=40.0$ sec. The dashed curves were obtained using Eq. (43) without any reduction in N_p. The circled points were obtained using Eq. (46) with $N_{\rm eff}=0.7N_p$. The number of noise realizations used was 1000 for both (a) and (b).

effective bandwidth [24] of the correlation outputs. The effective number of samples $N_{\rm eff}$ required in the present case, was found empirically to be about $0.7N_p$ for a large range of N_p. That $N_{\rm eff}/N_p$ should be a constant, as N_p is varied, is probably related to the stationarity of the noise.

Strictly speaking, nothing can be said about the validity of this fit for $\eta\gg 1$. For large thresholds (~ 7 or greater) the number of events would be small and, therefore, a large number of noise realizations (approximately 10^4 or 10^5) would be required in a Monte Carlo simulation to register such events. This, however, is not possible. A typical noise realization in such a simulation can have upwards of 10^5 samples. Since, pseudorandom number generators have a period of $\sim 10^8$ numbers, the number of realizations which can be used is limited to just a few thousand. We will, nonetheless, assume that such a fit will remain valid for sufficiently large values of η. Monte Carlo simulations also show that $F_{0,m}(z)$ is independent of the template chirp time ξ_m. This is also evident from Eq. (43). Therefore, we suppress m henceforth: $F_{0,m}(z)=F_0(z)$. Plots of the Monte Carlo simulations are shown in Figs. 3(a) and 3(b). In the rest of the paper,

$$F_{0,m}(z)=F_0(z)=\left(\int_0^z R(x)dx\right)^{N_{\rm eff}} \tag{45}$$

or

$$\ln F_0(z) = N_{\mathrm{eff}} \ln(1 - e^{-z^2/2})$$
$$\approx -N_{\mathrm{eff}} e^{-z^2/2} \quad \text{for} \quad z \gg 1,$$

which gives

$$F_0(z) \simeq \exp\left[-N_{\mathrm{eff}} \exp\left(-\frac{z^2}{2}\right)\right] \quad \text{for} \quad z \gg 1. \tag{46}$$

We find the approximate expression to be quite good and use it in our analysis.

We believe that the effective number of samples found here would remain the same even when post-Newtonian templates are used. This is because N_{eff} appears to depend on the correlation noise power spectral density via a quantity, such as its effective bandwidth, which is quite insensitive to the details of its shape. The use of higher order wave forms will not produce a large change in such a quantity. On the other hand, a change in the detector noise itself should definitely lead to a change in N_{eff}.

In principle, the technique of using a Monte Carlo simulation for estimating the true distribution can be extended to the case of $F_{1,m}(z;\overline{\mu_s})$. In practice, this is quite cumbersome because of its dependence on both the signal and the template parameters. This would require the cataloging of a large number of distributions corresponding to various combinations of signal and template parameters. For sufficiently strong signals, however, it is possible to bypass this procedure.

Suppose that a signal $h(t;\overline{\mu_s})$ with a chirp time ξ is present in $x(t)$. The processed form of a signal produced by a template is observed to have a very sharp maximum. Therefore, it can be expected that in the presence of noise also, the maximum over the rectified output of a template would occur near the maximum of the processed form of the signal produced by that template. This would not be true, however, if the maximum of the processed form (the observed strength S_{obs}) is not large enough. In the absence of a signal and for a typical value of $N_p \sim 10^5$, we see from Figs. 3(a) and 3(b) that noise alone can produce $\lambda_m \lesssim 4.5$ with a non-negligible probability. This noise-induced maximum can occur anywhere at random within the time series of the rectified output. Therefore, S_{obs} should at least be greater than about 5.0 so that the presence of the signal is able to restrict the location of λ_m.

Actually, in the case of a strong signal also, a scatter over a few neighboring points will always be present. We see from Monte Carlo simulations that this scatter is always within two or three samples around the expected position. But such samples will be strongly correlated and will effectively act as one. We can safely say, therefore, that the location of λ_m coincides with that of the maximum of the processed form of the signal produced by the template. However, this would be true for only those templates in the neighborhood of the signal for which S_{obs} is sufficiently large ($\gtrsim 5.0$). Also, for the kind of thresholds which will occur in this paper, an observed strength of $\lesssim 5.0$ will not be significant as far as the detection probability of the signal is concerned. We mark the boundary of such a neighborhood around a signal chirp time ξ by $\xi_p < \xi < \xi_q$.

Let i_m be the time sample for which the processed form, produced by a template with chirp time ξ_m, attains its maximum value. Thus, the distribution function of λ_m will now be the same as that of the i_m^{th} sample, $X_{i_m}(\xi_m)$, of the rectified output. In this approximation, $\mathcal{F}_1(z;\overline{\mu_s})$ becomes the distribution of the maximum over $\{X_{i_k}(\xi_k)\}$, the template chirp times in this set being located within $[\xi_p, \xi_q]$. We now make the assumption that $X_{i_m}(\xi_m)$ and $X_{i_n}(\xi_n)$ are statistically independent random variables for any two template chirp times $\xi_m, \xi_n \in \{\xi_j\}$. This assumption is quite strong and can be expected to hold only when the template chirp times are placed far apart. Under this assumption, $\{X_{i_k}(\xi_k)\}$ becomes a set of statistically independent random variables. The distribution of the maximum over such a set has already been discussed at length earlier. It will just be the product of the distribution functions of $X_{i_k}(\xi_k)$ in the presence of a signal. Thus,

$$\mathcal{F}_1(z;\overline{\mu_s})$$
$$= \Pr\{\Lambda \leqslant z; \text{ when a signal, with parameters } \overline{\mu_s}, \text{ is present}\}$$
$$= \prod_{k=p}^{q} \int_0^z \mathrm{Ri}(x,S_k)\,dx, \tag{47}$$

where

$$S_k = S\mathcal{H}(\xi,\xi_k)$$
$$= S\mathcal{H}(\xi-\xi_k) \tag{48}$$

is the observed strength of the signal in the template ξ_k.

The detection probability of the signal $h(t;\overline{\mu_s})$ can now be determined. Since only the observed strengths of the signal enter into the calculation of $\mathcal{F}_1$, this function depends on only the parameters ξ and S of the signal. This will, therefore, hold for the detection probability also. Given a threshold η, we denote the detection probability of the signal by $Q_d(\eta;S,\xi)$:

$$Q_d(\eta;S,\xi) = 1 - \prod_{k=p}^{q} \int_0^{\eta} \mathrm{Ri}(x,S_k)\,dx. \tag{49}$$

We will now explore the effect of reducing the number of templates used in the above formula. Suppose that ξ lies between the two consecutive template chirp times ξ_m and ξ_{m+1} and that the signal strength, $S=10.0$. Let $\xi-\xi_m=\xi_{m+1}-\xi=0.10$ sec. Thus, the template chirp times are 0.20 sec apart. Statistical correlations between λ_m and λ_{m+1} would be negligible for such a separation (we justify this later) and Eq. (49) will, thus, be valid. The observed strengths required in this equation can be calculated from the ambiguity function. For the templates ξ_m and ξ_{m+1}, $S_m=S_{m+1}=10.0\mathcal{H}(0.10)=8.2$. If we also assume that $\xi_{m+2}-\xi_{m+1}=\xi_m-\xi_{m-1}=0.20$ sec, then $S_{m+2}=S_{m-1}=10.0\mathcal{H}(0.20+0.10)=6.0$. For a typical value of the threshold, $\eta=7.8$, the value of $[\int_0^{7.8}\mathrm{Ri}(x,8.2)dx]^2=0.1$ while $[\int_0^{7.8}\mathrm{Ri}(x,6.0)dx]^2=0.92$. Thus, the inclusion of ξ_{m+2} and

ξ_{m-1} will lead to an increase of the detection probability from 0.9 to $1-0.1\times 0.92=0.908$, an entirely negligible change. The correction because of the templates ξ_{m-2} and ξ_{m+3} (for $\xi_{m-1}-\xi_{m-2}=\xi_{m+3}-\xi_{m+2}=0.20$ sec) can be calculated similarly: $\mathcal{H}(0.40+0.10)=0.5$ and hence, $S_{m+3}=S_{m-1}=10.0\mathcal{H}(0.5)=5.0$. Therefore, $[\int_0^{7.8}\mathrm{Ri}(x,5.0)dx]^2\approx 0.99$ leading to a reduction of just 0.1% in the detection probability if these templates are neglected. For larger detection probabilities the error because of the exclusion of ξ_{m-1} and ξ_{m+2} becomes larger but it still remains negligible. For instance, if η were brought down to $\eta=7.5$, $[\int_0^{7.5}\mathrm{Ri}(x,8.2)dx]^2=0.045$ (which gives a detection probability of $1-0.045=0.955$), but now $[\int_0^{7.5}\mathrm{Ri}(x,6.0)dx]^2\approx 0.85$, which implies a reduction of 0.7% in the detection probability if these templates are neglected. Thus, when the template chirp times are far apart, we are completely justified in keeping just the two templates, ξ_m and ξ_{m+1}, in Eq. (49). It should be noted that if only a single template were used in Eq. (49), the error in the detection probability would be too large. Again, taking an observed strength of 8.2 and a threshold of 7.8, the detection probability obtained in this case would be just $1-\int_0^{7.8}\mathrm{Ri}(x,8.2)dx=0.68$. This should be compared with the value of 0.9 obtained when the two-template formula was used.

If, however, the template chirp times were closely spaced, the statistical correlations among the maxima over rectified outputs will not be negligible. In such a case the true detection probability of a signal will be reduced. We used Monte Carlo simulations to estimate the true detection probability for a signal with $\xi=(\xi_m+\xi_{m+1})/2$ and various values of $\xi_{m+1}-\xi_m$. The results are shown in Figs. 4(a) and 4(b). We have also shown the results obtained when only ξ_{m+1} and ξ_m are used in Eq. (49) for the calculation of the detection probability, i.e., when

$$Q_d(\eta;S,\xi)=1-\int_0^{\eta}\mathrm{Ri}(x,S_m)dx\int_0^{\eta}\mathrm{Ri}(x,S_{m+1})dx. \tag{50}$$

We find that Eq. (50) indeed provides an overestimate of the detection probability when the difference between the template chirp times is small [Fig. 4(a)]. The detection probability estimated from the simulation, for a spacing of 0.030 sec, is about 12% less when Eq. (50) gives a value of 0.95. In our analysis, this value for the detection probability will be used as a fiducial value. Hence, the use of the two-template formula would be erroneous if it is used for template chirp times spaced more finely than $\sim$0.030 sec. Also shown in Fig. 4(a) is a plot of the detection probability obtained when only one template is used in Eq. (49) with an observed strength S_m (or S_{m+1}). We see that now the detection probability is *underestimated*. In a sense, such as the effective number of samples in the case of F_0, statistical correlations will now lead to an effective number of templates. However, this effective number would depend on the spacing of the templates. For this reason, it is difficult to incorporate it in our calculations.

For closely spaced templates it can be expected that templates other than ξ_m and ξ_{m+1} would also contribute significantly to the detection probability because the observed strength would still be high in these templates. Thus, if more

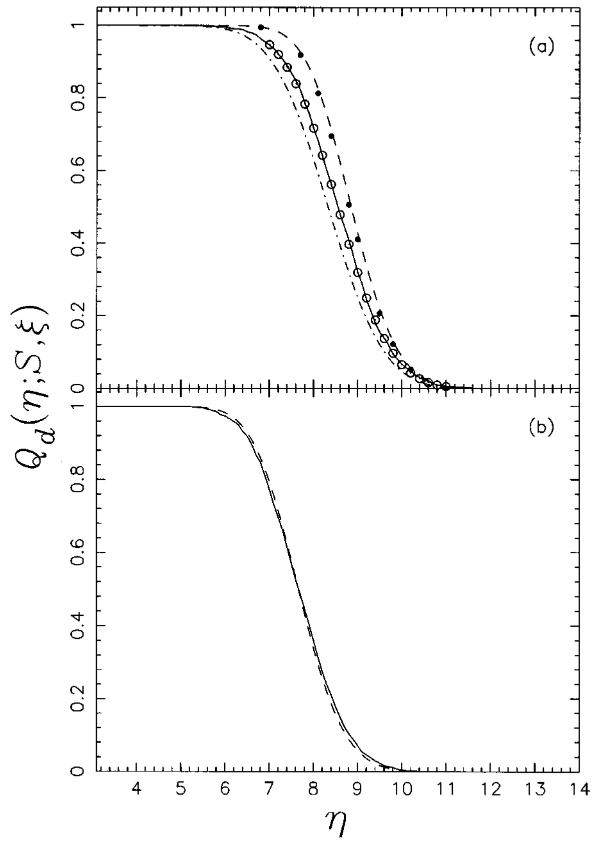

FIG. 4. The detection probability of a signal. Solid curves are Monte Carlo estimates. Dashed curves were obtained using Eq. (50) while for the dash-dot-dash curve in (a), only one template (ξ_m) was used in Eq. (49). In (a), the open circles and the filled circles were obtained using Eq. (51) with $\rho=0.75$ and $\rho=0.0$, respectively. The signal chirp time in both (a) and (b) is $\xi=(\xi_m+\xi_{m+1})/2$. For (a), $\xi_m=4.0$ sec, $\xi_{m+1}=4.03$ sec. The signal strength was chosen as $S=8.5$ and the signal chirp time is $\xi=4.015$ sec. For (b), $\xi_m=4.0$ sec, $\xi_{m+1}=4.40$ sec, $\xi=4.20$ sec, and the signal strength $S=10.0$. The number of noise realizations used were 2000 and 1000, for (a) and (b), respectively.

templates were included in the above simulation, the true detection probability should rise. On the other hand, these templates would, at the same time, be highly dependent statistically and may not be able to contribute much to the detection probability. We repeated the simulation used to obtain Fig. 4(a) with six templates instead of two but having the same spacing. The signal was placed in the middle of the third and the fourth templates and had the same strength as before. We found *no significant rise* in the detection probability.

The asymptotic form of a Rician density function, for a threshold close to the observed strength, is a Gaussian with the observed strength as the mean [Eq. (40)]. Thus, each of the maxima over the rectified outputs can be considered to have approximately a Gaussian distribution. The distribution of Λ can, therefore, be approximated by that of the maximum over correlated Gaussian variables. It appears that an exact expression for this distribution has not been obtained for more than two variables. But the latter is precisely the case of interest to us. An elementary integration yields the following approximation to the two-template detection probability when the observed strengths in the two templates are equal ($S_{m+1}=S_m=S_{\rm obs}$):

$$Q_d(\eta;S,\xi)\simeq\sqrt{\frac{2}{\pi}}\int_0^\eta dx\ \exp[-(x-S_{\rm obs})^2/2]\times\left[1-{\rm erfc}\left((x-S_{\rm obs})\sqrt{\frac{1-\rho}{1+\rho}}\right)\right], \quad (51)$$

where ρ is the covariance between the two maxima and

$${\rm erfc}(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty dt\ \exp(-t^2/2). \quad (52)$$

For $\rho=0$, we should get an approximation to Eq. (50). Indeed, as shown in Fig. 4(a), a good fit is obtained which is also valid for thresholds that are not close to the observed strength. The Monte Carlo curve can also be fit almost exactly by Eq. (51) when $\rho=0.75$. The covariance ρ should depend on the intrinsic ambiguity function but this dependence does not appear to be straightforward. Moreover, since rectification mixes the signal and the noise nonlinearly, ρ would also depend on the observed strength of the signal. Further work to obtain ρ analytically is in progress. For the present we will continue to use Eq. (50) in our analysis.

As we will show later, an overestimation of the detection probability will lead to an underestimation of the computational cost of a one-step search. The effect on a two-step search is similar but it will be elaborated upon in Sec. III B. Thus, the use of Eq. (50) will lead to a lower limit on the computational cost. Similarly, the one-template formula will lead to an overestimation of the computational cost since this formula underestimates the true detection probability. In any case, both an upper and a lower bound can be put on the computational cost of a one-step search. However, it is more convenient to use Eq. (50) since it can also be used for large spacings. While if the one-template equation were used, a shift from the two- to one-template formula would have to be performed when the spacings being dealt with become small. This is cumbersome since it is difficult to demarcate the regimes where one formula would be better than the other.

We also observe, in Fig. 4(b), that the agreement between Eq. (50) and the estimated distribution becomes better as the templates are drawn farther apart. This is an indication that statistical correlations between the rectified outputs have become negligible. This agreement is observed to hold for a smaller separation of $\xi_{m+1}-\xi_m=0.20$ sec also which justifies our use of this value in the argument above. We found that the effect of statistical correlations becomes noticeable when $\delta\lesssim 0.080$ sec. The same simulations also show that the scatter in the location of λ_m is mostly over two to three samples around $X_{i_m}(\xi_m)$ in both the rectified outputs.

Another feature of Eq. (50) which should be noted is the following. For sufficiently large spacings ($\gtrsim 0.03$ sec), this formula assigns the minimum detection probability, for a given strength, to the signal which lies in the middle of two consecutive templates (i.e., $\xi=(\xi_m+\xi_{m+1})/2$]. This is in conformity with the expected variation of the detection probability since such a signal would be furthest from the templates on either side. For spacings smaller than $\lesssim 0.03$ sec, however, this formula leads to the above kind of a signal having the highest detection probability. This is because of the fact that, for small spacings, the drop in the intrinsic ambiguity function is not much across the templates neighboring ξ_m and ξ_{m+1} [$\mathcal{H}(0.045)\simeq 0.91$, for instance]. Hence, templates other than the two adjacent ones (ξ_{m+2}, ξ_{m-1}, and others) can also contribute, in Eq. (49), to the detection probability of signals lying between ξ_m and ξ_{m+1}. Indeed, the expected behavior of the detection probability for such signals is recovered when these extra templates are used. All this would be applicable only if the rectified outputs of these templates were statistically independent of each other. However, as discussed above, the assumption of statistical independence would not remain valid for templates spaced so closely. The failure of Eq. (50) to assign detection probabilities in the correct order, in such a case, is perhaps less serious than neglecting statistical correlations. Hence, as long as its use is confined to ''centered'' signals, Eq. (50) appears to be a good tool to use.

To summarize, we can now calculate the detection probability of a signal from Eq. (50), given a bank of templates $\{\xi_j\}$ and a threshold η. We also have a formula, Eq. (45), to calculate the distribution of the maximum over a single rectified output. While the former was derived under some approximations, the latter is exact and was estimated from Monte Carlo simulations. These formulas will be used unaltered in the case of a two-step search. However, the estimation of the distribution function of Λ in the absence of a signal, $\mathcal{F}_0(\eta)$, requires a clearer specification of the bank of templates. We proceed to do that below.

E. One-step search

We now state the *one-step template placement criteria*: The bank of templates should be chosen in such a way that (i) every wave form, having a strength S greater than a given minimum strength $S_{\rm min}$, should have a detection probability greater than a given minimum detection probability $Q_{d,\rm min}$, and (ii) the false alarm should stay below a specified level, $Q_{0,\rm max}$. A solution in terms of η and $\{\xi_j\}$, satisfying both criteria, need not always exist. For instance, a signal having $S=S_{\rm min}=6.0$ will not be detected with a detection probability of $Q_{d,\rm min}=0.95$ if the false alarm is kept such that there is only one false event, on the average, in a year. These numbers can be computed by using the formulas below.

The detection probability would be smallest for signals having a strength $S_{\rm min}$ and chirp time $\xi=(\xi_m+\xi_{m+1})/2$ for $\xi_m\in\{\xi_j\}$. Such signals will have a detection probability given by $Q_d(\eta;S_{\rm min},(\xi_m+\xi_{m+1})/2)$, which can be calculated using Eq. (50). To satisfy criterion (i) above, all that needs to be done, given a threshold η, is to ensure that all such minimum detection probability signals have,

$$Q_d(\eta;S_{\rm min},(\xi_m+\xi_{m+1})/2)=Q_{d,\rm min}. \quad (53)$$

The observed strengths that will be required in the above formula are

$$S_m=S_{m+1}=S_{\rm min}\mathcal{H}\left(\frac{\xi_{m+1}-\xi_m}{2}\right). \quad (54)$$

Thus, only $\xi_{m+1}-\xi_m$ enters into the calculation of the detection probability and not ξ_m and ξ_{m+1} separately. This implies that $\xi_{m+1}-\xi_m$ should be constant throughout a template bank. We call this quantity the *spacing* of the templates and denote it by δ. The whole template bank can now be constructed, using δ, as $\xi_k=\xi_{\min}+k\delta\,(k=0,1,\ldots)$ till $\xi_{\max}$ is reached. It should be noted that the template bank is now characterized by a single parameter, namely, the spacing.

The spacing of the templates depends on the value of $S_{\min}$. For instance, if $\eta=7.9$ (a typical value), and $Q_{d,\min}=0.95$, the observed strength required for a signal lying in the middle of two templates is $S_{\rm obs}\simeq 8.6$. For a signal having $S=8.8$, this would mean that the value of the intrinsic ambiguity function be $\approx 8.6/8.8=0.98$. From Fig. 2 it is clear that the difference between the chirp times of the signal and a template should then be $\simeq 0.015$ sec. Thus, the template chirp times would be placed $\simeq 0.030$ sec apart. For a larger strength of 10.0, the value of the intrinsic ambiguity function required would be 0.86. Hence, the spacings obtained now would be larger.

In order to calculate the false alarm, the distribution function $\mathcal{F}_0(z)$ of Λ is required. In Sec. II D, the cumulative distribution functions (F_0 and $F_{1,m}$) of λ_m were discussed. There, the problems associated with obtaining the distribution of the maximum, over a set of statistically dependent variables, were outlined. In the case of Λ, essentially the same problems are again encountered. This is because the set of random variables $\{\lambda_j\}$, whose maximum is Λ, would, in general, be a statistically dependent set. However, as in the case of F_0, we can first obtain an expression for $\mathcal{F}_0$ by assuming that $\{\lambda_j\}$ is a statistically independent set, and then, explore the effect of statistical correlations on this expression by using Monte Carlo simulations. Actually, the assumption of statistical independence here would be, in a sense, an extension of the assumption which was used in obtaining Eq. (49). There, statistical independence was assumed among samples of rectified outputs where only one sample was used from each rectified output. Further, the positions of those points were related to each other by the fact that the same signal was being observed. Here, all such restrictions would be dropped.

Having made the above assumption, the cumulative distribution function $\mathcal{F}_0(z)$ is easily obtained as

$$\mathcal{F}_0(z)=\prod_{i=1}^{N_T} F_0(z), \tag{55}$$

where

$$N_T=\frac{1}{\delta}(\xi_{\max}-\xi_{\min}) \tag{56}$$

is the number of templates in the template bank. Note that, as stated earlier in Sec. II C, $\mathcal{F}_0(z)$ now depends on δ (via N_T). We include this dependence explicitly in the false alarm which we denote, for a given threshold η, by $Q_0(\eta;N_T)$,

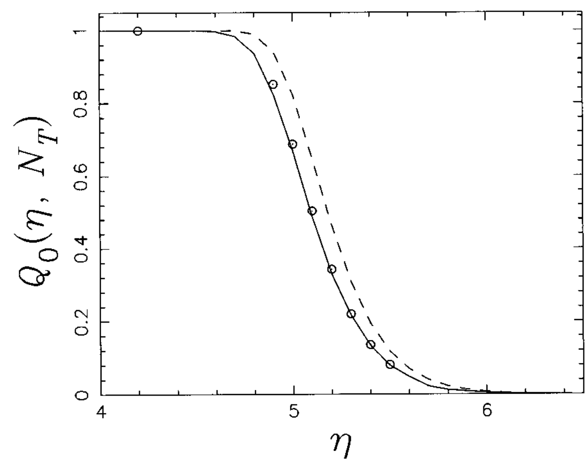

FIG. 5. Monte Carlo estimates of the false alarm for three templates having a spacing of 0.020 sec. The padding used was $N_p\Delta=108.0$ sec. The solid curve is the Monte Carlo estimate while the dashed curve was obtained using Eq. (56) and $N_{\rm eff}=0.7N_p$. The circled points were obtained using the same formula but with $N_{\rm eff}=0.47N_p$.

$$Q_0(\eta;N_T)=1-\mathcal{F}_0(\eta)$$
$$=1-\prod_{i=1}^{N_T} F_0(\eta) \tag{57}$$
$$\approx 1-\exp[-N_T N_{\rm eff}\exp(-\eta^2/2)]. \tag{58}$$

For typical values of the spacing, $N_T\sim 1000$. If a false alarm of $\simeq 8\times 10^{-6}$ is required over a padding of $N_p\Delta=256.0$ sec (corresponding to an average of one false event per year), the threshold needed according to Eq. (58) would be $\eta\approx 7.93$. This is higher than the threshold of 7.5 estimated in [7]. The discrepancy is because of the use of a Gaussian distribution by these authors instead of the Rayleigh distribution used here.

Unlike the case of λ_m, which is the maximum over a *single* rectified output, Monte Carlo simulations for the estimation of the true $\mathcal{F}_0(z)$ are more difficult to perform. This would require not only the use of a large number of templates, but also several values of the spacing between them. We, nonetheless, performed a Monte Carlo simulation to estimate $Q_0(\eta;N_T)$ for a small value of $N_T=3$ and a spacing $\delta=0.020$ sec. The value of δ was kept small to ensure that statistical dependencies were able to play a significant role. The results are shown in Fig. 5. For comparison, the curve obtained from Eq. (58) is also plotted. The curious property of their being an effective number of samples, for each rectified output, surfaced again here. We find that Eq. (58) provides an excellent fit to the Monte Carlo curve if the value of $N_{\rm eff}$ in that formula is changed from $0.7N_p$ (the value used for F_0) to $0.47N_p$. Of course, it can also be interpreted as an effective number of templates. We are unable to test the fit for higher thresholds since the number of noise realizations that can be used for such a simulation is not large enough. We will assume, as in the case of $F_0(z)$, that the fit to the Monte Carlo curve will remain good for higher thresholds also. Note that the reduction required in the number of samples would be dependent on the spacing of the templates and hence, it is more difficult to incorporate it in a calculation. However, the necessity of such an effort should be decided by the use that Eq. (57) would be put to, while setting up a one-step search.

A simple algorithm for setting up a one-step search, given $S_{\min}$, $Q_{d,\min}$, and $Q_{0,\max}$, would be the following. A value for the spacing is chosen (starting from a large value, say $\delta=0.10$ sec) and N_T is found. Then a threshold η is found, from Eq. (58), such that the false alarm becomes $Q_{0,\max}$. The detection probability $Q_d(\eta;S_{\min},\xi_{\min}+\delta/2)$ is found. If it exceeds $Q_{d,\min}$, then stop or else reduce the spacing and repeat the above process. In this algorithm, the threshold found in the second step should be accurate to within $\sim 4\%$ so as not to make a significant change in the detection probability that is calculated in the next step. For instance, if $S_m=S_{m+1}=8.6$, then $\eta=7.9$ gives a detection probability of 0.95. If η were changed by 4%, i.e. $\eta=8.2$, then the detection probability would fall to 0.90. This error is significant because, usually, a minimum detection probability greater than 0.9 is desired.

Fortunately, the threshold, for a given false alarm, is not very sensitive to which of the two curves in Fig. 5 is used for its determination. This is clear, by inspection, in the region where the curves are steep. For larger values of η, we can estimate the error by using the fit to the Monte Carlo curve provided by Eq. (58) with $N_{\rm eff}=0.47N_p$. When $\eta \gg 1$, Eq. (58) can be further approximated as

$$Q_0(\eta;N_T)\approx N_T N_{\rm eff}\exp(-\eta^2/2). \tag{59}$$

In order to study the effect of the change in the effective number of samples, let $N_{\rm eff}=xN_p$ for $0<x\leqslant 1$. Then it follows from the equation above that for a fixed false alarm, the change $\Delta\eta$ in the threshold for a change Δx in x is $\Delta\eta\approx\Delta x/(\eta x)$. In the case of the above simulation, x changes from $x=0.7$ to $x=0.47$. Therefore, $\Delta x=0.7-0.47=0.23$, and for $\eta=7.9$, we get $\Delta\eta\simeq 0.04$. This change is negligible as far as its effect on detection probability is concerned. For instance, if a detection probability of 0.95 is required for a threshold $\eta=7.9$, the observed strength required (for a signal lying in the middle of two templates) is $S_{\rm obs}\approx 8.6$. If η were now to be increased by 0.04, i.e., $\eta=7.94$, the detection probability falls by only $\simeq 0.6\%$ to 0.944. In this sense, even the correction of 0.7 used for $N_{\rm eff}$ is not significant. Suppose, $x=1.0$, then $\Delta x=0.53$ which leads to $\Delta\eta=0.07$. In the above example, the detection probability would now fall by $\simeq 1\%$ which is again insignificant. However, this correction was obtained by performing a simulation with a single template and, therefore, it was quite easy to incorporate it in our calculations. The argument presented above shows that Eq. (57) for the false alarm is a good approximation for our analysis even for closely spaced templates.

Since the threshold can be determined quite accurately, the main source of error in the algorithm given above is the detection probability. We had seen that Eq. (50) provided an overestimation of the detection probability when templates were spaced closely. In the algorithm, an overestimation of the detection probability for a given threshold would result in a spacing that would be larger than the true spacing. This is because a detection probability of $Q_{d,\min}$ can now be provided by a lower observed strength and hence a wider spacing. Thus, the algorithm would stop at a larger δ than is actually required. This would lead to an underestimation of the computational requirements since the number of templates would turn out to be lower.

Given a false alarm $Q_{0,\max}$ that can be tolerated, the threshold η can be obtained provided N_T, or equivalently, the spacing δ of the templates, is known. But at the same time, δ cannot be determined unless η is specified since the detection probability, which (along with $S_{\min}$) determines δ, also depends on η. One way to find a solution is to use the algorithm presented above. Another solution to this problem was obtained in [7] by making an *a priori* estimate of N_T which was taken to be ~ 1000. This allows the circular chain of dependencies to be broken. The final value of N_T turned out to be consistent with the assumed one. Such an approach can be justified if N_T is large since, as can be seen from Eq. (58), $\Delta\eta\approx\Delta N_T/(\eta N_T)$, which makes the threshold insensitive to N_T in such a regime.

III. TWO-STEP HIERARCHICAL SEARCH

In a one-step search, the operation which makes the heaviest demand on number crunching capability is the correlation of $x(t)$ with a large number of templates. The primary motivation behind implementing a hierarchical search is to obtain a reduction in the number of templates used in this operation. In order to quantify the efficacy of a hierarchical search, however, this as yet loosely defined concept needs to be cast into a rigorous form. We present below one such structure for a two-step hierarchical search. This structure is based on the properties of a one-step search which were described above. It should be noted that it need not be unique. It is based on our experience with a one-step search.

A. A two-step hierarchical search: Description

The basic idea behind a two-step search is the use of two banks of templates. One of them has template chirp times placed farther apart than those of the other. A one-steplike search would be conducted with the finely spaced templates but only around ''promising'' candidate chirp times, namely, those templates in the coarser bank for which the maximum over their rectified output exceeds a threshold. This threshold would be kept lower than the one which would be used with the finely spaced templates.

In the case of a one-step search the template bank could be fixed using the one-step template placement criteria and it turned out that the spacing between the templates was a constant. This was essentially because of the dependence of the intrinsic ambiguity function on only the difference in signal and template chirp times. A similar set of conditions can be imposed on a two-step search also. We will do this formally in the next section but it can be expected that the template banks will again have uniform spacings, although one of them would be less than the other. We, thus, construct a two-step search as follows.

The maximum over a rectified output is, first, computed for each chirp time in a bank $B_1=\{\xi_j\}$. We call B_1, the *first-stage* template bank and denote the spacing of this bank by $\delta^{(1)}$. The number of templates used in B_1 is $n_t^{(1)}$,

$$n_t^{(1)}=\frac{1}{\delta^{(1)}}(\xi_{\max}-\xi_{\min}). \tag{60}$$

If for some template in B_1 having a chirp time ξ_m, it happens that the maximum over its rectified output λ_m crosses $\eta^{(1)}$ then we call this event a *crossing* of $\eta^{(1)}$ produced by the chirp time ξ_m. Given such a crossing, the next step involves using a template bank $B_{2,m}$ with a spacing $\delta^{(2)}$ that is smaller than $\delta^{(1)}$. We take $\delta^{(1)}/\delta^{(2)}=n$ to be an integer. This keeps the two banks of templates commensurate with each other. It is possible to envisage a two-step search where this condition is relaxed and n is allowed to take nonintegral values also. However, an incommensurability in the two banks would require the storage of more template wave forms than those for commensurate banks. Also, there does not seem to be any significant advantage when the computational costs are considered because allowing incommensurability marginally reduces the minimum in the computational cost (Sec. III B).

The set of chirp times used in $B_{2,m}$ will be located symmetrically around ξ_m (except when $\xi_m=\xi_{\min}$ or $\xi_{\max}$, but these can be ignored). It will be convenient, therefore, to index the chirp times in $B_{2,m}$ with both positive and negative integers. Thus, the set of chirp times used in $B_{2,m}$ can be constructed as

$$\xi_p=\xi_m+p\,\delta^{(2)}, \tag{61}$$

where $-n+1\leqslant p\leqslant n-1$. However, the range of p need not be made as wide as this. For instance, it could be $-n/2\leqslant p\leqslant n/2$ also. For the former, the number of template chirp times in $B_{2,m}$ would be $2(n-1)$ while for the latter it will be n.

Since $\eta^{(1)}$ would, in general, be kept quite low, the probability of more than one crossing will not be negligible. In general, for every crossing of $\eta^{(1)}$, a fixed number, M $[n\leqslant M\leqslant 2(n-1)]$, of templates will be employed as described above. Since the number of crossings that appear in our final results is small (typically, ~ 2), the choice of M within the above-mentioned range does not make too much of a difference to the computational cost. We choose $M=2(n-1)$ for our analysis, the maximum of the range, in order to maximize our chances of detection. Thus, the templates, in $B_{2,m}$, with chirp times ξ_{n-1} and ξ_{-n+1}, will have a separation of $\delta^{(2)}$ from the templates corresponding to ξ_{m+1} and ξ_{m-1}, respectively.

Let n_c be the number of crossings that are produced by the first-stage templates in B_1. Then the total number of *second-stage* templates that will be used will be $n_c M$. Adjacent crossings will reduce this number since there would be some second-stage templates in common for such crossings. We assume, however, that adjacent crossings have a negligible probability compared to nonadjacent ones. Finally, the overall maximum over the rectified outputs of the second-stage templates employed is found. We denote it by $\Lambda^{(2)}$. If $\Lambda^{(2)}$ crosses a threshold $\eta^{(2)}$ $(>\eta^{(1)})$, a detection is announced. Thus, $\Lambda^{(2)}$ is the test statistic for a two-step search configured as above.

Neglecting adjacent crossings is justified if $n_c\ll n_t^{(1)}$ For instance, if $n_t^{(1)}=200$ and $n_c=2$, then the number of ways in which adjacent crossings can occur is 199 while for nonadjacent crossings, the number of ways is 19 701. Thus, the probability of an adjacent crossing is $\sim 1/100$ that of a nonadjacent crossing. It turns out that the condition $n_c\ll n_t^{(1)}$ is satisfied in our final results.

The dependence of the distributions of $\Lambda^{(2)}$ will now be on a set of two spacings and two thresholds. As a consequence, both detection probability and false alarm will also depend on the same. Before proceeding further, we emphasize that $\Lambda^{(2)}$ need not be equal to the test statistic that is obtained by using a one-step search. The same holds for $\eta^{(2)}$ and $\delta^{(2)}$ also. A one-step search with a threshold $\eta^{(2)}$ and a spacing $\delta^{(2)}$ will not, in general, satisfy the one-step template placement criteria. However, statistical correlations step in to effectively decouple the detection probability and false alarm from the presence of a hierarchy which allows $\eta^{(2)}$ and $\delta^{(2)}$ to be valid solutions for a one-step search.

B. Determination of thresholds and spacings

We impose on the two-step search described above, conditions similar to the one-step template placement criteria of Sec. II E. The *two-step template placement criteria* are: (i) Every signal with a strength greater than a given minimum strength $S_{\min}$ should produce, with a probability $Q_{d,\min}$, at least one crossing among the two templates which lie on either side of it. It should also be detected with a probability of $Q_{d,\min}$ when the second-stage templates corresponding to the above crossings are employed; (ii) The false alarm should be less than a specified level $Q_{0,\max}$. This false alarm is for the overall search and does not refer to a specific level of the hierarchy. As in the case of a one-step search, a solution in terms of thresholds and spacings need not exist for all combinations of $S_{\min}$, $Q_{d,\min}$, and $Q_{0,\max}$.

Our choice of only two adjacent templates for the first-stage crossing can be justified using the following example. Suppose that $\delta^{(1)}$, the first-stage spacing, is 0.300 sec which is a typical value. Let a signal be present with a chirp time ξ such that $\xi=(\xi_m+\xi_{m+1})/2$, where ξ_m and ξ_{m+1} belong to B_1. Let $\eta^{(1)}=6.0$, $\eta^{(2)}=7.9$, and the signal strength $S=8.8$. These are again representative values. The observed strength of the signal in the templates ξ_m and ξ_{m+1} would be $8.8\mathcal{H}(0.150)=6.56$. The probability of at least one crossing among ξ_m and ξ_{m+1} can be found using Eq. (50). It is, $1-(\int_0^{6.0}\mathrm{Ri}(x,6.56)dx)^2=0.93$. The observed strength of the signal in the template ξ_{m-1} would be $8.8\mathcal{H}(0.150+0.300)=4.51$. If the templates ξ_{m-1} and ξ_{m+2} were also included, then the probability of at least one crossing among these four templates would be $1-(\int_0^{6.0}\mathrm{Ri}(x,6.56)dx)^2(\int_0^{6.0}\mathrm{Ri}(x,4.51)dx)^2=0.94$. Thus, there is an increase of only $\sim 1\%$. These extra events would be those in which at least one crossing was produced among ξ_{m-1} and ξ_{m+2} but none was produced in ξ_m or ξ_{m+1}.

Consider the case where only ξ_{m-1} produces a crossing. In such a case, a search with second-stage templates (and a higher threshold $\eta^{(2)}$) would be performed around ξ_{m-1}. Assume that the observed strength in the template $\xi_{n-1}\in B_{2,m-1}$, the extreme ''right'' among the second stage templates used, is also almost the same ($=6.56$) as that in the template ξ_m. Since there was no crossing for ξ_m, the only way the signal can be detected now would be if a crossing of $\eta^{(2)}$ is produced in ξ_{n-1}. The probability of this event is $1-\int_0^{7.9}\mathrm{Ri}(x,6.56)dx=0.2$ which is quite small. Thus, most of the extra crossings obtained, when four templates are used instead of two, would actually be spurious since they will not lead to the final detection of the signal. The relevant crossings are, therefore, only those that are produced in ξ_m or ξ_{m+1}. It should be ensured that the probability of such crossings is sufficiently high for a given strength. This is precisely what is required in criterion (i) above. We, therefore, continue to use Eq. (50) for the first stage.

Since the second-stage spacing in our final results will turn out to be ~ 0.030 sec, statistical correlations will not be negligible as far as the calculation of the detection probability is concerned. However, just as in the case of a one-step search, the use of this formula would lead to an overestimate of $\delta^{(2)}$ for a given $\eta^{(2)}$. This will lead to an underestimate of $\delta^{(1)}/\delta^{(2)}$. This quantity can be expected to be an estimate of the reduction in computational requirements, as compared to a one-step search, that a two-step search can bring about. Hence, the use of Eq. (50) in the second stage will give a lower bound on the efficacy of a hierarchical search. If this lower bound turns out to be significant, a further, more detailed analysis can be carried out.

We emphasize here that, for a two-step search, the actual detection probability of a signal will be *less* than $Q_{d,\min}$. In fact, it is the probability of a composite event which involves two different template banks and crossings of two different thresholds. The event whose probability is being sought here is that, in the presence of a signal, there be at least one crossing in either of the two consecutive first-stage templates between which the signal lies and that this be followed by the test statistic $\Lambda^{(2)}$ crossing $\eta^{(2)}$. Suppose that a signal, with a strength $S_{\min}$ and chirp time ξ, is present in $x(t)$. Let $\xi_k \in B_1$ be such that $\xi_k<\xi<\xi_{k+1}$. Also, let ξ_l be the template, in the second-stage templates around either ξ_k or ξ_{k+1}, such that $\xi_l<\xi<\xi_{l+1}$. Then, the *true detection probability* of this signal would be dominated by the probability of the event: ($\lambda_k>\eta^{(1)}$ *or* $\lambda_{k+1}>\eta^{(1)}$) *and* ($\lambda_l>\eta^{(2)}$ *or* $\lambda_{l+1}>\eta^{(2)}$). Now, there are two possibilities. The first is that $n=\delta^{(1)}/\delta^{(2)}$ is odd. In this case, if $\xi=(\xi_k+\xi_{k+1})/2$ then it will also be true that $\xi=(\xi_l+\xi_{l+1})/2$. Such a signal can be expected to have the least true detection probability for the strength $S_{\min}$ since it lies in the middle of consecutive templates in both the banks. If we *assume statistical independence* between the maxima over rectified outputs, then the *minimum* true detection probability $Q^t_{d,\min}$ would be $Q^2_{d,\min}$. The second possibility is that n is even. In this case, if the signal chirp time $\xi=(\xi_k+\xi_{k+1})/2$, then ξ will actually coincide with ξ_l in the second stage. Such a signal will not have the minimum true detection probability. In fact, it is easy to see that for this case there is no signal which occurs exactly in the middle of two consecutive templates in both the banks. Thus, under the same assumption as above, $Q^t_{d,\min}$ would be strictly greater than $Q^2_{d,\min}$ Thus, in general, $Q^t_{d,\min}\geqslant Q^2_{d,\min}$ for a two-step search. In our case $Q_{d,\min}=0.95$; therefore, $Q^t_{d,\min}\geqslant 0.9$. However, in the presence of significant statistical correlations among the different rectified outputs, $Q^t_{d,\min}$ would be almost equal to $Q_{d,\min}$. This is so because a crossing of the second-stage threshold would be more likely to be accompanied by a crossing of the first-stage threshold now than would have been the case if the rectified outputs were statistically independent.

If a crossing of $\eta^{(2)}$ (which is quite large) were to be induced by noise alone, it would imply that the noise ''resembles'' the template wave form very closely. Therefore, as in the case of an actual signal, one can expect that such a noise realization would also induce a crossing of the first-stage threshold in a nearby first-stage template. This need not be true when the templates are far apart. However, for small spacings (~ 0.030 sec) it can be expected that the presence of a hierarchy will not be an impediment to a false alarm.

In order to check this, we performed a Monte Carlo simulation with a set of five templates spaced 0.025 sec apart and the input data consisting of noise alone. Since the crossing of a large threshold (~ 7) is an extremely rare event, we had to work with a lower value of $\eta^{(2)}=5.7$ in order to register a sizable number of the events that are of interest here. The first stage threshold $\eta^{(1)}$ was kept at 5.2. These two thresholds are obtained when the value of $Q_{0,\max}$ is kept such that there are, on the average, 10^6 false events in a year while $Q_{d,\min}=0.95$. The calculation of these thresholds assumed statistical independence. The same calculation also yields the spacing used in this simulation (the method used here will be outlined below). The extreme-right and -left templates can be taken as first-stage templates while the remaining serve as second-stage templates. The template chirp times were chosen small enough so that a large number of independent noise realizations (4000) could be generated.

Given a maximum among the second-stage templates which crosses $\eta^{(2)}$, we found that the probability of a crossing of $\eta^{(1)}$ in either of the first-stage templates is ≈ 0.9. On the other hand, if the rectified outputs had been statistically independent, the corresponding probability would have been just that of a crossing of $\eta^{(1)}$ among either of the first-stage templates and would be ≈ 0.08. The few events that failed to make it through the hierarchy were those in which the crossing of $\eta^{(2)}$ occured in the middle template. The probability of such events would be small, especially when the number of second-stage templates becomes larger. Thus, it is a good approximation that statistical correlations almost wipe out the presence of a hierarchy. Since this is true for a low $\eta^{(2)}$, it would also be true for a larger threshold as a crossing would then indicate a stronger match between the noise realization and the template. Therefore, Eq. (57) should remain valid for the calculation of $\eta^{(2)}$ where the number of templates to be used would now be the *total number* of second-stage templates. Since we present our results for weak signal strengths, for which the spacing $\delta^{(2)}$ would be small, we will use Eq. (57) in our analysis. An immediate implication of this is that the second-stage threshold and spacing are now determined independently of the corresponding first-stage quantities. Also, this simulation shows that the detection probability of a *signal* will be much higher than $Q^2_{d,\min}$ when the spacings are small.

The template placement criteria given above admit many solutions for given values of $Q_{d,\min}$, $S_{\min}$, and $Q_{0,\max}$. All solutions will not, however, lead to the same computational requirement in terms of the number of templates that will be employed ultimately. An estimation of the computational requirement of a given solution is complicated by the fact that the number of templates used is a random variable. This is so because the number of crossings n_c of $\eta^{(1)}$ depends on the particular realization of the noise present in $x(t)$. We choose to quantify the computational requirement of a given two-step search by the average number of templates $n_t^{\rm av}$ that will be required on the whole. We denote the average value of n_c in the absence of a signal by $n_c^{\rm av}$. The computational requirement can then be expressed as

$$n_t^{\rm av}=(n_c^{\rm av}+q)\times M+n_t^{(1)}, \qquad (62)$$

where the quantity q is the average number of signals that may occur in a given length of data. This number will depend on the event rate of the signals. For coalescing binary signals, the expected event rate is quite low ($\sim$ 3/yr up to a distance of 200 Mpc [25]). Taking into account the antenna pattern factor, if we assume an event rate of even 50–100 per year and about 10^5 data trains in a year, $q\sim10^{-3}$. We take $q=0$. In Eq. (62), we have neglected, as before, the probability of adjacent crossings in the absence of a signal. Under the assumption of statistical independence made above, $n_c^{\rm av}$ can be obtained as,

$$n_c^{\rm av}=Q_0(\eta^{(1)})\times n_t^{(1)}, \qquad (63)$$

where $Q_0(\eta^{(1)})$ is the probability of a crossing for a single template, $Q_0(\eta)=1-F_0(\eta)$. As an example, for $N_p=224\times2048$ and $N_{\rm eff}=0.7N_p$, $n_c^{\rm av}/n_t^{(1)}=0.005$ for $\eta^{(1)}=6.0$ and $n_c^{\rm av}/n_t^{(1)}=0.7$ for $\eta^{(1)}=5.0$. We see that the number of crossings is quite sensitive to the threshold. Note that Eq. (63) should provide an accurate result because $\delta^{(1)}$ can be expected to be quite large and, thus, the effect of statistical correlations would be small.

From Eq. (62), we see that in order to reduce the computational requirement, $n_t^{(1)}$ should be made small or equivalently, the first-stage spacing $\delta^{(1)}$ should be made large. However, an increase in $\delta^{(1)}$ will lower the observed strength $S_{\rm obs}$ of a signal having a chirp time $\xi=(\xi_m+\xi_{m+1})/2$, for $\xi_m\in B_1$. This would imply a decrease in the probability of a crossing induced by such a signal in the first stage and hence, a violation of criterion (i) above. To avert this, $\eta^{(1)}$ would have to be lowered too. From Figs. 3(a) and 3(b) we see that $F_0(z)$ has an almost step-functionlike behavior below a critical value of z. If, in the course of increasing $\delta^{(1)}$, $\eta^{(1)}$ became less than this critical value, the value of $n_c^{\rm av}$ would rise quite fast so much so that $n_t^{\rm av}$ would actually *increase* with an increase of $\delta^{(1)}$ beyond this point. Thus, there should exist a solution for the thresholds and spacings in a two-step search, for which the computational requirement is minimized. This optimum solution can be found by a simple extension of the algorithm that was presented for a one-step search.

The Algorithm

(i) Given $S_{\rm min}$, $Q_{d,\rm min}$, $Q_{0,\rm max}$, and the padding N_p, a one-step template bank and threshold is set up using the algorithm presented in Sec. II E.

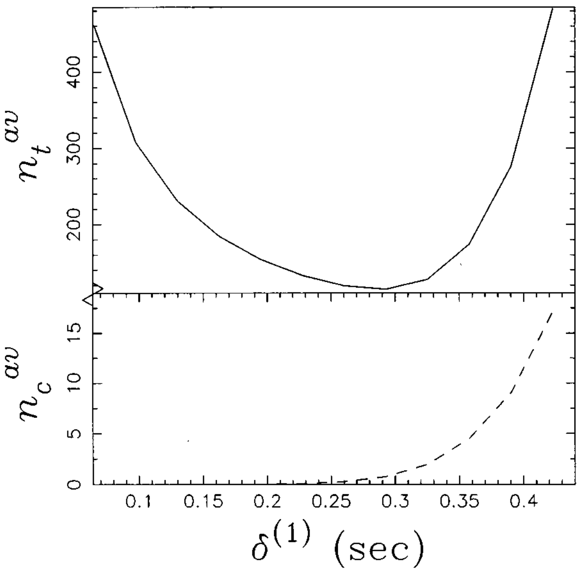

FIG. 6. The average number of templates $n_t^{\rm av}$ required in a two-step search as a function of the first-stage spacing $\delta^{(1)}$. For this curve, $Q_{d,\rm min}=0.95$, $S_{\rm min}=9.0$, $N_p\Delta=224.0$ sec, and $Q_{0,\rm max}=7.10\times10^{-6}$. This false alarm corresponds to an average of one false event per year. The range of chirp times used was $\xi_{\rm min}=2.0$ sec and $\xi_{\rm max}=32.0$ sec. The dashed curve is a plot of the average number of first-stage crossings $n_c^{\rm av}$ in the absence of a signal.

(ii) A trial value of $\delta^{(1)}$ is chosen as $\delta^{(1)}=j\times\delta^{(2)}$ where $j\geqslant2$ is an integer. For each trial value of $\delta^{(1)}$, $\eta^{(1)}$ is calculated so that $Q_d(\eta^{(1)};S_{\rm min},\xi_{\rm min}+\delta^{(1)}/2)=Q_{d,\rm min}$. The average computational requirement $n_t^{\rm av}$ is then calculated using Eqs. (63) and (62). The value of $\delta^{(1)}$ is increased by incrementing j, starting from a suitable initial value, until the minimum of $n_t^{\rm av}$ is reached.

In Fig. 6, a representative plot of the average computational requirement as obtained using this algorithm is shown as a function of the first-stage spacing $\delta^{(1)}$. Also shown is $n_c^{\rm av}$. Observe the sharp rise in $n_c^{\rm av}$ and consequently, in $n_t^{\rm av}$.

Since the value of $Q_{0,\rm max}$ would, in general, be kept quite small ($\sim10^{-6}$), Eq. (58) can be further approximated to yield the following expression for $\eta^{(2)}$:

$$\eta^{(2)}=\left[2\ln\left(\frac{N_TN_{\rm eff}}{Q_{0,\rm max}}\right)\right]^{1/2}, \qquad (64)$$

where N_T is the total number of second-stage templates. A false alarm that yields not more than one false event per year on the average would be

$$Q_{0,\rm max}=N_p\Delta/(365.0\times24.0\times3600.0). \qquad (65)$$

From Eq. (64) and the above expression, it follows that $\eta^{(2)}$ is independent of N_p.

For small spacings of $\delta^{(2)}\sim0.030$ sec, the number of templates that will be required in the one step-search constructed in step (i) above would be ~1000. Thus, the typical threshold $\eta^{(2)}$ that would be required is ~7.9 for a false alarm that leads to one false event per year on the average. The observed strength required to attain a detection probability of 0.95 for such thresholds is ~ 8.6. If the detection of signals having a strength $\simeq 8.6$ is desired with the above probability, then it is clear that an almost continuous set of template chirp times would be required since otherwise the observed strength would become <8.6. Of course, this would require an infinite amount of computing power. Thus, there does not exist a solution to the template placement criteria for such a set of values for $Q_{d,\min}, Q_{0,\max}$ and $S_{\min} \leqslant 8.6$. We call such a limiting value of $S_{\min}$ as the *minimum visible strength* for a one-step search. In actual practice, we find that the minimum visible strength is a little higher at 8.75. This is because $\eta^{(2)}$ also increases as $\delta^{(2)}$ is reduced.

The padding N_p was kept fixed throughout the discussion above. This parameter of a two-step search is, however, decisive in an estimation of the computational *power* required.

C. Computational power required for an on-line analysis

For the on-line detection of a signal, it is required that the processing of a given segment of data be completed within the time required to gather the next one [4]. In a two-step search, the processing required consists of the computation of rectified outputs, finding the maxima over them and, finally, obtaining the test statistic $\Lambda^{(2)}$. The construction of a single rectified output involves, first, performing correlations between the time series of the detector output and two templates, $\bar{h}_0$ and $\bar{h}_{\pi/2}$. For a given N_p, each time series will have $N = N_p + \xi_{\max}/\Delta$ samples, where Δ is the sampling interval. If an FFT is used, this implies doing $6N\ln_2 N$ floating point operations (flops) [20]. The correlations will be followed by $3N_p$ flops for the squaring and summation required for the calculation of a rectified output. The maximization over a single rectified output would involve, at most, N_p flops. Thus, the total number of flops required per template chirp time n_{flop} is

$$n_{\text{flop}} = 6N\ln_2 N + 4N_p . \tag{66}$$

The total number of flops required for the whole template bank N_{flop} is, therefore,

$$N_{\text{flop}} = n_t^{\text{av}} n_{\text{flop}} \tag{67}$$

on the average. We neglect the relatively small number of flops involved in the calculation of $\Lambda^{(2)}$. Thus, for an on-line implementation of a two-step search, N_{flop} operations would have to be performed in $N_p\Delta$ sec. The average computational power required $C_{\text{on line}}$ is, then,

$$C_{\text{on line}} = \frac{N_{\text{flop}}}{N_p\Delta} \times 10^{-6} \text{ MFlops}, \tag{68}$$

where ''MFlops'' stands for a million floating point operations per second.

An increase in N_p leads to an increase in $F_0(\eta)$ for a given threshold. Consequently, the number of false crossings in the first stage would increase, for given $\delta^{(1)}$ and $\eta^{(1)}$, with an increase in N_p. Since n_t^{av} starts to rise when the number of crossings is $\gtrsim 1$, the minimum of n_t^{av} for a larger N_p will be achieved at a larger value of $\eta^{(1)}$. At the same time, the requirement that the probability of a crossing be $Q_{d,\min}$ will force $\delta^{(1)}$ to a smaller value since the required observed strength would now be higher. The overall effect is an increase in n_t^{av} as well as N_{flop}. On the other hand, an increase in N_p will result in a longer time in which the required processing has to be completed. Thus, given $S_{\min}$, $Q_{d,\min}$, and $Q_{0,\max}$, there would exist an optimum N_p at which $C_{\text{on line}}$ is minimized.

TABLE I. Minimum $C_{\text{on line}}$ as a function of N for: $S_{\min}=8.8$, $\xi_{\min}=2.0$ sec, $\xi_{\max}=32.0$ sec, $Q_{d,\min}=0.95$. $\eta^{(2)}=7.92$, $\delta^{(2)}=0.0325$ sec.

$N\times\Delta$(sec)	$C_{\text{on line}}$(MFlops)	$\eta^{(1)}$	$\delta^{(1)}$(sec)	n_t^{av}	n_c^{av}
64.0	41.3 (392.9)	5.58	0.358	97	1
128.0	32.6 (279.6)	5.75	0.325	107	1
256.0	31.6 (253.7)	5.92	0.293	115	1
512.0	33.6 (249.3)	6.11	0.260	124	1
1024.0	36.7 (253.2)	6.11	0.260	134	1

We compute the value of $C_{\text{on line}}$ as a function of N, for two different ranges of the chirp time. For each range, the minimum values of n_t^{av} are found for a few representative values of N_p, keeping $S_{\min}$ fixed. This process is then repeated for progressively lower values of $S_{\min}$ till $\delta^{(2)}$ becomes ~ 0.030 sec. We quote our results for such values of $S_{\min}$ (note that these values are not the minimum visible strengths). Table I contains the results for $\xi_{\min}=2.0$ sec and $\xi_{\max}=32.0$ sec. In this table, $S_{\min}=8.8$. In Table II, $\xi_{\max}=138.0$ sec and $S_{\min}=9.0$. In each table, the minimum value of n_t^{av} is computed, using the algorithm (for small spacings) presented in the previous section, for several values of N. The value of $C_{\text{on line}}$ is then found at each such minimum. We also list the corresponding values of $\eta^{(1)}$, $\delta^{(1)}$, n_t^{av}, and n_c^{av} (the last two are rounded to the nearest whole number). Note that the value of $\eta^{(2)}$, and hence the value of $\delta^{(2)}$, is independent of N_p as shown earlier. They are presented in the captions of the tables. The numbers in parentheses in the second column are the computing powers required for an on-line one-step search. The values of N are chosen as powers of 2 because an FFT is most efficient at these values [20]. The value of $Q_{0,\max}$ is always chosen to give an average of one false event per year [Eq. (65)].

From Table I we see that the value for $N\Delta$ at which $C_{\text{on line}}$ is minimized is 256.0 sec. This implies a padding of $N_p\Delta = 224.0$ sec. However, this minimum is quite broad. In fact, it may be preferable to use $N\Delta = 128.0$ sec since memory requirements would become smaller for computations involving a shorter time series. The corresponding padding in this case is $N_p\Delta = 96.0$ sec which is three times larger

TABLE II. Minimum $C_{\text{on line}}$ as a function of N for: $S_{\min}=9.0$, $\xi_{\min}=2.0$ sec, $\xi_{\max}=138.0$ sec, $Q_{d,\min}=0.95$. $\eta^{(2)}=8.10$, $\delta^{(2)}=0.0335$ sec.

$N\times\Delta$(sec)	$C_{\text{on line}}$(MFlops)	$\eta^{(1)}$	$\delta^{(1)}$(sec)	n_t^{av}	n_c^{av}
256.0	234.2 (2092.1)	5.84	0.335	455	3
512.0	172.9 (1400.8)	6.03	0.301	502	3
1024.0	167.1 (1245.5)	6.21	0.268	545	3
2048.0	175.2 (1211.5)	6.21	0.268	588	6

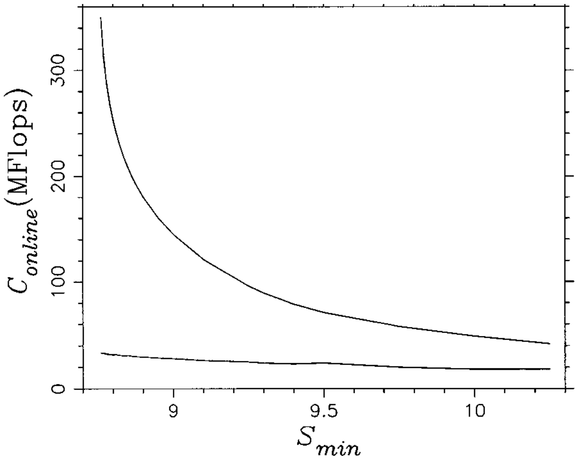

FIG. 7. The computing power required, as a function of $S_{\min}$, for an on-line one-step search (top) and an on-line two-step search (bottom). For both the curves, $N\Delta = 256.0$ sec, $\xi_{\max} = 32.0$ sec, $Q_{d,\min} = 0.95$, and $Q_{0,\max} = 7.10 \times 10^{-6}$.

than $\xi_{\max}$. Similarly, the optimum padding for the larger range of chirp times is also $\sim 3\,\xi_{\max}$. In [4], the optimum padding for a one-step search was found as a trade off between the *number of flops* required for a single data segment and the number of segments that would have to be analyzed in a long period (~ 1 yr). Here, we have used the computing *power* required for an on-line search as a discriminant.

We call the ratio of the computing power required for an on-line one-step search to that required for an on-line two-step search as the *computational advantage* of a two-step search. From Table I, the computational advantage at the minimum of $C_{\text{on line}}$ is 8.0. In Table II, the corresponding number is 7.5. The number of crossings n_c will have a variance given by $n_t^{(1)} Q_0(\eta^{(1)})[1 - Q_0(\eta^{(1)})]$. For the entry from Table I considered above, the rms deviation in $n_c M$ will be $\simeq 12$. Thus, the value of $n_t^{\text{av}} \simeq 127$ and the computational advantage falls to $\simeq 7.2$. This is not a large change. Thus, a two-step search offers a large reduction in the computing power required for an on-line detection while providing a useful combination of detection and false alarm probabilities.

In Fig. 7, we show the computing power required for an on-line two-step search, and the corresponding on-line one-step search, as a function of $S_{\min}$. The value of $\xi_{\max} = 32.0$ sec. For this range of chirp times and the values of $Q_{d,\min}$ and $Q_{0,\max}$ which were used above, it was noted earlier that $S_{\min} \sim 8.6$ is the minimum strength which would be detectable. At this value of $S_{\min}$, the computational requirement would become infinitely large since a template would be required for each value of the chirp time. This is evident in the sharp rise in the computing power for the one-step search as $S_{\min}$ approaches the minimum visible strength. The computational advantage of a two-step search increases to a value of ≈ 13 near this limit. However, the second-stage spacing becomes quite small for such low values of $S_{\min}$ which implies that the statistical correlations among the rectified outputs can no longer be ignored. The formula used for the detection probability would, therefore, be very erroneous in such a case. A more careful analysis, taking statistical correlations into account, is needed when the value of $S_{\min}$ becomes close to the minimum visible strength.

The assumption of statistical independence of rectified outputs is quite good when applied to the first stage. Therefore, the values obtained for $\eta^{(1)}$ and $\delta^{(1)}$ are accurate. However, the spacings obtained for the second-stage template banks are quite small which implies that statistical correlations would now be significant. As far as the false alarm is concerned, it is used in the algorithm only for the determination of $\eta^{(2)}$. We have shown in Sec. II E that the determination of the threshold is insensitive to the presence of such correlations. As in the case of a one-step search, the main source of error here is the use of Eq. (50) in the second stage for the calculation of detection probability.

It was pointed out in the previous section that in the presence of significant statistical correlations, the use of Eq. (50) for the second stage provides an underestimate of $\delta^{(1)}/\delta^{(2)}$. This is approximately the computational advantage of a two-step search. Thus, the computational advantage of a two-step search could be more than the value of $\simeq 8$ obtained above. However, it is difficult to obtain a clean estimate of this error.

For instance, consider the case when only one template is used in Eq. (49) for the detection probability. It was observed in Fig. 4(a) that such a formula gave a detection probability that was lower than the true value. Its use in the algorithm, for the calculation of second-stage detection probability, should therefore give an *overestimate* for $\delta^{(1)}/\delta^{(2)}$. However, a straightforward comparison is difficult because the minimum visible strength for a one-template formula turns out to be significantly higher (~ 9.5). When this larger value of $S_{\min}$ is used with a two-template formula, the typical $\delta^{(2)}$ obtained is large enough (~ 0.10 sec) that statistical correlations should be insignificant and the answer obtained should be correct. But a one-template formula leads, for the same strength, to small spacings (~ 0.020 sec) in which case it is definitely preferable to use this formula rather than Eq. (50).

The probability of adjacent crossings was neglected as compared to nonadjacent crossings. This is true when $n_c \ll n_t^{(1)}$. The values obtained for n_c^{av} clearly satisfy this condition. Note that $\delta^{(1)}$ is large enough for statistical correlations to be negligible and hence Eq. (63) to be valid.

IV. CONCLUSIONS

We have investigated the performance of a two-step hierarchical search for the detection of Newtonian wave forms from coalescing binaries. The noise power spectral density used in the analysis is that of the initial LIGO. A rigorous formalism to describe a two-step search was presented which employs the detection probability of a signal in an essential way to set up the bank of templates and thresholds. The formulas for detection probability and the false alarm used in this formalism were obtained under the assumption of statistical independence of time samples in the rectified output of templates. However, Monte Carlo simulations were used to study the effect of statistical correlations on these formulas. We found that, in the absence of a signal, it is straightforward to incorporate such effects for the case of a single rectified output by the use of an *effective number* of statistically independent samples. The threshold, for a given false alarm, was found to be insensitive to correlations.

Statistical correlations, however, affect the detection probability in a complex manner which makes it difficult to incorporate these effects in a simple way. Strictly speaking, neglecting correlations gives us a lower bound on the computational advantage of a two-step search. However, the bound itself is close to the actual value if the spacing between templates is not too small (we take this value as $\gtrsim 0.030$ sec).

(i) We find that a two-step search allows a large reduction, by at least a factor of $\simeq 8$, in the computing power required for an on-line detection as compared to a one-step search for Newtonian signals. For an on-line detection of signals having a strength of ~ 8.8 (detection probability $\simeq 0.95$ and an average of one false event per year) the computing power required, for a two-step search, is 167 MFlops when the range of chirp times is taken as $\xi_{\rm min}=2.0$ sec and $\xi_{\rm max}=138.0$ sec. We expect our results to hold good since the second-stage spacings are ~ 0.030 sec. For weaker signals the spacings turn out to be much smaller, in which case statistical correlations will play a very significant role. The formula used for the detection probability would then be suspect. However, if we apply this formula for smaller spacings, the reduction achieved in computational power turns out to be much larger. This can be seen from Fig. 7. But these cases merit a more thorough investigation in which statistical correlations are considered more carefully. For closely spaced templates the use of two templates in the calculation of the detection probability, Eq. (50), leads to an overestimate while the use of a single template leads to an underestimate. Either of these can be used in the algorithm that we have presented for a two-step search. If the one-template formula were used (only in the second stage), the minimum strength that would be required to achieve a detection probability of 0.95, for a second-stage threshold of 7.9, would increase to 9.5 from the 8.6 obtained with Eq. (50). However, the computational advantage of a two-step search, when the one-template formula is used, again turns out to be a factor of ~ 10.

(ii) The one-step search formalism presented here was restricted to the family of Newtonian wave forms. This family has essentially a single important parameter, namely, ξ. There is no problem though in extending this formalism to include multiparameter signal families such as the post-Newtonian signals. In fact, this is very much required since it has now been established [9] that the quadrupole approximation will not furnish templates that are good enough for the detection of coalescing binary signals.

The extension is straightforward because the key idea in this formalism, namely, the template placement criteria, is not restricted to a particular signal family. The same would hold for the second-stage template placement criteria also. It may be expected that the computational advantage of a two-step search would be larger when the number of parameters increases because we may expect the hierarchical search to yield an advantage for each parameter independently. The combined computational advantage would be the product of the computational advantage for each parameter.

(iii) The values obtained for the second-stage spacing (~ 0.030 sec) imply that, in the middle of two consecutive templates, the intrinsic ambiguity function has a value of ~ 0.97. This is consistent with the value used in [8] for a one-step search.

In terms of detection probability, the argument used in [8] reduces to assigning a detection probability of unity to signals having a strength above the threshold and zero to signals with a strength below it. This is justified since the detection probability does indeed fall very rapidly as the observed strength goes below the threshold. For instance, if the threshold were 8.0 then an observed strength of 7.5 corresponds to a detection probability of 0.55. In terms of distances, a strength of 8.0 corresponds to ~ 38.0 Mpc while 7.5 corresponds to 40.5 Mpc for a 1.4–1.4 $M_\odot$ binary. Thus, the detectability of events would fall to only half the actual number of events, within a short distance from that corresponding to the threshold.

It should be noted that the canonical value of 0.9 for the intrinsic ambiguity function that is usually used for the determination of the spacing is valid only for large signal strengths (when a detection probability of 0.95 is desired). For instance, when the threshold in a one-step search is 7.9, a signal having an observed strength of 8.6 is detectable with a probability of 0.95. But if the observed strength is taken as $0.9\times 8.6/0.97$, the detection probability would fall to 0.80.

In itself, the reduction of $S_{\rm min}$ from a value of, say, 10.0 to ~ 8.8 leads to an increase of the volume in which sources can be detected by a factor of 1.5. If a one-step search were used, the computing power required for the detection of the weaker signals (with $Q_d=0.95$) would be much larger than that for the stronger signals. Therefore, the question arises whether it is worth devoting a lot of extra resources in order to obtain such an increase in the volume of detectability. However, a two-step search would require only a very small increase in computing power for the same increase in event rate. Moreover, as mentioned in (i), it may bring down the computing requirement significantly in the case of higher order templates. This may happen even for large strengths (~ 10).

(iv) The spacings in both a two-step as well as a one-step search are primarily governed by the intrinsic ambiguity function. If the bandwidth of the detector were to be made larger, as would be the case for the VIRGO and advanced LIGO detectors, the falloff in the intrinsic ambiguity function, as the spacing between template and signal chirp times is increased, would be faster. We believe, however, that the computational advantage of a two-step search over a one-step search would still remain the same. This is because the effect on both the first-stage and second-stage spacings should be similar. In absolute terms, the computing powers required may be greater for both the one-step and two-step search.

ACKNOWLEDGMENTS

We are grateful to Professor B. F. Schutz for pointing out an error in the calculation of the false alarm for a two-step search. Special thanks are due from S.D.M. to Dr. Soma Mukherjee for help with the statistical literature. S.D.M. was supported by the Council of Scientific and Industrial Research (CSIR) of India.

[1] K.S. Thorne, in *300 Years of Gravitation*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987).
[2] A. Abramovici, W.E. Althouse, R.W.P. Drever, Y. Gursel, S. Kawamura, F.J. Raab, D. Shoemaker, L. Sievers, R.E. Spero, K.S. Thorne, R.E. Vogt, R. Weiss, S.E. Whitcomb, and M.E. Zucker, Science **256**, 325 (1992).
[3] C. Bradaschia *et al.*, Nucl. Instrum. Methods Phys. Res. A **289**, 518 (1990).
[4] B.F. Schutz, in *The Detection of Gravitational Radiation*, edited by D. Blair (Cambridge University Press, Cambridge, England, 1989), pp. 406-427.
[5] C.W. Helstrom, *Statistical Theory of Signal Detection*, 2nd ed. (Pergamon Press, London, 1968).
[6] B.S. Sathyaprakash and S.V. Dhurandhar, Phys. Rev. D **44**, 3819 (1991).
[7] S.V. Dhurandhar and B.S. Sathyaprakash, Phys. Rev. D **49**, 1707 (1994).
[8] B. Owen, Phys. Rev. D **53**, 6749 (1996).
[9] T.A. Apostolatos, Phys. Rev. D **54**, 2421 (1996).
[10] Workshop on Coalescing Binaries, *Conference on Astrophysical Sources of Gravitational Waves*, Pennsylvania State University, 1995 (unpublished).
[11] K. Kokkotas, A. Królak, and G. Tsegas, Class. Quantum Grav. **11**, 1901 (1994).
[12] W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.I. Vetterling, *Numerical Recipes* (Cambridge University Press, Cambridge, England, 1986).
[13] L.S. Finn and D.F. Chernoff, Phys. Rev. D **47**, 2198 (1993).
[14] LIGO www site at http://www.ligo.caltech.edu (Physics Monitoring).
[15] J.P.A. Clarke and D.M. Eardley, Astrophys. J. **215**, 311 (1977).
[16] C. Cutler and E.E. Flanagan, Phys. Rev. D **49**, 2658 (1994).
[17] B.F. Schutz and M. Tinto, Mon. Not. R. Astron. Soc. **224**, 131 (1987).
[18] S.V. Dhurandhar and M. Tinto, Mon. Not. R. Astron. Soc. **234**, 663 (1988).
[19] P. Jaranowski and A. Krolak, Phys. Rev. D **49**, 1723 (1994).
[20] O.E. Brigham, *Fast Fourier Transform and its Applications* (Prentice-Hall, Englewood Cliffs, NJ, 1988).
[21] R.N. Bracewell, *Fourier Transform and its Applications*, 2nd ed. (McGraw-Hill, New York, 1986).
[22] J.W. Goodman, *Statistical Optics* (John Wiley, New York, 1985).
[23] A. Stuart and J.K. Ord, *Kendall's Advanced Theory of Statistics*, 5th ed. (Oxford University Press, New York, 1987), Chap. 15.
[24] S.V. Dhurandhar and B.F. Schutz, Phys. Rev. D **50**, 2390 (1994).
[25] R. Narayan, T. Piran, and A. Shemi, Astrophys. J. **379**, L17 (1991); E.S. Phinney, *ibid.* **380**, L17 (1992).