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The local Lorentz gauge represents a natural coordinate frame for an observer to analyze the effect of gravitational waves on detectors, and has been widely used to describe the response of resonant bars. Its application to laser interferometers has thus far been restricted to the long-wavelength regime, in which the separation between the test masses is much less than the wavelength of the gravitational waves. In this paper we show that the local Lorentz gauge can be used for calculations of geodesic deviations of the masses even when their separation is comparable to or greater than the wavelength of the gravitational waves. We find that a complete description of the gravitational waves in this gauge requires taking into account three different effects: displacements of the test masses, the gravitational redshift of light propagating between the masses, and variations in the rates of stationary clocks, all of which are induced by the gravitational wave. Only when taken together do these three effects represent a quantity which is translationally invariant and which can be observed in experiments. This translationally invariant quantity is identical to the response function calculated in the transverse traceless gauge.

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I. INTRODUCTION

Searches for gravitational waves are now conducted with laser interferometers in which the test masses for sensing gravitational waves are separated by distances of several kilometers [1,2]. Variations in the proper distance between these test masses, which might be caused by gravitational waves, are measured with light. Calculations of the test mass response to gravitational waves require fixing the gauge, i.e. one has to choose a particular coordinate system to describe the gravitational wave. There are two widely known choices for fixing the gauge. In the first approach, the gauge is fixed by choosing a transverse traceless (TT) form for the metric perturbations which represent the gravitational wave. Such coordinates are often called transverse-traceless coordinates. In the second approach, one fixes the gauge by choosing a fiducial observer and by building an orthonormal coordinate system in the vicinity of this observer. The corresponding coordinates are called local Lorentz coordinates. The two coordinate systems and the differences in the description of gravitational waves associated with their choice are discussed in textbooks on general relativity [3,4].

In the TT gauge, the coordinates of an inertial test mass are not changing in response to the gravitational wave, and the changes in the distance between two such test masses occur entirely due to changes in the metric. Being a global coordinate system, the TT gauge does not have spatial limitations and therefore calculations of the test mass response to gravitational waves in this gauge are valid for arbitrarily large distances between the masses. In the local Lorentz gauge, the test masses are moving in response to gravitational waves and the changes in the distance between them occur due to the physical displacement of the masses. However, thus far the calculations of the test mass response in this gauge have been limited to the long-wavelength regime, i.e. when the separations between the test masses are much less than the wavelength of the gravitational wave. In this paper we go beyond the long-wavelength regime by taking into account the gravitational redshift of photons propagating between the masses and the changes in the rate of clocks which are used to measure the photon round-trip times. We show that the results are valid for arbitrarily large separations between the masses and agree with those obtained in the TT gauge.

Historically, the coordinates of a local observer have been used primarily for calculations of the response of bar detectors and there were several attempts to apply this gauge to laser interferometers. The role of the local Lorentz gauge as a natural coordinate frame associated with the detector was emphasized in a number of papers [5–8]. Some of the calculations in these studies relied on a Fermi normal expansion as a means of building the coordinate frame of a local observer. As a result, explicit coordinate transformations which connected the TT gauge with the local Lorentz gauge have been constructed and analyzed [9–11]. These coordinate transformations played an essential role in these early studies and were typically used as a starting point for calculations of the test mass response to gravitational waves. It was later realized that the calculations can proceed in the TT gauge and the coordinate transformations can be applied to the final result [12]. The approach we take in this paper is somewhat different: we do not use the TT gauge as a basis for our calculations, nor do we rely on coordinate transformations which connect one gauge with the other. We calculate the effect of the gravitational wave on the test masses entirely in the coordinates of a local observer.

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The presentation in this paper is such that only a few concepts from differential geometry are used, and whenever possible the formulas are given in their Newtonian forms. In this way, we assume the point of view of a Newtonian physicist [3] conducting experiments in a laboratory environment on Earth, and interpreting the outcomes in familiar Newtonian terms, even though the effects themselves belong to general relativity.

II. COORDINATES OF THE TT GAUGE

We begin with a brief overview of the TT gauge. This digression will allow us to introduce the test mass response function which will be needed later for comparison. Subsequent calculations, however, do not rely on the TT gauge.

In the TT gauge the metric which describes a plane-polarized gravitational wave propagating in flat spacetime is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1+h & & \\ & & 1-h & \\ & & & 1 \end{pmatrix}, \tag{1}$$

where $h = h(t + z/c)$ represents the amplitude of the "+" polarization [3]. The corresponding 4-dimensional interval takes the form

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 + h(t + z/c)(dx^2 - dy^2). \tag{2}$$

In these coordinates the gravitational wave is moving in the negative z direction and its wave front is parallel to the xy plane. For all anticipated astrophysical sources, the amplitude of gravitational waves upon their arrival at Earth is expected to be extremely small: $|h| \sim 10^{-21}$ or even less. We therefore will frequently use perturbation expansions in powers of h, keeping only first order terms. For any quantity which is already first order in h we can neglect all terms of order h in its argument because those represent second order corrections.

A special property of the TT coordinates is that inertial test masses, which are initially at rest in these coordinates, remain at rest throughout the entire passage of the gravitational wave [3,4]. Here the words "at rest" only mean that the coordinates of test masses do not change, the masses actually move under the influence of gravitational waves as can be seen from changes in the proper distances between them. A convenient way to analyze variations in the proper distance is by means of a "bouncing photon" [13]. In this approach, a photon is launched from one test mass to be bounced back by the other, as shown in Fig. 1, and its round-trip time is measured with a stationary clock. We assume here for simplicity that the test masses are located along the x axis of the coordinate system.

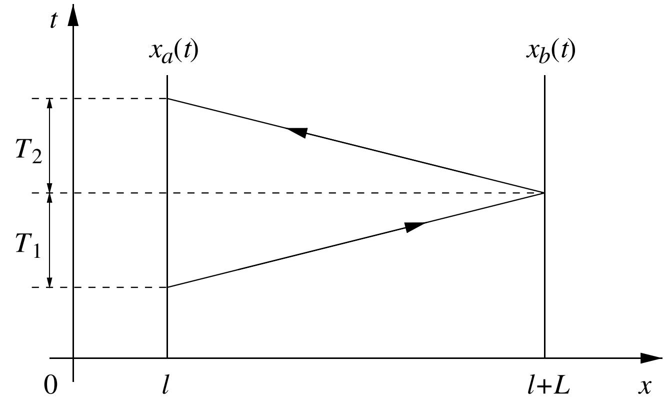

FIG. 1. Bouncing photon in the coordinates of the TT gauge.

The absence of x and y dependences in the metric implies that photon momentum in these directions is conserved. However, momentum in the z direction is not. As a result, photons launched in the xy plane will deflect out of this plane. Here we can safely neglect this effect because the photon deflection into the z direction will be at most of order h. Therefore, to first order in h we can neglect the dz^2 term and approximate $h(t + z/c) \approx h(t)$ in the interval, Eq. (2). Then for photons launched in the x direction the interval becomes particularly simple,

$$ds^2 = -c^2dt^2 + [1 + h(t)]dx^2. \tag{3}$$

The condition for a null trajectory ($ds = 0$) gives us the coordinate velocity of the photon:

$$v^2 \equiv \left(\frac{dx}{dt}\right)^2 = \frac{c^2}{1 + h(t)}, \tag{4}$$

which is a convenient quantity for calculations of the photon propagation times between the test masses. Knowing that the coordinates of the test masses do not change under the influence of the gravitational wave, $x_a = l$ and $x_b = l + L$, we find that the duration of the forward trip is

$$T_1(t) = \int_l^{l+L} \frac{dx}{v(t')}, \tag{5}$$

where the retardation time t' is a parameter along the trajectory of the bouncing photon: $x = x(t')$. To first order in h this integral can be approximated as

$$T_1(t) = T + \frac{1}{2c}\int_l^{l+L} h(t')dx, \tag{6}$$

where $T = L/c$ is the light transit time in the absence of the gravitational wave. Here the retardation time t', as an argument of h, can be approximated by its unperturbed value: $t' = t - (l + L - x)/c$. Similarly, the duration of the return trip is

$$T_2(t) = T + \frac{1}{2c}\int_{l+L}^{l} h(t')(-dx), \tag{7}$$

where the retardation time is given by $t' = t - (x - l)/c$.

The round-trip time can then be found by adding $T_2(t)$ and $T_1[t - T_2(t)]$. The latter can be approximated by $T_1(t - T)$ because the difference between the exact and the approximate values is second order in h. Therefore, to first order in h, the duration of the round-trip is

$$T_{\rm r.t.}(t) = T_1(t - T) + T_2(t). \tag{8}$$

Deviations of this round-trip time from its unperturbed value ($2T$) are given by

$$\delta T(t) = \frac{1}{2c}\int_{l}^{l+L}\left[h\left(t - 2T + \frac{x-l}{c}\right) + h\left(t - \frac{x-l}{c}\right)\right]dx. \tag{9}$$

Even though l explicitly enters this equation, δT does not depend on l. This observation implies that the choice of the origin for this coordinate system does not affect δT. In other words, the result is translationally invariant.

The deviations of the round-trip time, Eq. (9), can also be written in the Fourier or Laplace domain. Laplace transformations are commonly used to analyze linear responses of interferometric gravitational-wave detectors [14] and sometimes are easier to interpret than their time domain equivalents. Define the Laplace transform of $h(t)$ by

$$\tilde{h}(s) = \int_0^\infty e^{-st}h(t)dt, \tag{10}$$

where s is related to the frequency of the gravitational wave (f) via $s = 2\pi i f$. Then the Laplace domain version of Eq. (9) can be written as

$$\frac{\delta\tilde{T}(s)}{T} = C(s)\tilde{h}(s), \tag{11}$$

where $C(s)$ represents the response of test masses to gravitational waves:

$$C(s) = \frac{1 - e^{-2sT}}{2sT}. \tag{12}$$

A number of derivations of this result, some quite different from ours, can be found in the literature, for example, in Refs. [15–18] and more recently in Refs. [14,19]. Calculations which allow arbitrary orientations of test masses with respect to incoming gravitational waves can be found in Refs. [20,21].

There are several reasons why the above picture is not satisfactory from a physical point of view, even though it is mathematically sound. The main problem with the coordinates of the TT gauge is that they generally cannot be realized in experiments. They would be difficult to implement in a laboratory environment on Earth because the coordinate grid would have to be changing in unison with the passing gravitational wave, an effect commonly known as "breathing of the frame." They may, however, be realized in space with a network of freely falling satellites. For ground-based gravitational-wave detectors, the interpretation of the results derived in the TT gauge is not straightforward because these coordinates are not feasible. We therefore turn our attention to the coordinates of a local observer which form a reference frame naturally associated with a detector of gravitational waves.

III. COORDINATES OF A LOCAL OBSERVER

An observer in a laboratory environment on Earth typically uses a coordinate system in which spacetime is locally flat [13], and the distance between close points is given simply by the difference in their coordinates in the usual sense of Newtonian physics [3]. In this reference frame, gravitational waves manifest themselves through the tidal forces which they exert on the masses. To describe the tidal forces we consider a test mass which is free to move in the field of a gravitational wave. For simplicity, we assume that the gravitational wave is propagating along the z axis, and that the x and y directions of the coordinate system match the polarization of the gravitational wave. Then the tidal acceleration of the test mass [3] in the plane of the wave front of the gravitational wave is given by

$$\ddot{x} = +\frac{1}{2}\ddot{h}x, \tag{13}$$

$$\ddot{y} = -\frac{1}{2}\ddot{h}y, \tag{14}$$

where $h = h(t + z/c)$. Equivalently, one can say that there is a gravitational potential [5,9,22]:

$$\Phi(\mathbf{r}, t) = -\frac{1}{4}\ddot{h}(t + z/c)(x^2 - y^2), \tag{15}$$

which generates the tidal forces, and that the motion of the test mass is governed by Newton's law:

$$\ddot{\mathbf{r}} = -\nabla\Phi + \mathbf{a}_{\rm p}, \tag{16}$$

where $\mathbf{a}_{\rm p}$ is a post-Newtonian correction (see Appendix A). This correction is needed for a complete description of test mass motion in the direction of the gravitational-wave propagation (z direction). Here we neglect the test mass motion in this direction because it will be perpendicular to the trajectory of the photon bouncing off this mass, and, to first order in h, it will not affect the photon trajectory or its propagation time.

The potential Φ is not static and therefore the energy of the test mass is not conserved. In particular, a test mass at rest can acquire energy from the passing gravitational wave—a notion proved by Bondi *et al.* with a thought experiment involving a bead sliding on a rigid rod [23].

The correspondence between Newtonian theory and general relativity is given by the formula

$$g_{00} = -1 - \frac{2}{c^2}\Phi, \tag{17}$$

a mathematical relationship which encompasses the physics of gravitational redshifts. In the following calculations we will use perturbation expansions and therefore will rely on the assumption that $|\Phi|/c^2 \ll 1$. To satisfy this condition, we require that the spatial coordinates x and y do not extend indefinitely. Such a limitation, however, will not restrict us in any way. Indeed, for gravitational waves with the largest expected amplitudes ($|h| \sim 10^{-21}$) and the highest frequencies within the bandwidth of laser interferometric detectors (~ 10 kHz), this limitation implies that $|x|, |y| \ll 10^{14}$ m, which is always satisfied in a laboratory environment on Earth.

The solution to Eqs. (13) and (14) is usually found using the perturbation method. Assume that the test mass is initially at rest in the $z = 0$ plane. To first order in h, the displacements of the test mass caused by the gravitational wave are given by

$$\delta x(t) = +\frac{1}{2}x_0 h(t), \tag{18}$$

$$\delta y(t) = -\frac{1}{2}y_0 h(t), \tag{19}$$

where x_0 and y_0 are the initial (unperturbed) coordinates of the test mass [3]. In this regard the local Lorentz gauge is very different from the TT gauge in which the test mass coordinates are not changing under the influence of the gravitational wave.

IV. REQUIREMENT OF TRANSLATIONAL INVARIANCE

An interesting feature of the local Lorentz gauge is the coordinate dependence of tidal forces—they can be changed by a mere shift of the origin of the coordinate system:

$$x \to x + X, \quad \text{and} \quad y \to y + Y. \tag{20}$$

The same can be said about test mass displacements [Eqs. (18) and (19)]. This is the earliest indication that the translational symmetry may not be explicit in these coordinates. However, at this point the dependence on the origin seems to be quite harmless, and we can entertain the notion that it can be removed simply by considering the *relative* motion of test masses.

As before, we probe the geometry of spacetime with a bouncing photon. Consider two test masses with coordinates x_a and x_b, and assume that the photon is launched from one test mass and bounces off the other. Let the unperturbed values for the test mass coordinates be

$$x_a = l, \quad \text{and} \quad x_b = l + L, \tag{21}$$

and the unperturbed propagation time between the masses be

$$T = \frac{L}{c}. \tag{22}$$

From Eq. (18) we find that the displacements of the test masses under the influence of the gravitational wave are

$$\delta x_a(t) = \frac{1}{2} l h(t), \tag{23}$$

$$\delta x_b(t) = \frac{1}{2}(l + L)h(t). \tag{24}$$

If we define the relative displacement as

$$\delta L(t) = \delta x_b(t) - \delta x_a(t), \tag{25}$$

we would obtain the following result:

$$\delta L(t) = \frac{1}{2} L h(t), \tag{26}$$

which is obviously independent of l and therefore independent of the choice of the origin for these coordinates, as we expected. Note, however, that the corresponding change in the photon round-trip time ($\delta T = 2\delta L/c$), written in the Laplace domain as

$$\frac{\delta\tilde{T}(s)}{T} = \tilde{h}(s), \tag{27}$$

would be different from the one obtained in the TT gauge, Eq. (11). This well-known observation simply reflects the fact that Eq. (26) is an approximation. As we will see, several physical effects have been neglected in this simplified picture.

Until recently Eq. (27) was regarded as a good approximation to the exact result, Eq. (11). For a long time, searches for gravitational waves have been conducted with metal bar detectors and prototype laser interferometers with relatively small sizes (a few meters). For these detectors the separation between the test masses is usually much less than the wavelength of the gravitational waves $|sT| \ll 1$ and therefore $C(s) \approx 1$, which makes Eq. (11) equivalent to Eq. (27). The situation changed with the arrival of large-scale laser interferometers. In these detectors the test masses for sensing gravitational waves are separated by distances of several kilometers and the long-wavelength regime ($|sT| \ll 1$) becomes hard to justify. Furthermore, recent studies [24] have shown that these interferometers are capable of detecting gravitational waves with wavelengths comparable to their arm-lengths ($|sT| \sim 1$) thus operating entirely outside the long-wavelength regime. In what follows we do not assume the long-wavelength approximation and therefore consider test masses which are separated by arbitrarily large distances.

V. REQUIREMENT OF CAUSALITY

For large separations between the masses the definition for relative displacement, Eq. (25), becomes unphysical. Causality requires that the displacement of one test mass be compared with the displacement of the other at a later time to allow for a finite delay from the light propagation. The relative displacements of the test masses defined in this way will, in general, be different for the forward and return trips:

$$\delta L_1(t) = \delta x_b(t) - \delta x_a(t - T_1), \tag{28}$$

$$\delta L_2(t) = \delta x_b(t - T_2) - \delta x_a(t), \tag{29}$$

where T_1 and T_2 are the corresponding photon propagation times, as shown in Fig. 2. Within Eqs. (28) and (29) these propagation times can be replaced with their nominal value (T) because the test mass displacements, $\delta x_{a,b}$, are already first order in h. The total change in the distance between the masses in one photon round-trip is therefore given by

$$\begin{aligned}\delta L_{\text{r.t.}}(t) &= \delta L_1(t - T) + \delta L_2(t)\\ &= 2\delta x_b(t - T) - \delta x_a(t) - \delta x_a(t - 2T).\end{aligned} \tag{30}$$

An explicit formula for this length change, written in terms of the amplitude of the gravitational wave, is

$$\delta L_{\text{r.t.}}(t) = (l + L)h(t - T) - \frac{1}{2}lh(t) - \frac{1}{2}lh(t - 2T). \tag{31}$$

Note that $\delta L_{\text{r.t.}}$ is not translationally invariant despite the fact that it represents the relative displacement of the test masses. This is the price one has to pay for satisfying the causality condition.

Changes in the distance between the masses lead to changes in the round-trip time for the bouncing photon,

$$\frac{\delta_x T(t)}{T} = h(t - T) - \mu[h(t) - 2h(t - T) + h(t - 2T)]. \tag{32}$$

Here we introduce the dimensionless parameter

$$\mu = \frac{l}{2L}. \tag{33}$$

The presence of this parameter in subsequent formulas will indicate a loss of translational invariance. The Laplace domain version of Eq. (32) can be written in a manner similar to Eq. (11), namely

$$\frac{\delta_x \tilde{T}(s)}{T} = D_x(s)\tilde{h}(s), \tag{34}$$

where $D_x(s)$ is the corresponding response function

$$D_x(s) = e^{-sT} - \mu(1 - e^{-sT})^2. \tag{35}$$

Clearly $D_x(s)$ depends on the choice of the origin for this coordinate system. By introducing the light propagation delays, we lost the translational symmetry which was explicit in Eq. (26). At first, this loss of translational invariance may seem quite natural. After all, the potential explicitly depends on coordinates, which in classical mechanics usually means that the symmetry with respect to translations is lost. However, this contradicts our physical intuition which maintains that all locations on the wave front of the plane gravitational wave are equivalent. Therefore, physical quantities must be the same no matter where on this plane they are measured, even though the potential explicitly discriminates between different locations. We will soon see that this is indeed the case and that translational invariance is restored, but only when another physical effect is added to the picture: the gravitational redshift of light propagating between the masses.

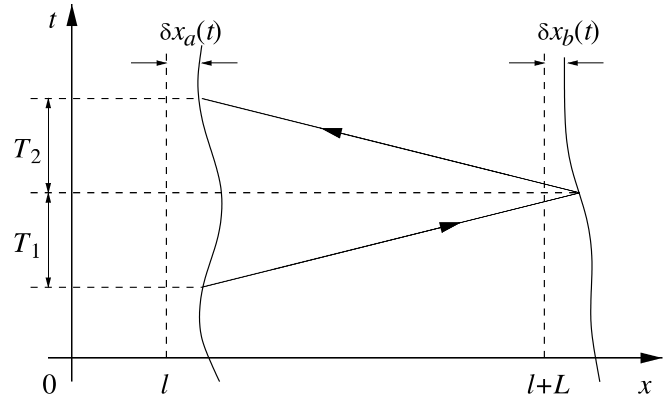

FIG. 2. Bouncing photon in the coordinates of a local observer.

VI. DISTRIBUTED GRAVITATIONAL REDSHIFT

We have calculated variations in the photon round-trip time from the motion of the test masses induced by the gravitational wave. In this calculation, we implicitly assumed that the propagation of the photon between the test masses is uniform, as if it were moving in flat spacetime. The presence of tidal forces indicates that the spacetime is curved, and therefore, the bouncing photon will experience a gravitational redshift [see Eq. (17)]. There will be two such effects in the following calculations. The first will require spatial separation between the test masses and will be called the distributed gravitational redshift; the second will occur at a single point in space and therefore will be called the localized gravitational redshift.

The distributed gravitational redshift can be calculated as follows. Consider the interval for photons propagating along the x axis,

$$ds^2 = g_{00}c^2dt^2 + dx^2, \tag{36}$$

where g_{00} is the time component of the metric, Eq. (17). As was the case with the TT gauge, deflection of the photon trajectories into the z direction caused by the gravitational wave gives rise to second order terms in the interval and therefore can be neglected (see also Appendix B). Then the condition for a null trajectory ($ds = 0$) gives us the coordinate velocity of the photon:

$$v^2 \equiv \left(\frac{dx}{dt}\right)^2 = c^2 + 2\Phi(x, t). \tag{37}$$

To first order in h, the velocity can be approximated by

$$v \approx \pm c\left[1 + \frac{1}{c^2}\Phi(x, t)\right], \tag{38}$$

where "+" and "−" correspond to the forward and return trip, respectively.

Knowing the coordinate velocity of the photon, we can define the propagation times for the forward and return trips as

$$T_1(t) = \int_{x_a(t-T_1)}^{x_b(t)} \frac{dx}{v}, \quad \text{and} \quad T_2(t) = \int_{x_b(t-T_2)}^{x_a(t)} \frac{dx}{v}. \tag{39}$$

Deviations of the test masses from their nominal positions caused by the gravitational wave make the boundaries of these integrals vary with time:

$$x_a(t) = l + \delta x_a(t), \tag{40}$$

$$x_b(t) = l + L + \delta x_b(t), \tag{41}$$

where $\delta x_{a,b}$ are given by Eqs. (23) and (24). Separating the boundary terms, we obtain

$$T_1(t) = \frac{\delta L_1(t)}{c} + \int_l^{l+L} \frac{dx}{v}, \tag{42}$$

$$T_2(t) = \frac{\delta L_2(t)}{c} + \int_{l+L}^{l} \frac{dx}{v}. \tag{43}$$

Note that we have already considered the contribution of the varying boundaries to the photon propagation times; their combined effect is given by $\delta_x T$ in Eq. (32). At this point we only need to calculate the times for photon propagation between the fixed boundaries: l and $l + L$. These propagation times will be denoted here by $\Delta T_{1,2}$ to be distinguished from $T_{1,2}$.

For the forward trip, the propagation time between the fixed boundaries is

$$\Delta T_1(t) = \int_l^{l+L} \frac{dx}{v(x, t')} \approx T - \frac{1}{c^3}\int_l^{l+L} \Phi(x, t')dx, \tag{44}$$

where t' is the retardation time which corresponds to the unperturbed photon trajectory: $t' = t - (l + L - x)/c$. Similarly, the propagation time between the fixed boundaries for the return trip is

$$\Delta T_2(t) = T - \frac{1}{c^3}\int_{l+L}^{l} \Phi(x, t')(-dx), \tag{45}$$

where the retardation time is given by $t' = t - (x - l)/c$. The round-trip time for photons traveling between the fixed boundaries can be found by adding $\Delta T_2(t)$ and $\Delta T_1(t - T)$. Deviations of this round-trip time from its unperturbed value ($2T$) are given by

$$\delta_v T(t) = -\frac{1}{c^3}\int_l^{l+L} \Phi\left(x, t - 2T + \frac{x - l}{c}\right)dx - \frac{1}{c^3}\int_l^{l+L} \Phi\left(x, t - \frac{x - l}{c}\right)dx. \tag{46}$$

After replacing the potential with its explicit form, Eq. (15), we obtain a formula for $\delta_v T$ in terms of the amplitude of the gravitational wave:

$$\delta_v T(t) = \frac{1}{4c^3}\int_l^{l+L}\left[\ddot{h}\left(t - 2T + \frac{x - l}{c}\right) + \ddot{h}\left(t - \frac{x - l}{c}\right)\right]x^2 dx. \tag{47}$$

This contribution to the round-trip propagation time comes from changes in the coordinate velocity of the bouncing photon, which accumulate over the photon trajectory. It will be called here the distributed gravitational redshift.

Equation (47) is similar to Eq. (9), in that both formulas represent cumulative effects of the gravitational wave. However, unlike Eq. (9), which is translationally invariant, Eq. (47) is not, as can be seen from the presence of the x^2 factor in the integrand. A better way to analyze the loss of translational invariance would be to rewrite the result in the Laplace domain:

$$\frac{\delta_v \tilde{T}(s)}{T} = D_v(s)\tilde{h}(s), \tag{48}$$

where $D_v(s)$ is the corresponding response function:

$$D_v(s) = \frac{1}{2sT}(1 - e^{-2sT}) - e^{-sT} + \mu(1 - e^{-sT})^2 + \mu^2(1 - e^{-2sT})sT. \tag{49}$$

The terms proportional to μ and μ^2 represent the dependence of the response function on the choice of the origin for this coordinate system.

We can now combine the variations in the photon propagation time which are caused by the motion of the test masses with those caused by the distributed gravitational redshift. The resulting round-trip time is

$$T_{\mathrm{r.t.}} = 2T + \delta_x T + \delta_v T. \tag{50}$$

Thus far the combined effect of the gravitational wave is given by

$$D_x(s) + D_v(s) = \left(\frac{1}{2sT} + \mu^2 sT\right)(1 - e^{-2sT}). \tag{51}$$

By adding the two response functions we cancel the terms proportional to μ. However, the terms proportional to μ^2

remain. As will be shown next, these terms are related to the localized gravitational redshift.

VII. LOCALIZED GRAVITATIONAL REDSHIFT

The third contribution to the photon round-trip time is also related to gravitational redshift, although it is somewhat different from the distributed effect described above. The photon round-trip time $T_{\mathrm{r.t.}}$ as given in Eq. (50) refers to the lapse of coordinate time. The physical time measured with stationary clocks is different, except for the clock at the origin which coincidentally reads the coordinate time. In general, the time t^* registered by a stationary clock is related to the coordinate time t by

$$dt^{*2} = -g_{00}(\mathbf{r}, t)dt^2, \tag{52}$$

where $\mathbf{r}$ is the location of this clock. Note that the trajectories of the bouncing photon begin and end at the first test mass, which implies that direct measurements of round-trip times must be made with a clock attached to this test mass. To first order in h we can neglect the motion of the clock associated with the motion of the mass, and assume that the clock is at rest at $x = l$. The presence of the time-dependent gravitational potential affects the rate of this clock, causing it to register the round-trip time as

$$\begin{aligned} T^*_{\mathrm{r.t.}}(t) &= \int_{t-T_{\mathrm{r.t.}}}^{t} \sqrt{-g_{00}(l, t')}dt' \\ &\approx T_{\mathrm{r.t.}}(t) + \frac{1}{c^2}\int_{t-T_{\mathrm{r.t.}}}^{t} \Phi(l, t')dt'. \end{aligned} \tag{53}$$

Because the integrand is first order in h, the lower boundary of integration can be replaced with its unperturbed value, $t - 2T$. Therefore, to first order in h, variations in the round-trip time due to this effect are given by

$$\begin{aligned} \delta_t T(t) &\approx \frac{1}{c^2}\int_{t-2T}^{t} \Phi(l, t')dt' \\ &= -\frac{l^2}{4c^2}[\dot{h}(t) - \dot{h}(t - 2T)]. \end{aligned} \tag{54}$$

This contribution to the round-trip propagation time comes from the nonuniformity of time flow which occurs at a given place. It will be called here the localized gravitational redshift. In the Laplace domain the effect of the localized gravitational redshift can be written as

$$\frac{\delta_t \tilde{T}(s)}{T} = D_t(s)\tilde{h}(s), \tag{55}$$

where $D_t(s)$ is the corresponding response function

$$D_t(s) = -\mu^2(1 - e^{-2sT})sT. \tag{56}$$

Addition of this response function to Eq. (51) will cancel the μ^2 terms, giving us a translationally invariant result.

We can now conclude that the change in the round-trip time caused by the gravitational wave consists of three contributions:

$$\delta T = \delta_x T + \delta_v T + \delta_t T, \tag{57}$$

which correspond to displacements of test masses, changes in the coordinate velocity of bouncing photons, and variations in the rate of stationary clocks. Their combined effect is given by the sum:

$$D_x(s) + D_v(s) + D_t(s) = \frac{1 - e^{-2sT}}{2sT}, \tag{58}$$

which is translationally invariant. Furthermore, the sum gives us a response function identical to $C(s)$, Eq. (12). With the distributed and localized gravitational redshifts taken into account, calculations of the observable photon round-trip time in the local Lorentz gauge yield the same result as calculations in the TT gauge.

VIII. ROUND-TRIP PHASE SHIFT OF LIGHT

Thus far, we have considered the bouncing photon as a particle, assuming that there is a beginning and an end to the photon round trips. In practice, measurements of photon propagation times are done with optical interferometry in which photons are represented by continuous electromagnetic waves. We will therefore briefly describe how the above calculations can be modified to become applicable to continuous waves. Assume that the light is represented by a plane monochromatic wave with frequency ω and wave number k. In the absence of gravitational waves, the light wave is given explicitly by $\exp[i(\omega t \mp kx)]$. Then the photon trajectory introduced above would describe advancement of a surface of constant phase, whereas the photon velocity becomes the phase velocity of the wave. In this approach, the quantity of interest is the round-trip phase, or more precisely, its variation caused by the gravitational wave.

The first contribution to the round-trip phase variation comes from the motion of the test masses:

$$\delta\psi_x = -k\delta L_{\mathrm{r.t.}} = -\omega\delta_x T, \tag{59}$$

where $\delta L_{\mathrm{r.t.}}$ is the change in the distance between the test masses, Eq. (31), and $\delta_x T$ is the corresponding change in the round-trip time, Eq. (32). The second contribution comes from the change in the phase velocity of the wave:

$$\delta\psi_v = -\omega\delta_v T, \tag{60}$$

where $\delta_v T$ is the corresponding variation in the round-trip time. Here we give a brief derivation of this result based on simple physical arguments. Another derivation based on a solution of the eikonal equation is given in Appendix B.

In the presence of the gravitational wave, the frequency and wave number are no longer constant, they become functions of position and time: $\Omega(x, t)$ and $K(x, t)$. Then the dispersion relation for the electromagnetic wave is

$$\Omega^2 = v^2K^2, \tag{61}$$

where v is the phase velocity of the wave previously introduced as the coordinate velocity of the photon, Eq. (37). To first order in h, this dispersion relation can be approximated as

$$\Omega - cK = \frac{k}{c}\Phi. \tag{62}$$

For an electromagnetic wave moving in the positive x direction, an infinitesimal phase shift is given by $(\Omega dt - Kdx)$. The accumulated phase shift can be found by integrating this quantity along the trajectory of a given wave front. In doing so we would find that the accumulated phase shift vanishes by virtue of $dx/dt = \Omega/K$, i.e. traveling with a wave front implies following a surface of constant phase for which no phase change ensues. We must remember however that the effect of the gravitational wave on the light is not described by the total phase shift along the photon trajectory. Rather, it is given by the phase difference between perturbed and unperturbed electromagnetic waves. This phase difference can be found by integrating the infinitesimal phase shift along the unperturbed photon trajectory:

$$\delta\psi_v = \int_C (\Omega dt - Kdx). \tag{63}$$

Here C stands for the unperturbed photon trajectory: $dx/dt = \pm c$, which extends to the unperturbed test mass locations: $x_a = l$ and $x_b = l + L$. Using the dispersion relation, Eq. (62), it is easy to show that

$$\delta\psi_v = \frac{k}{c^2}\int_C \Phi dx, \tag{64}$$

which is equivalent to Eq. (60) as can be seen from the definition of $\delta_v T$ in Eq. (46).

We can now add this phase change to the phase change produced by the motion of the test masses, Eq. (59). There is no need to worry about the difference between k and K in the definition of $\delta\psi_x$. The displacements of test masses are first order in h and therefore any correction to k would result in second order terms. Thus, the combined effect is given by

$$\delta\psi_W = \delta\psi_x + \delta\psi_v = -\omega(\delta_x T + \delta_v T). \tag{65}$$

As we already know [see Eqs. (50) and (51)], this phase change is not translationally invariant and therefore cannot be observed in experiment. This is because $\delta\psi_W$ represents the difference between the round-trip phases for two electromagnetic waves: with and without the gravitational wave. Such a phase change cannot be observed because the two waves do not coexist in the same spacetime. To form an observable quantity we need to compare the round-trip phase variation of the traveling wave with that of a reference wave which can coexist with the traveling wave, for example, the source itself.

In flat spacetime, the phase of the source would simply be ωt, and the phase shift of the source $2\omega T$. In the presence of the gravitational wave, the phase of the source becomes ωt^*, where t^* is the proper time at the location of the source. Then the phase shift of the source is given by

$$\omega[t^*(t) - t^*(t - 2T)] = \omega \int_{t-2T}^{t}\left[1 + \frac{1}{c^2}\Phi(l, t')\right]dt'. \tag{66}$$

The change in this phase shift which is caused by the gravitational wave is

$$\delta\psi_S = \omega\delta_t T, \tag{67}$$

where $\delta_t T$ is given by Eq. (54).

We can now compare the phase change of the traveling wave, Eq. (65), with that of a stationary source, Eq. (67). The difference between the phase of the wave front for the electromagnetic wave returning to the source and the phase of the source at that moment is

$$\delta\psi = \delta\psi_W - \delta\psi_S. \tag{68}$$

More explicitly, this phase difference is given by

$$\delta\psi = -\omega(\delta_x T + \delta_v T + \delta_t T). \tag{69}$$

As we already know [see Eqs. (57) and (58)], this phase is translationally invariant and therefore represents an observable quantity. It is not surprising that this phase deviation is related to the deviation of the photon round-trip time, Eq. (57), by

$$\delta\psi = -\omega\delta T. \tag{70}$$

This formula could have been guessed from a simple dimensional analysis and the requirement of translational invariance, except perhaps for the minus sign. The derivation above serves to explain the physical meaning of the relative phase shift and its constituent parts. In short, the motion of the test masses and the distributed gravitational redshift contribute to the phase shift of the traveling wave, whereas the localized gravitational redshift contributes to the phase shift of the stationary source.

IX. CONCLUDING REMARKS

We have introduced the three effects caused by gravitational waves one by one for simplicity. Of course, one can take a more direct approach and derive these three effects concurrently. This can be done by starting with an abstract definition for the proper time of the photon round-trip,

$$T^*_{\text{r.t.}} = \int dt^*, \tag{71}$$

and then by proceeding with the integration as follows:

$$\int dt^* = \int \sqrt{-g_{00}(l, t')}dt' = \int_{C^*} \frac{\sqrt{-g_{00}(l, t')}}{v(x, t')}dx, \tag{72}$$

where C^* stands for the unperturbed photon trajectory which extends to the perturbed test mass positions: $x_a(t)$

and $x_b(t)$. By evaluating various terms in the integral, Eq. (72) to first order in h, one would reproduce the three components of the round-trip time variation described above.

The requirement of translational invariance played an important role throughout this paper. It allowed us to tell the difference between physical and unphysical terms in the response of test masses to gravitational waves. The associated coordinate transformations, Eq. (20), are a particular case of transformations known as *changes of the origin*, which in general relativity are usually accomplished with the help of Fermi-Walker transports [13]. Following the Newtonian style of our presentation, we viewed these transformations as *translations* and required that they represent a symmetry. This symmetry owes its existence to the planeness of the gravitational wave [23].

We have shown that the coordinates of the local Lorentz gauge can be used for calculating geodesic deviations even for large separations between the test masses. The response of test masses to gravitational waves in this gauge acquires contributions from three different effects: the motion of the test masses and the distributed and localized gravitational redshifts. Only when taken together do these effects yield the observable quantity. The approach followed in this paper allows us to calculate physical quantities directly in the coordinates of the local observer which form a natural reference frame associated with detectors of gravitational waves.

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APPENDIX A: METRIC TRANSFORMATION AND GEODESIC EQUATION

For completeness, we present here the first-order transformation rules which connect the coordinates of a local observer with the coordinates of the TT gauge. Denote the coordinates of a local observer by x^μ, where $\mu = 0, 1, 2, 3$ ($x^0 = ct$) and the metric in these coordinates by $g_{\mu\nu}$. Also, denote the coordinates of the TT gauge by $\bar{x}^\mu$ and the corresponding metric by $\bar{g}_{\alpha\beta}$. The components of the metric in the TT gauge, Eq. (1), can be written as

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & -h & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A1}$$

where $\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$ is the Minkowski metric and $h = h(\bar{t} + \bar{z}/c)$. The coordinate transformation $\bar{x}^\mu \to x^\mu$ induces the following transformation of the metric:

$$g_{\mu\nu} = \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu}\bar{g}_{\alpha\beta}. \tag{A2}$$

By definition, $g_{\mu\nu}$ becomes the Minkowski metric at the origin, and all its first derivatives vanish at this point. There are a number of metrics which satisfy these conditions. Here we consider one such choice [5,6,9]. It can be obtained with coordinate transformations, which to first order in h, are given by

$$\bar{t} = t - \frac{1}{4c^2}\dot{h}(x^2 - y^2), \tag{A3}$$

$$\bar{x} = x - \frac{1}{2}hx, \tag{A4}$$

$$\bar{y} = y + \frac{1}{2}hy, \tag{A5}$$

$$\bar{z} = z + \frac{1}{4c}\dot{h}(x^2 - y^2). \tag{A6}$$

The corresponding metric tensor can be obtained by performing the induced transformation, Eq. (A2). To first order in h, the result is

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{2}{c^2}\begin{pmatrix} \Phi & 0 & 0 & \Phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi & 0 & 0 & \Phi \end{pmatrix}, \tag{A7}$$

where Φ is a function of the new coordinates:

$$\Phi = -\frac{1}{4}\ddot{h}(t + z/c)(x^2 - y^2). \tag{A8}$$

In Newtonian theory this function becomes the generating potential for tidal forces induced by the gravitational wave. It is interesting to note that although the transformation rules, Eqs. (A3)–(A6), are approximate, the metric, Eq. (A7), is an exact solution of Einstein's equations [25,26]. A brief discussion of the relationship between the metric in the local Lorentz gauge and the exact solution can be found in Ref. [22].

Geodesic motion of a test mass in spacetime with metric, Eq. (A7), can be described as follows. Introduce two new coordinates:

$$u = ct + z, \quad \text{and} \quad w = ct - z. \tag{A9}$$

Note that the potential Φ depends on x, y, and u but not w. In these coordinates, the Lagrangian for the test mass is

$$\mathcal{L} = \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 - \frac{du}{d\tau}\frac{dw}{d\tau} - \frac{2}{c^2}\left(\frac{du}{d\tau}\right)^2\Phi, \tag{A10}$$

with the usual constraint: $\mathcal{L} = -c^2$. Thus, $d^2u/d\tau^2 = 0$ which implies that $u(\tau) = u_0 + c\tau$. The other equations of motion are

$$\frac{d^2x}{d\tau^2} = -\frac{\partial\Phi}{\partial x}, \qquad \frac{d^2y}{d\tau^2} = -\frac{\partial\Phi}{\partial y}, \tag{A11}$$

and

$$\frac{d^2w}{d\tau^2} = -\frac{4}{c}\left(\frac{\partial\Phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\Phi}{\partial y}\frac{dy}{d\tau}\right) - 2\frac{\partial\Phi}{\partial u}. \tag{A12}$$

This last equation leads to

$$\frac{d^2z}{d\tau^2} = -\frac{\partial\Phi}{\partial z} + \frac{2}{c}\frac{d\Phi}{d\tau}. \tag{A13}$$

To first order in h, we can approximate τ with t and thus obtain the equation of motion in the semi-Newtonian form:

$$\ddot{\mathbf{r}} = -\nabla\Phi + \frac{2}{c}\frac{d\Phi}{dt}\hat{\mathbf{z}}, \tag{A14}$$

where the last term represents the post-Newtonian correction. Note that this equation is valid for arbitrarily large values of $\mathbf{r}$.

APPENDIX B: EIKONAL EQUATION

Propagation of an electromagnetic wave in curved spacetime is described by the eikonal Ψ [27] which satisfies the equation

$$g^{\mu\nu}\frac{\partial\Psi}{\partial x^\mu}\frac{\partial\Psi}{\partial x^\nu} = 0, \tag{B1}$$

where $g^{\mu\nu}$ is the contravariant metric tensor,

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{2}{c^2}\begin{pmatrix} -\Phi & 0 & 0 & \Phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi & 0 & 0 & -\Phi \end{pmatrix}. \tag{B2}$$

For light propagating along the x axis, the z dependence of the eikonal can be neglected. Indeed, the deflection of light into the z direction is represented by $\partial\Psi/\partial z$ which is first order in h. This term enters the eikonal equation either squared or multiplied by Φ and therefore leads to second order corrections. Neglecting all such terms, we reduce the eikonal equation to

$$\left(1 - \frac{2}{c^2}\Phi\right)\left(\frac{\partial\Psi}{\partial ct}\right)^2 = \left(\frac{\partial\Psi}{\partial x}\right)^2. \tag{B3}$$

Taking the square root of both sides of this equation and keeping only the terms which are first order in h, we obtain

$$\left(\frac{\partial}{\partial ct} \pm \frac{\partial}{\partial x}\right)\Psi = \frac{1}{c^2}\Phi\frac{\partial\Psi}{\partial ct}, \tag{B4}$$

where $\pm$ corresponds to wave propagation in the positive and negative x directions. For simplicity, we consider the eikonal for photons propagating between the fixed boundaries: $x_a = l$ and $x_b = l + L$. The motion of the test masses can be added separately, as we have done above.

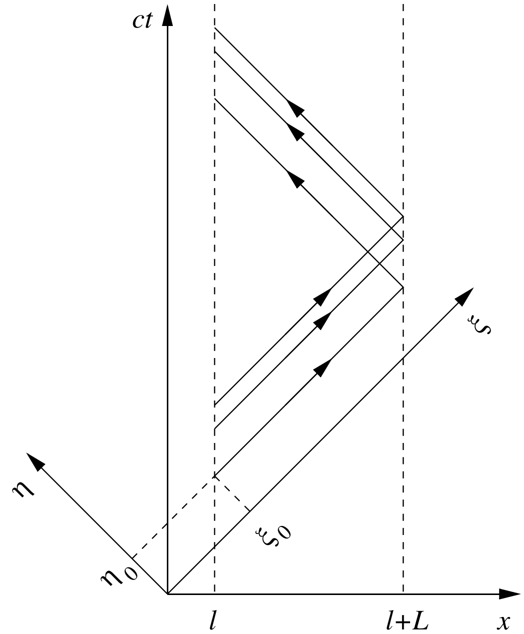

FIG. 3. The world lines of bouncing photons in the ξ and η coordinates.

The large unperturbed value of the eikonal satisfies Eq. (B4) in the absence of the gravitational wave ($\Phi = 0$), and is given by $\omega t \mp kx$ up to an additive constant. Therefore, we look for a solution of the form:

$$\Psi_1 = \omega t - kx + kl + \delta\Psi_1, \tag{B5}$$

$$\Psi_2 = \omega t + kx - k(l + 2L) + \delta\Psi_2, \tag{B6}$$

where $\delta\Psi_{1,2}$ are first order perturbations, and the subscripts 1 and 2 correspond to the forward and return trip, respectively. For convenience, we introduce two new coordinates:

$$\xi = (ct + x)/2, \tag{B7}$$

$$\eta = (ct - x)/2, \tag{B8}$$

which naturally parametrize the photon world lines, as shown in Fig. 3. In these coordinates, the first order eikonal perturbations satisfy the equations:

$$\frac{\partial}{\partial\xi}\delta\Psi_1 = \frac{k}{c^2}\Phi, \tag{B9}$$

$$\frac{\partial}{\partial\eta}\delta\Psi_2 = \frac{k}{c^2}\Phi, \tag{B10}$$

which allow direct integration,

$$\delta\Psi_1(\xi, \eta) = \frac{k}{c^2}\int_{\xi_0}^{\xi}\Phi(\xi', \eta)d\xi' + f_1(\eta), \tag{B11}$$

$$\delta\Psi_2(\xi,\eta) = \frac{k}{c^2}\int_{\eta_0}^{\eta}\Phi(\xi,\eta')d\eta' + f_2(\xi), \tag{B12}$$

where $f_1(\eta)$ and $f_2(\xi)$ are arbitrary at this point. The coordinates ξ_0 and η_0 correspond to the location of the source and the emission time for the bouncing photon. Transforming back to coordinates x and t, we obtain the solution:

$$\delta\Psi_1(x,t) = \frac{k}{c^2}\int_l^x \Phi\left(x', t - \frac{x-x'}{c}\right)dx' + f_1(x,t), \tag{B13}$$

$$\delta\Psi_2(x,t) = \frac{k}{c^2}\int_x^{l+L} \Phi\left(x', t + \frac{x-x'}{c}\right)dx' + f_2(x,t). \tag{B14}$$

The function f_1 is defined by the boundary value of the eikonal Ψ_1. The eikonal at the location of the source must be $\Psi_1(l,t) = \omega t^*$. To first order in h, this boundary value can be approximated as

$$\Psi_1(l,t) = \omega\int_0^t \sqrt{-g_{00}(l,t')}dt' \tag{B15}$$

$$\approx \omega\int_0^t\left[1 + \frac{1}{c^2}\Phi(l,t')\right]dt', \tag{B16}$$

which according to Eq. (B5) must be the same as $\omega t + f_1(l,t)$. We thus find f_1 at the location of the source. Knowing that f_1 is a function of $ct - x$, we can extend it to the entire x axis:

$$f_1(x,t) = \frac{k}{c}\int_0^{t-[(x-l)/c]}\Phi(l,t')dt'. \tag{B17}$$

The function f_2 is defined by the boundary value of the eikonal Ψ_2. Continuity of the eikonal at the turning point requires that

$$\delta\Psi_1(l+L,t) = \delta\Psi_2(l+L,t). \tag{B18}$$

We thus find f_2 at the turning point. Knowing that f_2 is a function of $ct + x$, we can extend it to the entire x axis:

$$f_2(x,t) = \frac{k}{c}\int_0^{t-2T+[(x+l)/c]}\Phi(l,t')dt' + \frac{k}{c^2}\int_l^{l+L}\Phi\left(x', t - 2T + \frac{x+x'-2l}{c}\right)dx'. \tag{B19}$$

The phase shift acquired by the electromagnetic wave in one round-trip is given by the difference between the values of the eikonal at the beginning and the end of the propagation:

$$\delta\psi_v(t) = \delta\Psi_2(l,t) - \delta\Psi_1(l,t-2T). \tag{B20}$$

Simple algebra shows that this definition leads to

$$\delta\psi_v(t) = \frac{k}{c^2}\int_l^{l+L}\Phi\left(x, t - \frac{x-l}{c}\right)dx + \frac{k}{c^2}\int_l^{l+L}\Phi\left(x, t - 2T + \frac{x-l}{c}\right)dx, \tag{B21}$$

which is the expanded form of Eq. (64). Similarly, the phase shift acquired by the static source is given by

$$\delta\psi_S(t) = \delta\Psi_1(l,t) - \delta\Psi_1(l,t-2T). \tag{B22}$$

This definition leads to

$$\delta\psi_S(t) = \frac{k}{c}\int_{t-2T}^{t}\Phi(l,t')dt', \tag{B23}$$

which is the expanded form of Eq. (67).

Finally, we give explicit formulas for Ω and K in terms of the gravitational potential. These two quantities can be derived from the eikonal according to

$$\Omega = \frac{\partial\Psi}{\partial t}, \quad \text{and} \quad K = -\frac{\partial\Psi}{\partial x}. \tag{B24}$$

For example, in the forward propagation

$$\Omega(x,t) = \omega + \frac{k}{c}\Gamma(x,t) + \frac{k}{c}\Phi\left(l, t - \frac{x-l}{c}\right), \tag{B25}$$

$$K(x,t) = k + \frac{k}{c^2}\Gamma(x,t) + \frac{k}{c^2}\Phi\left(l, t - \frac{x-l}{c}\right) - \frac{k}{c^2}\Phi(x,t), \tag{B26}$$

where Γ represents the purely nonstationary component of the gravitational redshift:

$$\Gamma(x,t) = \frac{1}{c}\int_l^x \frac{\partial}{\partial t}\left[\Phi\left(x', t - \frac{x-x'}{c}\right)\right]dx'. \tag{B27}$$

Note that K can also be written as

$$K(x,t) = \frac{1}{c}\Omega(x,t) - \frac{k}{c^2}\Phi(x,t), \tag{B28}$$

which leads directly to the dispersion relation, Eq. (62).

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