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Abstract

We show that, unlike the Yao-Yao graph YY_6, the Theta-Theta graph $\Theta\Theta_6$ defined by six cones is a spanner for sets of points in convex position. We also show that, for sets of points in non-convex position, the spanning ratio of $\Theta\Theta_6$ is unbounded.

1 Introduction

Let S be a set of n points in the plane and let $G = (S, E)$ be a weighted geometric graph with vertex set S and a set E of (directed or undirected) edges between pairs of points, where the weight of an edge $uv \in E$ is equal to the Euclidean distance $|uv|$ between u and v. The *length* of a path in G is the sum of the weights of its constituent edges. The distance $d_G(u, v)$ in G between two points $u, v \in S$ is the length of a shortest path in G between u and v. The graph G is called a *t-spanner* if any two points $u, v \in S$ at distance $|uv|$ in the plane are at distance $d_G(u, v) \leq t \cdot |uv|$ in G. The smallest integer t for which this property holds is called the *spanning ratio* of G.

The Yao graph $Y_k(S)$ and the Theta graph $\Theta_k(S)$ are defined for a fixed integer $k > 0$ as follows. Partition the plane into k equiangular cones by extending k equally-separated rays starting at the origin, with the first ray in the direction of the positive x-axis. Then translate the cones to each point $u \in S$, and connect u to a "nearest" neighbor in each cone. The difference between Yao and Theta graphs is in the way the "nearest" neighbor is defined. For a fixed point $u \in S$ and a cone $\mathcal{C}(u)$ with apex u, a Yao edge $\overrightarrow{uv} \in \mathcal{C}(u)$ minimizes the Euclidean distance $|uv|$ between u and v, whereas a Theta edge $\overrightarrow{uv} \in \mathcal{C}(u)$ minimizes the *projective distance* $\|uv\|$ from u to v, which is the Euclidean distance between u and the orthogonal projection of v on the bisector of $\mathcal{C}(u)$. Ties are arbitrarily broken.

Each of the graphs Θ_k and Y_k has out-degree k, but in-degree $n-1$ in the worst case (consider, for example, the case of $n-1$ points uniformly distributed on the circumference of a circle centered at the n^{th} point: for any $k \geq 6$, the center point has in-degree $n-1$). This is a significant drawback in certain wireless networking applications where a wireless node can communicate with only a limited number of neighbors. To reduce the in-degrees, a second filtering step can be applied to the set of incoming edges in each cone. This filtering step eliminates, for each each point $u \in S$ and each cone with apex u, all but a "shortest" incoming edge. The result of this filtering step applied on Θ_k (Y_k) is the Theta-Theta (Yao-Yao) graph $\Theta\Theta_k$ (YY_k). Again, the definition of "shortest" differs for Yao and Theta graphs: a shortest Yao edge $\overrightarrow{vu} \in \mathcal{C}(u)$ minimizes $|vu|$, and a shortest Theta edge $\overrightarrow{vu} \in \mathcal{C}(u)$ minimizes $\|vu\|$. Again, ties are arbitrarily broken.

Yao and Theta graphs (and their Yao-Yao and Theta-Theta sparse variants) have many important applications in wireless networking [1], motion planning [9] and walkthrough animations [15].

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We refer the readers to the books by Li [18] and Narasimhan and Smid [21] for more details on their uses, and to the comprehensive survey by Eppstein [14] for related topics on geometric spanners. Many such applications take advantage of the spanning and sparsity properties of these graphs, which have been extensively studied. Molla [20] showed that Y_2 and Y_3 may not be spanners, and her examples can be used to show that Θ_2 and Θ_3 are not spanners either. On the other hand, it has been shown that, for any $k \geq 4$, Y_k and Θ_k are spanners: Y_4 is a 54.6-spanner [13] and Θ_4 is a 17-spanner [4]; Y_5 is a 3.74-spanner [2] and Θ_5 is a 9.96-spanner [6]; Y_6 is a 5.8-spanner [2] and Θ_6 is a 2-spanner [3]; for $k \geq 7$, the spanning ratio of Y_k is $\frac{1+\sqrt{2-2\cos(2\pi/k)}}{2\cos(2\pi/k)-1}$ [5] and the spanning ratio of Θ_k is $\frac{1}{1-2\sin(\pi/k)}$ [22]; improved bounds on the spanning ratio of Y_k for odd $k \geq 5$, and for Θ_k for even $k \geq 6$, also exist [7].

In contrast with Yao and Theta graphs, our knowledge of Yao-Yao and Theta-Theta graphs is more limited. Li et al. [19] proved that YY_k is connected for $k > 6$ and provided substantial experimental evidence suggesting that YY_k is a spanner for large k values. This conjecture has been partly confirmed by Bauer and Damian [11] who showed that, for $k \geq 6$, YY_{6k} is a spanner with spanning ratio 11.76. This spanning ratio has been improved to 7.82 in [10] for a more general class of graphs called *canonical k-cone graphs*, which include both YY_{6k} and $\Theta\Theta_{6k}$, for $k \geq 6$. The same paper establishes a spanning ratio of 16.76 for YY_{30} and $\Theta\Theta_{30}$. Recent breakthroughs show that YY_{2k}, for any $k \geq 42$, is a spanner with spanning ratio $6.03 + O(k^{-1})$ [17], and YY_k for odd $k \geq 3$ is not a spanner [16]. For small values $k \leq 5$, Damian et al. [12] show that YY_4 is not a spanner, and Barba et al. [2] show that YY_5 is not a spanner, and their constructions can also be used to show that $\Theta\Theta_4$ and $\Theta\Theta_5$ are not spanners. Molla [20] showed that YY_6 is also not a spanner, even for sets of points in convex position. This paper fills in one of the gaps in our knowledge of Theta-Theta graphs and shows that $\Theta\Theta_6$ is an 8-spanner for sets of points in convex position, but has unbounded spanning ratio for sets of points in non-convex position.

2 Definitions

Throughout the paper, S is a fixed set of n points in the plane and $k > 1$ is a fixed integer. The graphs Y_k and Θ_k use a set of k equally-separated rays starting at the origin. These rays define k equiangular cones $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$, each of angle $\theta = 2\pi/k$, with the lower ray of $\mathcal{C}_1$ extending in the direction of the positive x-axis. Refer to Figure 1. We assume that each cone is half-open and half-closed, meaning that it includes the clockwise bounding ray, but it excludes the counterclockwise bounding ray. Let $\mathcal{C}_i(a)$ denote a copy of $\mathcal{C}_i$ translated to a, for each $a \in S$ and each $i = 1, \ldots, k$.

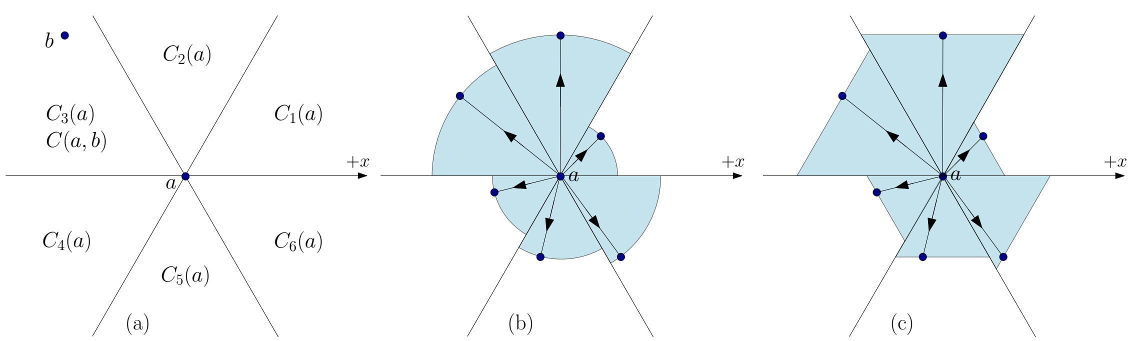

Figure 1: (a) Cones defining Y_6 and Θ_6 (b) Y_6 edges minimize Euclidean distances (c) Θ_6 edges minimize projective distances.

The directed graphs $\overrightarrow{Y_k}$ and $\overrightarrow{\Theta_k}$ are constructed as follows. In each cone $C_i(a)$, for each $i = 1, \ldots, k$

and each $a \in S$, extend a directed edge from a to a "nearest" point b that lies in $C_i(a)$. Yao and Theta graphs differ only in the way "nearest" is defined. A point b is "nearest" to a in Y_k if it minimizes the Euclidean distance $|ab|$, whereas b is "nearest" to a in Θ_k if it minimizes the projective distance $\|ab\|$. See Figure 2a,b for simple graph examples illustrating these definitions.

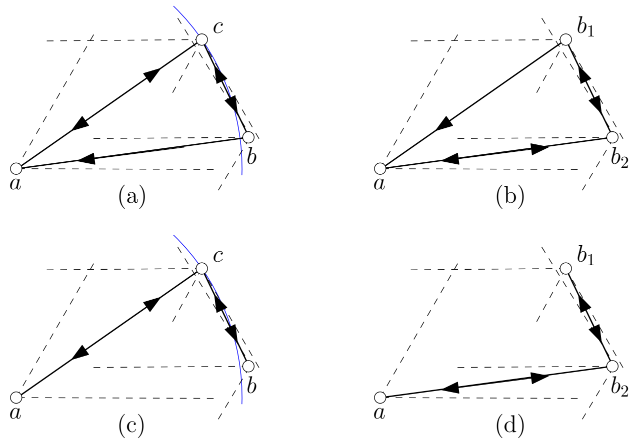

Figure 2: Graph examples (a) Y_6 (b) Θ_6 (c) YY_6 (d) $\Theta\Theta_6$.

The Yao-Yao graph $\overrightarrow{YY_k} \subseteq \overrightarrow{Y_k}$ and Theta-Theta graph $\overrightarrow{\Theta\Theta_k} \subseteq \overrightarrow{\Theta_k}$ are obtained by applying a filtering step to the set of incoming edges at each vertex in $\overrightarrow{Y_k}$ and $\overrightarrow{\Theta_k}$, respectively. Specifically, for each $a \in S$ and each $i = 1, \ldots, k$, these graphs retain a "shortest" incoming edge that lies in $\mathcal{C}_i(a)$ and and discard the rest of incoming edges, if any. Recall that a "shortest" Yao edge $\overrightarrow{ba} \in \mathcal{C}_i(a)$ minimizes $|ba|$, whereas a "shortest" Theta edge $\overrightarrow{ba} \in \mathcal{C}_i(a)$ minimizes $\|ba\|$. Figure 2c(d) depicts the graph YY_6 ($\Theta\Theta_6$) after this filtering step has been applied to the graph Y_6 (Θ_6) from Figure 2a(b).

3 Background: YY_6 is not a Spanner

Molla [20] gave an example of a set of points in convex position for which YY_6 is not a spanner. We briefly review her construction here and show that the result does not hold for $\Theta\Theta_6$. The construction begins with a strip of equilateral triangles between two horizontal lines with vertices $\{a_1, a_2, \ldots, a_n\}$ on the lower line (which we call the a-line) and $\{b_1, b_2, \ldots, b_n\}$ on the upper line (which we call the b-line). See the left of Figure 3a. Next the a-line is rotated clockwise about a_1 and the b-line is rotated counterclockwise about b_1 by a small angle $\alpha > 0$, to guarantee that $|a_{i-1}a_i| < |b_{i-1}a_i|$ and $|b_{i-1}b_i| < |a_ib_i|$, for $i = 2, \ldots, n$. The points are also slightly perturbed to ensure that $\mathcal{C}_2(a_i)$ and $\mathcal{C}_5(b_i)$ are all empty, for $i = 1, \ldots, n$. The result is depicted in the right of Figure 3a.

The graphs Y_6 and YY_6 induced by the set of points $S = \{a_1, \ldots a_4\} \cup \{b_1, \ldots, b_4\}$ are depicted in Figure 3b. Note that, with the exception of a_1b_1, YY_6 includes none of the Y_6 edges incident on both the a-line and the b-line. This is because, for $i > 1$, $\overrightarrow{b_{i-1}a_i}$ and $\overrightarrow{a_{i-1}a_i}$ both lie in $\mathcal{C}_3(a_i)$ and YY_6 maintains only the shorter of the two, which is $\overrightarrow{a_{i-1}a_i}$. Similarly, $\overrightarrow{a_ib_i}$ and $\overrightarrow{b_{i-1}b_i}$ both lie in $\mathcal{C}_4(b_i)$ and YY_6 maintains only the shorter of the two, which is $\overrightarrow{b_{i-1}b_i}$. This shows that the shortest path in YY_6 between a_n and b_n is a Hamiltonian path of length at least $2n - 1$, which grows arbitrarily large with n. It follows that YY_6 is not a spanner.

For the same point set S, the graphs Θ_6 and $\Theta\Theta_6$ are depicted in Figure 3c. Note that, if projective distances are used, then $\|a_{i-1}a_i\| > \|b_{i-1}a_i\|$ and $\|b_{i-1}b_i\| > \|a_ib_i\|$, for $i = 2, \ldots, n$. These properties force $\Theta\Theta_6$ to maintain $\overrightarrow{b_{i-1}a_i} \in \mathcal{C}_3(a_i)$ and $\overrightarrow{a_ib_i} \in \mathcal{C}_4(b_i)$, for each $i = 2, \ldots, n$. The

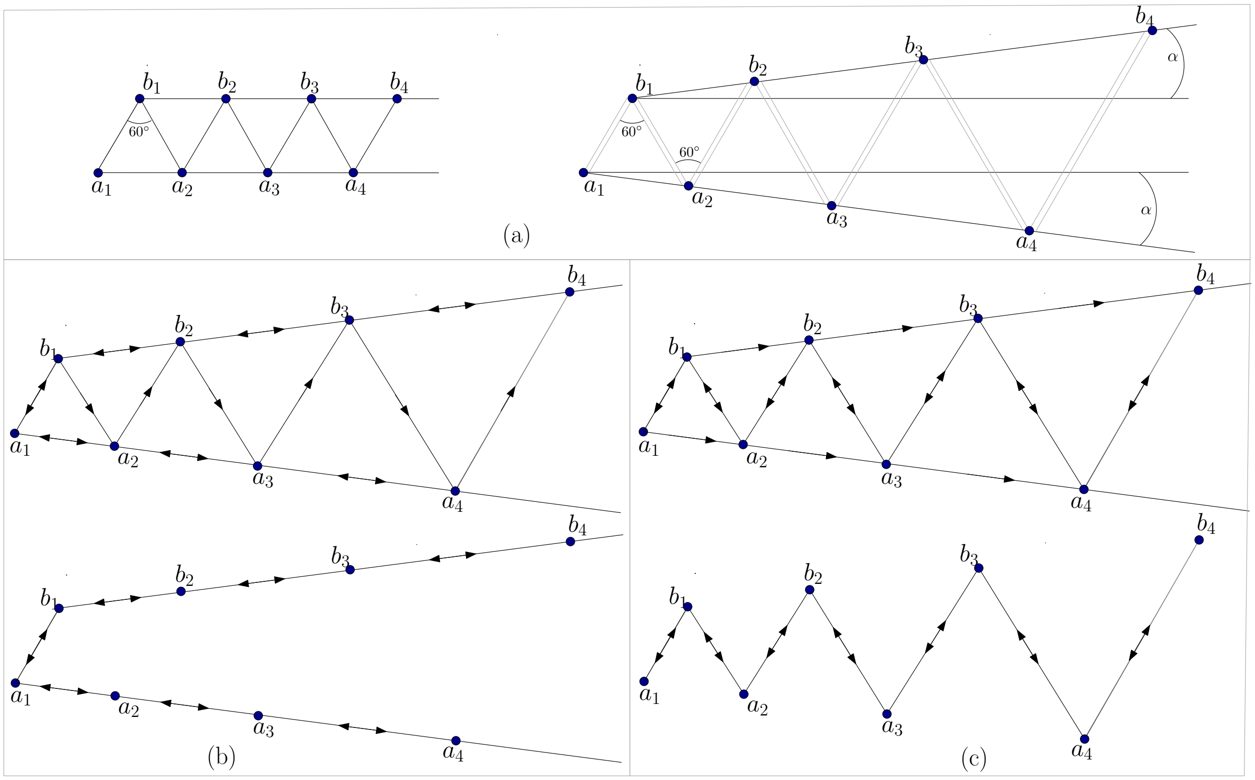

Figure 3: (a) Point set $\{a_1, \dots a_4\} \cup \{b_1, \dots, b_4\}$ (b) Graphs Y_6 (top) and YY_6 (bottom) (c) Graphs Θ_6 (top) and $\Theta\Theta_6$ (bottom).

result is the zig-zag path depicted in Figure 3c which shows that, for this particular point set, $\Theta\Theta_6$ is a spanner. In the next section we show that $\Theta\Theta_6$ is a spanner for any set of points in convex position.

4 $\Theta\Theta_6$ is a Spanner for Points in Convex Position

It has been established in [20] (and revisited in Section 3 of this paper) that YY_6 is not a spanner for sets of points in convex position. In this section we show that, unlike YY_6, the graph $\Theta\Theta_6$ is an 8-spanner for sets of points in convex position (in the next section we will show that this result does not hold for sets of points in non-convex position). This is the first result that marks a difference in the spanning properties of YY-graphs and $\Theta\Theta$-graphs.

Throughout this section, we assume that S is a set of points in convex position. For simplicity, we also assume that the points in S are in general position, meaning that no two points lie on a line parallel to one of the rays that define the cones. This implies that there is a unique nearest point in each cone of Θ_6 and $\Theta\Theta_6$. We begin with a few definitions.

For any $a, b \in S$, let $\mathcal{C}(a,b)$ denote the cone with apex a that contains b. For any ordered pair of vertices a and b, let $\mathcal{T}(a,b)$ be the *canonical triangle* delimited by the rays bounding $\mathcal{C}(a,b)$ and the perpendicular through b on the bisector of $\mathcal{C}(a,b)$. See Figure 4a. For a fixed point $a \in S$ and $i \in \{1, \dots, k\}$, let $p_{\Theta_6}(a,i)$ denote the path in Θ_6 that starts at a and follows the Θ_6-edges that lie in cones C_i. See, for example, the path $p_{\Theta_6}(a,1)$ depicted in Figure 4. Note that this path is monotone with respect to the bisector of C_i. This along with the fact that the point set S is finite implies that the path itself is finite and well defined. We say that two edges ab and cd *cross* if they share a point other than an endpoint (a, b, c or d).

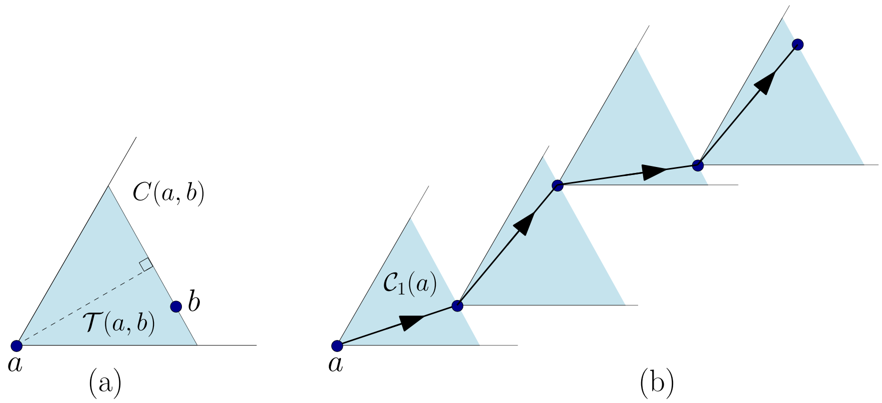

Figure 4: (a) Canonical triangle $\mathcal{T}(a,b)$ (b) Path $p_{\Theta_6}(a,1)$.

The *half-*Θ_6*-graph* introduced in [3] takes only "half" the edges of Θ_6, those belonging to non-consecutive cones. Thus, the Θ_6-graph is the union of two half-Θ_6-graphs: one that includes all Θ_6-edges that lie in cones C_1, C_3, C_5, and one that includes all Θ_6-edges that lie in cones C_2, C_4, C_6. Bonichon et al. [3] show that half-Θ_6 is a *triangular-distance*[1] Delaunay triangulation, computed as the dual of the Voronoi diagram based on the triangular distance function. This, combined with Chew's proof that any triangular-distance Delaunay triangulation is a 2-spanner [8], yields the following result.

Theorem 1 [3] *The half-*Θ_6*-graph is a plane* 2*-spanner.*

Next we introduce two preliminary lemma that will be useful in proving the main result of this section.

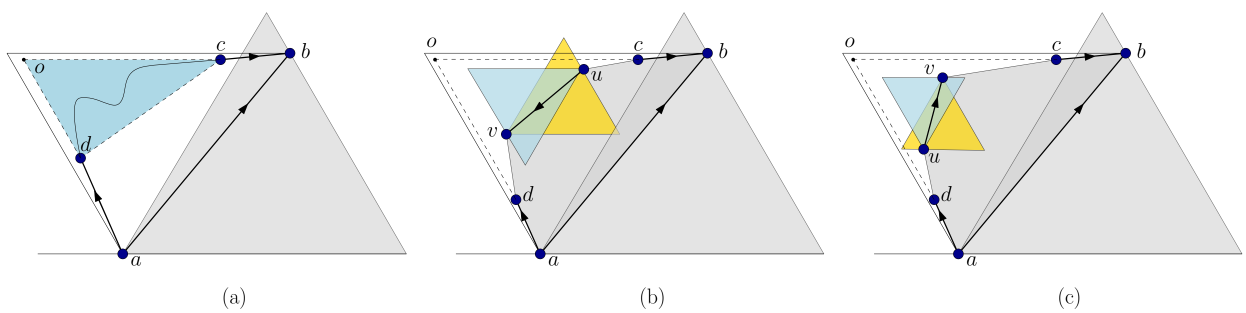

Figure 5: Lemma 2 (a) There is a path $p_{\Theta\Theta_6}(d,c)$ that lies inside $\triangle cod$ (b) If $\overrightarrow{uv} \in \Theta_6$, $\overrightarrow{uv} \in \mathcal{C}_4(u)$, then $\overrightarrow{uv} \in \Theta\Theta_6$ (c) If $\overrightarrow{uv} \in \Theta_6$, $\overrightarrow{uv} \in \mathcal{C}_2(u)$, then $\overrightarrow{uv} \in \Theta\Theta_6$.

Lemma 2 *Let S be a set of points in convex position and let $a,b,c,d \in S$ be distinct points such that $b \in \mathcal{C}_1(a)$ and $\overrightarrow{ab} \in \Theta_6 \setminus \Theta\Theta_6$; $c \in \mathcal{C}_4(b)$ and $\overrightarrow{cb} \in \Theta\Theta_6$; $d \in \mathcal{C}_2(a)$ and $\overrightarrow{ad} \in \Theta_6$. Let o be the intersection point between the upper ray of $\mathcal{C}_4(c)$ and the left ray of $\mathcal{C}_2(d)$. Then there is a path in $\Theta\Theta_6$ between c to d that lies in $\triangle cod$ and is no longer than $|oc| + |od|$.*

Proof Note that, since the points in S are in convex position, the point o exists and lies outside the convex quadrilateral $abcd$. Refer to Figure 5a. Consider the paths $p_c = p_{\Theta_6}(c,4)$ and $p_d = p_{\Theta_6}(d,2)$.

[1]The *triangular distance* from a point a to a point b is the side length of the smallest equilateral triangle centered at a that touches b and has one horizontal side.

Since p_c and p_d are in the same half-Θ_6 graph, Theorem 1 tells us that p_c and p_d do not cross. This implies that p_c and p_d meet in a point $e \in \triangle cod$. Let $p(c,e)$ be the piece of p_c extending from c to e, and $p(d,e)$ the piece of p_d extending from d to e. Note that $p(c,d) = p(c,e) \cup p(d,e)$ is a convex path that lies inside $\triangle cod$, which implies that $|p(c,d)| < |oc| + |od|$.

To complete the proof, it remains to show that p_{cd} is a path in $\Theta\Theta_6$. To do so, we consider an arbitrary edge $\overrightarrow{uv} \in p(c,d) \in \Theta_6$, and show that $\overrightarrow{uv} \in \Theta\Theta_6$. Assume first that $\overrightarrow{uv} \in p(c,e)$, meaning that $v \in \mathcal{C}_4(u)$. Refer to Figure 5b. The convexity property of S implies that no points may lie in $\mathcal{T}(v,u)$ and above u. Ignoring the piece of $\mathcal{T}(v,u)$ that extends above u, the rest of $\mathcal{T}(v,u)$ lies inside $\mathcal{T}(u,v) \cup abcuvd \cup T(a,b)$. This region, however, is empty of points in S: $\mathcal{T}(u,v)$ is empty of points in S because $\overrightarrow{uv} \in \Theta_6$; $abcuvd$ is a convex polygon empty of points in S, by the convexity property of S; and $\mathcal{T}(a,b)$ is empty of points in S, because $\overrightarrow{ab} \in \Theta_6$. It follows that $\mathcal{T}(v,u)$ is empty of points in S and therefore $\overrightarrow{uv} \in \Theta\Theta_6$.

The arguments for the case when $\overrightarrow{uv} \in p(d,e)$ are similar: in this case, $v \in \mathcal{C}_2(u)$; no points in S may lie in $\mathcal{T}(v,u)$ and left of $\mathcal{C}_2(u)$; ignoring the piece of $\mathcal{T}(v,u)$ that extends left of $\mathcal{C}_2(u)$, the rest of $\mathcal{T}(v,u)$ lies inside $\mathcal{T}(u,v) \cup abcvud \cup T(a,b)$, which is empty of points in S. It follows that $\mathcal{T}(v,u)$ is empty of points in S and therefore $\overrightarrow{uv} \in \Theta\Theta_6$. ∎

Lemma 3 *For any edge $\overrightarrow{ab}$ in the Θ_6-graph induced by a set of points S in convex position, there is a path between a and b in $\Theta\Theta_6$ no longer than $4|ab|$.*

Proof Assume without loss of generality that $\overrightarrow{ab} \in \mathcal{C}_1(a)$ and let α be the angle formed by ab with the lower ray of $\mathcal{C}_1(a)$. Let i_1 (h_1) be the intersection point between the upper ray of $\mathcal{C}_1(a)$ and the horizontal (perpendicular) through b. Refer to Figure 6a. Let i_2 (h_2), i_3 (h_3) and i_4 (h_4) be copies of i_1 (h_1) rotated counterclockwise by $\pi/3$, $2\pi/3$ and $2\pi/3 + \alpha$, respectively. Note that $|ah_1| < |ab|$ and $|bi_1| = 2|i_1h_1|$. We show that there is a convex path $p(a,b) \in \Theta\Theta_6$ between a and b that lies inside the convex region $\mathcal{R} = abi_2i_3i_4$ (shaded in Figure 6a). The length of such a path is

$$\begin{aligned} |p(a,b)| &< |bi_2| + |i_2i_3| + |i_3i_4| + |i_4a| \\ &= 2|i_1h_1| + |i_1i_2| + |i_2i_3| + |i_3i_4| + |i_4a| \\ &< 2|i_1h_1| + 4|i_1i_2| < 4(|i_1h_1| + |i_1i_2|) = 4|ah_1| \\ &< 4|ab| \end{aligned}$$

It remains to prove the existence of such a path $p(a,b) \in \Theta\Theta_6$. If $\overrightarrow{ab} \in \Theta\Theta_6$, then $p(a,b) = ab$ and the lemma trivially holds. Otherwise, there is $\overrightarrow{c_1b} \in \Theta\Theta_6$, with $c_1 \in \mathcal{C}(b,a)$. By definition $\|c_1b\| < \|ab\|$, which implies that c_1 lies in $\mathcal{C}_2(a)$ or $\mathcal{C}_6(a)$. Assume without loss of generality that $c_1 \in \mathcal{C}_2(a)$; the case where $c_1 \in \mathcal{C}_6(a)$ is symmetric. Because $\mathcal{C}_2(a)$ is non-empty, Θ_6 includes an edge $\overrightarrow{ab_2} \in \mathcal{C}_2(a)$. Refer to Figure 6b. If b_2 and c_1 coincide, let $p(b_2,c_1)$ be the empty path; otherwise, $p(b_2,c_1) \in \Theta\Theta_6$ is the path established by Lemma 2, which lies in a triangular region inside $\mathcal{T}(a,c_1)$ (shaded in Figure 6b). If $\overrightarrow{ab_2} \in \Theta\Theta_6$, then

$$p(a,b) = ab_2 \oplus p(b_2,c_1) \oplus c_1b$$

is a convex path (by the convexity property of S) from a to b in $\Theta\Theta_6$ that lies inside $\mathcal{R}$, so the lemma holds. Here $\oplus$ denotes the concatenation operator. If $\overrightarrow{ab_2} \notin \Theta\Theta_6$, then there is $\overrightarrow{c_2b_2} \in \Theta\Theta_6$, with $c_2 \in \mathcal{C}(b_2,a)$. By definition $\|c_2b_2\| < \|ab_2\|$, which implies that $c_2 \in \mathcal{C}_3(a)$. Because $\mathcal{C}_3(a)$ is non-empty, Θ_6 includes an edge $\overrightarrow{ab_3} \in \mathcal{C}_3(a)$. If b_3 and c_2 coincide, let $p(b_3,c_2)$ be the empty path;

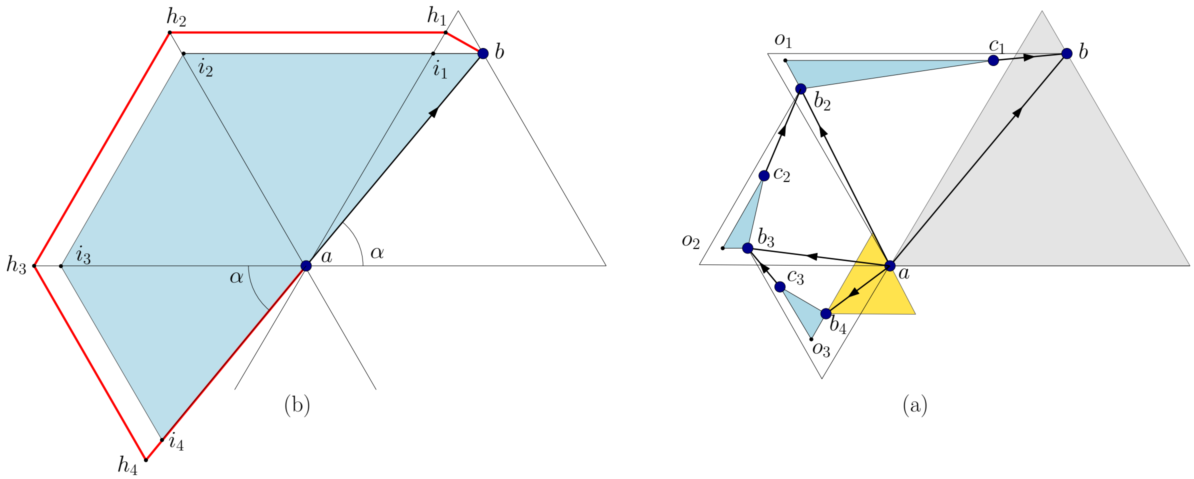

Figure 6: Lemma 3 (a) Any convex path from a to b that lies in the shaded area is no longer than $4|ab|$ (b) Path in $\Theta\Theta_6$ from a to b: $\overrightarrow{ab} \in \Theta_6 \setminus \Theta\Theta_6$, $\overrightarrow{c_1b}, \overrightarrow{c_2b_2}, \overrightarrow{c_3b_3}, \overrightarrow{ab_4} \in \Theta\Theta_6$.

otherwise, $p(b_3, c_2) \in \Theta\Theta_6$ is the path established by Lemma 2, which lies inside $\mathcal{T}(a, c_2)$ (shaded in Figure 6b). If $\overrightarrow{ab_3} \in \Theta\Theta_6$, then

$$p(a, b) = ab_3 \oplus p(b_3, c_2) \oplus c_2b_2 \oplus p(b_2, c_1) \oplus c_1b$$

is a convex path from a to b in $\Theta\Theta_6$ that lies inside $\mathcal{R}$, so the lemma holds. If $\overrightarrow{ab_3} \notin \Theta\Theta_6$, then there is $\overrightarrow{c_3b_3} \in \Theta\Theta_6$, with $c_3 \in \mathcal{C}(b_3, a)$. By definition $\|c_3b_3\| < \|ab_3\|$, which implies that $c_3 \in \mathcal{C}_4(a)$. Because $\mathcal{C}_4(a)$ is non-empty, Θ_6 includes an edge $\overrightarrow{ab_4} \in \mathcal{C}_4(a)$. If b_4 and c_3 coincide, let $p(b_4, c_3)$ be the empty path; otherwise, $p(b_4, c_3) \in \Theta\Theta_6$ is the path established by Lemma 2, which lies inside $\mathcal{T}(a, c_3)$. The convexity property of S implies that the region of $\mathcal{T}(b_4, a)$ that extends right of the line supporting ab is empty of points in S. Ignoring this region, the rest of $\mathcal{T}(b_4, a)$ lies in $\mathcal{T}(a, b_4) \cup \mathcal{T}(a, b_3)$, which is also empty of points in S. It follows that $\mathcal{T}(b_4, a)$ is empty of points in S, therefore $\overrightarrow{ab_4} \in \Theta\Theta_6$. These together imply that

$$p(a, b) = ab_4 \oplus p(b_4, c_3) \oplus c_3b_3 \oplus p(b_3, c_2) \oplus c_2b_2 \oplus p(b_2, c_1) \oplus c_1b$$

is a convex path from a to b in $\Theta\Theta_6$ that lies inside $\mathcal{R}$. This completes the proof.

Lemmas 1 and 3 together yield the main result of this section.

Theorem 4 *The $\Theta\Theta_6$-graph induced by a set of points in convex position is an 8-spanner.*

The following lemma establishes a lower bound of 4 on the spanning ratio of $\Theta\Theta_6$ for convex point sets. In addition, it shows that the bound 4 of Lemma 3 on the spanning ratio of $\Theta\Theta_6$-paths spanning Θ_6-edges is tight.

Lemma 5 *The spanning ratio of the $\Theta\Theta_6$-graph induced by a set of points in convex position is at least 4.*

Proof We construct a set of points S that satisfies the claim of this lemma. Let a be an arbitrary point in the plane and let b_i be the point at unit distance from a that lies on the counterclockwise ray of $\mathcal{C}_i(a)$, for $i = 1, \ldots, 4$. Refer to Figure 7a. Perturb the points infinitesimally so that b_1 lies strictly inside $\mathcal{C}_1(a)$ and b_i lies strictly inside $\mathcal{C}_i(a) \cap \mathcal{T}(a, b_{i-1})$, for $i = 2, 3, 4$. We ignore

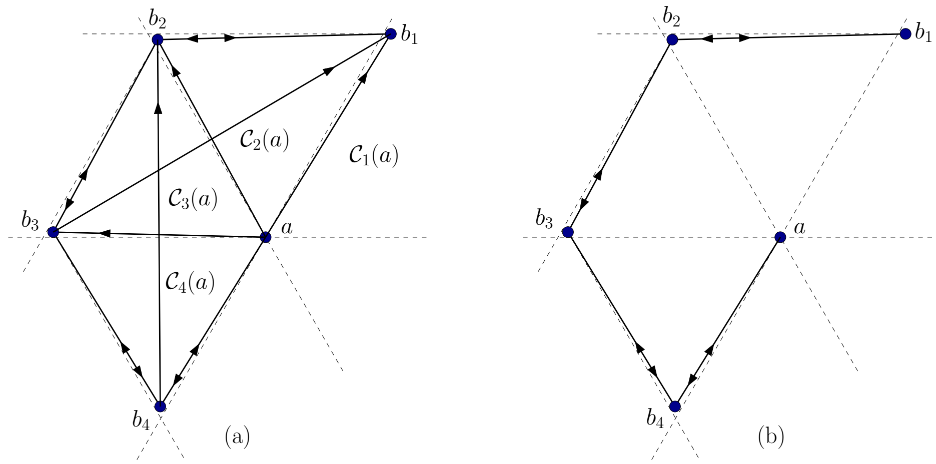

Figure 7: Set $S = \{a, b_1, b_2, b_3, b_4\}$ of points in convex position (a) Θ_6-graph (b) $\Theta\Theta_6$-graph.

this infinitesimal quantity from our calculations and assume that $|ab_i| = 1$, for $i = 1, \dots, 4$ and $|b_i b_{i+1}| = 1$, for $i = 1, 2, 3$.

Let $S = \{a, b_1, b_2, b_3, b_4\}$. The Θ_6-graph and $\Theta\Theta_6$-graph induced by S are depicted in Figure 7a and Figure 7b, respectively. Note that $\overrightarrow{ab_1} \in \Theta_6$, however $\overrightarrow{ab_1} \notin \Theta\Theta_6$ and $p_{\Theta\Theta_6}(a, b_1) = ab_4 \oplus b_4b_3 \oplus b_3b_2 \oplus b_2b_1$ is a shortest path in $\Theta\Theta_6$ between a and b of length 4. This proves the claim of this lemma. It also shows that the bound of Lemma 3 is tight.

5 $\Theta\Theta_6$ is not a Spanner for Points in Non-Convex Position

In this section we show that there exist sets of points in non-convex position for which $\Theta\Theta_6$ has unbounded spanning ratio and therefore it is not a spanner. We show how to construct a set $S = \{a_i, b_i, c_i, d_i : i = 1, 2, \dots, n\}$ of $4n$ points with this property.

Let a_1 and b_1 be points in the plane such that a_1b_1 forms a $\pi/3$-angle with the horizontal through a_1. Let r_a (r_b) be the ray with origin a_1 (b_1) pointing in the direction of the positive x-axis. Fix a small positive real value $0 < \alpha < 2$, and rotate r_a (r_b) clockwise (counterclockwise) about a_1 (b_1) by angle α. Let $a_2, a_3, \dots a_n$ be points along r_a, and $b_2, b_3, \dots b_n$ points along r_b, such that $\angle b_{i-1}a_ib_i = \pi/3$ for each $i = 2, \dots, n$, and $\angle a_ib_ia_{i+1} = \pi/3$ for each $i = 1, 2, \dots, n-1$. Refer to Figure 8a (which shows α enlarged for clarity). Note that at this point $\mathcal{C}_2(a_i)$ and $\mathcal{C}_5(b_i)$ share the line segment a_ib_i, for each $i = 1, 2, \dots, n$. Fix an arbitrary real value

$$\delta < \frac{|a_1a_2| \sin \alpha}{2}. \tag{1}$$

Keep a_1 in place and shift the remaining points rightward alongside their supporting rays r_a and r_b such that the horizontal distance between the right boundary ray of $\mathcal{C}_2(a_i)$ and the left boundary ray of $\mathcal{C}_5(b_i)$ is δ, for each i. Refer to Figure 8b. Finally, let c_i (d_i) be a copy of b_i (a_i) shifted upward (downward) by 2δ, for $i = 1, 2, \dots, n$. Thus b_ic_i and a_id_i are vertical line segments of length $|b_ic_i| = |a_id_i| = 2\delta$.

The following property is key to establishing an unbounded spanning ratio for $\Theta\Theta_6(S)$.

Property 6 *For each $i = 1, 2, \dots, n-1$, the point c_i lies in a small triangular region at the intersection between $\mathcal{C}_2(a_i)$, $\mathcal{C}_2(a_{i+1})$ and $\mathcal{C}_4(b_{i+1})$.*

To establish this property, fix an arbitrary $i \in \{1, 2, \dots, n-1\}$. Let y be the intersection point between the right ray of $\mathcal{C}_2(a_i)$ and the left ray of $\mathcal{C}_2(a_{i+1})$. See Figure 8b, which depicts the instance

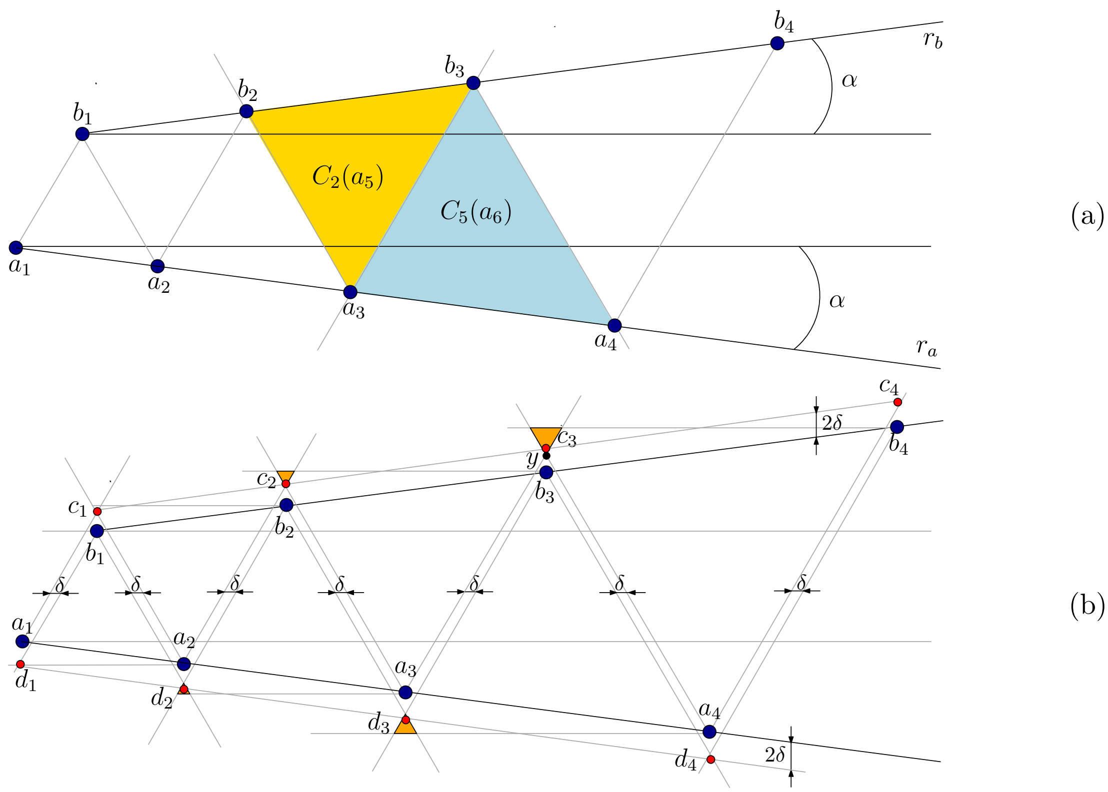

Figure 8: (a) Initial point configuration (b) Shifted point positions.

$i = 3$. Note that $|b_i y|$ is equal to the height of an equilateral triangle of side length 2δ, which is $\delta\sqrt{3} < 2\delta = |b_i c_i|$. This means that c_i lies vertically above y, therefore $c_i \in \mathcal{C}_2(a_i) \cap \mathcal{C}_2(a_{i+1})$. To establish that $c_i \in \mathcal{C}_4(b_{i+1})$, it suffices to show that c_i lies below the horizontal line through b_{i+1}. The distance from b_i to this line is $|b_i b_{i+1}| \sin\alpha > |a_1 a_2| \sin\alpha > 2\delta$ (cf. Equation 1). This along with the fact that $|b_i c_i| = 2\delta$ implies that c_i lies below the horizontal line through b_{i+1}. This settles Property 6.

Symmetric arguments establish the following property.

Property 7 *For each $i = 2, \ldots, n-1$, the point d_i lies in a small triangular region at the intersection between $\mathcal{C}_5(b_{i-1})$, $\mathcal{C}_5(b_i)$ and $\mathcal{C}_3(a_{i+1})$. If $i = 1$, $d_i \in \mathcal{C}_5(b_i) \cap \mathcal{C}_3(a_{i+1})$.*

We use Properties 6 and 7 in identifying the set of edges in $\Theta_6(S)$ and $\Theta\Theta_6(S)$. Fix an arbitrary $i \in \{1, \ldots, n\}$. The edges in $\Theta_6(S)$ outgoing from a_i are: $\overrightarrow{a_i b_i} \in \mathcal{C}_1(a_i)$; $\overrightarrow{a_i c_{i-1}} \in \mathcal{C}_2(a_i)$, if $i > 1$; $\overrightarrow{a_i d_{i-1}} \in \mathcal{C}_3(a_i)$, if $i > 1$ (note that Property 7 implies that $\|a_i d_{i-1}\| < \|a_i b_{i-1}\|$); $\overrightarrow{a_i d_i} \in \mathcal{C}_5(a_i)$; and $\overrightarrow{a_i a_{i+1}} \in \mathcal{C}_6(a_i)$, if $i < n$. Refer to Figure 9a, which depicts the instance $i = 3$. (Note that the cone $\mathcal{C}_4(a_i)$ is empty of points in S.) The edges in $\Theta_6(S)$ outgoing from d_i are: $\overrightarrow{d_i b_{i+1}} \in \mathcal{C}_1(d_i)$, if $i < n$; $\overrightarrow{d_i a_i} \in \mathcal{C}_2(d_i)$; $\overrightarrow{d_i d_{i-1}} \in \mathcal{C}_3(d_i)$, if $i > 1$; and $\overrightarrow{d_i a_{i+1}} \in \mathcal{C}_6(d_i)$, if $i < n$. (Note that the cones $\mathcal{C}_4(d_i)$ and $\mathcal{C}_5(d_i)$ are empty of points in S.) The edges in $\Theta_6(S)$ outgoing from b_i are: $\overrightarrow{b_i b_{i+1}} \in \mathcal{C}_1(b_i)$, if $i < n$; $\overrightarrow{b_i c_i} \in \mathcal{C}_2(b_i)$; $\overrightarrow{b_i c_{i-1}} \in \mathcal{C}_4(b_i)$, if $i > 1$ (note that Property 6 implies that $\|b_i c_{i-1}\| < \|b_i a_i\|$); $\overrightarrow{b_i d_i} \in \mathcal{C}_5(b_i)$; and $\overrightarrow{b_i a_{i+1}} \in \mathcal{C}_6(b_i)$, if $i < n$. (Note that the cone $\mathcal{C}_3(b_i)$ is empty of points in S.) Finally, the edges in $\Theta_6(S)$ outgoing from c_i are: $\overrightarrow{c_i b_{i+1}} \in \mathcal{C}_1(c_i)$, if $i < n$; $\overrightarrow{c_i c_{i-1}} \in \mathcal{C}_4(c_i)$, if $i > 1$; $\overrightarrow{c_i b_i} \in \mathcal{C}_5(c_i)$; and $\overrightarrow{c_i a_{i+2}} \in \mathcal{C}_6(c_i)$, if $i < n - 1$. (Note that the cones $\mathcal{C}_2(c_i)$ and $\mathcal{C}_3(c_i)$ are empty of points in S.) Figure 9b depicts the graph $\Theta_6(S)$, for $n = 4$.

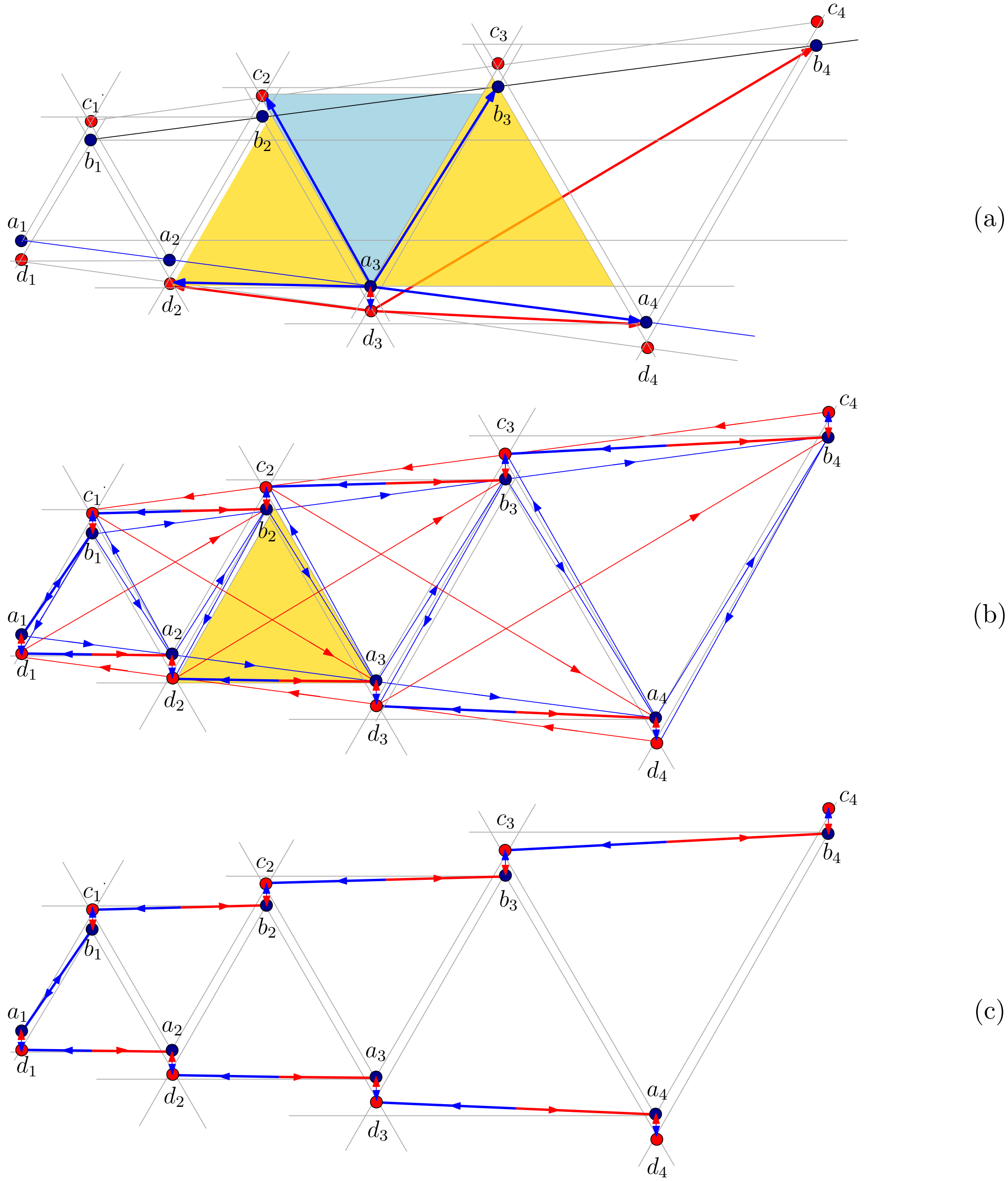

Figure 9: (a) Edges in Θ_6 outgoing from a_3 and d_3 (b) $\Theta_6(S)$ (c) $\Theta\Theta_6(S)$.

We now turn our attention to the set of incoming edges at each vertex in $\Theta_6(S)$. From among the four (three) edges directed into a_i and lying in $\mathcal{C}_3(a_i)$ for $i > 2$ ($i = 2$), the edge $\overrightarrow{d_{i-1}a_i}$ has the shortest projective distance: $\|\overrightarrow{d_{i-1}a_i}\| < \|\overrightarrow{b_{i-1}a_i}\| < \|\overrightarrow{a_{i-1}a_i}\|$, and this latter quantity is in turn smaller than $\|\overrightarrow{c_{i-2}a_i}\|$, for $i > 2$. This implies that $\Theta\Theta_6(S)$ keeps $\overrightarrow{d_{i-1}a_i}$ and eliminates the other three (two) edges, for $i > 2$ ($i = 2$). Note that any cone with apex a_i other than $\mathcal{C}_3(a_i)$ contains at most one edge directed into a_i, which continues to exist in $\Theta\Theta_6$.

For each i, the two edges directed into d_i that lie in $\mathcal{C}_2(d_i)$ satisfy $\|\overrightarrow{a_id_i}\| < \|\overrightarrow{b_id_i}\|$, therefore $\overrightarrow{b_id_i}$ gets eliminated from $\Theta\Theta_6(S)$ in favor of $\overrightarrow{a_id_i}$. Similarly, for $i < n$, $\overrightarrow{d_{i+1}d_i} \in \mathcal{C}_6(d_i)$ gets eliminated from $\Theta\Theta_6(S)$ in favor of $\overrightarrow{a_{i+1}d_i} \in \mathcal{C}_6(d_i)$. There are no edges in $\Theta_6(S)$ directed into d_i that lie in any of the cones $\mathcal{C}_1(d_i)$, $\mathcal{C}_3(d_i)$, $\mathcal{C}_4(d_i)$ and $\mathcal{C}_5(d_i)$.

For $i > 1$, the four edges directed into b_i that lie in $\mathcal{C}_4(b_i)$ satisfy $\|\overrightarrow{c_{i-1}b_i}\| < \|\overrightarrow{a_ib_i}\| < \|\overrightarrow{b_{i-1}b_i}\| < \|\overrightarrow{d_{i-1}b_i}\|$. This implies that $\Theta\Theta_6(S)$ keeps $\overrightarrow{c_{i-1}b_i}$ and eliminates the other three edges. The only other edge directed into b_i is $\overrightarrow{c_ib_i} \in \mathcal{C}_2(b_i)$. For $i = 1$, the two edges directed into b_i are $\overrightarrow{a_ib_i} \in \mathcal{C}_4(b_i)$ and $\overrightarrow{c_ib_i} \in \mathcal{C}_2(b_i)$, which remain in place in $\Theta\Theta_6(S)$. Finally, $\Theta\Theta_6(S)$ eliminates $\overrightarrow{a_{i+1}c_i} \in \mathcal{C}_5(c_i)$ in favor of $\overrightarrow{b_ic_i} \in \mathcal{C}_5(c_i)$, and $\overrightarrow{c_{i+1}c_i} \in \mathcal{C}_1(c_i)$ in favor of $\overrightarrow{b_{i+1}c_i} \in \mathcal{C}_1(c_i)$, for $i < n$. For $i = n$, the only edge directed into c_i is $\overrightarrow{b_ic_i}$.

The resulting $\Theta\Theta_6$-graph is the path depicted in Figure 9c. The edge set of $\Theta\Theta_6(S)$ is $\{a_1b_1\} \cup \{a_id_i, b_ic_i : i = 1, 2, \ldots, n\} \cup \{d_ia_{i+1}, c_ib_{i+1} : i = 1, 2, \ldots, n-1\}$.

For an arbitrarily small α value, we have $|a_nb_n| \approx |a_1b_1|$. A shortest path $\xi_{\Theta\Theta_6}(a_n, b_n)$ in this graph between a_n and b_n has length

$$\begin{aligned}
|\xi_{\Theta\Theta_6}(a_n, b_n)| &> |a_1b_1| + \sum_{i=1}^{n-1}(|a_id_i| + |d_ia_{i+1}|) + \sum_{i=1}^{n-1}(|b_ic_i| + |c_ib_{i+1}|) \\
&> |a_1b_1| + \sum_{i=1}^{n-1}|a_ia_{i+1}| + \sum_{i=1}^{n-1}|b_ib_{i+1}| \quad \text{(by triangle inequality)} \\
&> (2n-1) \cdot |a_1b_1|
\end{aligned}$$

This shows that the spanning ratio of $\Theta\Theta_6(S)$ is $\Omega(n)$, therefore we have the following result.

Theorem 8 *The $\Theta\Theta_6$-graph is not a spanner.*

6 Conclusions

This paper establishes the first result showing a difference in the spanning properties of two related classes of sparse graphs, namely Yao-Yao and Theta-Theta. Previous results show YY_k and $\Theta\Theta_k$ are not spanners for $k \leq 5$, and are spanners for some values of $k > 6$. In this paper we show that, unlike YY_6, the graph $\Theta\Theta_6$ is a spanner for sets of points in convex position. We also show that, for sets of points in non-convex position, $\Theta\Theta_6$ is not a spanner. The spanning ratios of YY_k and $\Theta\Theta_k$, for all even k in the range $[8, 28]$ and for some even values of k (those that are not multiples of 6) in the range $[32, 82]$, remain unknown.

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