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A new integrable two-component system with cubic nonlinearity

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In this paper, a new integrable two-component system, $m_t = [m(u_x v_x - uv + uv_x - u_x v)]_x$, $n_t = [n(u_x v_x - uv + uv_x - u_x v)]_x$, where $m = u - u_{xx}$ and $n = v - v_{xx}$, is proposed. Our system is a generalized version of the integrable system $m_t = [m(u_x^2 - u^2)]_x$, which was shown having cusped solution (cuspon) and *W/M*-shape soliton solutions by Qiao [J. Math. Phys. **47**, 112701 (2006). The new system is proven integrable not only in the sense of Lax-pair but also in the sense of geometry, namely, it describes pseudospherical surfaces. Accordingly, infinitely many conservation laws are derived through recursion relations. Furthermore, exact solutions such as cuspons and W/M-shape solitons are also obtained. © *2011 American Institute of Physics*. [doi:10.1063/1.3530865]

I. INTRODUCTION

The Camassa–Holm (CH) equation[1,2]

$$m_t + 2u_x m + u m_x = 0, \quad m = u - u_{xx} + k \tag{1.1}$$

has attracted much attention in the past 2 decades. The CH equation has multipeaked solitons and is completely integrable in the sense of Lax-pair and bi-Hamiltonian structure. A lot of works concerning about its well-posedness, blow up, exact solutions, and algebraic and geometric formulations have been fulfilled (see Refs. 3–8 and references therein). More interestingly, geometric interpretation for the CH equation has been provided from several different points of view. For the standard CH equation $m_t + 2u_x m + u m_x = 0$ with $m = u - u_{xx}$, it describes the geodesic flow with respect to the H^1 right-invariant metric on the diffeomorphism group of the circle,[9] and in the case of $m = u - u_{xx} + k$, $k \neq 0$, the CH equation is able to take on the geodesic equation with respect to the H^1 right-invariant metric on the Bott-Virasoro group.[10] The apparent similarity for both cases was further explained by Constantin and Kolev in Ref. 11. In Ref. 12, Constantin *et al.* noticed that the Riemannian exponential map provides a smooth chart on the diffeomorphism group, but not on the Bott-Virasoro group. In Ref. 13, Chou and Qu proved that the CH equation arises from a nonstretching invariant plane curve flow in the centro-affine differential geometry. It was shown by Reyes[14] that the CH equation describes pseudospherical surfaces.

A two-component extension to the CH equation is the following system:[15–17]

$$\begin{aligned} m_t + 2u_x m + um \pm \rho\rho_x = 0,& \\ \rho_t + (\rho u)_x = 0,& \end{aligned} \tag{1.2}$$

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where $m = \sigma u - u_{xx}$, $\sigma = 0, 1$. They are referred to the two-component CH system and Hunter–Saxton system corresponding to $\sigma = 1$ and $\sigma = 0$, respectively. Since those two systems have Lax-pair and bi-Hamiltonian structure, they are completely integrable. Guha and Olver[18] verified that the system (1.2) with "+" describes geodesic flows with respect to the H^1 metric on the semidirect product space $\mathrm{Diff}^s(S^1) \ltimes C^\infty(S^1)$. Well-posedness and wave breaking phenomena for the system (1.2) with certain initial values have been studied. [19–22].

One may notice that the nonlinear terms in the CH system, the Hunter–Saxton system, and their two-component extensions are quadratic. An interesting question is whether there exist integrable systems admitting peaked solitons (peakons) and nonlinear cubic terms. Indeed, two integrable equations with peaked solitons and cubic nonlinearity have been found, which are

$$m_t + \left[m\left(u^2 - u_x^2\right)\right]_x = 0, \tag{1.3}$$

and

$$m_t + u^2 m_x + 3uu_x m = 0. \tag{1.4}$$

Equation (1.3) was found to admit tri-Hamiltonian structure by Olver and Rosenau,[15] and recently Qiao gave its Lax-pair and cusp soliton solutions and first time proposed a new kind of solitons—*W/M*-shape solitons.[23] Equation (1.4) (named Novikov's equation) arises from symmetry classification of nonlinear partial differential equations (PDEs) with cubic nonlinearity (see Ref. 24 for the derivation of this equation).

The PDE notion describing pseudospherical surfaces was introduced first by Chern and Tenenblat.[25] They showed that several well-known nonlinear evolution equations solved by the inverse scattering method, such as Korteweg-de Vries, modified Korteweg-de Vries, Schrödinger, and sine-Gordon equations, describe pseudospherical surfaces. Furthermore, they provided a complete classification to evolution equations of the form

$$u_t = F(u, u_1, \cdots, u_n),$$

where $u_j = \partial^j u / \partial x^j$ and F is a smooth function of the indicated variables, when they describe pseudospherical surfaces. In the subsequent works, their approach was applied to perform other types of evolution equations, which describe pseudospherical surfaces (see Ref. 26 and references therein).

One of the remarkable properties of the CH equation is the existence of peakons.[1] It was noted in Refs. 27, 28, and 29 that the peakons replicate a feature of the waves for great height waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves. Another type of interesting solutions, called cusped solutions (cuspons) for integrable equations, frequently appeared in a hydrodynamical context. It is worthy of noticing that the only known explicit solution without a flat free surface for steady water waves is Gerstners family of waves, the limiting forms of which admit a cusp at their crest.[30,31] There are even three-dimensional traveling waves (whose free surface is genuinely nonflat and two-dimensional) which also present cusps.[32] It was found by Qiao[23] that Eq. (1.3) has cuspons and W/M-shape soliton solutions.

The purpose of this paper is to propose the following two-component extension of Eq. (1.3); that is,

$$\begin{aligned} m_t &= [m(u_x v_x - uv + uv_x - u_x v)]_x, \\ n_t &= [n(u_x v_x - uv + uv_x - u_x v)]_x, \end{aligned} \tag{1.5}$$

where $m = u - u_{xx}$ and $n = v - v_{xx}$. Apparently, it reduces to (1.3) when $v = u$. We will show that this system is integrable, namely, it has Lax-pair and is also geometrically integrable. As a consequence of geometric integrability, its conservation laws are constructed by expanding the pseudopotential. Finally, the cuspons and W/M-shape solitons of system (1.5) are obtained.

II. LAX-PAIR OF SYSTEM (1.5)

It was shown in Ref. 23 that Eq. (1.3) has the Lax-pair

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(\lambda, m) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.1}$$

and

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V(\lambda, u, m) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.2}$$

where

$$U(\lambda, m) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2} \end{pmatrix}, \quad m = u - u_{xx},$$

$$V(\lambda, u, m) = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}\lambda m(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}\lambda m(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}.$$

Similarly we have the following result.

Theorem 2.1: System (1.5) has the following Lax-pair

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2} & \lambda m \\ \lambda n & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.3}$$

and

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = \begin{pmatrix} \frac{1}{4}\lambda^{-2} + \frac{1}{2}Q & \frac{1}{2}\lambda^{-1}(u + u_x) + \lambda m Q \\ \frac{1}{2}\lambda^{-1}(v - v_x) + \lambda n Q & -\frac{1}{4}\lambda^{-2} - \frac{1}{2}Q \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.4}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$, $Q = u_x v_x - uv + uv_x - u_x v$.

Remark 1.1: Equation (1.3) has bi-Hamiltonian structure,[15,23] namely, it can be written as

$$m_t = J\frac{\delta H_0}{\delta m} = K\frac{\delta H_1}{\delta m},$$

where

$$J = -\partial m \partial^{-1} m \partial,$$
$$K = \partial^3 - \partial,$$
$$H_0 = 2\int_{\mathbb{R}} um dx,$$
$$H_1 = \frac{1}{4}\int_{\mathbb{R}} (u^4 + 2u^2 u_x^2 - \frac{1}{3}u_x^4) dx.$$

III. GEOMETRIC INTEGRABILITY OF SYSTEMS (1.5)

According to Ref. 25, a PDE

$$E(x, t, u, u_x, u_t, \cdots, u_{x^n t^m}) = 0 \tag{3.1}$$

for a real function $u(x, t)$ with two independent variables t and x is said to describe pseudospherical surfaces if it is a necessary and sufficient condition for the existence of smooth function f_{ij}, $1 \leq i \leq 3$, $1 \leq j \leq 2$, depending on u and its derivatives such that the 1-forms $d\omega_i = f_{i1}dx + f_{i2}dt$, $1 \leq i \leq 3$, satisfy the structure equations of a surface with a constant Gaussian curvature equal to -1; that is,

$$d\omega_1 = \omega_3 \wedge \omega_2,$$
$$d\omega_2 = \omega_1 \wedge \omega_3,$$
$$d\omega_3 = \omega_1 \wedge \omega_2, \tag{3.2}$$

where $u = u(x, t)$ is a solution of $E = 0$.

Definition 3.1: Equation (3.1) is geometrically integrable if it describes a nontrivial one-parameter family of pseudospherical surfaces.

We now have the following result.

Theorem 3.1: System (1.5) describes pseudospherical surfaces.

Proof: Consider the following three 1-forms ω_i, $i = 1, 2, 3$:

$$\begin{aligned}
\omega_1 &= \lambda[e^{(1-\lambda)x}n + e^{(\lambda-1)x}m]dx + [2\lambda(e^{(\lambda-1)x}m + e^{(1-\lambda)x}n)Q \\
&\quad +\lambda^{-1}\left(e^{(\lambda-1)x}(u+u_x) + e^{(1-\lambda)x}(v-v_x)\right)]dt, \\
\omega_2 &= \lambda dx + (\frac{1}{2}\lambda^{-2} + Q)dt, \\
\omega_3 &= \lambda[e^{(1-\lambda)x}n - e^{(\lambda-1)x}m]dx + [2\lambda(e^{(1-\lambda)x}n - e^{(\lambda-1)x}m)Q \\
&\quad +\lambda^{-1}\left(e^{(1-\lambda)x}(v-v_x) - e^{(\lambda-1)x}(u+u_x)\right)]dt.
\end{aligned} \tag{3.3}$$

Through a direct computation, we know that the structure equations (3.2) hold whenever $u(x, t)$ and $v(x, t)$ are solutions of system (1.5).

Proposition 3.1: (Ref. 33) Let $E = 0$ be a differential equation of pseudospherical type with associated 1-forms $d\omega_i = f_{i1}dx + f_{i2}dt$, $1 \leq i \leq 3$. Then $E = 0$ is the integrability condition of the $sl(2, \mathbb{R})$-valued linear problem

$$dv = \Omega v, \tag{3.4}$$

where Ω is the matrix-valued 1-form

$$\Omega = Xdx + Tdt = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}. \tag{3.5}$$

Consequently, if $E = 0$ is geometrically integrable, it is the integrability condition of a 1-parameter family of linear problems.

It follows from the 1-forms (3.3) and (3.4) that the integrability condition of the $sl(2, \mathbb{R})$ linear problem

$$\begin{aligned}
v_x &= Xv, \\
v_t &= Tv,
\end{aligned}$$

gives the system (1.5). The expression (3.5) implies that the matrices X and T are

$$X = \frac{1}{2}\begin{pmatrix} \lambda & 2\lambda e^{(\lambda-1)x}m \\ 2\lambda e^{(1-\lambda)x}n & -\lambda \end{pmatrix},$$

$$T = \frac{1}{2}\begin{pmatrix} \frac{1}{2}\lambda^{-2} + Q & e^{(\lambda-1)x}\left(\lambda^{-1}(u+u_x) + 2\lambda mQ\right) \\ e^{(1-\lambda)x}\left(\lambda^{-1}(v-v_x) + 2\lambda nQ\right) & -\frac{1}{2}\lambda^{-2} - Q, \end{pmatrix}. \tag{3.6}$$

Therefore we have gotten the following result.

Corollary 3.1: System (1.5) is geometrically integrable.

IV. PSEUDOPOTENTIALS AND CONSERVATION LAWS OF SYSTEM (1.5)

It was known that the equations of pseudospherical type admit quadratic pseudopotentials.[33,34] Let $\Gamma = \psi_1/\psi_2$, $\gamma = \psi_2/\psi_1$, where ψ_1 and ψ_2 satisfy (2.3) and (2.4). One can easily verify that

$$d\Gamma = -\frac{1}{2}(\tilde{\omega}_3 - \tilde{\omega}_1) + \tilde{\omega}_2\Gamma - \frac{1}{2}(\tilde{\omega}_3 + \tilde{\omega}_1)\Gamma^2, \tag{4.1}$$

and

$$d\gamma = \frac{1}{2}(\tilde{\omega}_3 + \tilde{\omega}_1) - \tilde{\omega}_2\gamma + \frac{1}{2}(\tilde{\omega}_3 - \tilde{\omega}_1)\gamma^2 \tag{4.2}$$

are completely integrable, where

$$\tilde{\omega}_1 = \lambda(n+m)dx + \big[\frac{1}{2}\lambda^{-1}(v - v_x + u + u_x) + \lambda(n+m)Q\big]dt,$$

$$\tilde{\omega}_2 = dx + \big(\frac{1}{2}\lambda^{-2} + Q\big)dt,$$

$$\tilde{\omega}_3 = \lambda(n-m)dx + \big[\frac{1}{2}\lambda^{-1}(v - v_x - u - u_x) + \lambda(n-m)Q\big]dt.$$

It follows from (4.2) that γ satisfies the following two equations:

$$\gamma_x = -\lambda m\gamma^2 - \gamma + \lambda n, \tag{4.3}$$

and

$$\begin{aligned}\gamma_t = \; & -\big[\frac{1}{2}\lambda^{-1}(u+u_x) + \lambda m Q\big]\gamma^2 - \big(\frac{1}{2}\lambda^{-2} + Q\big)\gamma \\ & + \frac{1}{2}\lambda^{-1}(v - v_x) + \lambda n Q.\end{aligned} \tag{4.4}$$

A direct computation shows that γ satisfies the conservation law

$$(m\gamma)_t = [\frac{1}{2}\lambda^{-2}(u+u_x)\gamma + (\gamma m + \frac{1}{2}\lambda^{-1})Q]_x. \tag{4.5}$$

Based on Eqs. (4.3) and (4.5), using the standard algorithm we are able to obtain infinitely many of conservation laws of system (1.5) by expanding the pseudopotential γ. As usual, two sets of expansions for γ will be used. The first one is

$$\gamma = \sum_{n=0}^{\infty} \gamma_n \lambda^{-n}. \tag{4.6}$$

Substituting (4.6) into (4.3), we arrive at the following systems of γ_n:

$$\begin{aligned} & n - m\gamma_0^2 = 0, \\ & \gamma_{0,x} + 2m\gamma_0\gamma_1 + \gamma_0 = 0, \\ & \gamma_{1,x} + m(2\gamma_0\gamma_2 + \gamma_1^2) + \gamma_1 = 0, \\ & \gamma_{2,x} + m(2\gamma_0\gamma_3 + 2\gamma_1\gamma_2) + \gamma_2 = 0, \\ & \qquad \cdots, \\ & \gamma_{n,x} + \sum_{k=0}^{n+1} \gamma_k\gamma_{n-k} + \gamma_n = 0. \end{aligned} \tag{4.7}$$

Solving the system, we get the first few solutions given by

$$\begin{aligned} & \gamma_0 = \sqrt{\frac{n}{m}}, \\ & \gamma_1 = -\frac{1}{2m}\big(\frac{\gamma_{0,x}}{\gamma_0} + 1\big), \\ & \gamma_2 = -\frac{1}{2m}\big[\big(\frac{\gamma_1}{\gamma_0}\big)_x - \frac{1}{4m\gamma_0}\big(\frac{\gamma_{0,x}}{\gamma_0} + 1\big)^2\big], \\ & \qquad \cdots, \end{aligned}$$

which lead to the first two conservation functionals

$$H_{-1} = \int_{\mathbb{R}} \sqrt{mn}\,dx,$$

$$H_{-2} = \int_{\mathbb{R}} \frac{1}{\sqrt{mn}}\big(\frac{\gamma_{0,x}}{\gamma_0} + 1\big)^2 dx.$$

These Hamiltonians will generate the negative members of the integrable hierarchy associated with the spectral problem (2.3).

Next, we consider the expansion of γ in the form

$$\gamma = \sum_{n=0}^{+\infty} \Gamma_n \lambda^n.$$

Substituting it into (4.3) yields the following system:

$$\begin{aligned}
&\Gamma_{0,x} + \Gamma_0 = 0,\\
&\Gamma_{1,x} + \Gamma_1 = n - m\Gamma_0^2,\\
&\Gamma_{2,x} + \Gamma_2 = -2m\Gamma_0\Gamma_1,\\
&\qquad\quad \cdots,\\
&\Gamma_{n,x} + \Gamma_n = -m\sum_{k=0}^{n-1}\Gamma_k\Gamma_{n-1-k}.
\end{aligned}\tag{4.8}$$

Similarly, we can obtain the first few conservation functionals

$$\begin{aligned}
H_1 &= \int_{\mathbb{R}} m(v - v_x)dx,\\
H_2 &= \int_{\mathbb{R}} (u + u_x)(v - v_x)^2 m dx,\\
H_3 &= \int_{\mathbb{R}} (u + u_x)(v - v_x) m\Gamma_3 dx.
\end{aligned}$$

These Hamiltonians will generate the positive members of the integrable hierarchy associated with the spectral problem (2.3).

V. EXACT SOLUTIONS OF SYSTEM (1.5)

We are now seeking traveling wave solutions of (1.5) by setting

$$u = U(x - ct), \quad v = V(x - ct), \tag{5.1}$$

where c is the wave speed. Let $\xi = x - ct$, then $u = U(\xi)$ and $v = V(\xi)$ satisfy the system

$$\begin{aligned}
&[m(u_\xi v_\xi + uv_\xi - vu_\xi - uv + c)]_\xi = 0,\\
&[n(u_\xi v_\xi + uv_\xi - vu_\xi - uv + c)]_\xi = 0.
\end{aligned}$$

After integration, we arrive at the system

$$\begin{aligned}
&m(u_\xi v_\xi + uv_\xi - vu_\xi - uv + c) = a,\\
&n(u_\xi v_\xi + uv_\xi - vu_\xi - uv + c) = b,
\end{aligned}\tag{5.2}$$

where a and b are integration constants. We consider two possibilities:

Case (i): $ab = 0$. In this case, u and v must satisfy the equation

$$u_\xi v_\xi + uv_\xi - vu_\xi - uv + c = 0,$$

which can be written as

$$(e^\xi u)_\xi (e^{-\xi} v)_\xi = -c. \tag{5.3}$$

One can readily see that Eq. (5.3) has the following solution:

$$\begin{aligned}
u &= c_1 e^{|\xi|} + c_2 e^{-|\xi|},\\
v &= d_1 e^{|\xi|} + d_2 e^{-|\xi|},
\end{aligned}\tag{5.4}$$

where c_i and d_i, $i = 1, 2$, are constants satisfying the constraint

$$2c_1d_2(1 + \text{sign }\xi) + 2d_1c_2(1 - \text{sign }\xi) - c = 0.$$

We are not interested in such solutions since $u \longrightarrow \infty$, $v \longrightarrow \infty$ as $\xi \longrightarrow \infty$.

Case (ii): $a, b \neq 0$. It follows from (5.2) that

$$\frac{a}{m} = \frac{b}{n},$$

which implies that $w = av - bu$ satisfies

$$w - w_{\xi\xi} = 0.$$

It has the general solution

$$w = c_3e^{\xi} + c_4e^{-\xi},$$

where c_3 and c_4 are constants. Therefore, we have

$$v = \frac{b}{a}u + c_3e^{\xi} + c_4e^{-\xi}.$$

If we require that

$$\lim_{|\xi|\longrightarrow\infty} u = A, \quad \lim_{|\xi|\longrightarrow\infty} v = B,$$

where A and B are constants. It implies that

$$B = \frac{b}{a}A, c_3 = c_4 = 0.$$

Consequently, $v = \frac{b}{a}u$ and u satisfies

$$\frac{b}{a}m(u_\xi^2 - u^2) + cm + a = 0,$$

where $m = U - U_{\xi\xi}$. Let $d = (ac)/b$, $\beta = -a^2/b$, then U satisfies the equation

$$[d - (U^2 - U'^2)]^2 = \alpha - 4\beta U,$$

where α is an integration constant. Solving it, one gets

$$U'^2 = U^2 - d \pm \sqrt{\alpha - 4\beta U}.$$

Noticing $\lim_{\xi\longrightarrow\infty} U = A$, $\lim_{\xi\longrightarrow\infty} U_\xi = 0$, we immediately have

$$\beta = A(d - A^2), \quad \alpha = (d - A^2)(d + 3A^2).$$

Thus U satisfies the equation

$$[d - (U^2 - U'^2)]^2 = (d - A^2)(d + 3A^2 - 4AU). \tag{5.5}$$

Using a similar discussion as shown in Ref. 23, we know that Eq. (5.5) has the following solution

$$U = A[\frac{5}{3} - (3z + 2)(z - \sqrt{z^2 - \frac{4}{9}})],$$

where $z = \cosh(\frac{|\xi|}{2} - \ln 2) - \frac{1}{3}$, $c = \frac{11}{3}A^2$, which are the W/M-shape solitons corresponding to $A > 0$ or $A < 0$.

Equation (5.5) also has the following cusp solitons:

$$U = A[\frac{5}{3} - (3z + 2)(z - \sqrt{z^2 - \frac{4}{9}})],$$

where $z=\cosh(\frac{x}{2}-\frac{11}{6}t^2)-\frac{1}{3}$. Let $A=\pm 1$, then the solution reads as

$$U(X)= \pm\left(2-3\cosh^2 X+(\cosh X+\frac{1}{3})\sqrt{3(3\cosh X+1)(\cosh X-1)}\right),$$
$$X=\frac{x}{2}-\frac{11}{6}t.$$

It has the following properties:

$$U(0)=\mp 1,\quad U'(0^+)\pm=\infty,\quad U'(0^-)=\mp\infty,$$

which imply that $U(X)$ is indeed a cusp soliton.

VI. CONCLUDING REMARKS

In this paper, we present a new integrable two-component cubic system, which is a natural extension to Eq. (1.3). It is shown that system (1.5) admits the Lax-pair, and thus it is integrable. Moreover, we show that the system is geometrically integrable, namely, it describes pseudospherical surfaces. As a consequence, its infinite number of conservation laws are able to be constructed through some recursion relations. The explicit solutions of Eq. (1.3), including cuspons and W/M-shape solitions are also derived. Following the idea in Ref. 35, we may extend Eq. (1.3) to an integrable hierarchy since all Hamiltonians, both positive and negative orders, are found through the above goemetric procedure.

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