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Zhijun Qiao
The University of Texas Rio Grande Valley, zhijun.qiao@utrgv.edu

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Negative-order Korteweg–de Vries equations

Zhijun Qiao*
Department of Mathematics, The University of Texas-Pan American, 1201 W. University Drive, Edinburg, Texas 78539, USA

Engui Fan†
School of Mathematical Sciences and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, P.R. China

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In this paper, based on the regular Korteweg–de Vries (KdV) system, we study negative-order KdV (NKdV) equations, particularly their Hamiltonian structures, Lax pairs, conservation laws, and explicit multisoliton and multikink wave solutions thorough bilinear Bäcklund transformations. The NKdV equations studied in our paper are differential and actually derived from the first member in the negative-order KdV hierarchy. The NKdV equations are not only gauge equivalent to the Camassa-Holm equation through reciprocal transformations but also closely related to the Ermakov-Pinney systems and the Kupershmidt deformation. The bi-Hamiltonian structures and a Darboux transformation of the NKdV equations are constructed with the aid of trace identity and their Lax pairs, respectively. The single and double kink wave and bell soliton solutions are given in an explicit formula through the Darboux transformation. The one-kink wave solution is expressed in the form of tanh while the one-bell soliton is in the form of sech, and both forms are very standard. The collisions of two-kink wave and two-bell soliton solutions are analyzed in detail, and this singular interaction differs from the regular KdV equation. Multidimensional binary Bell polynomials are employed to find bilinear formulation and Bäcklund transformations, which produce N-soliton solutions. A direct and unifying scheme is proposed for explicitly building up quasiperiodic wave solutions of the NKdV equations. Furthermore, the relations between quasiperiodic wave solutions and soliton solutions are clearly described. Finally, we show the quasiperiodic wave solution convergent to the soliton solution under some limit conditions.

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I. INTRODUCTION

The Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

was proposed by Korteweg and de Vries in fluid dynamics [1], starting from the observation and subsequent experiments by Russell [2]. There are many excellent sources that detail the highly interesting background and historical development of the KdV equation, which bring it to the forefront of modern mathematical physics. In 1967, Gardner, Greener, Kruskal, and Miura found the inverse-scattering transformation method to solve the Cauchy problem of the KdV equation with sufficiently rapidly decaying initial data [3]. Soon thereafter, Lax explained the magical isospectral property of the time-dependent family of Schrödinger operators by what is now called *the Lax pair* and introduced the KdV hierarchy through a recursive procedure [4]. In the same year a sequence of infinitely many polynomial conservation laws was obtained with the help of the Miura transformation [5,6].

There are some tools to view the KdV equation as a completely integrable system by Gardner and Zakharov and Faddeev [7,8]. The bilinear derivative method was developed by Hirota to find N-soliton solutions of the KdV equation [9]. The KdV hierarchy was constructed by Lax [10] through a recursive approach and further studied by Gel'fand and Dikii [11]. The extension of the inverse-scattering method to periodic initial data, based on both the inverse spectral theory and algebrogeometric methods, was developed by Novikov, Dubrovin, Lax, Its, Matveev *et al.* [12–15]. For more recent reviews on the KdV equation one may refer, for instance, to Refs. [16–25].

All the work done in the above-mentioned publications dealt with the positive-order KdV hierarchy, which includes the KdV equation as a special member. However, there was little work on the NKdV hierarchy. Verosky [26] studied symmetries and negative powers of recursion operator and gave the following negative-order KdV (NKdV) equation,

$$v_t = w_x, \quad w_{xxx} + 4vw_x + 2v_x w = 0, \tag{1.1}$$

and Lou [27] presented additional symmetries based on the invertible recursion operator of the KdV system and particularly provided the following NKdV equation (called the NKdV-1 equation thereafter):

$$v_t = 2uu_x, \quad u_{xx} + vu = 0, \iff \left(\frac{u_{xx}}{u}\right)_t + 2uu_x = 0, \tag{1.2}$$

which can be reduced from the NKdV equation (1.1) under the following transformation:

$$w = u^2, \quad v = -\frac{u_{xx}}{u}. \tag{1.3}$$

Moreover, the second part of NKdV-1 equation (1.2) is a linear Schrödinger equation or Hill equation,

$$u_{xx} + vu = 0.$$

*qiao@utpa.edu
†Corresponding author: faneg@fudan.edu.cn

Fuchssteiner [28] pointed out the gauge-equivalent relation between the NKdV equation (1.1) and the Camassa-Holm (CH) equation [29],

$$m_t + m_x u + 2mu_x = 0, \quad m = u - u_{xx},$$

through some hodograph transformation, and, later, Hone proposed the associate CH equation, which is actually equivalent to NKdV equation (1.1), and gave soliton solutions through the KdV system [30]. Zhou generalized the Kupershmidt deformation and proposed a kind of the mixed KdV hierarchy, which contains NKdV equation (1.1) as a special case [31].

Very recently, Qiao and Li [32] gave a unifying formulation of the Lax representations for both negative- and positive-order KdV hierarchies and, furthermore, studied all possible traveling wave solutions, including soliton, kink wave, and periodic wave solutions, of the integrable NKdV-1 equation (1.2), which possesses the following Lax pair:

$$\begin{aligned} L\psi &\equiv \psi_{xx} + v\psi = \lambda\psi, \\ \psi_t &= \tfrac{1}{2}u^2\lambda^{-1}\psi_x - \tfrac{1}{2}uu_x\lambda^{-1}\psi. \end{aligned} \tag{1.4}$$

The most interesting fact is that the NKdV-1 equation has both soliton and kink solutions, which is the first integrable example, within our knowledge, to have such a property in soliton theory.

Studying negative-order integrable hierarchies plays an important role in the theory of peaked solitons (peakons) and cusp solitons (cuspons). For instance, the well-known CH peakon equation is actually produced through its negative-order hierarchy while its positive-order hierarchy includes the remarkable Harry-Dym–type equation [33]. The Degasperis-Procesi (DP) peakon equation [34] can also be generated through its negative-order hierarchy [35]. Both the CH equation and the DP equation are typical integrable peakon and cuspon systems with nonlinear quadratic terms [29,33,36–38]. Recently, some nonlinear cubic integrable equations have also been found to have peakon and cuspon solutions [39–42].

In this paper, we study the NKdV hierarchy and, in particular, focus on the NKdV equation (1.1) and the NKdV-1 equation (1.2). Actually, as in Refs. [27,43], the NKdV equation (1.1) can embrace other possible differential-integro forms according to the kernel of the operator $K = \frac{1}{4}\partial_x^3 + \frac{1}{2}(v\partial_x + \partial_x v)$. Here we just list the NKdV-1 equation (1.2) as it is differential and we find that the first negative-order KdV equation is also equivalent to a nonlinear quartic integrable system,

$$uu_{xxt} - u_{xx}u_t - 2u^3u_x = 0,$$

with both classic soliton and kink wave solutions.

The purpose of this paper is to investigate integrable properties, the N-soliton and N-kink solutions of the NKdV equation (1.1) and NKdV-1 equation (1.2). In Sec. II, the trace identity technique is employed to construct the bi-Hamiltonian structures of the NKdV hierarchy. In Sec. III, we show that the NKdV equation (1.1) is related to the Kupershmidt deformation and the Ermakov-Pinney systems and is also able to reduced to the NKdV-1 equation (1.2) under a transformation. The relation between the solution of the NKdV equation (1.1) and that of NKdV-1 equation (1.2) is given. In Sec. IV, a Darboux transformation of the NKdV equation (1.1) is provided with the help of its Lax pairs. In Sec. V, as a direct application of the Darboux transformation, the kink wave and bell soliton solutions are explicitly given, and the collision of two soliton solutions is analyzed in detail through two solitons. In Sec. VI, an extra auxiliary variable is introduced to bilinearize the NKdV equation (1.1) through binary Bell polynomials. In Sec. VII, the bilinear Bäcklund transformations are obtained and Lax pairs are also recovered. In Sec. VIII, we will give a kind of Darboux covariant Lax pair, and in Sec. IX, infinitely many conservation laws of the NKdV equation (1.1) are presented through its Lax equation and a generalized Miura transformation. All conserved densities and fluxes are recursively given in an explicit formula. In Sec. X, a direct and unifying scheme is proposed for building up quasiperiodic wave solutions of the NKdV equation (1.1) in an explicit formula. Furthermore, the relations between quasiperiodic wave solutions and soliton solutions are clearly described. Finally, we show the quasiperiodic wave solution convergent to the soliton solution under the assumption of small amplitude.

II. HAMILTONIAN STRUCTURES OF THE NKDV HIERARCHY

To find the Hamiltonian structures of the NKdV hierarchy, let us rederive the NKdV hierarchy in matrix form.

A. The NKdV hierarchy

Consider the Schrödinger-KdV spectral problem

$$\psi_{xx} + v\psi = \lambda\psi, \tag{2.1}$$

where λ is an eigenvalue, ψ is the eigenfunction corresponding to the eigenvalue λ, and v is a potential function.

Let $\varphi_1 = \psi$, $\varphi_2 = \psi_x$, and then the spectral problem (2.1) becomes

$$\varphi_x = U\varphi = \begin{pmatrix} 0 & 1 \\ \lambda - v & 0 \end{pmatrix}\varphi, \tag{2.2}$$

where $\varphi = (\varphi_1,\varphi_2)^T$ is a two-dimensional vector of eigenfunctions.

The Gateaux derivative of spectral operator U in direction ξ at point v is

$$U'[\xi] = \frac{d}{d\varepsilon}U(v + \varepsilon\xi)|_{\varepsilon=0} = \begin{pmatrix} 0 & 0 \\ -\xi & 0 \end{pmatrix}, \tag{2.3}$$

which is injective and linear with respect to the variable ξ.

The Lenard recursive sequence $\{G_m\}$ of the spectral problem (2.1) is defined by

$$\begin{aligned} G_{-1} &\in Ker K = \{G|KG = 0\}, \\ G_0 &\in Ker J = \{G|JG = 0\}, \\ KG_{m-1} &= JG_m, \quad m = 0,-1,-2\ldots, \end{aligned} \tag{2.4}$$

which directly produces the NKdV hierarchy

$$v_t = KG_{m-1} = JG_m, \quad m = 0,-1,-2\ldots, \tag{2.5}$$

where

$$K = \tfrac{1}{4}\partial_x^3 + \tfrac{1}{2}(v\partial_x + \partial_x v), \quad J = \partial_x, \tag{2.6}$$

and K is exactly a recursion operator of the well-known KdV hierarchy

$$v_t = K^n v_x, \quad n = 0,1,2,\dots.$$

The first equation ($m = 0$) in the NKdV hierarchy (2.5) is a trivial equation,

$$v_t = JG_0 = 0, \quad JG_0 = KG_{-1} = 0.$$

The second equation ($m = -1$) in the NKdV hierarchy (2.5) takes

$$v_t = G_{-1,x}, \quad KG_{-1} = 0,$$

which is exactly the NKdV equation (1.1) but replacing $G_{-1} = w$.

In a similar way to that in Ref. [32], we construct a zero-curvature representation for the NKdV hierarchy.

Proposition 1. Let U be the spectral matrix defined in (2.2), and then, for an arbitrarily smooth function $G \in C^\infty(\mathbb{R})$, the operator equation

$$V_x - [U,V] = U'[KG] - \lambda U'[JG] \tag{2.7}$$

admits a matrix solution

$$V = V(G) = \begin{pmatrix} -\frac{1}{4}G_x & \frac{1}{2}G \\ -\frac{1}{4}G_{xx} - \frac{1}{2}vG + \frac{1}{2}\lambda G & \frac{1}{4}G_x \end{pmatrix}\lambda^{-1},$$

which is a linear function with respect to G, and the Gateaux derivative is defined by (2.3).

Theorem 1. Suppose that $\{G_j, \quad j = -1,-2,\dots\}$ is the first Lenard sequence defined by (2.4), and $V_j = V(G_j)$ is a corresponding solution to the operator equation (2.7) for $G = G_j$. With V_j being its coefficients, a mth matrix polynomial in λ is constructed as follows:

$$W_m = \sum_{j=1}^{m} V_j \lambda^{-m+j}.$$

We then conclude that the NKdV hierarchy (2.5) admits zero curvature representation

$$U_t - W_{m,x} + [U,W_m] = 0,$$

which is equivalent to

$$\begin{aligned} \varphi_x &= U\varphi = \begin{pmatrix} 0 & 1 \\ \lambda - v & 0 \end{pmatrix}\varphi, \\ \varphi_t &= W_m\varphi \\ &= \sum_{j=1}^{m} \begin{pmatrix} -\frac{1}{4}G_{j,x} & \frac{1}{2}G_j \\ -\frac{1}{4}G_{j,xx} - \frac{1}{2}vG_j + \frac{1}{2}\lambda G_j & \frac{1}{4}G_{j,x} \end{pmatrix}\lambda^{-m+j-1}\varphi. \end{aligned} \tag{2.8}$$

This theorem actually provides a unified formula of the Lax pairs for the whole NKdV hierarchy (2.5).

According to theorem 1, the NKdV equation (1.1) admits a Lax pair with parameter λ

$$\begin{aligned} L\psi &\equiv \psi_{xx} + v\psi = \lambda\psi, \\ \psi_t &= \tfrac{1}{2}w\lambda^{-1}\psi_x - \tfrac{1}{4}w_x\lambda^{-1}\psi, \end{aligned}$$

or, equivalently,

$$\begin{aligned} L\psi &= \left(\partial_x^2 + v\right)\psi = \lambda\psi, \\ M\psi &= \left(4\partial_x^2\partial_t + 4v\partial_t + 2w\partial_x + 3w_x\right)\psi = 0. \end{aligned} \tag{2.9}$$

The NKdV equation (1.1) also possesses a Lax pair without the parameter

$$\begin{aligned} L\psi &= \left(\partial_x^2 + v\right)\psi = 0, \\ M\psi &= \left(4\partial_x^2\partial_t + 4v\partial_t + 2w\partial_x + 3w_x\right)\psi = 0. \end{aligned} \tag{2.10}$$

Especially, taking the constraint $v = -u_{xx}/u$ and $w = u^2 \in Ker K$, we then further get the NKdV equation (1.2) and its Lax pair (1.4).

B. Hamiltonian structures

Proposition 2. For the spectral problem (2.2), assume that V is a solution to the following stationary zero curvature equation with the given homogeneous rank [24],

$$V_x = [U,V] \equiv UV - VU. \tag{2.11}$$

There then exists a constant β such that

$$\frac{\delta}{\delta v}\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = \left(\lambda^{-\beta}\frac{\partial}{\partial\lambda}\lambda^{\beta}\right)\left\langle V, \frac{\partial U}{\partial v}\right\rangle \tag{2.12}$$

holds, where $\langle\cdot,\cdot\rangle$ stands for the trace of the product of two matrices.

Let $\{G_m, \ m = -1,-2\dots\}$ be the negative-order Lenard sequence recursively given through (2.4) and

$$G_\lambda = \sum_{m=-\infty}^{-1} G_m\lambda^{-m} \tag{2.13}$$

be a series with respect to λ. Assume that $V_\lambda = V(G_\lambda)$ is the matrix solution for the operator equation (2.9) corresponding to $G = G_\lambda$. So, V_λ can be written as

$$V_\lambda = \sum_{m=-\infty}^{-1} V_m\lambda^{-m}.$$

We then have the following proposition.

Proposition 3. V_λ satisfies the following Lax form:

$$V_{\lambda,x} = [U,V_\lambda].$$

Proof. By (2.4), we have

$$\begin{aligned} &(K - \lambda J)G_\lambda \\ &= \sum_{m=-\infty}^{-1} KG_m\lambda^{-m} - \sum_{m=-\infty}^{-1} JG_m\lambda^{-m+1} \\ &= KG_{-1}\lambda^{-1} + \sum_{m=-\infty}^{-1}(KG_{m-1} - JG_m)\lambda^{-m} = 0. \end{aligned}$$

Therefore, proposition 1 implies

$$\begin{aligned} V_{\lambda,x} - [U,V_\lambda] &= U'[KG_\lambda] - \lambda U'[JG_\lambda] \\ &= U'[KG_\lambda - \lambda JG_\lambda] = 0. \end{aligned}$$

■

We next discuss the Hamiltonian structures of the hierarchy (2.5). It is crucial to find infinitely many conserved densities.

Theorem 2.

(1) The hierarchy (2.5) possesses the bi-Hamiltonian structures

$$v_t = K\frac{\delta H_{m-1}}{\delta v} = J\frac{\delta H_m}{\delta v}, \quad m = -1, -2\ldots, \tag{2.14}$$

where the Hamiltonian functions H_m are implicitly given through the following formulas:

$$H_{-1} = G_{-1} \in Ker K, \quad H_m = \frac{G_m}{m}, \quad m = -1, -2\ldots. \tag{2.15}$$

(2) The hierarchy (2.5) is integrable in the Liouville sense.

(3) The Hamiltonian functions $\{H_m\}$ are conserved densities of the whole hierarchy (2.5) and, therefore, they are in involution in pairs for the Poisson bracket

$$\{H_n, H_m\} = \left(\frac{\delta H_n}{\delta v}, J\frac{\delta H_m}{\delta v}\right) = \int \frac{\delta H_n}{\delta v} J \frac{\delta H_m}{\delta v} dx,$$

where $(\cdot,\cdot)$ stands for inner product of two functions.

Proof. A direction calculation leads to

$$\left\langle V_\lambda, \frac{\partial U}{\partial \lambda}\right\rangle = \frac{1}{2}G_\lambda, \quad \left\langle V_\lambda, \frac{\partial U}{\partial v}\right\rangle = -\frac{1}{2}G_\lambda.$$

By using the trace identity (2.12) and the expansion (2.13), we obtain

$$\begin{aligned}\frac{\delta}{\delta v}\left(\sum_{m=-\infty}^{-1} G_m \lambda^{-m}\right) &= \sum_{m=-\infty}^{-1} (m-1-\beta)G_{m-1}\lambda^{-m} \\ &\quad + (-1-\beta)G_{-1}, \quad m = -1, -2\ldots. \end{aligned} \tag{2.16}$$

If taking $G_{-1} \neq 0$ from (2.16) we find $\beta = -1$ and

$$\frac{\delta H_m}{\delta v} = G_{m-1}, \quad m = -1, -2\ldots, \tag{2.17}$$

where H_m are given by (2.15). Substituting (2.17) into (2.5) yields the bi-Hamiltonian structures (2.14).

We next consider infinitely many conserved densities to guarantee integrability of the hierarchy (2.16). Since J and K are skew-symmetric operators, we infer that

$$\mathcal{L}^* J = (J^{-1}K)^* J = -K^* = K = J\mathcal{L},$$

which implies

$$\begin{aligned}\{H_n, H_m\} &= \left(\frac{\delta H_n}{\delta v}, J\frac{\delta H_m}{\delta v}\right) = (\mathcal{L}^n G_{-1}, J\mathcal{L}^m G_{-1}) \\ &= (\mathcal{L}^n G_{-1}, \mathcal{L}^* J\mathcal{L}^{m-1} G_{-1}) = (\mathcal{L}^{n+1} G_{-1}, J\mathcal{L}^{m-1} G_0) \\ &= \{H_{n+1}, H_{m-1}\}, \quad m,n \leqslant -1.\end{aligned}$$

Repeating the above argument gives

$$\{H_n, H_m\} = \{H_m, H_n\} = \{H_{m+n}, H_{-1}\}. \tag{2.18}$$

On the other hand, we find

$$\begin{aligned}\{H_m, H_n\} &= (\mathcal{L}^m G_{-1}, J\mathcal{L}^n G_{-1}) \\ &= (J^*\mathcal{L}^m G_{-1}, \mathcal{L}^n G_{-1}) = -\{H_n, H_m\}.\end{aligned} \tag{2.19}$$

Combining (2.18) with (2.19) then leads to

$$\{H_m, H_n\} = 0,$$

which implies that $\{H_m\}$ are in involution, and, therefore, the hierarchy (2.14) are integrable in the Liouville sense.

Especially, under the constraint (1.3), we obtain bi-Hamilton structures of the NKdV equation (1.2)

$$v_t = K\frac{\delta H_{-1}}{\delta v} = J\frac{\delta H_0}{\delta u},$$

where two Hamiltonian functions are given by

$$H_0 = \tfrac{1}{3}u^3, \quad H_{-1} = -u^2,$$

which can also be written in a conserved density form in terms of an equivalence class,

$$H_0 \sim -\frac{1}{3}\int u^3 dx, \quad H_{-1} \sim -\int u^2 dx.$$

III. RELATIONS TO OTHER REMARKABLE SYSTEMS

In this section, we discuss relations of the NKdV hierarchy (2.5) with Kupershmidt deformation, soliton equations with self-consistent sources and Ermakov-Pinney systems.

Recently, a class of new integrable systems, known as the Kupershmidt deformation of soliton equations, have attracted much attention. This topic is the work of Kupershmidt and Karasu-Kalkani' [44–46]. A Kupershmidt nonholonomic deformation of the NKdV hierarchy (2.5) takes

$$v_t = JG_m + Jw, \quad m = 0, -1, -2, \ldots, \quad Kw = 0, \tag{3.1}$$

where two operators K and J are given by (1.4). The first flow ($m = 0$) of the hierarchy (3.1) then is exactly the NKdV equation (1.1),

$$v_t = w_x, \quad w_{xxx} + 4vw_x + 2v_x w = 0,$$

which may be regarded as a Kupershmidt nonholonomic deformation of the trivial equation for the NKdV hierarchy (2.5). Soliton equations with self-consistent sources have important physical applications; for example, the KdV equation with a self-consistent source describes the interaction of long and short capillary-gravity waves [47–50].

For N distinct λ_j of the spectral problem (2.1), the functional gradient of λ with respect to v is

$$\frac{\delta\lambda_j}{\delta v} = \psi_j^2,$$

and we then define the NKdV hierarchy with self-consistent sources by

$$\begin{aligned} &v_t = JG_m + \alpha J\frac{\delta\lambda}{\delta v} = JG_m + \alpha J\sum_{j=1}^N \psi_j^2, \\ &\psi_{j,xx} + (v+\lambda_j)\psi_j = 0, \\ &m = 0, -1, -2, \ldots; \quad j = 1, \ldots, N. \end{aligned} \tag{3.2}$$

Taking $m = -1$, the hierarchy (3.4) gives the NKdV equation with self-consistent sources,

$$\begin{aligned} &v_t = w_x + \alpha\partial_x \sum_{j=1}^N \psi_j^2, \quad w_{xxx} + 4vw_x + 2v_x w = 0, \\ &\psi_{j,xx} + (v+\lambda_j)\psi_j = 0, \quad j = 1, \ldots, N. \end{aligned}$$

Obviously, taking $N = 1$, $m = 0$, $\alpha = 1$, $v \to v + \lambda_1$ in the hierarchy (3.4), we then get NKdV equation (1.2),

$$v_t = \left(\psi_1^2\right)_x, \quad \psi_{1,xx} + v\psi_1 = 0.$$

The Ermakov-Pinney equation is a quite famous example of a nonlinear ordinary differential equation. Such a system has been shown to be relevant to a number of physical contexts, including quantum cosmology, quantum field theory, nonlinear elasticity, and nonlinear optics [51–58].

Theorem 3. (u,v) is a solution of NKdV-1 equation (1.2) if and only if (w,v) with $w = u^2$ is a solution of NKdV equation (1.1) under the transformation

$$u_{xx} + vu = 0,$$

which is actually a linear Schrödinger equation or Hill equation.

Theorem 4. (u,v) is a solution of the NKdV-1 equation (1.2) if and only if (w,v) is a solution of the NKdV equation (1.1) as ϕ is a solution of the Riccati equation

$$\phi_x + \phi^2 + v = 0,$$

while u is the Baker-Akhiezer function

$$u = \exp\left(\int_0^x \phi dx\right), \quad w = u^2.$$

Proposition 4. Suppose that (w,v) is a solution of the NKdV equation (1.1). Let $w = p_t = \psi^2$, $v = p_x$, then ψ satisfies a Ermakov-Pinney equation,

$$\psi_{xx} + v\psi = \frac{\mu}{\psi^3}, \tag{3.3}$$

where μ is an integration constant. Especially, if (u,v) is the solution of the NKdV-1 equation (1.2), let $u = \psi\exp(i\int \mu\psi^{-2}dx)$, then ψ also satisfies the Ermakov-Pinney equation (3.3).

Using the Muira transformation [26]

$$v = -\varphi_{xx} - \varphi_x^2,$$

the NKdV eqation (1.2) can be transformed to the sind-Gordon equation

$$\varphi_{xt} = \sinh\varphi.$$

IV. DARBOUX TRANSFORMATION OF NKDV EQUATIONS

In this section, we shall construct a Darboux transformation for the general NKdV equation (1.1) and then reduce it to the NKdV-1 equation (1.2).

A. Darboux transformation

A Darboux transformation is actually a special gauge transformation

$$\tilde\psi = T\psi \tag{4.1}$$

of solutions of the Lax pair (2.9), where T is a differential operator (for the Lax pair (2.10), the Darboux transformation with $\lambda = 0$ can be obtained). It requires that $\tilde\psi$ also satisfies the same Lax pair (2.9) with some $\tilde L$ and $\tilde M$, i.e.,

$$\tilde L\tilde\psi = \lambda\tilde\psi, \quad \tilde L = TLT^{-1}, \quad \tilde M\tilde\psi = 0, \quad \tilde M = TMT^{-1}. \tag{4.2}$$

Apparently, we have

$$[\tilde L,\tilde M] = T[L,M]T^{-1},$$

which implies that $\tilde L$ and $\tilde M$ are required to have the same forms as L and M, respectively, in order to make system (2.9) invariant under the gauge transformation (3.4). At the same time, the old potentials u and v in L, M will be mapped onto new potentials $\tilde u$ and $\tilde v$ in $\tilde L$, $\tilde M$. This process can be done continually and usually it may yield a series of multisoliton solutions.

Let us now set up a Darboux transformation for the system (2.9). Let $\psi_0 = \psi_0(x,t)$ be a basic solution of Lax pair (2.9) for λ_0, and use it to define the gauge transformation

$$\tilde\psi = T\psi, \tag{4.3}$$

where

$$T = \partial_x - \sigma, \quad \sigma = \partial_x \ln\psi_0. \tag{4.4}$$

From (2.9) and (4.4), one can see that σ satisfies

$$\sigma_x + \sigma^2 + v - \lambda = 0 \tag{4.5}$$

$$4\sigma_{xxt} + 12\sigma_x\sigma_t + 4v\sigma_t + 2w\sigma_x + 6\sigma\sigma_{xt} + 3w_{xx} = 0. \tag{4.6}$$

Proposition 5. The operator $\tilde L$ determined by (4.2) has the same form as L, that is,

$$\tilde L = \partial_x^2 + \tilde v,$$

where the transformation between v and $\tilde v$ is given by

$$\tilde v = v + 2\sigma_x. \tag{4.7}$$

The transformation: $(\psi,v) \to (\tilde\psi,\tilde v)$ is called a Darboux transformation of the first spectral problem of Lax pair (2.9).

Proof. According to (4.2), we just prove

$$\tilde LT = TL,$$

that is,

$$\left(\partial_x^2 + \tilde v\right)(\partial_x - \sigma) = (\partial_x - \sigma)\left(\partial_x^2 + v\right),$$

which is true through (4.5) and (4.7).

Proposition 6. Under the transformation (4.3), the operator $\tilde M$ determined by (4.2) has the same form as M, that is,

$$\tilde M = 4\partial_x^2\partial_t + 4\tilde v\partial_t - 2\tilde w\partial_x - 3\tilde w_x, \tag{4.8}$$

where the transformations between w, v and $\tilde w$, $\tilde v$ are given by

$$\tilde w = w + 2\sigma_t, \quad \tilde v = v + 2\sigma_x. \tag{4.9}$$

The transformation $(\psi,w,v) \to (\tilde\psi,\tilde w,\tilde v)$ is the Darboux transformation of the second spectral problem of Lax pair (2.9).

Proof. To see that $\tilde M$ has the form (4.8) the same as M, we just prove

$$\tilde MT = TM, \tag{4.10}$$

where

$$\tilde M = 4\partial_x^2\partial_t + f\partial_t + g\partial_x + h, \tag{4.11}$$

with three functions f, g, and h to be determined. Substituting $\tilde{M}$, M, L into (4.10) and comparing the coefficients of all distinct operators leads to the following.

Coefficient of operator $\partial_x\partial_t$:

$$f = 4v + 8\sigma_x = 4\tilde{v},$$

which holds by using (4.9).

Coefficient of operator ∂_x^2:

$$g = 2w + 4\sigma_t = 2\tilde{w},$$

which is implied from (4.9).

Coefficient of operator ∂_x:

$$\begin{aligned} h &= 8\sigma_{xt} + 5w_x - 2\sigma w + g\sigma \\ &= 6\sigma_{xt} + 3w_x + 2(\sigma_x + \sigma^2 + v)_t \\ &= 6\sigma_{xt} + 3w_x = 3\tilde{w}_x. \end{aligned}$$

Here we have used equations (4.5) and (4.9).

Coefficient of operator ∂_t:

$$-4\sigma_{xx} - f\sigma = 4v_x - 4v\sigma,$$

that is,

$$\sigma_{xx} + 2\sigma\sigma_x + v_x = 0.$$

which holds by using (4.5).

Coefficient of nonoperator:

$$4\sigma_{xxt} + f\sigma_t + g\sigma_x + \sigma h + 3w_{xx} - 3\sigma w_x = 0,$$

that is,

$$4\sigma_{xxt} + 12\sigma_x\sigma_t + 4v\sigma_t + 2w\sigma_x + 6\sigma\sigma_{xt} + 3w_{xx} = 0,$$

which is Eq. (4.6). We complete the proof. ■

Propositions 4 and 5 tell us that the transformations (4.3) and (4.9) send the Lax pair (2.9) to another Lax pair (4.2) in the same type. Therefore, both Lax pairs lead to the same NKdV equation (1.1). So we call the transformation $(\psi,w,v) \to (\tilde{\psi},\tilde{w},\tilde{v})$ a Darboux transformation of the NKdV equation (1.1). In summary, we arrive at the following theorem.

Theorem 5. A solution w, v of the NKdV equation (1.1) is mapped onto its new solution $\tilde{w}$, $\tilde{v}$ under the Darboux transformations (4.3) and (4.9).

B. Reduction of the Darboux transformation

To get the Darboux transformation for NKdV-1 equation (1.2), we consider two reductions of Darboux transformations (4.3) and (4.9).

Corollary 1. Let $\lambda = k^2 > 0$, then under the constraints $w = u^2$, $v = -u_{xx}/u$, the Darboux transformations (4.3) and (4.9) can be reduced to a Darboux transformation of the NKdV-1 equation (1.2), $(\psi,v,u) \to (\tilde{\psi},\tilde{v},\tilde{u})$, where

$$\tilde{\psi} = T\psi, \quad \tilde{v} = v + 2\sigma_x, \quad \tilde{u} = k^{-1}(u_x - \sigma u) = k^{-1}Tu. \tag{4.12}$$

Proof. For $\lambda > 0$, suppose that (v,u) is a solution of the NKdV-1 equation and ψ is an eigenfunction of Lax pair (1.4), and then we have

$$\lambda^{-1}(u\psi_x - u_x\psi) = \partial_x^{-1}(u\psi).$$

Therefore, the Lax pair (1.4) can be written as

$$\begin{aligned} \psi_{xx} + v\psi &= \lambda\psi, \\ \psi_t &= \tfrac{1}{2}u\lambda^{-1}(u\psi_x - u_x\psi) \\ &= \tfrac{1}{2}u\partial_x^{-1}(u\psi) = N(u,\lambda)\psi, \end{aligned} \tag{4.13}$$

where $N = N(u,\lambda) = \frac{1}{2}u\partial_x^{-1}u$.

According to proposition 6, the first spectral problem of Lax pair (4.13) is covariant under the transformation (4.12), that is,

$$\tilde{\psi}_{xx} + \tilde{v}\tilde{\psi} = \lambda\tilde{\psi}.$$

So we only need to prove that

$$\tilde{\psi}_t = N(\tilde{u},\lambda)\tilde{\psi}. \tag{4.14}$$

Substituting (4.12) into the left-hand side of (4.14) gives

$$\begin{aligned} \tilde{\psi}_t &= (\psi_t)_x - (\sigma\psi)_t = (N\psi)_x - \sigma N\psi - \left(\psi_0^{-1}N\psi_0\right)_x\psi, \\ &= \tfrac{1}{2}\big[(u_x - \sigma u)\partial_x^{-1}(u\psi) - \psi_0^{-1}\psi(u_x - \sigma u)\partial_x^{-1}(u\psi_0)\big], \\ &= \tfrac{1}{2}k\tilde{u}\big[\partial_x^{-1}(u\psi) + k^{-2}(u_x - \sigma u)\psi\big]. \end{aligned} \tag{4.15}$$

In the same way, substituting (4.12) into the right-hand side of (4.14) gives

$$\begin{aligned} N(\tilde{u},\lambda)\tilde{\psi} &= \tfrac{1}{2}\tilde{u}\partial_x^{-1}[k^{-1}(u_x - \sigma u)(\psi_x - \sigma\psi)], \\ &= \tfrac{1}{2}k^{-1}\tilde{u}\big[u_x\psi - \partial_x^{-1}(u_{xx}\psi) - \sigma u\psi \\ &\quad + \partial_x^{-1}\left(\psi_0^{-1}\psi_{0,xx}u\psi\right)\big], \\ &= \tfrac{1}{2}k^{-1}\tilde{u}\big[k^2\partial_x^{-1}(u\psi) + (u_x - \sigma u)\psi\big]. \end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16) implies that (4.14) holds. ■

Corollary 2. Let $\lambda = 0$, then under the constraints $w = u^2$, $v = -u_{xx}/u$, the Darboux transformation (4.3) and (4.9) can be reduced to Darboux transformation of the NKdV-1 equation (1.2): $(\psi,v,u) \to (\tilde{\psi},\tilde{v},\tilde{u})$, in which

$$\begin{aligned} &\tilde{v} = v + 2\sigma_x, \quad \tilde{\psi} = \psi - \psi_0^{-1}\sigma\partial_x^{-1}(\psi_0\psi), \\ &\tilde{u} = \begin{cases} \psi_0^{-1}\sigma, & u = 0, \\ u - \psi_0^{-1}\sigma\partial_x^{-1}(\psi_0 u), & u \neq 0, \end{cases} \end{aligned} \tag{4.17}$$

with $\sigma = \partial_x \ln(1 + \partial_x^{-1}\psi_0^2)$.

V. APPLICATIONS OF THE DARBOUX TRANSFORMATION

In this section, we shall apply the Darboux transformations (4.3) and (4.9) to obtain kink and bell types of explicit solutions for the NKdV equation (1.1).

A. The kink wave solutions

For the case of $\lambda = k^2 > 0$, we substitute $v = 0, w = 1$ into the Lax pair (2.9) and choose the following basic solution:

$$\psi = e^{\xi} + e^{-\xi} = 2\cosh\xi, \quad \xi = kx - \frac{1}{2k}t + \gamma, \tag{5.1}$$

where γ and k are two arbitrary constants.

Taking $\lambda = k_1^2$, (4.4) and (5.1) then lead to

$$\sigma_1 = \partial_x \ln \psi = k_1 \tanh \xi_1, \quad \xi_1 = k_1 x - \frac{1}{2k_1} t + \gamma_1.$$

The Darboux transformation (4.9) gives a bell-type solution for the NKdV equation (1.1),

$$\tilde{v}^I = 2\sigma_{1,x} = 2k_1^2 \mathrm{sech}^2 \xi_1, \quad \tilde{w}^I = 1 - 2\sigma_{1,t} = \tanh^2 \xi_1. \quad (5.2)$$

By using Darboux transformation (4.12), we get a kink-type wave solution for the NKdV equation (1.2),

$$\begin{aligned} \tilde{u}^I &= k_1^{-1}(u_x - \sigma u) = -\tanh \xi_1, \\ \xi_1 &= k_1 x - \frac{1}{2k_1} t + \gamma_1. \end{aligned} \quad (5.3)$$

Remark 1. There is a large difference between the traveling waves of the NKdV equation (1.2) and those of the classical KdV equation. For the NKdV equation (1.2), its one-wave solution is a negative-moving (i.e., from right to left) kink wave with velocity $-1/2k_1^2$, amplitude ± 1, and width $1/k_1$. Its amplitude is independent of velocity, and width is directly proportional to the velocity. For the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad (5.4)$$

the one-soliton solution is

$$u = \frac{k^2}{2} \mathrm{sech}^2 \frac{k(x - k^2 t)}{2}, \quad (5.5)$$

which is a bell-type positive-moving wave with velocity k^2, amplitude $k^2/2$, and width $1/k$, respectively. Its amplitude is directly proportional to velocity, and the width is inversely proportional to the velocity.

Let us now construct two-kink solutions to see the interaction of two-kink solutions. According to (4.4),

$$\tilde{\psi} = T\psi = (\partial_x - \sigma_1)(e^{\xi} + e^{-\xi}) \quad (5.6)$$

is also an eigenfunction of Lax pair (2.9). Taking $\lambda = k_2^2$, we have

$$\sigma_2 = -k_1 \tanh \xi_1 + \frac{k_1^2 - k_2^2}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2}. \quad (5.7)$$

Repeating the Darboux transformation (4.9) one more time, we get a two-soliton solution for the NKdV equation (1.1)

$$\begin{aligned} \tilde{v}^{II} &= \tilde{v}^I + 2\sigma_{2,x} = \frac{(k_1^2 - k_2^2)(k_2^2 \mathrm{sech}^2 \xi_2 - k_1^2 \mathrm{sech}^2 \xi_1)}{(k_1 \tanh \xi_1 - k_2 \tanh \xi_2)^2}, \\ \tilde{w}^{II} &= \tilde{w}^I - 2\sigma_{2,t} = \left(\frac{k_1 \tanh \xi_2 - k_2 \tanh \xi_1}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2} \right)^2. \end{aligned}$$

Therefore, we obtain a two-kink wave solution of the NKdV equation (1.2),

$$\tilde{\tilde{u}} = \frac{k_2 \tanh \xi_1 - k_1 \tanh \xi_2}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2}. \quad (5.8)$$

Let us use the two-kink wave solution (5.8) to analyze the interaction of the two one-soliton solutions (Fig. 1). Without loss of generality, we suppose $k_1 > k_2 > 0$, and then we have

$$\xi_2 = \frac{k_2}{k_1} \left[\xi_1 - \frac{k_1}{2} \left(\frac{1}{k_2^2} - \frac{1}{k_1^2} \right) t \right].$$

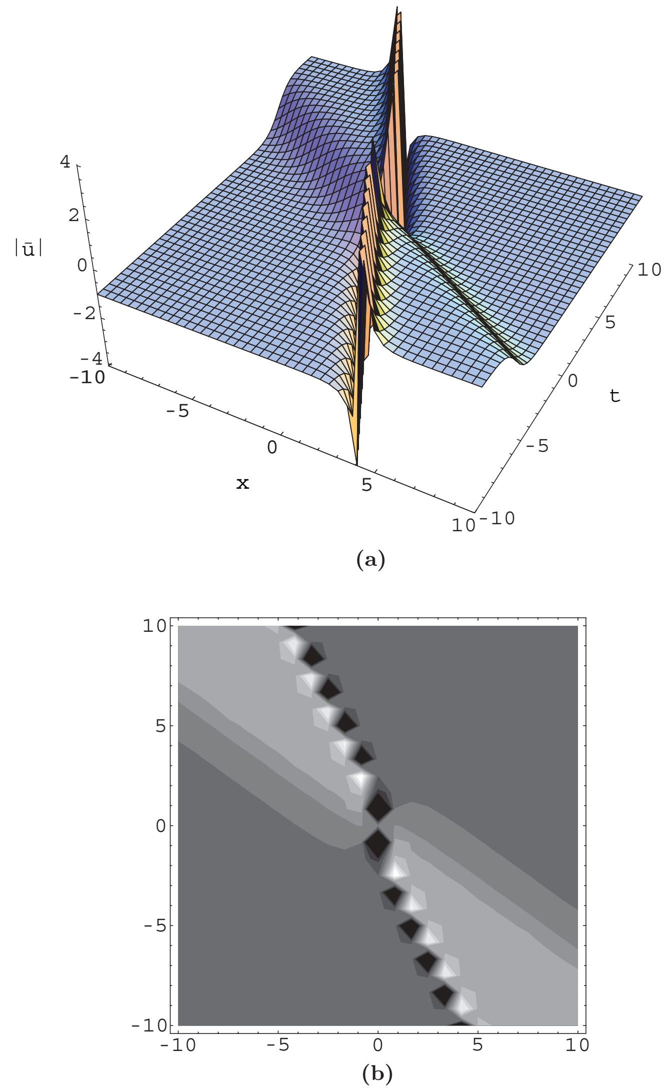

FIG. 1. (Color online) The two-kink wave solution $u(x,t)$ with parameters $k_1 = 1$, $k_2 = 0.6$. (a) Perspective view of the wave. (b) Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs.

Therefore, on the fixed line $\xi_1 = \mathrm{const}$, we get

$$\tanh \xi_2 \sim -1, \quad t \to +\infty,$$

and it follows (5.8) that

$$\begin{aligned} \tilde{\tilde{u}} &\sim \frac{k_2 \tanh \xi_1 + k_1}{k_1 \tanh \xi_1 + k_2} \\ &= \coth \left(\xi_1 - \frac{1}{2} \ln \frac{k_1 - k_2}{k_1 + k_2} \right), \quad t \to +\infty. \end{aligned} \quad (5.9)$$

In a similar way, one can get

$$\tanh \xi_2 \sim 1 \ \text{ as } t \to -\infty,$$

which are the main parts compared with terms 1 and $e^{2\xi_1}$, and it follows (3.19) that

$$\tilde{\tilde{u}} \sim \frac{k_2 e^{2\xi_1} - k_1}{k_1 e^{2\xi_1} - k_2} = \coth \left(\xi_1 + \frac{1}{2} \ln \frac{k_1 - k_2}{k_1 + k_2} \right), \quad t \to -\infty. \quad (5.10)$$

In a similar way, on the line $\xi_2 = \text{const}$, we will arrive at

$$\tilde{\tilde{u}} \sim \tanh\left(\xi_2 + \frac{1}{2}\ln\frac{k_1 - k_2}{k_1 + k_2}\right), \quad \text{as } t \to +\infty, \quad (5.11)$$

$$\tilde{\tilde{u}} \sim \tanh\left(\xi_2 - \frac{1}{2}\ln\frac{k_1 - k_2}{k_1 + k_2}\right), \quad \text{as } t \to -\infty. \quad (5.12)$$

Remark 2. From expressions (5.9) to (5.12), we see that the two-kink wave solution (5.8) is a singular solution, which is able to be decomposed into a singular kink-type solution and a smooth kink wave solutions. The expressions (5.10) and (5.12) show that the wave $\tanh\xi_2$ is on the left of the wave $\coth\xi_1$ before their interaction, while the expressions (5.9) and (5.11) show that the wave $\coth\xi_1$ is on the left of the wave $\tanh\xi_2$ after their interaction. The shapes of the two kink waves $\coth\xi_1$ and $\tanh\xi_2$ do not change except their phases. Their phases of the two waves $\coth\xi_1$ and $\tanh\xi_2$ are $\ln\frac{k_1-k_2}{k_1+k_2} > 0$ and $-\ln\frac{k_1-k_2}{k_1+k_2} < 0$, respectively, as the wave is negatively going along the x axis. A very interesting case is particular at $t = 0$: Collision of such two kink waves forms a smooth bell-type soliton and its singularity disappears (see Fig. 2).

After their interaction, it can be seen that the two kink waves resume their original shapes. At the right moment of interaction, the two kink waves are fused into a smooth bell-type soliton. The two-kink wave interactions possess the regular elastic-collision features and pass through each other, and their shapes keep unchanged with a phase shift after the interaction. Here, we also demonstrate a fact that the large-amplitude kink wave with faster velocity overtakes the small-amplitude one and, after collision, the smaller one is left behind.

B. The bell-type soliton solutions

(i) For the case of $\lambda = 0$ (i.e., without parameter λ), we substitute $v = -k^2$, $w = 0$ into the Lax pair (2.10), and choose the following basic solution as

$$\psi = e^{\xi} + e^{-\xi}, \quad \xi = kx + \frac{1}{2k}t,$$

where k is an arbitrary constant.

Taking $k = k_1$, (4.4) gives

$$\sigma = \sigma_1 = \partial_x \ln\psi = k_1\tanh\xi_1, \quad \xi_1 = k_1 x + \frac{1}{2k_1}t. \quad (5.13)$$

Using the Darboux transformation (4.9), we have a one-soliton solution for the NKdV equation (1.1),

$$\tilde{v} = v + 2\sigma_{1,x} = 2k_1^2\text{sech}^2\xi_1 - k_1^2, \quad \tilde{w} = -2\sigma_{1,t} = \text{sech}^2\xi_1. \quad (5.14)$$

So we get a one-soliton solution for the NKdV-1 equation (1.2) by using Darboux transformation (4.17)

$$\tilde{u} = \text{sech}\xi_1, \quad \xi_1 = k_1 x + \frac{1}{2k_1}t. \quad (5.15)$$

Remark 3. For the negative-order KdV equation (1.2), its one-soiton solution (5.15) is a smooth bell-type negative-moving wave, whose velocity, amplitude, and width are $1/2k_1^2$, ±1, and $1/k_1$, respectively. Its amplitude is independent of velocity, and the width is directly proportional to the velocity.

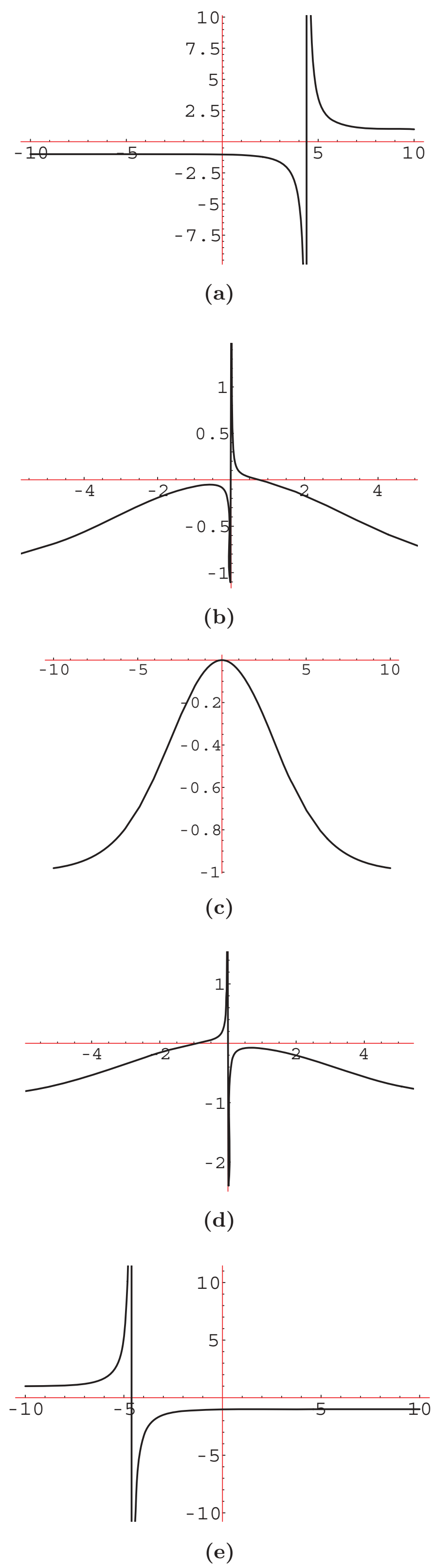

FIG. 2. (Color online) Interaction between singular soliton $\coth\xi_1$ and smooth soliton $\tanh\xi_2$ with the following parameters: (a) $t = -3$, (b) $t = -0.05$, (c) $t = 0$, (d) $t = 0.05$, and (e) $t = 3$.

(ii) For the case of $\lambda = -k^2$, we take a seed solution of $v = -2k^2, w = 1$ in the Lax pair (2.9) and choose the following basic solution as

$$\psi = e^{\xi} + e^{-\xi}, \quad \xi = kx - \frac{1}{2k}t + \gamma,$$

where k is an arbitrary constant.

Taking $k = k_1$ sends (5.1) to

$$\sigma = \sigma_1 = \partial_x \ln \psi = k_1 \tanh \xi_1, \quad \xi_1 = k_1 x - \frac{1}{2k_1}t + \gamma_1. \tag{5.16}$$

Using the Darboux transformation (3.12), we then get the one-soliton solution

$$\begin{aligned} \tilde{v}^I &= v + 2\sigma_{1,x} = -2k_1^2 \tanh^2 \xi_1 + \gamma_1, \\ \tilde{w}^I &= 1 - 2\sigma_{1,t} = 1 + \operatorname{sech}^2 \xi_1, \end{aligned} \tag{5.17}$$

which cannot satisfies the constraint (3.3), so $\sqrt{\tilde{w}^I}$ is not soliton for the NKdV equation (1.2).

Remark 4. For the NKdV equation (1.1), its one-soiton solution (5.14) is a smooth bell-type positive-moving wave, whose velocity, amplitude, and width are $1/2k_1^2, \pm 1$, and $1/k_1$, respectively. Its amplitude is independent of velocity, and the width is directly proportional to the velocity.

Let us construct a two-soliton solution of the NKdV equation (1.1). According to the gauge transformation (4.4),

$$\tilde{\psi} = T\psi = (\partial_x - \sigma_1)(e^{\xi} + e^{-\xi})$$

is also an eigenfunction of Lax (2.9). We have

$$\sigma_2 = -k_1 \tanh \xi_1 + \frac{k_1^2 - k_2^2}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2}.$$

Repeating the Darboux transformation (4.9) one more time, we obtain

$$\begin{aligned} \tilde{v}^{II} &= \tilde{v}^I + 2\sigma_{2,x} = \frac{\left(k_1^2 - k_2^2\right)\left(k_2^2 \operatorname{sech}^2 \xi_2 - k_1^2 \operatorname{sech}^2 \xi_1\right)}{(k_1 \tanh \xi_1 - k_2 \tanh \xi_2)^2}, \\ \tilde{w}^{II} &= \tilde{w}^I - 2\sigma_{2,t} = \left(\frac{k_1 \tanh \xi_2 - k_2 \tanh \xi_1}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2}\right)^2, \end{aligned}$$

which is the same for NKdV-1 equation (1.2). So we get two-soliton solution with (5.8)

$$\tilde{\tilde{u}} = \pm \frac{k_1 \tanh \xi_2 - k_2 \tanh \xi_1}{k_1 \tanh \xi_1 - k_2 \tanh \xi_2},$$

but here $\xi_j = k_j x - \frac{1}{2k_j}t$, $j = 1,2$.

VI. BILINEARIZATION OF THE NKDV EQUATION

The bilinear derivative method, developed by Hirota [9], has become a powerful approach to construct exact solutions of nonlinear equations. Once a nonlinear equation is written in a bilinear form by using some transformation, then multisolitary wave solutions or quasiperiodic wave solutions usually can be obtained [59–63]. However, unfortunately, this method is not as direct as many people might wish because the original equation is reduced to two or more bilinear equations under new variables called Hirota variables. Since there is no general rule to select Hirota variables, there is no rule to choose some essential formulas (such as exchange formulas) either. The construction of a bilinear Bäcklund transformation especially relies on a particular skill and appropriate exchange formulas. On the other hand, in recent years, Lambert and his coworkers have found a type of generalized Bell polynomial that plays an important role in seeking the characterization of bilinearized equations. Based on the Bell polynomials, they presented an alternative procedure to obtain parameter families of a bilinear Bäcklund transformation and Lax pairs for soliton equations in a quick and short way [64–66].

A. Multidimensional binary Bell polynomials

The main tool we use here is a class of generalized multidimensional binary Bell polynomials.

Definition 1. Let $n_k \geqslant 0$, $k = 1,\ldots,\ell$ denote arbitrary integers, $f = f(x_1,\ldots,x_\ell)$ be a C^∞ multivariable function, and then

$$Y_{n_1 x_1,\ldots,n_\ell x_\ell}(f) \equiv \exp(-f)\partial_{x_1}^{n_1} \ldots \partial_{x_\ell}^{n_\ell} \exp(f) \tag{6.1}$$

is a polynomial in the partial derivatives of f with respect to $x_1,\ldots,x_\ell$, which we call a multidimensional Bell polynomial (a generalized Bell polynomial or Y polynomial).

For the two-dimensional case, let $f = f(x,t)$, and then the associated Bell polynomials through (6.1) can produce the following representatives:

$$\begin{aligned} &Y_x(f) = f_x, \quad Y_{2x}(f) = f_{2x} + f_x^2, \\ &Y_{3x}(f) = f_{3x} + 3f_x f_{2x} + f_x^3, \\ &Y_{x,t}(f) = f_{x,t} + f_x f_t, \\ &Y_{2x,t}(f) = f_{2x,t} + f_{2x} f_t + 2f_{x,t} f_x + f_x^2 f_t, \\ &\ldots\ldots \end{aligned}$$

Definition 2. Based on the use of the above Bell polynomials (6.1), the multidimensional binary Bell polynomials ($\mathcal{Y}$ polynomials) are defined as follows:

$$\mathcal{Y}_{n_1 x_1,\ldots,n_\ell x_\ell}(g,h) = Y_{n_1 x_1,\ldots,n_\ell x_\ell}(f) \,\Big|_{f_{r_1 x_1,\ldots,r_\ell x_\ell} = \begin{cases} g_{r_1 x_1,\ldots,r_\ell x_\ell}, & r_1 + \cdots + r_\ell \text{ is odd}, \\ h_{r_1 x_1,\ldots,r_\ell x_\ell}, & r_1 + \cdots + r_\ell \text{ is even}, \end{cases}}$$

which is a multivariable polynomial with respect to all partial derivatives $g_{r_1 x_1,\ldots,r_\ell x_\ell}$ ($r_1 + \cdots + r_\ell$ is odd) and $h_{r_1 x_1,\ldots,r_\ell x_\ell}$ ($r_1 + \cdots + r_\ell$ is even), $r_k = 0,\ldots,n_k$, $k = 0,\ldots,\ell$.

The binary Bell polynomials also inherit the easily recognizable partial structures. The first few lower-order binary Bell polynomials are

$$\begin{aligned} &\mathcal{Y}_x(g) = g_x, \quad \mathcal{Y}_{2x}(g,h) = h_{2x} + g_x^2, \\ &\mathcal{Y}_{x,t}(g,h) = h_{xt} + g_x g_t, \\ &\mathcal{Y}_{3x}(g,h) = g_{3x} + 3g_x h_{2x} + g_x^3, \ldots. \end{aligned} \tag{6.2}$$

Proposition 7. The link between binary Bell polynomials $\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(g,h)$ and the standard Hirota bilinear expression $D_{x_1}^{n_1}\ldots D_{x_\ell}^{n_\ell}FG$ can be given by an identity

$$\begin{aligned}&\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(g=\ln F/G,h=\ln FG)\\&\quad=(FG)^{-1}D_{x_1}^{n_1}\ldots D_{x_\ell}^{n_\ell}FG,\end{aligned}\tag{6.3}$$

in which $n_1+n_2+\cdots+n_\ell\geqslant 1$ and operators $D_{x_1},\ldots,D_{x_\ell}$ are classical Hirota bilinear operators defined by

$$\begin{aligned}D_{x_1}^{n_1}\ldots D_{x_\ell}^{n_\ell}FG&=\left(\partial_{x_1}-\partial_{x_1'}\right)^{n_1}\ldots\left(\partial_{x_\ell}-\partial_{x_\ell'}\right)^{n_\ell}F(x_1,\ldots,x_\ell)\\&\quad\times G(x_1',\ldots,x_\ell')|_{x_1'=x_1,\ldots,x_\ell'=x_\ell}.\end{aligned}$$

In the special case of $F=G$, the formula (6.4) becomes

$$\begin{aligned}&F^{-2}D_{x_1}^{n_1}\ldots D_{x_\ell}^{n_\ell}GG\\&\quad=\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(0,q=2\ln G)\\&\quad=\begin{cases}0, & n_1+\cdots+n_\ell \text{ is odd,}\\ P_{n_1x_1,\ldots,n_\ell x_\ell}(q), & n_1+\cdots+n_\ell \text{ is even.}\end{cases}\end{aligned}\tag{6.4}$$

The first few P polynomial are

$$\begin{aligned}&P_{2x}(q)=q_{2x},\quad P_{x,t}(q)=q_{xt},\quad P_{4x}(q)=q_{4x}+3q_{2x}^2,\\&P_{6x}(q)=q_{6x}+15q_{2x}q_{4x}+15q_{2x}^3,\ldots.\end{aligned}\tag{6.5}$$

The formulas (6.4) and (6.5) will prove particularly useful in connecting nonlinear equations to their corresponding bilinear forms. This means that if a nonlinear equation is expressible by a linear combination of P polynomials, then the nonlinear equation can be transformed into a linear equation.

Proposition 8. The binary Bell polynomials $\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(v,w)$ can be separated into P polynomials and Y polynomials

$$\begin{aligned}&(FG)^{-1}D_{x_1}^{n_1}D_{x_\ell}^{n_\ell}F\cdot G\\&\quad=\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(g,h)|_{g=\ln F/G,h=\ln FG}\\&\quad=\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(g,g+q,)|_{g=\ln F/G,q=2\ln G}\\&\quad=\sum_{n_1+\cdots+n_\ell=\text{even}}\sum_{r_1=0}^{n_1}\cdots\sum_{r_\ell=0}^{n_\ell}\prod_{i=1}^{\ell}\binom{n_i}{r_i}\\&\qquad\times P_{r_1x_1,\ldots,r_\ell x_\ell}(q)Y_{(n_1-r_1)x_1,\ldots,(n_\ell-r_\ell)x_\ell}(v).\end{aligned}\tag{6.6}$$

The key property of the multidimensional Bell polynomials

$$Y_{n_1x_1,\ldots,n_\ell x_\ell}(g)|_{g=\ln\psi}=\psi_{n_1x_1,\ldots,n_\ell x_\ell}/\psi\tag{6.7}$$

implies that the binary Bell polynomials $\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(g,h)$ still can be linearized by means of the Hopf-Cole transformation $g=\ln\psi$, that is, $\psi=F/G$. The formulas (6.6) and (6.7) will then provide the shortest way to the associated Lax system of nonlinear equations.

B. Bilinearization

Theorem 6. Under the transformation

$$v=v_0+2(\ln G)_{2x},\quad w=w_0+2(\ln G)_{xt},$$

the NKdV equation (1.1) can be bilinearized into

$$\begin{aligned}&\left(D_x^4+12v_0D_x^2-D_xD_y\right)GG=0,\\&\left(2D_tD_x^3+6w_0D_x^2+D_tD_y\right)GG=0,\end{aligned}\tag{6.8}$$

where y is an auxiliary variable and u_0, v_0 are two constant solutions of the NKdV equation (1.1).

Proof. The invariance of the NKdV equation (1.1) under the scale transformation

$$x\to\lambda x,\quad t\to\lambda^\alpha t,\quad v\to\lambda^{-2}v,\quad w\to\lambda^{-\alpha-1}w$$

shows that the dimensions of the fields v and w are -2 and $-(\alpha+1)$, respectively. So we may introduce a dimensionless potential field q by setting

$$v=v_0+q_{2x},\quad w=w_0-q_{xt}.\tag{6.9}$$

Substituting the transformation (6.9) into Eq. (1.1), we can write the resulting equation in the following form:

$$q_{4x,t}+4q_{2x}q_{2x,t}+2q_{3x}q_{xt}+4v_0q_{2x,t}+2w_0q_{3x}=0,$$

which is regrouped as follows:

$$\begin{aligned}&\tfrac{2}{3}q_{4x,t}+2(q_{2x}q_{2x,t}+q_{xt}q_{3x})+\tfrac{1}{3}q_{4x,t}\\&\quad+2q_{2x}q_{2x,t}+4v_0q_{2x,t}+2w_0q_{3x}=0,\end{aligned}\tag{6.10}$$

where we will see that such an expression is necessary to get a bilinear form of the equation (1.1). Further integrating the equation (6.10) with respect to x yields

$$\begin{aligned}E(q)\equiv{}&\tfrac{2}{3}(q_{3x,t}+3q_{2x}q_{xt}+3w_0q_{2x})\\&+\tfrac{1}{3}\partial_x^{-1}\partial_t\left(q_{4x}+3q_{2x}^2+12v_0q_{2x}\right)=0.\end{aligned}\tag{6.11}$$

In order to write Eq. (6.11) in a local bilinear form, let us, first, get rid of the integral operator ∂_x^{-1}. To do so, we introduce an auxiliary variable y and impose a subsidiary constraint condition

$$q_{4x}+3q_{2x}^2+12v_0q_{2x}-q_{xy}=0.\tag{6.12}$$

Equation (6.10) then becomes

$$2(q_{3x,t}+3q_{2x}q_{xt}+3w_0q_{2x})+q_{yt}=0.\tag{6.13}$$

According to the formula (6.5), Eqs. (6.12) and (6.13) are then cast into a pair of equations in the form of P polynomials,

$$\begin{aligned}&P_{4x}(q)+12v_0P_{2x}(q)-P_{xy}(q)=0,\\&2P_{3x,t}(q)+6w_0P_{2x}(q)+P_{yt}(q)+3\gamma=0.\end{aligned}$$

Finally, by the property (6.4), making the following variable

$$q=2\ln G\Longleftrightarrow v=v_0+2(\ln G)_{2x},\quad w=w_0+2(\ln G)_{xt},$$

we change the above system to the following bilinear forms of the NKdV equation (1.1):

$$\begin{aligned}&\left(D_x^4+12v_0D_x^2-D_xD_y\right)GG=0,\\&\left(2D_tD_x^3+6w_0D_x^2+D_tD_y\right)GG=0,\end{aligned}\tag{6.14}$$

which is also simultaneously the bilinear system in y. This system is easily solved with multisoliton solutions by using the Hirota bilinear method. ■

Finally, we show that the NKdV-1 equation (1.1) can be directly bilinearized through a transformation, not Bell polynomials. Making a dependent-variable transformation,

$$v=v_0+2(\ln F)_{xx},\quad u=G/F,\tag{6.15}$$

we can change Eq. (1.2) into

$$\begin{aligned}&2(F_{xt}-F_xF_t)=G^2,\\&F_{xx}G-2F_xG_x+G_{xx}F+v_0FG=0,\end{aligned}$$

which is equivalent to the bilinear form

$$D_x D_t F F = G^2, \quad \left(D_x^2 + v_0\right) F G = 0. \tag{6.16}$$

It is obvious that the bilinear form of the NKdV-1 (6.16) is more simple than the bilinear form of NKdV (6.15).

C. N-soliton solutions

In the same procedure as the normal perturbation method, let us expand G in the power series of a small parameter ε as follows:

$$G = 1 + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)} + \varepsilon^3 g^{(3)} + \cdots .$$

Substituting the above equation into (6.8) and arranging each order of ε, we have

$$\begin{aligned} &\varepsilon : \left(D_x^4 + 12 v_0 D_x^2 - D_x D_y\right) g_1 1 = 0, \\ &\left(2 D_t D_x^3 + 6 w_0 D_x^2 + D_t D_y\right) g^{(1)} 1 = 0, \end{aligned} \tag{6.17}$$

$$\begin{aligned} &\varepsilon^2 : \left(D_x^4 + 12 v_0 D_x^2 - D_x D_y\right)(2 g^{(2)} 1 + g^{(1)} g^{(1)}) = 0, \\ &\left(2 D_t D_x^3 + 6 w_0 D_x^2 + D_t D_y\right)(2 g^{(2)} 1 + g^{(1)} g^{(1)}) = 0, \end{aligned} \tag{6.18}$$

$$\begin{aligned} &\varepsilon^3 : \left(D_x^4 + 12 v_0 D_x^2 - D_x D_y\right)(g^{(3)} 1 + g^{(1)} g^{(2)}) = 0, \\ &\left(2 D_t D_x^3 + 6 w_0 D_x^2 + D_t D_y\right)(g^{(3)} 1 + g^{(1)} g^{(2)}) = 0, \end{aligned} \tag{6.19}$$

$$\cdots\cdots$$

By employing the formulas mentioned above, the system (6.17) is equivalent to the following linear system:

$$\begin{aligned} &g^{(1)}_{xxxx} + 12 v_0 g^{(1)}_{xx} - g^{(1)}_{xy} = 0, \\ &2 g^{(1)}_{xxxt} + 6 w_0 g^{(1)}_{xx} + g^{(1)}_{yt} = 0, \end{aligned}$$

which has solution

$$g^{(1)} = e^{\xi}, \quad \xi = kx - \frac{2 k w_0}{k^2 + 4 v_0} t + (k^3 + 12 v_0 k) y + \sigma, \tag{6.20}$$

where k and σ are two arbitrary parameters.

Substituting (6.12) into (6.10) and (6.11) and choosing $g^{(2)} = g^{(3)} = \cdots = 0$, the G expansion then is truncated with a finite sum as

$$G = 1 + e^{\xi},$$

which gives regular one-soliton solution of the NKdV equation (1.1),

$$\begin{aligned} v &= v_0 + 2\partial_x^2 \ln(1 + e^{\xi}) = v_0 + \frac{k^2}{2} \operatorname{sech}^2 \xi/2, \\ w &= w_0 + 2\partial_t \partial_x \ln(1 + e^{\xi}) \\ &= w_0 + \frac{k^2 w_0}{k^2 + 4 v_0} \operatorname{sech}^2 \xi/2, \\ \xi &= kx - \frac{2 k w_0}{k^2 + 4 v_0} t + \gamma, \end{aligned} \tag{6.21}$$

where $\gamma = (k^3 + 12 v_0 k) y + \sigma$ and k, v_0, w_0 are constants.

Let $w_0 = 1$, $v_0 = 0$, and then the solution (6.21) reads as a kink-type solution of the NKdV-I equation (1.2),

$$u = \pm \tanh \xi/2, \quad \xi = kx - \frac{2}{k} t + \gamma.$$

In a similar way, taking

$$g^{(1)} = e^{\xi_1} + e^{\xi_2}, \quad \xi_j = k_j x - \frac{2 k_j w_0}{k_j^2 + 4 v_0} t + \gamma_j, \quad j = 1,2,$$

we get a two-soliton wave solution,

$$\begin{aligned} v &= v_0 + 2\partial_x^2 \ln(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}}) \\ w &= w_0 - 2\partial_t \partial_x \ln(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}}), \\ A_{12} &= \ln\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 . \end{aligned} \tag{6.22}$$

In general, we can get a N-soliton solution of the NKdV equation (1.1)

$$v = v_0 + 2\partial_x^2 \ln\left(\sum_{\mu_j = 0,1} \exp \sum_{j=1}^{N} \mu_j \xi_j + \sum_{1 \leqslant j \leqslant N}^{N} \mu_j \mu_l A_{jl}\right),$$

$$w = w_0 - \partial_t \partial_x \ln\left(\sum_{\mu_j = 0,1} \exp \sum_{j=1}^{N} \mu_j \xi_j + \sum_{1 \leqslant j \leqslant N}^{N} \mu_j \mu_l A_{jl}\right),$$

$$A_{jl} = \ln\left(\frac{k_j - k_l}{k_j + k_l}\right)^2 ,$$

where the notation $\sum_{\mu_j = 0,1}$ represents all possible combinations $\mu_j = 0,1$, and $\xi_j = k_j x - \frac{2 k_j w_0}{k_j^2 + 4 v_0} t + \gamma_j$, $j = 1,2,\ldots,N$.

In the following, we discuss the soliton solutions for the NKdV-1 equation by using the bilinear equation (6.16). Let us expand F and G in the power series of a small parameter ε as follows:

$$\begin{aligned} F &= 1 + f^{(2)} \varepsilon^2 + f^{(4)} \varepsilon^4 + f^{(6)} \varepsilon^6 + \cdots \\ G &= g^{(1)} \varepsilon + g^{(3)} \varepsilon^3 + g^{(5)} \varepsilon^5 + \cdots . \end{aligned}$$

Substituting the above equation into (6.16) and arranging each order of ε, we have

$$\begin{aligned} &g^{(1)}_{xx} + v_0 g^{(1)} = 0, \\ &g^{(3)}_{xx} + v_0 g^{(3)} = -\left(D_x^2 + v_0\right) f^{(2)} g^{(1)}, \\ &g^{(5)}_{xx} + v_0 g^{(5)} = -\left(D_x^2 + v_0\right)(f^{(2)} g^{(3)} + f^{(4)} g^{(1)}), \\ &\cdots\cdots \end{aligned} \tag{6.23}$$

$$\begin{aligned} 2 f^{(2)}_{xt} &= (g^{(1)})^2, \\ 2 f^{(4)}_{xt} &= 2 g^{(1)} g^{(3)} - D_x D_t f^{(2)} f^{(2)}, \\ 2 f^{(6)}_{xt} &= 2 g^{(1)} g^{(5)} + 2 (g^{(3)})^2 - 2 D_x D_t f^{(3)} f^{(3)}, \\ &\cdots\cdots \end{aligned} \tag{6.24}$$

Let $v_0 = -k^2$. It follows from the first equation of (6.23) and (6.24) that

$$g^{(1)} = e^{\xi}, \quad f^{(2)} = \frac{1}{4} e^{2\xi}, \quad \xi = kx + \frac{1}{2k} t + \gamma. \tag{6.25}$$

Substituting (6.25) into the second equation of (6.23) leads to

$$g^{(3)}_{xx} - k^2 g^{(3)} = 0,$$

from which we may take $g^{(3)} = 0$ and further choose $g^{(5)} = \cdots = 0$, $f^{(4)} = \cdots = 0$. So F and G are truncated with a finite sum as

$$F = 1 + \tfrac{1}{4}e^{2\xi}, \quad G = e^{\xi}.$$

Finally, the formula (6.14) gives one-soliton solution of the NKdV-1 equation (1.2),

$$v = 2k^2 \mathrm{sech}^2\xi - k^2, \quad u = \mathrm{sech}\xi.$$

VII. BILINEAR BÄCKLUND TRANSFORMATION

In this section, we search for the bilinear Bäcklund transformation and Lax pair of the NKdV equation (1.1).

A. Bilinear Bäcklund transformation

Theorem 7. Suppose that F is a solution of the bilinear equation (6.8), and if G satisfies

$$\begin{aligned} &\left(D_x^2 - \lambda\right)FG = 0, \\ &\left[D_t D_x^2 + 2w_0 D_x + (4v_0 + 3\lambda)D_t\right]FG = 0, \end{aligned} \tag{7.1}$$

then G is another solution of Eq. (6.8).

Proof. Let

$$q = 2\ln G, \quad \tilde{q} = 2\ln F$$

be two different solutions of Eq. (6.10). Introducing two new variables

$$h = (\tilde{q} + q)/2 = \ln(FG), \quad g = (\tilde{q} - q)/2 = \ln(F/G),$$

makes the function E invariant under the two fields $\tilde{q}$ and q,

$$\begin{aligned} E(\tilde{q}) - E(q) &= E(h+g) - E(h-g) \\ &= 8v_0 g_{xt} + 4w_0 g_{2x} + 2g_{3x,t} + 4h_{2x}g_{x,t} \\ &\quad + 4h_{x,t}g_{2x} + 4\partial_x^{-1}(h_{2x}g_{2x,t} + h_{2x,t}g_{2x}) \\ &= 2\partial_x(\mathcal{Y}_{2x,t}(g,h) + 4v_0\mathcal{Y}_t(g) - 2w_0\mathcal{Y}_x(g)) \\ &\quad + R(g,h) = 0, \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} R(g,h) = &-2\partial_x\left[\left(h_{2x} + g_x^2\right)g_t\right] + 4h_{2x}g_{xt} - 4h_{2x,t}g_x \\ &+ 4\partial_x^{-1}(h_{2x}g_{2x,t} + h_{2x,t}g_{2x}). \end{aligned}$$

This two-field invariant condition can be regarded as a natural ansatz for a bilinear Bäcklund transformation and may produce some required transformations under additional appropriate constraints.

In order to decouple the two-field condition (7.2), let us impose a constraint to express $R(g,h)$ in the form of the x derivative of $\mathcal{Y}$ polynomials. The simple possible choice of the constraint may be

$$\mathcal{Y}_{2x}(g,h) = h_{2x} + g_x^2 = \lambda, \tag{7.3}$$

which directly leads to

$$R(g,h) = 2\lambda g_{xt} + 4h_{2x}g_{xt} - 4h_{2x,t}g_x - 4g_x^2 g_{xt} = 6\lambda g_{xt}, \tag{7.4}$$

where $h_{2x,t} = -2g_x g_{xt}$ and $h_{2x} = \lambda - g_x^2$ are used.

Using the relations (7.2)–(7.4), we derived a coupled system of $\mathcal{Y}$ polynomials

$$\begin{aligned} &\mathcal{Y}_{2x}(g,h) - \lambda = 0, \\ &\mathcal{Y}_{2x,t}(g,h) + (4v_0 + 3\lambda)\mathcal{Y}_t(g) + 2w_0\mathcal{Y}_x(g) = 0, \end{aligned} \tag{7.5}$$

where we prefer the second equation to be expressed in the form of conserved quantity without integration with respect to x. This is very useful to construct conservation laws. Apparently, the identity (6.2) directly sends the system (7.5) to the following bilinear Bäcklund transformation

$$\begin{aligned} &\left(D_x^2 - \lambda\right)FG = 0, \\ &\left[D_t D_x^2 + 2w_0 D_x + (4v_0 + 3\lambda)D_t\right]FG = 0, \end{aligned} \tag{7.6}$$

where we have integrated the second equation in the system (7.5) with respect to x, and w_0 is the corresponding integration constant. ■

B. Inverse-scattering formulation

Theorem 8. The NKdV equation (1.1) admits a Lax pair

$$\begin{aligned} &\psi_{2x} + v\psi = \lambda\psi, \\ &4\psi_{2x,t} + 4v\psi_t - 2w\psi_x - 3w_x\psi = 0. \end{aligned} \tag{7.7}$$

Proof. By the transformation $v = \ln\psi$, it follows from the formulas (6.5) and (6.6) that

$$\begin{aligned} &\mathcal{Y}_t(g) = \psi_t/\psi, \quad \mathcal{Y}_x(g) = \psi_x/\psi, \\ &\mathcal{Y}_{2x}(g,h) = q_{2x} + \psi_{2x}/\psi, \\ &\mathcal{Y}_{2x,t}(g,h) = 2q_{xt}\psi_x/\psi + q_{2x}\psi_t/\psi + \psi_{2x,t}/\psi, \end{aligned}$$

which makes the system (7.5) linearized into a Lax pair with parameter λ,

$$L\psi \equiv \left(\partial_x^2 + q_{2x}\right)\psi = \lambda\psi, \tag{7.8}$$

$$M\psi \equiv \left[\partial_t\partial_x^2 + (4v_0 + q_{2x})\partial_t + 2(q_{xt} + w_0)\partial_x + 3\lambda\partial_t\right]\psi, \tag{7.9}$$

or, equivalently,

$$\psi_{2x} + v\psi = \lambda\psi, \quad 4\psi_{2x,t} + 4v\psi_t - 2w\psi_x - 3w_x\psi = 0,$$

where Eq. (7.8) is used to get the second equation. One can easily verify from Eqs. (7.8) and (7.9) that the integrability condition

$$[L,M] = q_{4x,t} + 4(v_0 + q_{2x})q_{2x,t} + 2q_{3x}(q_{xt} + w_0) = 0$$

exactly gives the NKdV equation (1.1) through replacing $v_0 + q_{2x}$ and $w_0 + q_{xt}$ with v and w, respectively. ■

VIII. DARBOUX COVARIANT LAX PAIR

In this section, we will give a kind of Darboux covariant Lax pair, whose form is invariant under the gauge transformation (4.3).

Theorem 9. The NKdV equation (1.1) possesses the following Darboux covariant Lax pair:

$$L\psi = \lambda\psi, \quad M_{\mathrm{cov}}\psi = 0, \quad M_{\mathrm{cov}} = M + 3\partial_x L,$$

under the gauge transformation $\tilde{\psi} = T\psi$. This is actually equivalent to the Lax pair (2.9).

Proof. In Sec. IV, we have shown that the gauge transformation (4.1) maps the operator $L(q)$ onto a similar operator

$$\tilde{L}(\tilde{q}) = TL(q)T^{-1},$$

which satisfies the following covariance condition:

$$\tilde{L}(\tilde{q}) = L(q+\Delta q), \quad \tilde{q} = q + \Delta q, \quad \text{with } \Delta q = 2\ln\phi.$$

We next want to find a third-order operator $M_{\rm cov}(q)$ with appropriate coefficients, such that $M_{\rm cov}(q)$ is mapped by gauge transformation (4.3) onto a similar operator $\tilde{M}_{\rm cov}(\tilde{q})$, which satisfies the covariance condition

$$\tilde{M}_{\rm cov}(\tilde{q}) = M_{\rm cov}(q+\Delta q), \quad \tilde{q} = q+\Delta q.$$

Suppose that ϕ is a solution of the following Lax pair:

$$\begin{aligned} &L\psi = \lambda\psi, \quad M_{\rm cov}\psi = 0,\\ &M_{\rm cov} = 4\partial_t\partial_x^2 + b_1\partial_x + b_2\partial_t + b_3, \end{aligned} \tag{8.1}$$

where b_1, b_2, and b_3 are functions to be determined. We require that the transformation T is necessary to map the operator $M_{\rm cov}$ to the similar one,

$$TM_{\rm cov}T^{-1} = \tilde{M}_{\rm cov}, \quad \tilde{L}_{2,\rm cov} = 4\partial_t\partial_x^2 + \tilde{b}_1\partial_x + \tilde{b}_2\partial_t + \tilde{b}_3, \tag{8.2}$$

where $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{b}_3$ satisfy the covariant condition

$$\tilde{b}_j = b_j(q) + \Delta b_j = b_j(q+\Delta q), \quad j = 1,2,3. \tag{8.3}$$

It follows from (8.1) and (5.3) that

$$\Delta b_1 = \tilde{b}_1 - b_1 = 4\sigma_t, \quad \Delta b_2 = \tilde{b}_2 - b_2 = 8\sigma_x, \tag{8.4}$$

$$\Delta b_3 = \tilde{b}_3 - b_3 = \sigma\Delta b_1 + 8\sigma_{xt} + b_{1,x}, \tag{8.5}$$

and σ satisfy

$$4\sigma_{2x,t} + \tilde{b}_1\sigma_x + \tilde{b}_2\sigma_t + \sigma\Delta b_3 + b_{3,x} = 0. \tag{8.6}$$

According to the relation (8.4), it remains to determine b_1, b_2, and b_3 in the form of polynomial expressions in terms of q derivatives,

$$b_j = F_j(q,q_x,q_y,q_{xy},q_{2x},q_{2y},q_{2x,y},\ldots), \quad j = 1,2,3,$$

such that

$$\begin{aligned} \Delta F_j = &F_j(q+\Delta q, q_x+\Delta q_x, q_t+\Delta q_t,\ldots)\\ &- F_j(q,q_x,q_t,\ldots) = \Delta b_j, \end{aligned} \tag{8.7}$$

with $\Delta q_{kx,lt} = 2(\ln\phi)_{kx,lt}$, $k,l = 1,2,\ldots$, and Δb_j being given through the relations (8.4)–(8.6).

Expanding the left-hand side of Eq. (8.7), we obtain

$$\begin{aligned} \Delta b_1 = \Delta F_1 = &F_{1,q}\Delta q + F_{1,q_x}\Delta q_x + F_{1,q_y}\Delta q_y\\ &+ F_{1,q_{xy}}\Delta q_{xt} + \cdots = 4\sigma_t = 2\Delta q_{xt}, \end{aligned}$$

which implies that we can determine b_1 up to a arbitrary constant c_1, namely

$$b_1 = F_1(q_{xt}) = 2q_{xt} + c_1, \tag{8.8}$$

where c_1 is an arbitrary constant. Proceeding in the same way, we deduce the function b_2 as follows:

$$b_2 = F_2(q_{2x}) = 4q_{2x} + c_2, \tag{8.9}$$

where c_2 is an arbitrary constant.

We see from the relation (8.5) that Δb_3 contains the term $b_{1,x} = q_{2x,t}$, which should be eliminated such that Δb_3 admits the form (8.7). By the Lax pair (8.1), we have the following relation:

$$q_{2x,t} = -\sigma_{xt} - 2\sigma\sigma_t. \tag{8.10}$$

Substituting (8.8) and (8.10) into (8.5) yields

$$\Delta b_3 = 4\sigma\sigma_t + 8\sigma_{xt} + 2q_{2x,t} = 6\sigma_{xt} = 3\Delta q_{2x,t}.$$

If choosing

$$b_3 = F_3(q_{2x,t}) = 3q_{2x,t} + c_3, \tag{8.11}$$

the third condition

$$\Delta F_3 = F_{3,q}\Delta q + F_{3,q_x}\Delta q_x + F_{3,q_t}\Delta q_t \cdots = \Delta b_3$$

can be satisfied, where c_3 is an arbitrary constant.

Letting $c_1 = -2v_0$, $c_2 = 0$, $c_3 = w_0$ in (8.8), (8.9), and (8.11), it then follows from (8.1) that we have the following Darboux covariant evolution equation:

$$M_{\rm cov}\psi = 0, \quad M_{\rm cov} = 4\partial_t\partial_x^2 + 2q_{xt}\partial_x + 4q_{2x}\partial_t + 3q_{2x,t},$$

which coincides with Eq. (8.6). Moreover, the relation between the two operators $L_{2,\rm cov}$ and L_2 are related through

$$M_{\rm cov} = M + 3\partial_x L.$$

The compatibility condition of the Darboux covariant Lax pair (8.1) exactly gives the NKdV equation (1.1) in the Lax representation

$$\begin{aligned} [M_{\rm cov},L] &= q_{4x,t} + 4(v_0 + q_{2x})q_{2x,t} + q_{3x}(q_{xt} + w_0)\\ &= v_{xxx} + 4vw_x + 2v_xw = 0. \end{aligned}$$

∎

In the above-repeated procedure, we are able to obtain higher-order operators, which are also Darboux covariant with respect to T, to produce higher-order members of the negative-order KdV hierarchy.

IX. CONSERVATION LAWS OF NKDV EQUATIONS

In this section, we will derive the conservation laws in a local form for the NKdV equation (1.1) based on a generalized Miura transformation.

Theorem 10. The NKdV equation (1.1) possesses the following infinitely many conservation laws:

$$F_{n,t} + G_{n,x} = 0, \quad n = 1,2,\ldots, \tag{9.1}$$

where the conversed densities F_n are recursively given by recursion formulas explicitly,

$$\begin{aligned} &F_0 = v_{xx} - v^2, \quad F_1 = -v_{xxx} + 2vv_x,\\ &F_n = I_{n,xx} - \sum_{k=0}^{n} I_kI_{n-k} + \sum_{k=0}^{n-2} I_kI_{n-2-k,x},\\ &n = 2,3,\ldots, \end{aligned} \tag{9.2}$$

and the fluxes G_n are

$$G_0 = 2wI_0 = 2wv, \quad G_1 = 2wI_1 = -2wv_x, \\ G_n = 2wI_n, \quad n = 2,3,\ldots. \tag{9.3}$$

Proof. For the simplicity, let us select $v_0 = w_0 = 0$ in the transformation (6.9). We introduce a new potential function,

$$q_{2x} = \eta + \varepsilon\eta_x + \varepsilon^2\eta^2, \tag{9.4}$$

where ε is a constant parameter. Substituting (9.4) into the Lax equation (7.10) leads to

$$0 = [L,M] = (1 + \varepsilon\partial_x + 2\varepsilon^2\eta)[-4(\eta + \varepsilon^2\eta^2)\eta_t \\ - 2(q_x - \varepsilon\eta)_t\eta_x + \eta_{2x,t}],$$

which implies that $v = q_{2x}$, $w = -q_{xt}$ given by (6.9) are a solution of the NKdV equation (1.1) if η satisfies the following equation:

$$-4(\eta + \varepsilon^2\eta^2)\eta_t - 2(q_x - \varepsilon\eta)_t\eta_x + \eta_{2x,t} - 4\eta_t = 0. \tag{9.5}$$

On the other hand, it follows from (9.4) that

$$[(q_x - \varepsilon\eta)_t]_x = -(\eta + \varepsilon^2\eta^2)_t.$$

Therefore, Eq. (9.5) can be rewritten as

$$(\eta_{2x} - \eta^2)_t + [2\eta(\varepsilon^2\eta - q_x)_t]_x = 0$$

or a divergent-type form,

$$(\eta_{2x} + 2\varepsilon^2\eta\eta_x - \eta^2)_t + (2\eta w)_x = 0, \tag{9.6}$$

by replacing $-q_{xt} = w$.

To proceed, inserting the expansion

$$\eta = \sum_{n=0}^{\infty} I_n(q,q_x,q_t\ldots)\varepsilon^n \tag{9.7}$$

into Eq. (9.4) and equating the coefficients for power of ε, we obtain the recursion relations to calculate I_n in an explicit form,

$$I_0 = q_{2x} = v, \quad I_1 = -I_{0,x} = -v_x, \\ I_n = -I_{n-1,x} - \sum_{k=0}^{n-2} I_k I_{n-2-k}, \quad n = 2,3,\ldots. \tag{9.8}$$

Substituting (9.7) into (9.6) and simplifying terms in the power of ε provide us infinitely many conservation laws,

$$F_{n,t} + G_{n,x} = 0, \quad n = 1,2,\ldots,$$

where the conversed densities F_n and the fluxes G_n are from (9.2) and (9.3), respectively. ■

Here, we already give recursion formulas (9.7) and (9.8) to show how to generate conservation laws (9.6) based on the first few explicitly provided. Apparently, the first equation in conservation laws (9.6)

$$v_{xxt} - 2vv_t + 2wv_x + 2w_xv = 0$$

is exactly the NKdV equation (1.1)

$$v_t + w_x = 0, \quad w_{xxx} + 4vw_x + 2wv_x = 0,$$

which is reduced to the NKdV equation (1.2) under the constraints $v = -u_{xx}/u$ and $w = u^2$.

In conclusion, the NKdV equation (1.1) is completely integrable and admits the bilinear Bäcklund transformation, the Lax pair, and infinitely many local conservation laws.

X. QUASIPERIODIC SOLUTIONS OF THE NKDV EQUATION

In this section, we study quasiperiodic wave solutions of the NKdV equation (1.1) by using the bilinear Bäcklund transformation (7.1) and bilinear formulas derived in Sec. IX. In fact, quasiperiodic solutions, also called algebrogeometric solutions or finite gap solutions, are often obtained based on the inverse spectral theory and algebrogeometric method [21,33,67–76]. The algebrogeometric theory, however, needs Lax pairs and is also involved in complicated analysis procedures on the Riemann surfaces. It is rather difficult to directly determine the characteristic parameters of waves, such as frequencies and phase shifts, for a function with given wave numbers and amplitudes. Based on the Hirota forms, Nakamura proposed a convenient way to find a kind of explicit quasiperiodic solution of nonlinear equations [77]. For example, it does not need any Lax pair and Riemann surface for the given nonlinear equation and is also able to find the explicit construction of multiperiodic wave solutions. The method relies on the existence of the Hirota bilinear form as well as arbitrary parameters appearing in the Riemann matrix [59,78,79].

A. Multidimensional Riemann θ functions

Let us, first, begin with some preliminary work about multidimensional Riemann θ functions and their quasiperiodicity. The multidimensional Riemann θ function is defined by

$$\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},s|\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \exp\{2\pi i\langle\boldsymbol{\zeta}+\boldsymbol{\varepsilon},\boldsymbol{n}+\boldsymbol{s}\rangle \\ - \pi\langle\boldsymbol{\tau}(\boldsymbol{n}+\boldsymbol{s}),\boldsymbol{n}+\boldsymbol{s}\rangle\}, \tag{10.1}$$

where $\boldsymbol{n} = (n_1,\ldots,n_N)^T \in \mathbb{Z}^N$ is an integer value vector and $\boldsymbol{s} = (s_1,\ldots,s_N)^T, \boldsymbol{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_N)^T \in \mathbb{C}^N$ is a complex parameter vector; $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_N)^T$, $\zeta_j = \alpha_j x + \beta_j t + \delta_j$, $\alpha_j,\beta_j,\delta_j \in \Lambda_0$, $j = 1,2,\ldots,N$ are complex phase variables, where x,t are ordinary real variables and θ is a Grassmann variable. The inner product of two vectors $\boldsymbol{f} = (f_1,\ldots,f_N)^T$ and $\boldsymbol{g} = (g_1,\ldots,g_N)^T$ is defined by

$$\langle\boldsymbol{f},\boldsymbol{g}\rangle = f_1g_1 + f_2g_2 + \cdots + f_Ng_N.$$

The matrix $\boldsymbol{\tau} = (\tau_{ij})$ is a positive definite and real-valued symmetric $N\times N$ matrix. The entries τ_{ij} of the periodic matrix $\boldsymbol{\tau}$ can be considered as free parameters of the θ function (10.1).

In this paper, we choose τ to be purely imaginary matrix to make the θ function (10.1) real valued. In definition (10.1) for the case of $\boldsymbol{s} = \boldsymbol{\varepsilon} = \boldsymbol{0}$, we denote $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau}) = \vartheta(\boldsymbol{\zeta},\boldsymbol{0},\boldsymbol{0}|\boldsymbol{\tau})$ for simplicity. Therefore, we have $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\boldsymbol{0}|\boldsymbol{\tau}) = \vartheta(\boldsymbol{\zeta}+\boldsymbol{\varepsilon},\boldsymbol{\tau})$.

Remark 4. The above periodic matrix $\boldsymbol{\tau}$ differs from the one in the algebrogeometric approach discussed in Refs. [15–21], where it is usually constructed on a compact Riemann surface Γ with genus $N\in\mathbb{N}$. One may see that the entries in the matrix $\boldsymbol{\tau}$ are not free and are difficult to be explicitly given. ■

Definition 3. A function $g(\boldsymbol{x},t)$ on $\mathbb{C}^N\times\mathbb{C}$ is said to be quasiperiodic in t with fundamental periods $T_1,\ldots,T_k\in\mathbb{C}$

if $T_1,\ldots,T_k$ are linearly dependent over $\mathbb{Z}$ and there exists a function $G(\boldsymbol{x},\boldsymbol{y})\in\mathbb{C}^N\times\mathbb{C}^k$ such that

$$\begin{aligned}&G(\boldsymbol{x},y_1,\ldots,y_j+T_j,\ldots,y_k)=G(\boldsymbol{x},y_1,\ldots,y_j,\ldots,y_k),\\&\text{for all}\quad y_j\in\mathbb{C},\quad j=1,\ldots,k.\\&G(\boldsymbol{x},t,\ldots,t,\ldots,t)=g(\boldsymbol{x},t).\end{aligned}$$

In particular, $g(\boldsymbol{x},t)$ becomes periodic with T if and only if $T_j=m_jT$. ■

Let us, first, see periodicity of the θ function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$.

Proposition 9. Let $\boldsymbol{e_j}$ be the jth column of the $N\times N$ identity matrix I_N; let τ_j be the jth column of $\boldsymbol{\tau}$ and τ_{jj} the (j,j) entry of $\boldsymbol{\tau}$ [80]. The θ function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$ then has the periodic properties

$$\vartheta(\boldsymbol{\zeta}+\boldsymbol{e_j}+i\boldsymbol{\tau_j},\boldsymbol{\tau})=\exp(-2\pi i\zeta_j+\pi\tau_{jj})\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau}).$$

The θ function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$ which satisfies the condition (5.4) is called a multiplicative function. We regard the vectors $\{\boldsymbol{e_j},\ j=1,\ldots,N\}$ and $\{i\boldsymbol{\tau_j},\ j=1,\ldots,N\}$ as periods of the θ function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$ with multipliers 1 and $\exp(-2\pi i\zeta_j+\pi\tau_{jj})$, respectively. Here, only the first N vectors are actually periods of the θ function $\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$, but the last N vectors are the periods of the functions $\partial^2_{\zeta_k,\zeta_l}\ln\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau})$ and $\partial_{\zeta_k}\ln[\vartheta(\boldsymbol{\zeta}+\boldsymbol{e},\boldsymbol{\tau})/\vartheta(\boldsymbol{\zeta}+\boldsymbol{h},\boldsymbol{\tau})]$, $k,l=1,\ldots,N$.

Proposition 10. Let $\boldsymbol{e_j}$ and $\boldsymbol{\tau_j}$ be defined as above in proposition 2. The meromorphic functions $f(\boldsymbol{\zeta})$ are as follows:

$$\text{(i)}\quad f(\boldsymbol{\zeta})=\partial^2_{\zeta_k\zeta_l}\ln\vartheta(\boldsymbol{\zeta},\quad\boldsymbol{\tau}),\quad\boldsymbol{\zeta}\in C^N,\quad k,l=1,\ldots,N,$$

$$\text{(ii)}\quad f(\boldsymbol{\zeta})=\partial_{\zeta_k}\ln\frac{\vartheta(\boldsymbol{\zeta}+\boldsymbol{e},\boldsymbol{\tau})}{\vartheta(\boldsymbol{\zeta}+\boldsymbol{h},\boldsymbol{\tau})},\quad\boldsymbol{\zeta},\ \boldsymbol{e},\ \boldsymbol{h}\in C^N,\quad j=1,\ldots,N,$$

and then, in cases (i) and (ii), it holds that

$$f(\boldsymbol{\zeta}+\boldsymbol{e_j}+i\boldsymbol{\tau_j})=f(\boldsymbol{\zeta}),\quad\boldsymbol{\zeta}\in C^N,\quad j=1,\ldots,N,$$

which implies that $f(\boldsymbol{\zeta})$ is a quasiperiodic function.

B. Bilinear formulas of θ functions

To construct a kind of explicitly quasiperiodic solutions of the NKdV equation (1.1), we propose some important bilinear formulas of multidimensional Riemann θ functions, whose derivations are similar to the case of super bilinear equations [79], so we just list them without proofs.

Theorem 11. Suppose that $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon}',\boldsymbol{0}|\boldsymbol{\tau})$ and $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\boldsymbol{0}|\boldsymbol{\tau})$ are two Riemann θ functions, in which $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_N)$, $\boldsymbol{\varepsilon}'=(\varepsilon_1',\ldots,\varepsilon_N')$, and $\boldsymbol{\zeta}=(\zeta_1,\ldots,\zeta_N)$, $\zeta_j=\alpha_jx+\omega_jt+\delta_j$, $j=1,2,\ldots,N$. The operators D_x, D_t, and S then exhibit the following perfect properties when they act on a pair of θ functions:

$$\begin{aligned}&D_x\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon}',\boldsymbol{0}|\boldsymbol{\tau})\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\boldsymbol{0}|\boldsymbol{\tau})\\&\quad=\sum_{\boldsymbol{\mu}}\partial_x\vartheta(2\boldsymbol{\zeta},\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon},-\boldsymbol{\mu}/2|2\boldsymbol{\tau})|_{\boldsymbol{\zeta}=\boldsymbol{0}}\vartheta(2\boldsymbol{\zeta},\boldsymbol{\varepsilon}'+\boldsymbol{\varepsilon},\boldsymbol{\mu}/2|2\boldsymbol{\tau}),\end{aligned}\tag{10.2}$$

where $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_N)$ and the notation $\sum_{\boldsymbol{\mu}}$ represents 2^N different transformations corresponding to all possible combinations $\mu_1=0,1;\ldots;\mu_N=0,1$.

In general, for a polynomial operator $H(D_x,D_t)$ with respect to D_x and D_t, we have the following useful formula:

$$\begin{aligned}&H(D_x,D_t)\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon}',\boldsymbol{0}|\boldsymbol{\tau})\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\boldsymbol{0}|\boldsymbol{\tau})\\&\quad=\sum_{\boldsymbol{\mu}}C(\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon},\boldsymbol{\mu})\vartheta(2\boldsymbol{\zeta},\boldsymbol{\varepsilon}'+\boldsymbol{\varepsilon},\boldsymbol{\mu}/2|2\boldsymbol{\tau}),\end{aligned}\tag{10.3}$$

in which, explicitly,

$$\begin{aligned}C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})=\sum_{\boldsymbol{n}\in\mathbb{Z}^N}&H(\boldsymbol{\mathcal{M}})\exp[-2\pi\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\mu}/2),\boldsymbol{n}-\boldsymbol{\mu}/2\rangle\\&-2\pi i\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon}\rangle].\end{aligned}\tag{10.4}$$

where we denote $\boldsymbol{\mathcal{M}}=(4\pi i\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\alpha}\rangle,\ 4\pi i\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\omega}\rangle)$.

Remark 6. The formulas (10.3) and (10.4) show that if the following equations are satisfied:

$$C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})=0,\tag{10.5}$$

for all possible combinations $\mu_1=0,1;\mu_2=0,1;\ldots;\mu_N=0,1$ [in other words, all such combinations are solutions of Eq. (10.5)], then $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon}',\boldsymbol{0}|\boldsymbol{\tau})$ and $\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\boldsymbol{0}|\boldsymbol{\tau})$ are N-periodic wave solutions of the bilinear equation

$$H(D_x,D_t)\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon}',\boldsymbol{0}|\boldsymbol{\tau})\vartheta(\boldsymbol{\zeta},\boldsymbol{\varepsilon},\boldsymbol{0}|\boldsymbol{\tau})=0.$$

We call the formula (10.5) constraint equations, whose number is 2^N. This formula actually provides us with a unified approach to construct multiperiodic wave solutions for supersymmetric equations. Once a supersymmetric equation is written in bilinear forms, then its multiperiodic wave solutions can be directly obtained by solving system (10.5).

Theorem 12. Let $C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})$ and $H(D_x,D_t)$ be given in theorem 10, and make a choice such that $\varepsilon_j'-\varepsilon_j=\pm1/2$, $j=1,\ldots,N$. Then

(i) If $H(D_x,D_t)$ is an symmetric operator, i.e.,

$$H(-D_x,-D_t)=H(D_x,D_t),$$

then $C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})$ vanishes automatically for the case when $\sum_{j=1}^N\mu_j$ is an odd number, namely

$$C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})|_{\boldsymbol{\mu}}=0,\quad\text{for}\quad\sum_{j=1}^N\mu_j=1,\quad\text{mod }2.$$

(ii) If $H(D_x,D_t)$ is a skew-symmetric operator, i.e.,

$$H(-D_x,-D_t)=-H(D_x,D_t),$$

then $C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})$ vanishes automatically for the case when $\sum_{j=1}^N\mu_j$ is an even number, namely

$$C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})|_{\boldsymbol{\mu}}=0,\quad\text{for}\ \sum_{j=1}^N\mu_j=0,\quad\text{mod }2.$$

Proposition 11. Let $\varepsilon_j'-\varepsilon_j=\pm1/2$, $j=1,\ldots,N$. Assume $H(D_x,D_t)$ is a linear combination of even and odd functions

$$H(D_x,D_t)=H_1(D_x,D_t)+H_2(D_x,D_t),$$

where H_1 is even and H_2 is odd. In addition, $C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})$ corresponding (10.8) is given by

$$C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})=C_1(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu})+C_2(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu}),$$

where

$$C_1(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} H_1(\boldsymbol{\mathcal{M}})\exp[-2\pi\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\mu}/2),\boldsymbol{n}-\boldsymbol{\mu}/2\rangle - 2\pi i\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon}\rangle],$$

$$C_2(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} H_2(\boldsymbol{\mathcal{M}})\exp[-2\pi\langle\boldsymbol{\tau}(\boldsymbol{n}-\boldsymbol{\mu}/2),\boldsymbol{n}-\boldsymbol{\mu}/2\rangle - 2\pi i\langle\boldsymbol{n}-\boldsymbol{\mu}/2,\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon}\rangle].$$

Then

$$C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu}) = C_2(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu}) \quad \text{for} \quad \sum_{j=1}^N \mu_j = 1, \quad \text{mod } 2,$$

$$C(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu}) = C_1(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\boldsymbol{\mu}), \quad \text{for} \sum_{j=1}^N \mu_j = 0, \quad \text{mod } 2.$$

Theorem 2 and corollary 1 are very useful to deal with coupled super-Hirota bilinear equations, which will be seen in the following Sec. X.

By introducing differential operators

$$\boldsymbol{\nabla} = (\partial_{\zeta_1},\partial_{\zeta_2},\ldots,\partial_{\zeta_N}),$$
$$\partial_x = \alpha_1\partial_{\zeta_1} + \alpha_2\partial_{\zeta_2} + \cdots + \alpha_N\partial_{\zeta_N} = \boldsymbol{\alpha}\cdot\boldsymbol{\nabla},$$
$$\partial_t = \beta_1\partial_{\zeta_1} + \beta_2\partial_{\zeta_2} + \cdots + \beta_N\partial_{\zeta_N} = \boldsymbol{\beta}\cdot\boldsymbol{\nabla},$$

we then have

$$\partial_x^k\partial_t^l\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau}) = (\boldsymbol{\alpha}\cdot\boldsymbol{\nabla})^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla})^l\vartheta(\boldsymbol{\zeta},\boldsymbol{\tau}), \quad k,l = 0,1,\ldots.$$

C. One-periodic waves and asymptotic analysis

Let us, first, construct one-periodic wave solutions of the NKdV equation (1.1) by using bilinear Báckland transformation (7.6). As a simple case of the θ function (10.1) with $N=1, s=0$, we choose F and G as follows:

$$F = \vartheta(\zeta,0,0|\tau) = \sum_{n\in\mathbb{Z}} \exp(2\pi i n\zeta - \pi n^2\tau),$$

$$G = \vartheta(\zeta,1/2,0|\tau) = \sum_{n\in\mathbb{Z}} \exp[2\pi i n(\zeta+1/2) - \pi n^2\tau] = \sum_{n\in\mathbb{Z}}(-1)^n \exp(2\pi i n\zeta - \pi n^2\tau), \tag{10.6}$$

where $\zeta = \alpha x + \beta t + \delta$ is the phase variable and $\tau > 0$ is a positive parameter.

By theorem 6, in Sec. IX, the operator $H_1 = D_x^2 - \lambda$ in bilinear equation (7.6) is symmetric, and its corresponding constraint equation in the formula (10.5) automatically vanishes for $\mu = 1$. Meanwhile, $H_2 = D_tD_x^2 - 2w_0D_x + (4v_0 + 3\lambda)D_t$ are skew symmetric, and its corresponding constraint equation automatically vanishes for $\mu = 0$. Therefore, the Riemann θ function (10.6) is a solution of the bilinear equation (7.6), provided the following equations:

$$\sum_{n\in\mathbb{Z}}\{[4\pi i(n-\mu/2)]^2\alpha^2 - \lambda\}\exp(-2\pi\tau(n-\mu/2)^2 + \pi i(n-\mu/2))|_{\mu=0} = 0,$$

$$\sum_{n\in\mathbb{Z}}\{[4\pi i(n-\mu/2)]^3\alpha^2\beta + 8\pi i(n-\mu/2)\alpha w_0 + 4\pi i(n-\mu/2)(4v_0+3\lambda)\beta\}\exp(-2\pi\tau(n-\mu/2)^2 + \pi i(n-\mu/2))|_{\mu=1} = 0 \tag{10.7}$$

hold.

We introduce the notations by

$$\rho = e^{-\pi\tau/2},$$
$$\vartheta_1(\zeta,\rho) = \vartheta(2\zeta,1/4,-1/2|2\tau) = \sum_{n\in\mathbb{Z}}\rho^{(2n-1)^2}\exp[4i\pi(n-1/2)(\zeta+1/4)],$$
$$\vartheta_2(\zeta,\rho) = \vartheta(2\zeta,1/4,0|2\tau) = \sum_{n\in\mathbb{Z}}\rho^{4n^2}\exp[4i\pi n(\zeta+1/4)],$$

Eq. (10.7) then can be written as a linear system about β and λ as follows:

$$\vartheta_2''\alpha^2 - \vartheta_2\lambda = 0, \quad \vartheta_1'''\alpha^2\beta + 2\vartheta_1'\alpha w_0 + (4v_0+3\lambda)\vartheta_1'\beta = 0, \tag{10.8}$$

where the derivative value of $\vartheta_j(\zeta,\rho)$ at $\zeta = 0$ is denoted by simple notations

$$\vartheta_j' = \vartheta_j'(0,\rho) = \left.\frac{d\vartheta_j(\zeta,\rho)}{d\zeta}\right|_{\zeta=0}, \quad j = 1,2.$$

It is not hard to see that the system (10.8) admits the following solution for the NKdV equation (1.1):

$$\lambda = \frac{\vartheta_2''\alpha^2}{\vartheta_2}, \quad \beta = \frac{-2\vartheta_1'\vartheta_2 w_0}{\vartheta_1'''\vartheta_2\alpha^2 + 4\vartheta_1'\vartheta_2 v_0 + 3\vartheta_1'\vartheta_2''\alpha^2}. \tag{10.9}$$

So we obtain the following one-periodic wave solution:

$$V = v_0 + 2\partial_x^2\ln\vartheta(\zeta,0,0|\tau), \quad W = w_0 + 2\partial_x\partial_t\ln\vartheta(\zeta,0,0|\tau), \tag{10.10}$$

where $\zeta = \alpha x + \beta t + \delta$ and parameter β is given by (10.9), while other parameters, α, τ, v_0, w_0, are arbitrary. Among the four parameters, α and τ completely dominate a one-periodic wave. In summary, the one-periodic wave (10.10) is one dimensional and has two fundamental periods 1 and $i\tau$ in phase variable ζ (see Fig. 3).

In the following theorem, we will see that the one-periodic wave solution (10.10) can be broken into soliton solution (6.21) under a long time limit and their relation can be established as follows.

Theorem 13. In the one-periodic wave solution (10.6), the parameter β is given by (10.9), and other parameters are chosen as

$$\alpha = \frac{k}{2\pi i}, \quad \delta = \frac{\gamma + \pi\tau}{2\pi i}, \tag{10.11}$$

where k_1 and γ are the same as those in (6.21). Then, under a small amplitude limit, the one-periodic wave solution (10.10) can be broken into the single soliton solutions (6.21), that is,

$$V \longrightarrow v, \quad W \longrightarrow w, \quad \text{as } \rho \to 0. \tag{10.12}$$

In particular, in the case of $v_0 = 0$, $w_0 = 1$, the one-periodic solution (10.5) tends to the kink-type soliton solution (5.2), that is,

$$V \longrightarrow \tilde{v}^I, \quad W \longrightarrow \tilde{w}^I, \quad \text{as } \rho \to 0. \tag{10.13}$$

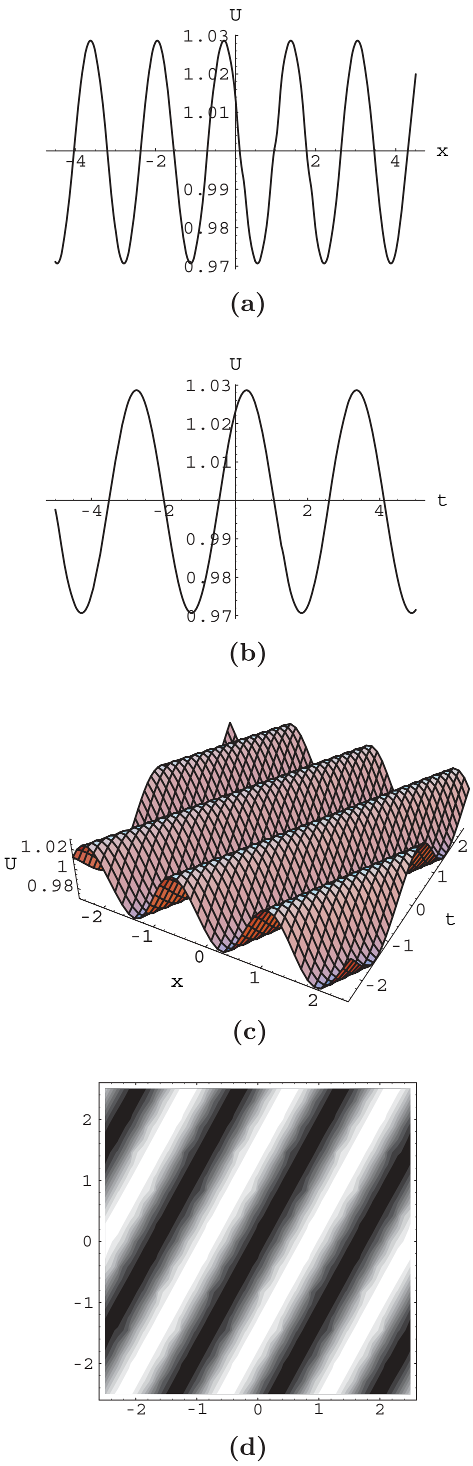

FIG. 3. (Color online) One-periodic wave for the NKdV equation (1.1) with parameters $\alpha = 0.6$, $\tau = 2$, $v_0 = 0.5$, $w_0 = 1$. (a) and (b) show that every one-periodic wave is periodic in both the x and y directions. (c) Perspective view of the wave. (d) Overhead view of the wave, with contour plot shown. The bright hexagons are crests and the dark hexagons are troughs.

Proof. Here we use the system (10.8) to analyze asymptotic properties of the one-periodic solution (10.10). Let us explicitly expand the coefficients of the system (10.8) as follows:

$$\begin{aligned} \vartheta_1' &= -4\pi\rho + 12\pi\rho^9 + \cdots, \\ \vartheta_1''' &= 16\pi^3\rho + 432\pi^3\rho^9 + \cdots, \\ \vartheta_2 &= 1 + 2\rho^4 + \cdots, \\ \vartheta_2'' &= 32\pi^2\rho^4 + \cdots. \end{aligned} \tag{10.14}$$

Suppose that the solution of the system (10.8)(10.8) has the following form:

$$\begin{aligned} \lambda &= \lambda_0 + \lambda_1\rho + \lambda_2\rho^2 + \cdots = \lambda_0 + o(\rho), \\ \beta &= \beta_0 + \beta_1\rho + \beta_2\rho^2 + \cdots = \beta_0 + o(\rho). \end{aligned} \tag{10.15}$$

Substituting the expansions (10.14) and (10.15) into the system (10.8) and letting $\rho \to 0$, we immediately obtain the following relation:

$$\lambda_0 = 0, \quad \beta_0 = \frac{-\alpha w_0}{-2\pi^2\alpha^2 + 2v_0}. \tag{10.16}$$

Combining (10.11) and (10.16) leads to

$$\lambda \longrightarrow 0,$$

$$2\pi i\beta \longrightarrow 2\pi i\beta_0 = \frac{-2\pi i\alpha w_0}{-2\pi^2\alpha^2 + 2v_0} = \frac{-2kw_0}{k^2 + 4v_0}, \text{ as } \rho \to 0,$$

or, equivalently, rewritten as

$$\begin{aligned} \hat{\xi} &= 2\pi i\zeta - \pi\tau = kx + 2\pi i\beta t + \gamma \\ &\longrightarrow kx - \frac{2kw_0}{k^2 + 4v_0}t + \gamma = \xi, \quad \text{as } \rho \to 0. \end{aligned} \tag{10.17}$$

It remains to verify that the one-periodic wave (10.11) has the same form as the one-soliton solution (6.21) under the limit $\rho \to 0$. Let us expand the function F in the following form:

$$F = 1 + \rho^2(e^{2\pi i\zeta} + e^{-2\pi i\zeta}) + \rho^8(e^{4\pi i\zeta} + e^{-4\pi i\zeta}) + \cdots.$$

It follows from (10.11) and (10.17) that

$$\begin{aligned} F &= 1 + e^{\hat{\xi}} + \rho^4(e^{-\hat{\xi}} + e^{2\hat{\xi}}) + \rho^{12}(e^{-2\hat{\xi}} + e^{3\hat{\xi}}) + \cdots \\ &\longrightarrow 1 + e^{\hat{\xi}} \longrightarrow 1 + e^{\xi}, \text{ as } \rho \to 0. \end{aligned} \tag{10.18}$$

So combining (10.11) and (10.18) yields

$$\begin{aligned} v &\longrightarrow v_0 + 2\partial_{xx}\ln(1 + e^{\xi}), \\ w &\longrightarrow w_0 + 2\partial_t\partial_x\ln(1 + e^{\xi}), \text{ as } \rho \to 0. \end{aligned}$$

Thus, we conclude that the one-periodic solution (10.10) may go to a bell-type soliton solutions (6.21) as the amplitude $\rho \to 0$. ■

D. Two-periodic waves and asymptotic properties

Let us now consider two-periodic wave solutions to the NKdV equation (1.1). For the case of $N = 2$, $\boldsymbol{s} = \boldsymbol{0}$, $\boldsymbol{\varepsilon} = \boldsymbol{1/2} = (1/2, 1/2)$ in the Riemann θ function (10.1), we choose F and G as follows:

$$F = \vartheta(\boldsymbol{\zeta}, \boldsymbol{0}, \boldsymbol{0}|\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \exp\{2\pi i\langle\boldsymbol{\zeta}, \boldsymbol{n}\rangle - \pi\langle\boldsymbol{\tau n}, \boldsymbol{n}\rangle\},$$

$$\begin{aligned} G &= \vartheta(\boldsymbol{\zeta}, \boldsymbol{1/2}, \boldsymbol{0}|\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \exp\{2\pi i\langle\boldsymbol{\zeta} + \boldsymbol{1/2}, \boldsymbol{n}\rangle - \pi\langle\boldsymbol{\tau n}, \boldsymbol{n}\rangle\} \\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^2} (-1)^{n_1+n_2} \exp\{2\pi i\langle\boldsymbol{\zeta}, \boldsymbol{n}\rangle - \pi\langle\boldsymbol{\tau n}, \boldsymbol{n}\rangle\}, \end{aligned} \tag{10.19}$$

where $\boldsymbol{n} = (n_1,n_2) \in Z^2$, $\boldsymbol{\zeta} = (\zeta_1,\zeta_2) \in \mathcal{C}^2$, $\zeta_i = \alpha_j x + \beta_j t + \delta_j$, $j = 1,2$, and $\boldsymbol{\alpha} = (\alpha_1,\alpha_2)$, $\boldsymbol{\beta} = (\beta_1,\beta_2) \in \mathcal{C}^2$. The matrix $\boldsymbol{\tau}$ is a positive definite and real-valued symmetric 2×2 matrix, that is,

$$\boldsymbol{\tau} = (\tau_{ij})_{2\times 2},\ \ \tau_{12} = \tau_{21},\ \ \tau_{11} > 0,\ \ \tau_{22} > 0,\ \ \tau_{11}\tau_{22} - \tau_{12}^2 > 0.$$

According to theorem 5, constraint equations associated with $H_1 = D_x^2 - \lambda$ and $H_2 = D_t D_x^2 - 2w_0 D_x + (4v_0 + 3\lambda)D_t$ automatically vanish for $(\mu_1,\mu_2) = (0,1),(1,0)$ and for $(\mu_1,\mu_2) = (0,0),(1,1)$, respectively. Hence, making the θ functions (10.19) satisfy the bilinear equation (7.6) gives the following constraint equations:

$$\begin{aligned}
&\sum_{n_1,n_2\in\mathbb{Z}} [-16\pi^2\langle \boldsymbol{n} - \boldsymbol{\mu}/2,\boldsymbol{\alpha}\rangle^2 - \lambda]\exp\{-2\pi\langle \boldsymbol{\tau}(\boldsymbol{n} - \boldsymbol{\mu}/2),\boldsymbol{n} - \boldsymbol{\mu}/2\rangle + \pi i \sum_{j=1}^{2}(n_j - \mu_j/2)\}|_{\boldsymbol{\mu}=(\mu_1,\mu_2)} = 0,\\
&\qquad \text{for } (\mu_1,\mu_2) = (0,0),\ (1,1) = 0,\\
&\sum_{n_1,n_2\in\mathbb{Z}} [-64\pi^3 i\langle \boldsymbol{n} - \boldsymbol{\mu}/2,\boldsymbol{\alpha}\rangle^2\langle \boldsymbol{n} - \boldsymbol{\mu}/2,\boldsymbol{\beta}\rangle + 8\pi i\langle \boldsymbol{n} - \boldsymbol{\mu}/2,\boldsymbol{\alpha}\rangle w_0 + 4\pi i\langle \boldsymbol{n} - \boldsymbol{\mu}/2,\boldsymbol{\beta}\rangle(4v_0 + 3\lambda)]\\
&\qquad \times\exp\{-2\pi\langle \boldsymbol{\tau}(\boldsymbol{n} - \boldsymbol{\mu}/2),\boldsymbol{n} - \boldsymbol{\mu}/2\rangle + \pi i \sum_{j=1}^{2}(n_j - \mu_j/2)\}|_{\boldsymbol{\mu}=(\mu_1,\mu_2)} = 0, \quad \text{for } (\mu_1,\mu_2) = (0,1),\ (1,0).
\end{aligned} \tag{10.20}$$

Next, let us introduce the following notations:

$$\rho_{kl} = e^{-\pi\tau_{kl}/2}, k,l = 1,2, \boldsymbol{\rho} = (\rho_{11},\rho_{12},\rho_{22})$$

$$\vartheta_j(\boldsymbol{\zeta},\boldsymbol{\rho}) = \vartheta(2\zeta,\boldsymbol{1/4}, -\boldsymbol{s}_j/2|2\tau) = \sum_{n_1,n_2\in Z} \exp\{4\pi i\langle \boldsymbol{\zeta} + \boldsymbol{1/4},\boldsymbol{n} - \boldsymbol{s}_j/2\rangle\}\prod_{k,l=1}^{2}\rho_{kl}^{(2n_k - s_{j,k})(2n_j - s_{j,l})},$$

$$\boldsymbol{s_j} = (s_{j,1},s_{j,2}),\quad j = 1,2,\ \ \boldsymbol{s}_1 = (0,1),\ \ \boldsymbol{s_2} = (1,0),\ \ \boldsymbol{s_3} = (0,0),\ \ \boldsymbol{s_4} = (1,1)$$

and then the system (10.20) can be rewritten as a linear system

$$(\boldsymbol{\alpha}\cdot\nabla)^2\vartheta_j - \lambda\vartheta_j = 0,\quad j = 3,4, \tag{10.21}$$

$$\begin{aligned}&(\boldsymbol{\beta}\cdot\nabla)(\boldsymbol{\alpha}\cdot\nabla)^2\vartheta_j + 2w_0(\boldsymbol{\alpha}\cdot\nabla)\vartheta_j + (4v_0 + 3\lambda)(\boldsymbol{\beta}\cdot\nabla)\vartheta_j\\ &\quad = 0,\quad j = 1,2,\end{aligned} \tag{10.22}$$

where ϑ_j represent the derivative values of functions $\vartheta_j(\boldsymbol{\zeta},\boldsymbol{\rho})$ at $\zeta_1 = \zeta_2 = 0$.

The system (10.22) admits a unique solution

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \left[\frac{\partial(f,g)}{\partial(\zeta_1,\zeta_2)}\right]^{-1}\begin{pmatrix}2w_0(\boldsymbol{\alpha}\cdot\nabla)\vartheta_1\\ 2w_0(\boldsymbol{\alpha}\cdot\nabla)\vartheta_2\end{pmatrix}, \tag{10.23}$$

where $\frac{\partial(f,g)}{\partial(\zeta_1,\zeta_2)}$ is the Wronskinan matrix given by

$$\frac{\partial(f,g)}{\partial(\zeta_1,\zeta_2)} = \begin{pmatrix}\partial_{\zeta_1} f & \partial_{\zeta_2} f\\ \partial_{\zeta_1} g & \partial_{\zeta_2} g\end{pmatrix},\quad f = [(\boldsymbol{\alpha}\cdot\nabla)^2 + 4v_0 + 3\lambda]\vartheta_1,$$

$$g = [(\boldsymbol{\alpha}\cdot\nabla)^2 + 4v_0 + 3\lambda]\vartheta_2.$$

With the help of the above (β_1,β_2), we are able to get a two-periodic wave solution to the NKdV equation (1.1),

$$V = v_0 + \partial_x^2 \ln\vartheta(\boldsymbol{\zeta},\boldsymbol{0},\boldsymbol{0}|\boldsymbol{\tau}),\ \ W = w_0 + \partial_x\partial_t\vartheta(\boldsymbol{\zeta},\boldsymbol{0},\boldsymbol{0}|\boldsymbol{\tau}), \tag{10.24}$$

where α_1, α_2, τ_{12}, δ_1, and δ_2 are arbitrary parameters, while other parameters, β_1,β_2 and τ_{11}, τ_{22}, are given by (10.23) and (10.21), respectively.

In summary, the two-periodic wave (10.24) is a direct generalization of two one-periodic waves (Fig. 4). Its surface pattern is two dimensional with two phase variables ζ_1 and ζ_2. The two-periodic wave (10.24) has four fundamental periods $\{e_1,e_2\}$ and $\{i\tau_1,i\tau_2\}$ in (ζ_1,ζ_2) and is spatially periodic in two directions ζ_1,ζ_2. Its real part is not periodic in the θ_1 direction, while its imaginary part and modulus are all periodic in both the x and t directions.

Finally, we study the asymptotic properties of the two-periodic solution (10.24). In a way similar to that in theorem 5, we figure out the relation between the two-periodic solution (10.24) and the two-soliton solution (6.22) as follows.

Theorem 14. Assume that (β_1,β_2) is a solution of the system (10.22), and in the two-periodic wave solution (10.24), parameters α_j, δ_j, and τ_{12} are chosen as

$$\alpha_j = \frac{k_j}{2\pi i},\quad \delta_j = \frac{\gamma_j + \pi\tau_{jj}}{2\pi i},\quad \tau_{12} = -\frac{A_{12}}{2\pi},\quad j = 1,2, \tag{10.25}$$

where $k_j,\gamma_j, j = 1,2$ and A_{12} are those given in (6.22). We then have the following asymptotic relations:

$$\begin{aligned}&\lambda \longrightarrow 0,\quad \zeta_j \longrightarrow \frac{\eta_j + \pi\tau_{jj}}{2\pi i},\quad j = 1,2,\\ &F \longrightarrow 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}},\quad \text{as } \rho_{11},\rho_{22} \to 0.\end{aligned} \tag{10.26}$$

So the two-periodic wave solution (10.24) just tends to the two-soliton solution (6.22) under a limit condition

$$V \longrightarrow v,\quad W \longrightarrow w,\quad \text{as } \rho_{11},\rho_{22} \to 0.$$

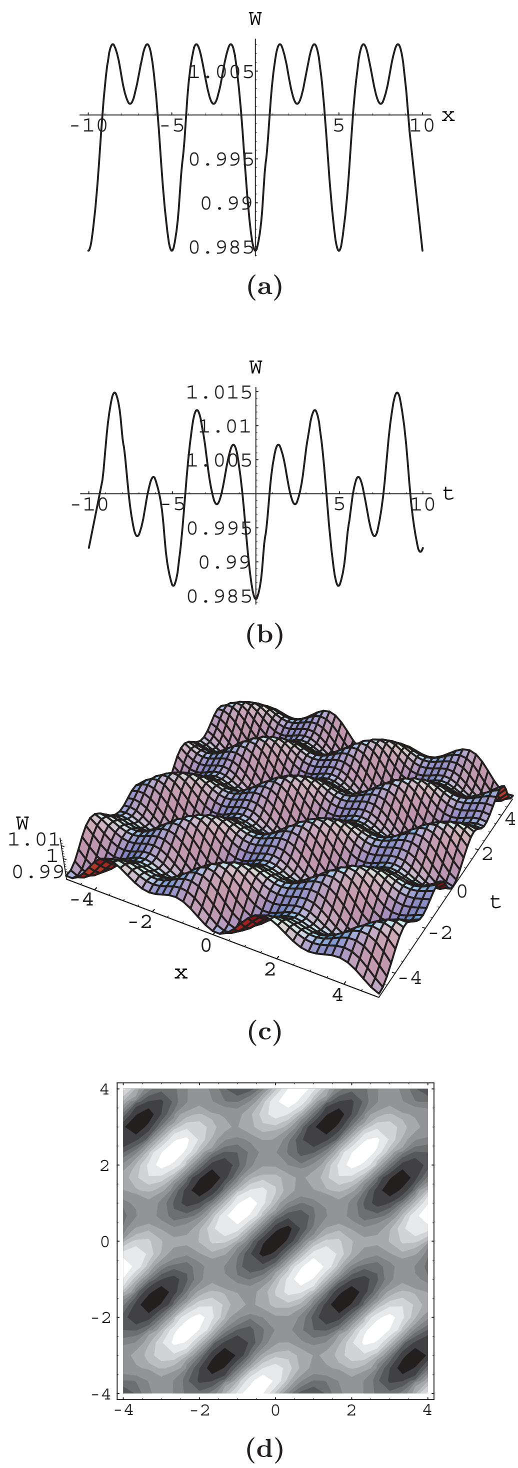

FIG. 4. (Color online) Two-periodic wave for the NKdV equation (1.1). (a) and (b) show that every one-periodic wave is periodic in both the x and y directions. (c) Perspective view of the wave. (d) Overhead view of the wave, with contour plot shown. The bright hexagons are crests and the dark hexagons are troughs.

Proof. Using (10.20), we may expand the function F in the following explicit form:

$$F = 1 + (e^{2\pi i\zeta_1} + e^{-2\pi i\zeta_1})e^{-\pi\tau_{11}} + (e^{2\pi i\zeta_2} + e^{-2\pi i\zeta_2})e^{-\pi\tau_{22}} + (e^{2\pi i(\zeta_1+\zeta_2)} + e^{-2\pi i(\zeta_1+\zeta_2)})e^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})} + \cdots .$$

Furthermore, adopting (10.25) and making a transformation, we infer that

$$\begin{aligned} F &= 1 + e^{\hat\zeta_1} + e^{\hat\zeta_2} + e^{\hat\zeta_1+\hat\zeta_2-2\pi\tau_{12}} + \rho_{11}^4 e^{-\hat\zeta_1} \\ &\quad + \rho_{22}^4 e^{-\hat\zeta_2} + \rho_{11}^4\rho_{22}^4 e^{-\hat\zeta_1-\hat\zeta_2-2\pi\tau_{12}} + \cdots \\ &\longrightarrow 1 + e^{\hat\zeta_1} + e^{\hat\zeta_2} + e^{\hat\zeta_1+\hat\zeta_2+A_{12}}, \quad \text{as } \rho_{11},\rho_{22} \to 0, \end{aligned}$$

where $\hat\zeta_j = \alpha_j x + \hat\beta_j t + \delta_j,\ j = 1,2,$ and $\hat\beta_j = 2\pi i\beta_j, j = 1,2.$

We now need to prove

$$\hat\beta_j \longrightarrow \frac{-2k_j w_0}{k_j^2 + 4v_0},\ \hat\zeta_j \longrightarrow \xi_j, \quad j = 1,2, \text{ as } \rho_{11},\rho_{22} \to 0. \tag{10.27}$$

As in the case of $N = 1$, the solution of the system (10.23) has the following form:

$$\begin{aligned} \beta_1 &= \beta_{1,0} + \beta_{1,1}\rho_{11} + \beta_{2,2}\rho_{22} + o(\rho_{11},\rho_{22}), \\ \beta_2 &= \beta_{2,0} + \beta_{2,1}\rho_{11} + \beta_{2,2}\rho_{22} + o(\rho_{11},\rho_{22}), \\ \lambda &= \lambda_0 + \lambda_1\rho_{11} + \lambda_2\rho_{22} + o(\rho_{11},\rho_{22}). \end{aligned} \tag{10.28}$$

Expanding functions $\vartheta_j,\ j = 1,2,3,4$ in Eqs. (10.21) and (10.22) with substitution of assumption (10.28), and letting $\rho_{11},\rho_{22} \longrightarrow 0$, we will obtain

$$\begin{aligned} &\lambda_0 = 0, \quad 16\pi i\big(-\pi^2\alpha_1^2 + v_0\big)\beta_{1,0} - 8\pi i w_0\alpha_1 = 0, \\ &16\pi i\big(-\pi^2\alpha_2^2 + v_0\big)\beta_{2,0} - 8\pi i w_0\alpha_2 = 0. \end{aligned} \tag{10.29}$$

Using (10.28) and (10.29), we conclude that

$$\begin{aligned} &\lambda = o(\rho_{11},\rho_{22}) \longrightarrow 0, \\ &\beta_j = \frac{-2k_j w_0}{k_j^2 + 4v_0} + o(\rho_{11},\rho_{22}) \longrightarrow \frac{-2k_j w_0}{k_j^2 + 4v_0}, \text{ as } \rho_{11},\rho_{22} \to 0, \end{aligned}$$

and therefore we have (10.26). So the two-periodic wave solution (10.24) tends to the two-soliton solution (6.22) as $\rho_{11},\rho_{22} \to 0.$ ■

In this paper, we only consider one- and two-periodic wave solutions of the NKdV equation (1.1). There are still certain computation difficulties in the calculation for the case of $N > 2$, which will be studied in the future.

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