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Intelligent OFDM telecommunication system. Part 2. Examples of complex and quaternion many-parameter transforms

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Abstract. In this paper, we propose unified mathematical forms of many-parametric complex and quaternion Fourier transforms for novel Intelligent OFDM-telecommunication systems (OFDM-TCS). Each many-parametric transform (MPT) depends on many free angle parameters. When parameters are changed in some way, the type and form of transform are changed as well. For example, MPT may be the Fourier transform for one set of parameters, wavelet transform for other parameters and other transforms for other values of parameters. The new Intelligent-OFDM-TCS uses inverse MPT for modulation at the transmitter and direct MPT for demodulation at the receiver.

1. Introduction

1.1. Jacobi parametrization of orthogonal transforms

One of the best-unknown MPT was developed by the 19^{th} century mathematician Jacobi [1]. We recall that Jacobi's sequential method reduces an orthogonal matrix $\mathbf{U}$ to identical matrix by applying orthogonal rotations to right of $\mathbf{U}$, $\mathbf{Q}=\mathbf{U}\cdot\mathbf{J}_N\left(\varphi_{pq}\right)$, where orthonormal Jacobi rotation

$$
\mathbf{J}_N^{(p,q)}(\varphi_{pq}) = \begin{array}{cc} & \begin{array}{ccccccc} & & p & & q & & \end{array} \\ \begin{array}{c} \\ \\ p \\ \\ q \\ \\ \\ \end{array} & \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c_{p,q} & \cdots & s_{p,q} & \cdots & 0 \\ \vdots & & \vdots & 1 & \vdots & & \vdots \\ 0 & \cdots & s_{p,q} & \cdots & -c_{p,q} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \end{array},
$$

is used to reduce the element U_{pq} or U_{qp} to zero. Jacobi rotation $\mathbf{J}_N^{(p,q)}\left(\varphi_{pq}\right)$ operates on p-th and q-th element of the p-th row of $\mathbf{U}=\left[U_{ik}\right]_{i,k=1}^N$ such that Q_{pq} becomes zero. For $Q_{pq}=0$ it must be required:

$-U_{pp}c + U_{pq}s = 0$. Hence, the expression for $\mathrm{tg}\left(\varphi_{pq}\right)$ become $\mathrm{tg}\left(\varphi_{pq}\right) = U_{pq} / U_{pp}$. This is equivalent to $(c,s) = \left(U_{pp} \Big/ \sqrt{U_{pp}^2 + U_{pq}^2},\ U_{pq} \Big/ \sqrt{U_{pp}^2 + U_{pp}^2} \right)$. For example,

$$\mathbf{Q}_N^{(1)} = \mathbf{U}_N \cdot \mathbf{J}_N^{(1,2)}(\varphi_{12}) = \begin{pmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{pmatrix} \left(\begin{array}{cc|cccc} c & s & & & & \\ s & -c & & & & \\ \hline & & 1 & & & \\ & & & 1 & & \\ & & & & \ldots & \\ & & & & & 1 \end{array} \right) = \begin{pmatrix} \square & \blacksquare & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{pmatrix},$$

where white boxes are nonzero elements and black box is the zero element. Further,

$$\mathbf{Q}_N^{(2)} = \mathbf{U}_N \cdot \mathbf{J}_N^{(1,2)}(\varphi_{12}) \mathbf{J}_N^{(1,3)}(\varphi_{13}) = \begin{pmatrix} \square & \blacksquare & \blacksquare & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{pmatrix}, \quad \ldots, \quad \mathbf{Q}_N^{(N-1)} = \mathbf{U}_N \cdot \mathbf{J}_N^{(1,2)}(\varphi_{12}) \mathbf{J}_N^{(1,3)}(\varphi_{13}) \cdots \mathbf{J}_N^{(1,N)}(\varphi_{1N}) = \begin{pmatrix} \square & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{pmatrix}.$$

But $\mathbf{Q}_N^{(N-1)}$ is an orthogonal matrix as the product of orthogonal matrices. For this reason, it can have only the following form: $\mathbf{Q}_N^{(N-1)} = \mathbf{U}_N \cdot \mathbf{J}_N(\varphi_{12}) \mathbf{J}_N(\varphi_{13}) \cdots \mathbf{J}_N(\varphi_{1N}) = \mathbf{U}_N \cdot \prod_{q=2}^{\rightarrow N} \mathbf{J}_N(\varphi_{1,q}) = [\pm 1] \oplus \mathbf{Q}_{N-1}$, where $\oplus$ is the symbol of direct matrix sum, $\mathbf{Q}_{N-1}$ is $(N-1)\times(N-1)$ orthogonal matrix opposite to $\mathbf{Q} = \mathbf{Q}_N$ that is $(N \times N)$ orthogonal matrix. Obviously, $\mathbf{Q}_N^{\left(\frac{(N-1)N}{2}\right)} = \mathbf{U}_N \prod_{p=1}^{\rightarrow N-1} \prod_{q=p+1}^{\rightarrow N} \mathbf{J}_N^{(p,q)}(\varphi_{p,q}) = \mathbf{diag}_N(\pm 1, \ldots, \pm 1) = \underbrace{= [\pm 1] \oplus [\pm 1] \oplus \ldots \oplus [\pm 1]}_{N}$. Hence, an orthogonal matrix $\mathbf{U}$ is composed of series of Jacobi rotations: $\mathbf{U}_N(\boldsymbol{\varphi}) = \prod_{p=1}^{\leftarrow N-1} \prod_{q=p+1}^{\leftarrow N} \mathbf{J}_N^{(p,q)} \varphi_{p,q})$, where $\boldsymbol{\varphi} = (\varphi_{1,2}, \varphi_{1,3}, \ldots, \varphi_{N-1,N})$ is $N(N+1)/2$-dimension vector of so-called the Jacobi angles φ_{pq}. Here $\prod_{i=0}^{\rightarrow n-1} \mathrm{T}^i = \mathrm{T}^0 \mathrm{T}^1 \cdots \mathrm{T}^{n-1}$ and $\prod_{i=0}^{\leftarrow n-1} \mathrm{T}^i = \mathrm{T}^{n-1} \cdots \mathrm{T}^1 \mathrm{T}^0$ are the right and left multiplications, respectively. Many-parameter representation $\mathbf{U}_N(\boldsymbol{\varphi}) = \prod_{p=1}^{\leftarrow N-1} \prod_{q=p+1}^{\leftarrow N} \mathbf{J}_N^{(p,q)}(\varphi_{p,q})$ is very important with theoretical point of view, but it is not very useful with digital processing point of view.

1.2. MPT in signal and image analysis

The concept of fast MPT in signal and image processing was printed by Andrews [2] in the form of tensor product of Jacobi (2×2)-matrices $\mathbf{J}_2(\varphi_i) = \begin{bmatrix} \cos\varphi_i & \sin\varphi_i \\ \sin\varphi_i & -\cos\varphi_i \end{bmatrix}$, $i = 1, 2, \ldots, n$:

$$\mathbf{CS}_{2^n}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \mathbf{J}_2(\varphi_1) \otimes \cdots \otimes \mathbf{J}_2(\varphi_n) = \begin{bmatrix} \cos\varphi_1 & \sin\varphi_1 \\ \sin\varphi_1 & -\cos\varphi_1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} \cos\varphi_n & \sin\varphi_n \\ \sin\varphi_n & -\cos\varphi_n \end{bmatrix}.$$

This tensor product is factorized into the ordinary product of sparse matrices

$$\mathbf{CS}_{2^n}(\varphi_1,\varphi_2,...,\varphi_n)=\prod_{r=1}^{n}\left[I_{2^{r-1}}\otimes\mathbf{J}_2(\varphi_r)\otimes I_{2^{n-r}}\right].$$

It is just the fast Andrews transform. In particular case, when $\varphi_1=\varphi_2=...=\varphi_n\equiv\pi/4$, we obtain ordinary Walsh transform $\mathbf{W}_{2^n}=\left(\sqrt{2}/2\right)^n\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\otimes\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. The same form has n-parameter Haar transform [3-6]: $\mathbf{HT}_{2^n}(\varphi_1,\varphi_2,...,\varphi_n)=\prod_{r=1}^{n}\left[\left(\mathbf{J}_2(\varphi_r)\otimes I_{2^{n-r}}\right)\oplus I_{2^n-2^{n-r+1}}\right]$. Obviously, $\mathbf{HT}_{2^n}\left(\frac{\pi}{4},\frac{\pi}{4},...,\frac{\pi}{4}\right)\equiv\mathbf{HT}_{2^n}=\prod_{i=1}^{n}[(\mathbf{W}_2\otimes I_{2^{n-i}})\oplus I_{2^n-2^{n-i+1}}]P_{2^n}$ is the ordinary Haar transform.

Let $p=p(k_r,s_r)=2^{r-1}k_r+s_r$, $q=q(k_r,s_r)=p(k_r,s_r)+2^{n-r}$ be the radix-$(2^{r-1},2^{n-r})$ representation of $p,q\in\left\{0,1,...,2^{n-1}-1\right\}$, where $k_r\in\left\{0,1,...,2^{r-1}-1\right\}$, $s_r\in\left\{0,1,...,2^{n-r}-1\right\}$. Then we can write matrices $\mathbf{CS}_{2^n}(\varphi_1,\varphi_2,...,\varphi_n)$ as

$$\mathbf{CS}_{2^n}(\varphi_1,\varphi_2,...,\varphi_n)=\prod_{r=1}^{n}\left[I_{2^{r-1}}\otimes\mathbf{J}_2(\varphi_r)\otimes I_{2^{n-r}}\right]=\prod_{r=1}^{n}\left[\prod_{k_r=0}^{2^{r-1}-1}\prod_{s_r=0}^{2^{n-r}-1}\mathbf{J}_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left(\varphi_r\right)\right].$$

Using different angles in every Jacobi matrix, we obtain $n\cdot 2^{n-1}$ - parameter Walsh-like transform:

$$\mathbf{CS}_{2^n}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,...,\boldsymbol{\varphi}_n)=\prod_{r=1}^{n}\left[\prod_{k_r=0}^{2^{r-1}-1}\prod_{s_r=0}^{2^{n-r}-1}\mathbf{J}_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left(\varphi_r^{(k_r,s_r)}\right)\right], \quad (1)$$

where

$\boldsymbol{\varphi}_1=(\varphi_1^{(0,0)},\varphi_1^{(0,1)},...,\varphi_1^{(0,2^{n-1}-1)})$, $\boldsymbol{\varphi}_2=(\varphi_2^{(0,0)},\varphi_2^{(0,1)},...,\varphi_2^{(0,2^{n-2}-1)};\varphi_2^{(1,0)},\varphi_2^{(1,1)},...,\varphi_2^{(1,2^{n-2}-1)})$,

$\boldsymbol{\varphi}_3=(\varphi_3^{(0,0)},\varphi_3^{(0,1)},...,\varphi_3^{(0,2^{n-3}-1)};\varphi_2^{(1,0)},\varphi_2^{(1,1)},...,\varphi_2^{(1,2^{n-3}-1)};\varphi_3^{(2,0)},\varphi_3^{(2,1)},...,\varphi_3^{(2,2^{n-3}-1)};\varphi_3^{(3,0)},\varphi_3^{(3,1)},...,\varphi_3^{(3,2^{n-3}-1)})$,

...,

$\boldsymbol{\varphi}_{n-1}=(\varphi_{n-1}^{(0,0)};\varphi_{n-1}^{(1,0)};...;\varphi_{n-1}^{(2^{n-1}-1,0)})$.

Recently, several authors [7]-[14] have proposed Jacobi parametrization of Golay and wavelet transforms.

1.3. Fractional and many-parameter ordinary Fourier transforms

The eigendecomposition (ED) is a tool of both practical and theoretical importance in digital signal and image processing. The ED transforms are defined by the following way. Let $\mathcal{U}$ be an arbitrary discrete orthogonal (or unitary) $(N\times N)$–transform, λ_k and $\left|\Psi_m(n)\right\rangle$, $m,n=0,1,\ldots,N-1$, be its eigenvalues and column-eigenvectors, respectively. Let $\mathbf{U}=\left[\left|\Psi_0(n)\right\rangle,\left|\Psi_1(n)\right\rangle,...,\left|\Psi_{N-1}(n)\right\rangle\right]$ be the matrix of eigenvectors of the $\mathcal{U}$–transform. Then $\mathbf{U}^{-1}\cdot\mathcal{U}\cdot\mathbf{U}=\mathbf{Diag}\left\{\lambda_0,\ldots,\lambda_{N\text{-}1}\right\}=\Lambda$. Hence, we have the following eigendecomposition: $\mathcal{U}=\left[u_k(n)\right]:=\sum_{m=0}^{N-1}\lambda_m\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|=\mathbf{U}\cdot\mathbf{Diag}\left(\lambda_0,\ldots,\lambda_{N\text{-}1}\right)\cdot\mathbf{U}^{-1}$.

Definition 1. For an arbitrary real numbers $a_0,\ldots,a_{N-1}$ we introduce the many-parameter $\mathcal{U}$–transform

$$\mathcal{U}^{(a_0,\ldots,a_{N-1})}:=\mathbf{U}\cdot\mathbf{Diag}\left(\lambda_0^{a_0},\ldots,\lambda_{N\text{-}1}^{a_{N\text{-}1}}\right)\cdot\mathbf{U}^{-1}. \quad (2)$$

If $a_0=\ldots=a_{N-1}\equiv a$ then this transform is called the fractional $\mathcal{U}$–transform. For this transform we have

$$\mathcal{U}^a:=\mathbf{U}\left\{\mathrm{diag}\left(\lambda_0^a,\ldots,\lambda_{N\text{-}1}^a\right)\right\}\mathbf{U}^{-1}=\mathbf{U}\,\Lambda^a\,\mathbf{U}^{-1}. \quad (3)$$

The zeroth-order fractional $\mathcal{U}$–transform is equal to the identity transform: $\mathcal{U}^0 = \mathbf{U}\Lambda^0\mathbf{U}^{-1} = \mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$ and the first-order fractional $\mathcal{U}$-transform operator is equal to the initial transform $\mathcal{U}^1 = \mathbf{U}\Lambda\mathbf{U}^{-1}$. The families $\left\{\mathcal{U}^{(\alpha_0,\dots,\alpha_{N-1})}\right\}_{(\alpha_0,\dots,\alpha_{N-1})\in\mathbf{R}^N}$ and $\left\{\mathcal{U}^a\right\}_{a\in\mathbf{R}}$ form many- and one-parameter continuous unitary groups with multiplications $\mathcal{U}^{(a_0,\dots,a_{N-1})}\mathcal{U}^{(b_0,\dots,b_{N-1})} = \mathcal{U}^{(a_0+b_0,\dots,a_{N-1}+b_{N-1})}$ and $\mathcal{U}^a\mathcal{U}^b = \mathcal{U}^{a+b}$, respectively.

Let $\mathcal{F}_N = \left[e^{-j\frac{2\pi}{N}kn}\right]_{k,n=0}^{N-1}$ be the discrete Fourier $(N\times N)$–transform (DFT). Relevant properties are that the square $\left(\mathcal{F}_N^2 f\right)(x) = f(-x)$ is the inversion operator, and that its fourth power $\left(\mathcal{F}_N^4 f\right)(x) = f(x)$ is the identity; hence $\mathcal{F}_N^3 = \mathcal{F}_N^{-1}$. The operator $\mathcal{F}_N$ thus generates the Fourier cyclic group of order 4: $\mathbf{Gr}_4(\mathcal{F}) = \left\{\mathcal{F}_N^a\right\}_{a\in\{0,1,2,3\}} = \left\{I,\mathcal{F}_N^1,\mathcal{F}_N^2,\mathcal{F}_N^3\right\}$. The idea of fractional powers of the Fourier operator $\mathcal{F}$ appears in the mathematical literature [15-22]. This idea is to consider the eigenvalue decomposition of the Fourier transform $\mathcal{F} = \sum_{n=0}^{\infty}\lambda_n\left|\Psi_n(x)\right\rangle\left\langle\Psi_n(\omega)\right|$ in terms of eigenvalues $\lambda_n = e^{jn\pi/2} = j^n$ and eigen-functions $\Psi_n(x)$ in the form of the Hermite functions. The family of FrFT $\left\{\mathcal{F}^a\right\}_{a\in[0,4)}$ (instead of $\left\{\mathcal{F}^a\right\}_{a\in\{0,1,2,3\}}$) is constructed by replacing the n-th eigenvalue $\lambda_n = e^{jn\pi/2}$ by its a-th power $\lambda_n^a = e^{jn\pi a/2}$, for a between 0 and 4.

The eigenvalues of the standard DFT matrix $\mathcal{F}_N$ are the fourth roots of unity, to be denoted by $\lambda_s \in \left\{e^{j\pi s/2}\right\}_{s=0}^{3} \in \{\pm 1, \pm j\}$ and $\left\{\Psi_m(n)\right\}_{m=0}^{N-1}$ are the discrete Hermite polynomials. This divides the space of *N*-point complex signals into four Fourier invariant subspaces whose dimensions N_s are the multiplicities of the eigenvalues λ_s, which have a modulo 4 recurrence in the dimension $N = 2^N = 4M$ given by $N_0 = M+1, N_1 = M-1,\ N_2 = M,\ N_3 = M$. Let $s(n):\{0,1,2,...,\mathrm{N}-1\}\to\{0,1,2,3\}$ be a peculiar function. It determines a distribution of eigenvalues along main diagonal $\mathbf{Diag}\left(e^{j\frac{\pi}{2}s(n)a}\right)$ in (3). This function takes $M+1$ times value 0, $M-1$ times value 1, and M times values 2 and 3.

Definition 2. The discrete classical and Bargmann fractional Fourier transforms are defined as

$$\mathcal{F}^a = \left[e_k^{(a)}(n)\right] := \mathbf{U}\left\{\mathbf{Diag}\left(e^{j\frac{\pi}{2}s(m)a}\right)\right\}\mathbf{U}^{-1} = \sum_{m=0}^{N-1} e^{j\frac{\pi}{2}s(m)a}\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|, \tag{4}$$

$$\mathcal{BF}^a = \left[be_k^{(a)}(n)\right] := \mathbf{U}\left\{\mathbf{Diag}\left(e^{j\frac{\pi}{2}ma}\right)\right\}\mathbf{U}^{-1} = \sum_{m=0}^{N-1} e^{j\frac{\pi}{2}ma}\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|, \tag{5}$$

Definition 3. The discrete classical-like and Bargmann-like many-parameter DFT we define by the following way

$$\mathcal{F}^{(\mathbf{a})} = \mathcal{F}^{(a_0,a_1,a_2,\dots,a_{N-1})} = \left[e_k^{(\mathbf{a})}(n)\right] = \mathbf{U}\left\{\mathrm{diag}\left(e^{j\frac{\pi}{2}s(m)a_m}\right)\right\}\mathbf{U}^{-1} = \sum_{m=0}^{N-1} e^{j\frac{\pi}{2}s(m)a_m}\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|, \tag{6}$$

$$\mathcal{BF}^{(\mathbf{a})} = \mathcal{BF}^{(a_0,a_1,a_2,\dots,a_{N-1})} = \left[be_k^{(\mathbf{a})}(n)\right] := \mathbf{U}\left\{\mathrm{diag}\left(e^{j\frac{\pi}{2}ma_m}\right)\right\}\mathbf{U}^{-1} = \sum_{m=0}^{N-1} e^{j\frac{\pi}{2}ma_m}\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|, \tag{7}$$

where $\mathbf{a} = (a_0,a_1,a_2,\dots,a_{N-1})$.

2. Quaternion MPT

2.1. Quaternion algebra

The space of quaternions denoted by $\mathbb{H}(\mathbf{R})$ were first invented by W.R. Hamilton in 1843 as an extension of the complex numbers into four dimensions [23-24]. General information on quaternions may be obtained from [26]-[27].

Definition 4. Numbers of the form ${}^4\mathbf{q} = a\mathbf{1}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$, where $a,b,c,d \in \mathbf{R}$ are called quaternions, where 1) 1 is the real unit; 2) $\mathbf{i},\mathbf{j},\mathbf{k}$ are three imaginary units.

We speck that quaternions ${}^4\mathbf{q}=a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ are written in the standard format. The addition and subtraction of two quaternions ${}^4\mathbf{q}_1 = a_1 + x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and ${}^4\mathbf{q}_2 = a_2 + x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$ are given by

$$\begin{aligned} {}^4\mathbf{q}_1 \pm {}^4\mathbf{q}_2 &= (a_1+b_1\mathbf{i}+c_1\mathbf{j}+d_1k)\pm(a_2+b_2\mathbf{i}+c_2\mathbf{j}+d_2k) = \\ &= (a_1\pm a_2)+(b_1\pm b_2)\mathbf{i}+(b_1\pm b_2)\mathbf{j}+(b_1\pm b_2)k. \end{aligned}$$

The product of quaternions for the standard format Hamilton defined according as:

$$\begin{aligned} {}^4\mathbf{q}_1 \circ {}^4\mathbf{q}_2 = {} & (a_1+b_1\mathbf{i}+c_1\mathbf{j}+d_1\mathbf{k})\circ(a_2+b_2\mathbf{i}+c_2\mathbf{j}+d_2\mathbf{k}) = \left(a_1a_2-b_1b_2-c_1c_2-d_1d_2\right)+ \\ & +\left(a_1b_2+b_1a_2+c_1d_2-d_1c_2\right)\mathbf{i}+\left(a_1c_2+c_1a_2+d_1b_2-b_1d_2\right)\mathbf{j}+\left(a_1d_2+d_1a_2+b_1c_2-c_1b_2\right)\mathbf{k}, \end{aligned}$$

where $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-1;$ 3) $\mathbf{i}\circ\mathbf{j}=-\mathbf{i}\circ\mathbf{j}=\mathbf{k},\ \mathbf{i}\circ\mathbf{k}=-\mathbf{k}\circ\mathbf{i}=\mathbf{j},\ \mathbf{j}\circ\mathbf{k}=-\mathbf{k}\circ\mathbf{j}=\mathbf{i}$.

The set of quaternions with operations of multiplication and addition forms 4-D algebra $\mathcal{A}_4\left(\mathbf{R}\,|\,1,\mathbf{i},\mathbf{j},\mathbf{k}\right) := \mathbf{R}+\mathbf{Ri}+\mathbf{Rj}+\mathbf{Rk}$ over the real field $\mathbf{R}$. Number component a and direction component ${}^3\mathbf{r}=b\mathbf{i}+c\mathbf{j}+d\mathbf{k}\in\mathbf{R}^3$ were called the *scalar* and 3-D *vector* parts of quaternion, respectively. Now these components are denoted as $S({}^4\mathbf{q})=a\in\mathbf{R}$ and $V(\mathbf{q})={}^3\mathbf{r}=b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$. A non–zero element ${}^3\mathbf{r}=bi+cj+dk$ is called *pure vector quaternion*. Hence, according to W. Hamilton every quaternion is the sum of a scalar number and a pure vector quaternion ${}^4\mathbf{q}= a+(b\mathbf{i}+c\mathbf{j}+d\mathbf{k})=a+{}^3\mathbf{r}=S(\mathbf{q})+V(\mathbf{q})$, where $a=S({}^4\mathbf{q})$ and $V({}^4\mathbf{q})={}^3\mathbf{r}$. Since $\mathbf{i}\circ\mathbf{j}=\mathbf{k}$, then a quaternion ${}^4\mathbf{q} = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}=(a1+b\mathbf{i})+$ $+(c\mathbf{j}+d\mathbf{i}\circ\mathbf{j})=(a+b\mathbf{i})+(c+d\mathbf{i})\circ\mathbf{j}=\mathbf{z}+\mathbf{w}\circ\mathbf{j}$ is the sum of two complex numbers $\mathbf{z}=a+b\mathbf{i}$, $\mathbf{w}=c+d\mathbf{i}$ with a new imaginary unit $\mathbf{j}$. So, every quaternion can be represented in several ways:

- as a 4-D hypercomplex number ${}^4\mathbf{q} = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$, $a,b,c,d\in\mathbf{R}$ (standard format);
- as a sum of a scalar and vector parts $\mathbf{q}=a+{}^3\mathbf{r}$ (1,3-D hypercomplex format);
- as a 2-D hypercomplex numbers ${}^{2,2}\mathbf{q}=\mathbf{z}+\mathbf{w}\circ\mathbf{j}$, $\mathbf{z},\mathbf{w}\in\mathbf{C}$ (2,2D complex format of $\mathbf{q}$).

The product of quaternions for the last two forms Hamilton defined as:

$$\begin{aligned} {}^4\mathbf{q}_1\circ{}^4\mathbf{q}_2 &= (\mathbf{z}_1+\mathbf{w}_1\circ\mathbf{j})\circ(\mathbf{z}_2+\mathbf{w}_2\circ\mathbf{j})=(\mathbf{z}_1\mathbf{z}_2-\mathbf{w}_1\bar{\mathbf{w}}_2)+(\mathbf{w}_1\bar{\mathbf{z}}_2+\mathbf{z}_1\mathbf{w}_2)\circ\mathbf{j}, \\ {}^4\mathbf{q}_1\circ{}^4\mathbf{q}_2 &= (a_1+\vec{\mathbf{r}}_1)\circ(a_2+\vec{\mathbf{r}}_2)=\left(a_1a_2-\left(\vec{\mathbf{r}}_1,\vec{\mathbf{r}}_2\right)\right)+\left(a_1\vec{\mathbf{r}}_2+a_2\vec{\mathbf{r}}_1+\left[\vec{\mathbf{r}}_1\times\vec{\mathbf{r}}_2\right]\right), \end{aligned}$$

where $S({}^4\mathbf{q}_1\circ{}^4\mathbf{q}_2)=a_1a_2-\left\langle{}^3\mathbf{r}_1\,|\,{}^3\mathbf{r}_2\right\rangle$, $V({}^4\mathbf{q}_1\circ{}^4\mathbf{q}_2)=a_1{}^3\mathbf{r}_2+a_2{}^3\mathbf{r}_1+{}^3\mathbf{r}_1\times{}^3\mathbf{r}_2$. Here $\left\langle{}^3\mathbf{r}_1\,|\,{}^3\mathbf{r}_2\right\rangle=b_1b_2+c_1c_2+d_1d_2$ and ${}^3\mathbf{r}_1\times{}^3\mathbf{r}_2=\mathbf{i}\left(c_1d_2-d_1c_2\right)-\mathbf{j}\left(b_1d_2-d_1b_2\right)+\mathbf{k}\left(b_1c_2-c_1b_2\right)$ are scalar and vector products, respectively.

The commutative property of multiplication does not hold for quaternion numbers: ${}^4\mathbf{q}_1\circ{}^4\mathbf{q}_2\neq{}^4\mathbf{q}_2\circ{}^4\mathbf{q}_1$. However, if the vector parts of quaternion numbers are parallel to each other ${}^3\mathbf{r}_1\,\|\,{}^3\mathbf{r}_2$, then their product is commutative.

Definition 5. Let ${}^4\mathbf{q} = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}\in\mathbb{H}(\mathbf{R})$ be a quaternion ($a,b,c,d\in\mathbf{R}$). Then

$$ {}^4\overline{\mathbf{q}} \;=\; \overline{a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}} = a-b\mathbf{i}-c\mathbf{j}-d\mathbf{k},\ {}^4\overline{\mathbf{q}} \;=\; \overline{a+{}^3\mathbf{r}} = a-{}^3\mathbf{r} $$

is the conjugate of ${}^4\mathbf{q}$, $N({}^4\mathbf{q}) = \|{}^4\mathbf{q}\| = \sqrt{a^2+b^2+c^2+d^2} = \sqrt{{}^4\overline{\mathbf{q}}\circ{}^4\mathbf{q}} = \sqrt{{}^4\mathbf{q}\circ{}^4\overline{\mathbf{q}}}$ is the norm of ${}^4\mathbf{q}$, and $tr({}^4\mathbf{q}) = 2a = {}^4\mathbf{q}+{}^4\overline{\mathbf{q}}$ is the trace of ${}^4\mathbf{q}$. Therefore ${}^4\mathbf{q}^2 - tr({}^4\mathbf{q})\,{}^4\mathbf{q} + N^2({}^4\mathbf{q}) = 0.$

Proposition 1. We have $\overline{{}^4\mathbf{q}_1\circ{}^4\mathbf{q}_2} = {}^4\overline{\mathbf{q}}_2\circ{}^4\overline{\mathbf{q}}_1$ and $N({}^4\mathbf{q}_1\circ{}^4\mathbf{q}_2) = N({}^4\mathbf{q}_1)\cdot N({}^4\mathbf{q}_2)$ f or every ${}^4\mathbf{q}_1, {}^4\mathbf{q}_2 \in \mathbb{H}(\mathbf{R})$. Note that $\|1\| = 1,\ \ \|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1.$

Definition 6. Quaternions $\left\{{}^4\boldsymbol{\rho}\,\middle|\,N({}^4\boldsymbol{\rho}) = 1\right\}$ of unit norm area called *unit quaternions.*

The unit quaternions $\boldsymbol{\rho}$ form a 3D hypersphere $\mathbb{S}^3 \subset \mathbb{H}(\mathbf{R}) \sim \mathbf{R}^4$. For each quaternion ${}^4\mathbf{q}$ with nonzero norm the following quaternion

$$ {}^4\boldsymbol{\rho} = \frac{{}^4\mathbf{q}}{\|{}^4\mathbf{q}\|} = \frac{a+{}^3\mathbf{r}}{\|{}^4\mathbf{q}\|} = \frac{a}{\|{}^4\mathbf{q}\|} + \frac{{}^3\mathbf{r}}{\|{}^4\mathbf{q}\|} = \frac{a}{\|{}^4\mathbf{q}\|} + \frac{\|{}^3\mathbf{r}\|}{\|{}^4\mathbf{q}\|}\frac{{}^3\mathbf{r}}{\|{}^3\mathbf{r}\|} = \frac{a}{\|{}^4\mathbf{q}\|} + \frac{\|{}^3\mathbf{r}\|}{\|{}^4\mathbf{q}\|}\,{}^3\boldsymbol{\mu} = $$

$$ = \cos\alpha + {}^3\boldsymbol{\mu}\sin\alpha = \cos\alpha + (\mu_1\mathbf{i}+\mu_2\mathbf{j}+\mu_3\mathbf{k})\sin\alpha $$

is an unit quaternion, where $\|{}^3\mathbf{r}\| = \sqrt{b^2+c^2+d^2}$, ${}^3\boldsymbol{\mu} = {}^3\mathbf{r}/\|{}^3\mathbf{r}\|$, $\cos\alpha = a/\|{}^4\mathbf{q}\|$, $\sin\alpha = \|{}^3\mathbf{r}\|/\|{}^4\mathbf{q}\|$, $\mu_1 = b/\|{}^3\mathbf{r}\|$, $\mu_2 = c/\|{}^3\mathbf{r}\|$, $\mu_3 = c/\|{}^3\mathbf{r}\|$ and $\boldsymbol{\mu} = \mu_1\mathbf{i}+\mu_2\mathbf{j}+\mu_3\mathbf{k}$. Obviously,

$$ {}^4\mathbf{q} = \|{}^4\mathbf{q}\|\cdot{}^4\boldsymbol{\rho} = \|{}^4\mathbf{q}\|\cdot\left[\cos\alpha + {}^3\boldsymbol{\mu}\sin\alpha\right] = \|{}^4\mathbf{q}\|\cdot\left[\cos\alpha + (\mu_1\mathbf{i}+\mu_2\mathbf{j}+\mu_3\mathbf{k})\sin\alpha\right]. $$

Introducing the polar coordinates on $\mathbb{S}^3$

$$ a = \|{}^4\mathbf{q}\|\cos\alpha,\ \ b = \left(\|{}^4\mathbf{q}\|\cos\gamma\right)\sin\alpha,\ c = \left(\|{}^4\mathbf{q}\|\sin\gamma\cos\theta\right)\sin\alpha,\ d = \left(\|{}^4\mathbf{q}\|\sin\gamma\sin\theta\right)\sin\alpha, $$

we may write

$$ {}^4\mathbf{q} \;=\; \|{}^4\mathbf{q}\|\left[\cos\alpha + (\mathbf{i}\cos\gamma + \mathbf{j}\sin\gamma\cos\theta + \mathbf{k}\sin\gamma\sin\theta)\sin\alpha\right] = \|{}^4\mathbf{q}\|\left[\cos\alpha + {}^3\boldsymbol{\mu}(\gamma,\theta)\sin\alpha\right], $$

where $\theta,\varphi \in [0,\pi]$, $\alpha \in [0,2\pi)$, ${}^3\boldsymbol{\mu}(\gamma,\theta = \mathbf{i}\cos\gamma + \mathbf{j}\sin\gamma\cos\theta + \mathbf{k}\sin\gamma\sin\theta \in \mathbf{S}^2$ is a pure unit quaternion.

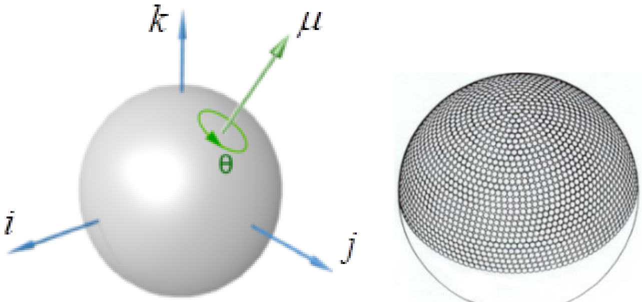

Figure 1. Each 3D vector $\vec{\boldsymbol{\mu}} \in \mathbb{S}^2$ of unit length can play a role of classical imaginary unit. For example, the special elements ${}^3\mathbf{i}, {}^3\mathbf{j}, {}^3\mathbf{k}$ are such elements.

In particular, for ${}^4\mathbf{q}_1 \equiv {}^3\mathbf{r}_1 = b_1\mathbf{i}+c_1\mathbf{j}+d_1\mathbf{k}$ and ${}^4\mathbf{q}_2 \equiv {}^3\mathbf{r}_2 = b_2\mathbf{i}+c_2\mathbf{j}+d_2\mathbf{k}_2$ we obtain

$$ {}^3\mathbf{r}_1\circ{}^3\mathbf{r}_2 \;=-\left\langle{}^3\mathbf{r}_1)\,|\,{}^3\mathbf{r}_2\right\rangle + \left[{}^3\mathbf{r}_1\times{}^3\mathbf{r}_2)\right],\ \ {}^3\mathbf{r}^2 = {}^3\mathbf{r}\circ{}^3\mathbf{r} = -\left\langle{}^3\mathbf{r}\,|\,{}^3\mathbf{r}\right\rangle = -\|{}^3\mathbf{r}\|^2, $$

and for a pure quaternion ${}^3\boldsymbol{\mu} \in \mathbb{S}^2 \subset \mathbf{R}^3$ with unity norm $\|{}^3\boldsymbol{\mu}\| = 1$ we have ${}^3\boldsymbol{\mu}^2 = -\|{}^3\boldsymbol{\mu}\|^2 = -1$, where $\mathbb{S}^2$ denotes the unit 2-D sphere in 3-D space $\mathbf{R}^3$. This unit-vector product identity represents the generalization of the complex-variable identity $i^2 = -1$. This means that, if in the ordinary theory of complex numbers there are only two different square roots of negative unity ($+i$ and $-i$) and they differ only in their signs, then in the quaternion theory there are infinite numbers of different square roots of negative unity ${}^{3}\boldsymbol{\mu}={}^{3}\boldsymbol{\mu}(\gamma,\theta)=\left(\mu_x\mathbf{i}+\mu_y\mathbf{j}+\mu_z\mathbf{k}\right)=\left(\cos\gamma\cdot\mathbf{i}+\sin\gamma\cos\theta\cdot\mathbf{j}+\sin\gamma\sin\theta\cdot\mathbf{k}\right)\in\mathbf{S}^2$, which gives ${}^{3}\boldsymbol{\mu}^2={}^{3}\boldsymbol{\mu}^2(\gamma,\theta)=-1$. Here ${}^{3}\boldsymbol{\mu}(\gamma,\theta)=\left(\cos\gamma,\sin\gamma\cos\theta,\sin\gamma\sin\theta\right)$ being still that point on the spherical surface, which has for its rectangular coordinates $\cos\gamma,\sin\gamma\cos\theta,\sin\gamma\sin\theta$ (see figure 1). In the feature we will omit left index: $\boldsymbol{\mu}(\gamma,\theta)\equiv{}^{3}\boldsymbol{\mu}(\gamma,\theta)$.

2.2. Quaternion-valued functions

The main subject of this section are quaternion-valued discrete exponential functions.

Definition 7. A functions ${}^{4}\mathbf{f}(n):[0,N-1]\to\mathbb{H}(\mathbf{R})$ are called quaternion-valued discrete functions. They have the following form: ${}^{4}\mathbf{f}(n)=f_0(n)+f_1(n)\mathbf{i}+f_2(n)\mathbf{j}+f_3(n)\mathbf{k}$.

The exponential map is $\exp({}^{4}\mathbf{q})=1+{}^{4}\mathbf{q}+\frac{{}^{4}\mathbf{q}^2}{2!}+....+\frac{{}^{4}\mathbf{q}^m}{m!}+...=\sum_{m=0}^{\infty}\frac{{}^{4}\mathbf{q}^m}{m!}$. Clearly for ${}^{4}\mathbf{q}\equiv a\in\mathbf{R}$, $\exp({}^{4}\mathbf{q})=e^a$ is the usual real exponent map on $\mathbf{R}$. In particular, if 0 in $\mathbf{R}$ is the null element, then $\exp(0)=1$. If ${}^{4}\mathbf{q}\equiv{}^{3}\mathbf{r}$ is a non-zero element in $\mathbf{R}^3$ then $\exp({}^{3}\mathbf{r})=\cos(\|{}^{3}\mathbf{r}\|)+\frac{{}^{3}\mathbf{r}}{\|{}^{3}\mathbf{r}\|}\sin(\|{}^{3}\mathbf{r}\|)$.

Theorem 1. For ${}^{4}\mathbf{q}=a+{}^{3}\mathbf{r}\in\mathbb{H}(\mathbf{R})$ $\exp(a+{}^{3}\mathbf{r})=e^a\exp({}^{3}\mathbf{r})=e^a\left(\cos(\|{}^{3}\mathbf{r}\|)+\frac{{}^{3}\mathbf{r}}{\|{}^{3}\mathbf{r}\|}\sin(\|{}^{3}\mathbf{r}\|)\right)$.

Obviously, $\|\exp({}^{3}\mathbf{r})\|=1$ and $\|\exp({}^{3}\mathbf{r})\|=\|\exp(a+{}^{3}\mathbf{r})\|=e^a$. In general case $\exp({}^{4}\mathbf{q}_1)\circ\exp({}^{4}\mathbf{q}_2)\neq$ $\neq\exp({}^{4}\mathbf{q}_2)\circ\exp({}^{4}\mathbf{q}_1)$ and $\exp({}^{4}\mathbf{q}_1+{}^{4}\mathbf{q}_2)\neq\exp({}^{4}\mathbf{q}_1)\circ\exp({}^{4}\mathbf{q}_2)\neq\exp({}^{4}\mathbf{q}_2)\circ\exp({}^{4}\mathbf{q}_1)$.

However, if the vector parts of quaternion numbers ${}^{4}\mathbf{q}_1=a_1+{}^{3}\mathbf{r}_1$ and ${}^{4}\mathbf{q}_2=a_2+{}^{3}\mathbf{r}_2$ are parallel to each other (*i.e.*, ${}^{3}\mathbf{r}_1\|{}^{3}\mathbf{r}_2$), then product $\exp({}^{4}\mathbf{q}_1)\circ\exp({}^{4}\mathbf{q}_2)$ is commutative $\exp({}^{4}\mathbf{q}_1)\circ\exp({}^{4}\mathbf{q}_2)=$ $=\exp({}^{4}\mathbf{q}_2)\circ\exp({}^{4}\mathbf{q}_1)$ and $\exp({}^{4}\mathbf{q}_1+{}^{4}\mathbf{q}_2)=\exp({}^{4}\mathbf{q}_1)\circ\exp({}^{4}\mathbf{q}_2)=\exp({}^{4}\mathbf{q}_2)\circ\exp({}^{4}\mathbf{q}_1)$.

2.3. Quaternion Fourier transforms

Before defining the quaternion Fourier transform, we briefly outline its relationship with Clifford Fourier transformations. Quaternions and Clifford hypercomplex number were first simultaneously and independently applied to quaternion-valued Fourier and Clifford-valued Fourier transforms by Labunets [28] and Sommen [29]-[31], respectively, at the 1981. The Labunets quaternion transforms were over quaternion with real and Galois coefficients (i.e., over $\mathbb{H}[\mathbf{R}]$ and $\mathbb{H}[\mathbf{GF}(p)]$). They generalize both classical and co-called number theoretical transforms (NNTs) and proposed for application to fast signal processing. Ernst [32] and Delsuc [33] in the late 1981s, seemingly without knowledge of the earlier works of Labunets and Sommen proposed bicomplex Fourier transforms over 4D commutative hypercomplex algebra of bicomplex numbers ($\mathbf{C}\oplus\mathbf{C}$). Note that the bicomplex algebra is quite different from the quaternion algebra; among general things, bicomplex multiplication is commutative, but quaternion one is noncommutative. For this reason, the Ernst and Delsuc transforms are direct sum of ordinary Fourier transforms (*i.e.*, duplex Fourier transform). They are a little bit similar in kind to quaternion Fourier transforms. Ernst and Delsuc's transforms were two-dimensional and proposed for application to nuclear magnetic resonance (NMR) imaging.

Two new ideas emerged in 1998-1999 in a paper by Labunets [34] and Sangwine [35]. These were, firstly, the choice of a general root ${}^{3}\boldsymbol{\mu}$ of -1 (a unit quaternion with zero scalar part) rather than a basis unit ($\mathbf{i},\mathbf{j}$ or $\mathbf{k}$) of the quaternion algebra, and secondly, the choice of a general roots ${}^{3}\boldsymbol{\mu}_0=\boldsymbol{\mu}_0(\gamma_0,\theta_0),{}^{3}\boldsymbol{\mu}_1=\boldsymbol{\mu}_1(\gamma_1,\theta_1),...,{}^{3}\boldsymbol{\mu}_{N-1}=\boldsymbol{\mu}_{N-1}(\gamma_{N-1},\theta_{N-1})$ of -1 (see cloud of imaginary units on figure 1) in Clifford algebra to create multi-parameter and fractional Fourier-Clifford transforms (with eigenvalues $e^{-\boldsymbol{\mu}_0(\gamma_0,\theta_0)},e^{-\boldsymbol{\mu}_1(\gamma_1,\theta_1)},...\,...,e^{-{}^{3}\boldsymbol{\mu}_{N-1}(\varphi_{N-1},\theta_{N-1})},...$).

Labunets, Rundblad-Ostheimer and Astola [36]-[40] used the classical and number theoretical quaternion Fourier and Fourier-Clifford transforms for fast invariant recognition of 2D, 3D and nD color and hyperspectral images, defined on Euclidean and non-Euclidean spaces. These publications give useful interpretation of quaternion and Cliffordean Fourier coefficients: they are relative quaternion- or Clifford-valued invariants of hyperspectral images with respect to Euclidean and non-Euclidean rotations and motions of physical and hyperspectral spaces. It removes the veil of mysticism and mystery from quaternion- and Clifford-valued Fourier coefficients. In the works of scientists F.Brackx, H. De Schepper, F. Sommen, and H. De Bie [41]-[48] mathematical theory of Fourier-Clifford transforms accepted the final completeness, beauty and elegance.

According to Theorem 1, for non-zero $\alpha \in \mathbf{R}$ and a non-zero ${}^4\mathbf{q} = a + {}^3\boldsymbol{\mu}$ C $\exp({}^4\mathbf{q}\alpha) = \exp\left((a + {}^3\boldsymbol{\mu})\alpha\right) =$ $= e^{a\alpha}\exp\left({}^3\mathbf{r}\alpha\right) = e^{a\alpha}\left(\cos(\|{}^3\boldsymbol{\mu}\|\alpha) + \frac{{}^3\boldsymbol{\mu}}{\|{}^3\boldsymbol{\mu}\|}\sin(\|{}^3\boldsymbol{\mu}\|\alpha)\right)$. In particular case, for ${}^4\mathbf{q} \equiv {}^3\boldsymbol{\mu} = \boldsymbol{\mu}(\gamma,\theta)$ we have $e^{\boldsymbol{\mu}(\gamma,\theta)\alpha} = \cos(\alpha) + \boldsymbol{\mu}(\varphi,\theta)\sin(\alpha)$. For $\alpha = \omega t$ and $\alpha = \alpha_k = 2\pi k/N$ $(k = 0,1,...,N-1)$ we obtain quaternion-valued discrete harmonics

$$e^{\boldsymbol{\mu}(\gamma_k,\theta_k)\frac{2\pi}{N}kn} = \boldsymbol{\varepsilon}_k^{kn} = \cos\left(\frac{2\pi}{N}kn\right) + \boldsymbol{\mu}(\gamma_k,\theta_k)\sin\left(\frac{2\pi}{N}kn\right),$$

where each quaternion harmonic $\boldsymbol{\varepsilon}_k^{-kn} = \exp\left(-2\pi\boldsymbol{\mu}(\gamma_k,\theta_k)kn/N\right)$ has its own imaginary unit $\boldsymbol{\mu}_k := \boldsymbol{\mu}(\gamma_k,\theta_k) = = \left(\cos\gamma_k \cdot \mathbf{i} + \sin\gamma_k \cos_k \cdot \mathbf{j} + \sin\gamma_k \sin\theta_k \cdot \mathbf{k}\right) \in \mathbf{S}^2$, $k = 0,1,...,N-1$.

Due to the non-commutative property of quaternion multiplication, there are two different types of quaternion Fourier transforms (QFTs). These QFTs are the left- and right-sided QFTs (LS-QFT and RS-QFT), respectively.

Definition 8. The direct discrete quaternion Fourier transforms of $\mathbf{f}(n):[0,N-1] \to \mathbb{H}(\mathbf{R})$ are defined as

$$ {}^4\mathbf{QF}\left(k\middle|\gamma_k,\theta_k\right) = \mathcal{QF}^{(\boldsymbol{\gamma},\boldsymbol{\theta})}\left\{{}^4\mathbf{f}(n)\right\} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} e^{-\boldsymbol{\mu}(\gamma_k,\theta_k)\frac{2\pi}{N}kn} \circ {}^4\mathbf{f}(n) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\boldsymbol{\varepsilon}_k^{-kn} \circ {}^4\mathbf{f}(n), \tag{8}$$

$$ {}^4\mathbf{FQ}\left(k\middle|\varphi_k,\theta_k\right) = \mathcal{FQ}^{(\boldsymbol{\varphi},\boldsymbol{\theta})}\left\{{}^4\mathbf{f}(n)\right\} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} {}^4\mathbf{f}(n) \circ e^{-\boldsymbol{\mu}(\varphi_k,\theta_k)\frac{2\pi}{N}kn} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} {}^4\mathbf{f}(n) \circ \boldsymbol{\varepsilon}_k^{-kn}, \tag{9}$$

where $\mathcal{QF}$, $\mathcal{FQ}$ are LS-QFT and RS-QFT, $\boldsymbol{\varphi} = (\gamma_0,\gamma_1,...,\gamma_{N-1})$, $\boldsymbol{\theta} = (\theta_0,\theta_1,...,\theta_{N-1})$.

We see, that ${}^4\mathbf{QF}\left(k\middle|\gamma_k,\theta_k\right)$ and ${}^4\mathbf{FQ}\left(k\middle|\gamma_k,\theta_k\right)$ depend on $2(N-2)$ parameters (γ_k,θ_k), $k \in \{1,2,...,N-1\} \setminus \{0, N/2\}$ if N is even and on $2(N-1)$ parameters (γ_k,θ_k), $k \in \{1,2,...,N-1\}$ if N is odd.

Definition 9. The inverse quaternion Fourier transforms are defined as

$$ {}^4\mathbf{f}(n) = \mathcal{QF}^{\dagger(\boldsymbol{\gamma},\boldsymbol{\theta})}\left\{\mathbf{QF}(k\,|\,\gamma_k,\theta_k)\right\} = 2\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\frac{\boldsymbol{\mu}(\gamma_k,\theta_k)}{\left[\boldsymbol{\mu}(\gamma_{-k},\theta_{-k}) + \boldsymbol{\mu}(\gamma_k,\theta_k)\right]_L} \circ \boldsymbol{\varepsilon}_k^{kn} \circ {}^4\mathbf{QF}\left(k\middle|\gamma_k,\theta_k\right), \tag{10}$$

$$ {}^4\mathbf{f}(n) = \mathcal{FQ}^{\dagger(\boldsymbol{\gamma},\boldsymbol{\theta})}\left\{\mathbf{FQ}(k\,|\,\gamma_k,\theta_k)\right\} = 2\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} {}^4\mathbf{QF}\left(k\middle|\gamma_k,\theta_k\right) \circ \boldsymbol{\varepsilon}_k^{kn} \circ \frac{\boldsymbol{\mu}(\gamma_k,\theta_k)}{\left[\boldsymbol{\mu}(\gamma_{-k},\theta_{-k}) + \boldsymbol{\mu}(\gamma_k,\theta_k)\right]_R}. \tag{11}$$

2.4. Quaternion fractional and many-parameter Fourier transforms

If ${}^3\boldsymbol{\mu}(\gamma_k,\theta_k) \equiv {}^3\boldsymbol{\mu}(\gamma,\theta)$, $\forall k = 0,1,...,N-1)$ then quaternion Fourier matrices $\mathcal{QF},\mathcal{FQ}$ contains commutative entries $e^{-{}^3\boldsymbol{\mu}(\gamma_k,\theta_k)\frac{2\pi}{N}kn}$. For this reason they have the same real-valued eigenfunction as ordinary DFT but with quaternion-valued eigenvalues $\left\{\pm1, \pm{}^3\boldsymbol{\mu}(\gamma,\theta)\right\} = \left\{e^{{}^3\boldsymbol{\mu}(\gamma,\theta)s\pi/2}\right\}_{s=0}^{3}$: $\mathcal{QF}\left|h_s(n)\right\rangle = \mathcal{FQ}\left|h_s(n)\right\rangle = {}^3\boldsymbol{\mu}(\gamma,\theta)\left|h_s(n)\right\rangle$, where $\left\{h_s(n)\right\}_{s=0}^{N-1}$ is the set of discrete Hermite functions.

Hence, we can define fractional and multi-parameter quaternion Fourier transforms by the following way.

Definition 10. For single parameter $\alpha\in\mathbf{Tor}^1_{2\pi}$ and multi-parameter $(\alpha_0,\dots,\alpha_{N-1})\in\mathbf{Tor}^N_{2\pi}$ we introduce fractional and multi-parameter quaternion Fourier transforms (FrQFT and MPQFT)

$$\mathcal{QF}^{\alpha}(\boldsymbol{\gamma},\boldsymbol{\theta})=\sum_{m=0}^{N-1}e^{{}^3\boldsymbol{\mu}(\gamma_m,\theta_m)\cdot s(m)\cdot\alpha}\left|h_m(k)\right\rangle\left\langle h_m(n)\right|=\mathcal{H}\mathbf{Diag}\left(e^{{}^3\boldsymbol{\mu}(\gamma_0,\theta_0)\cdot s(0)\cdot\alpha},e^{{}^3\boldsymbol{\mu}(\gamma_1,\theta_1)\cdot s(1)\cdot\alpha},\dots,e^{{}^3\boldsymbol{\mu}(\gamma_{N-1},\theta_{N-1})\cdot s(N-1)\cdot\alpha}\right)\mathcal{H}^{\dagger},$$

$$\mathcal{QF}^{\boldsymbol{\alpha}}(\boldsymbol{\gamma},\boldsymbol{\theta})=\mathcal{F}_{\mathcal{Q}}^{(\alpha_1,\alpha_2,\dots,\alpha_{N-1})}(\boldsymbol{\gamma},\boldsymbol{\theta})=\sum_{m=0}^{N-1}e^{{}^3\boldsymbol{\mu}(\varphi_m,\theta_m)\cdot s(m)\cdot\alpha_m}\left|h_m(k)\right\rangle\left\langle h_m(n)\right|=$$

$$=\mathcal{H}\cdot\mathbf{Diag}\left(e^{{}^3\boldsymbol{\mu}(\gamma_0,\theta_0)\cdot s(0)\cdot\alpha_0},e^{{}^3\boldsymbol{\mu}(\gamma_1,\theta_1)\cdot s(1)\alpha_1},\dots,e^{{}^3\boldsymbol{\mu}(\gamma_{N-1},\theta_{N-1})\cdot s(N-1)\cdot\alpha_{N-1}}\right)\cdot\mathcal{H}^{\dagger}.$$

Due to the non-commutative property of quaternion multiplication, there are left- and right-sided transforms (LS-FrQFTs, LS-MPQFTs and RS-FrQFTs, RS-MPQFTs).

Definition 11. The direct discrete LS-FrQFTs, LS-MPQFTs and RS-FrQFTs, RS-MPQFTs of $\mathbf{f}(n):[0,N-1]\to\mathbb{H}(\mathbf{R})$ are defined as

$$\left|\mathbf{QF}^{\alpha}\left(k\middle|\boldsymbol{\gamma},\boldsymbol{\theta}\right)\right\rangle=Q\mathcal{F}^{\alpha}(\boldsymbol{\gamma},\boldsymbol{\theta})\left|\mathbf{f}(n)\right\rangle=\sum_{m=0}^{N-1}e^{{}^3\boldsymbol{\mu}_m(\gamma_m,\theta_m)\cdot s(m)\cdot\alpha}\circ\left|h_m(k)\right\rangle\left\langle h_m(n)\,|\,\mathbf{f}(n)\right\rangle,$$

$$\left|\mathbf{FQ}^{\alpha}\left(k\,|\,\boldsymbol{\gamma},\boldsymbol{\theta}\right)\right\rangle=\mathcal{F}Q^{\alpha}(\boldsymbol{\gamma},\boldsymbol{\theta})\left|\mathbf{f}(n)\right\rangle=\sum_{m=0}^{N-1}\left[\left\langle h_m(n)\,|\,\mathbf{f}(n)\right\rangle\circ e^{{}^3\boldsymbol{\mu}_m(\gamma_m,\theta_m)\cdot s(m)\cdot\alpha}\right]\left|h_m(k)\right\rangle,$$

$$\left|\mathbf{QF}^{\boldsymbol{\alpha}}\left(k\middle|\boldsymbol{\gamma},\boldsymbol{\theta}\right)\right\rangle=Q\mathcal{F}^{(\alpha_0,\alpha_1,\dots,\alpha_{N-1})}(\boldsymbol{\gamma},\boldsymbol{\theta})\left|\mathbf{f}(n)\right\rangle=\sum_{m=0}^{N-1}e^{{}^3\boldsymbol{\mu}_m(\gamma_m,\theta_m)\cdot s(m)\cdot\alpha_m}\circ\left|h_m(k)\right\rangle\left\langle h_m(n)\,|\,\mathbf{f}(n)\right\rangle,$$

$$\left|\mathbf{FQ}^{\boldsymbol{\alpha}}\left(k\middle|\boldsymbol{\gamma},\boldsymbol{\theta}\right)\right\rangle=\mathcal{F}Q^{(\alpha_0,\alpha_1,\dots,\alpha_{N-1})}(\boldsymbol{\gamma},\boldsymbol{\theta})\left|\mathbf{f}(n)\right\rangle=\sum_{m=0}^{N-1}\left\langle h_m(n)\,|\,\mathbf{f}(n)\right\rangle\circ e^{{}^3\boldsymbol{\mu}_m(\gamma_m,\theta_m)\cdot s(m)\cdot\alpha_m}\left|h_m(k)\right\rangle,$$

According to physics and engineering tradition, it is sometimes convenient to refer to the quaternion constant $e^{{}^3\boldsymbol{\mu}_{\varphi,\theta}(m)\cdot s(m)\cdot\alpha_m}$ as a quaternion-valued phasor.

2.5. Fast quaternion Fourier transform

For fixed integer r $\left(r\in\{1,2,...,n\}\right)$ and $p_r,q_r\in\left\{0,1,...,2^{n-r}-1\right\}$ let

$$\Delta^{(p_r,q_r)}_{2^{n-r+1}}=\Delta^{(p_r,q_r)}_{2^{n-r+1}}\left(\varepsilon^{a(p_r)}_{p_r}\,|\,\varepsilon^{b(q_r)}_{q_r}\right)=\Delta^{(p_r)}_{2^{n-r}}\left(\varepsilon^{a(p_r)}_{p_r}\right)\oplus\Delta^{(q_r)}_{2^{n-r}}\left(\varepsilon^{b(q_r)}_{q_r}\right)=$$

$$=\mathbf{Diag}_{2^{n-r}}\left\{\overbrace{\underbrace{1,...,\varepsilon^{a(p_r)}_{p_r}}_{p_r},1,...,1}^{2^{n-r}}\right\}\oplus\mathbf{Diag}_{2^{n-r}}\left\{\overbrace{\underbrace{1,...,\varepsilon^{b(q_r)}_{q_r}}_{q_r},1,...,1}^{2^{n-r}}\right\}=\mathbf{Diag}_{2^{n-r+1}}\left\{\overbrace{\underbrace{1,...,\varepsilon^{a(p_r)}_{p_r}}_{p_r},1,...,1}^{2^{n-r}}\middle|\overbrace{\underbrace{1,...,\varepsilon^{b(q_r)}_{q_r}}_{p_r},1,...,1}^{2^{n-r}}\right\},$$

where $a=a(p_r),b=b(q_r)$ are integers depending on positions p_r and $q_r=q_r+2^{n-r}$, respectively, and $\varepsilon=\exp(2\pi j/2^n)$. Let farther $\Delta_{2^{n-r}}\left(\varepsilon^{p_r\cdot 2^{r-1}}_{p_r}\right):=\mathbf{Diag}_{2^{n-r}}\left(1,\varepsilon^{1\cdot 2^{r-1}},\varepsilon^{2\cdot 2^{r-1}},...,\varepsilon^{(2^{n-r}-1)\cdot 2^{r-1}}\right)=\prod_{p_r=0}^{2^{n-r}-1}\Delta^{(p_r)}_{2^{n-r}}\left(\varepsilon^{p_r\cdot 2^{r-1}}_{p_r}\right),$

$$I_{2^{n-r}}\oplus\Delta_{2^{n-r}}\left(\varepsilon^{p_r\cdot 2^{r-1}}_{p_r}\right)=\mathbf{Diag}_{2^{n-r}}(1,1,...,1)\oplus\mathbf{Diag}_{2^{n-r}}\left(1,\varepsilon^{1\cdot 2^{r-1}},\varepsilon^{2\cdot 2^{r-1}},...,\varepsilon^{(2^{n-r}-1)\cdot 2^{r-1}}\right)=$$

$$=\prod_{p_r=0}^{2^{n-r}-1}\Delta^{(p_r)}_{2^{n-r}}\left(1^{p_r}_{p_r}\right)\oplus\prod_{p_r=0}^{2^{n-r}-1}\Delta^{(p_r+2^{n-r})}_{2^{n-r}}\left(\varepsilon^{p_r\cdot 2^{r-1}}_{p_r+2^{n-r}}\right)=\prod_{p_r=0}^{2^{n-r}-1}\Delta^{(p_r,p_r+2^{n-r})}_{2^{n-r+1}}\left(1^{p_r}_{p_r}\,|\,\varepsilon^{p_r\cdot 2^{r-1}}_{p_r+2^{n-r}}\right).$$

Now we are going to use the radix-$(2^{r-1},2^{n-r})$ representation of $p,q\in\left\{0,1,...,2^{n-1}-1\right\}$: $p=p(k_r,s_r)=$ $=2^{r-1}k_r+s_r$, $q(k_r,s_r)=p(k_r,s_r)+2^{n-r}$, where $k_r\in\left\{0,1,...,2^{r-1}-1\right\}$, $s_r\in\left\{0,1,...,2^{n-r}-1\right\}$ and $r\in\{1,2,...,n\}$ in fast Fourier transform. We can write diagonal matrices of FFT (for all $r\in\{1,2,...,n\}$) as

$$I_{2^{r-1}}\otimes\left(I_{2^{n-r}}\oplus\Delta_{2^{n-r}}\left(\varepsilon_{p_r}^{p_r\cdot 2^{r-1}}\right)\right)=$$
$$=\prod_{k_r=0}^{2^{r-1}-1}\prod_{s_r=0}^{2^{n-r}-1}\left[\Delta_{2^n}^{(p(k_r,s_r))}(1^{s_r}_{p(s_r,k_r)})\oplus\Delta_{2^n}^{(q(k_r,s_r))}(\varepsilon^{s_r\cdot 2^{n-r}}_{q(k_r,s_r)})\right]=\prod_{k_r=0}^{2^{r-1}-1}\prod_{s_r=0}^{2^{n-r}-1}\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}(1^{s_r}_{p(k_r,s_r)}\mid\varepsilon^{s_r\cdot 2^{n-r}}_{q(k_r,s_r)}). \tag{12}$$

The fast DFT has the following form

$$\mathcal{F}=\frac{1}{\sqrt{2^n}}\prod_{r=1}^{n}\left(\left[I_{2^{r-1}}\otimes\left(I_{2^{n-r}}\oplus\Delta_{2^{n-r}}\left(\varepsilon_{p_r}^{p_r\cdot 2^{r-1}}\right)\right)\right]\cdot\left[I_{2^{r-1}}\otimes\begin{bmatrix}1&1\\1&-1\end{bmatrix}\otimes I_{2^{n-r}}\right]\right)=$$
$$=\frac{1}{\sqrt{2^n}}\prod_{r=1}^{n}\left[I_{2^{r-1}}\otimes\left[\begin{array}{c:c}I_{2^{n-r}}&\cdot\\ \hdashline \cdot&\Delta_{2^{n-r}}\end{array}\right]\left[\begin{array}{c:c}I_{2^{n-r}}&I_{2^{n-r}}\\ \hdashline I_{2^{n-r}}&-I_{2^{n-r}}\end{array}\right]\right]=\frac{1}{\sqrt{2^n}}\prod_{r=1}^{n}\left[I_{2^{r-1}}\otimes\left[\begin{array}{c:c}I_{2^{n-r}}&I_{2^{n-r}}\\ \hdashline \Delta_{2^{n-r}}I_{2^{n-r}}&-\Delta_{2^{n-r}}I_{2^{n-r}}\end{array}\right]\right], \tag{13}$$

or in details (using (12))

$$\mathcal{F}=\prod_{r=1}^{n}\left(\left[\prod_{k_r=0}^{2^{r-1}-1}\prod_{s_r=0}^{2^{n-r}-1}\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left(1^{s_r}_{p(k_r,s_r)}\middle|\varepsilon^{s_r\cdot 2^{n-r}}_{q(k_r,s_r)}\right)\right]\cdot\left[\prod_{k_r=0}^{2^{r-1}-1}\prod_{s_r=0}^{2^{n-r}-1}\mathbf{J}_{2^n}^{p(k_r,s_r),q(k_r,s_r)}\left(\pi/4\right)\right]\right)=$$
$$=\prod_{r=1}^{n}\left(\prod_{s_r=0}^{2^{n-r}-1}\prod_{k_r=0}^{2^{r-1}-1}\left[\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left(1^{s_r}_{p(k_r,s_r)}\middle|\varepsilon^{s_r\cdot 2^{n-r}}_{q(k_r,s_r)}\right)\cdot\mathbf{J}_{2^n}^{p(k_r,s_r),q(k_r,s_r)}\left(\pi/4\right)\right]\right). \tag{14}$$

The first generalization of (17) is based on Jacobi matrices $\mathbf{J}_{2^n}^{(p,q)}(\varphi_r^{(p,q)})$ instead of $\mathbf{J}_N^{p,q}(\pi/4)$:

$$\mathcal{F}\left(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,...,\boldsymbol{\varphi}_n\right)=\prod_{r=1}^{n}\left(\prod_{s_r=0}^{2^{n-r}-1}\prod_{k_r=0}^{2^{r-1}-1}\left[\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left(1^{s_r}_{p(k_r,s_r)}\middle|\varepsilon^{s_r\cdot 2^{n-r}}_{q(k_r,s_r)}\right)\cdot\mathbf{J}_{2^n}^{p(k_r,s_r),q(k_r,s_r)}\left(\varphi_r^{(p,q)}\right)\right]\right). \tag{15}$$

Obviously, this transform is $n\cdot 2^{n-1}$ - parameter Fourier-like transform. The second generalized is based on arbitrary phasors in $\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left(1^{s_r}_{p(k_r,s_r)}\middle|\varepsilon^{s_r\cdot 2^{n-r}}_{q(k_r,s_r)}\right)\to\Delta_{2^n}^{(p,q)}\left(e^{j\beta_r^p}\mid e^{j\beta_r^q}\right)$:

$$\mathcal{F}\left(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,...,\boldsymbol{\varphi}_n;\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,...,\boldsymbol{\beta}_n\right)=\prod_{r=1}^{n}\left(\prod_{s_r=0}^{2^{n-r}-1}\prod_{k_r=0}^{2^{r-1}-1}\left[\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left(e^{j\beta_r^p}\mid e^{j\beta_r^q}\right)\cdot\mathbf{J}_{2^n}^{p(k_r,s_r),q(k_r,s_r)}\left(\varphi_r^{(p,q)}\right)\right]\right), \tag{16}$$

It is $2n\cdot 2^{n-1}$ - parameter Fourier-like transform.

We are going to use this expression for obtaining many-parameter quaternion Fourier-like transform. Indeed, if ${}^3\boldsymbol{\mu}(\varphi_k,\theta_k)\equiv{}^3\boldsymbol{\mu}(\varphi,\theta),\ \forall k=0,1,...,2^n-1$), then quaternion Fourier matrices $\mathcal{QF},\mathcal{FQ}$ contains commutative quaternion-valued entries $e^{-{}^3\boldsymbol{\mu}(\varphi,\theta)\frac{2\pi}{2^n}kn}=\boldsymbol{\varepsilon}^{kn}$ (where $\boldsymbol{\varepsilon}=e^{{}^3\boldsymbol{\mu}(\varphi,\theta)\frac{2\pi}{2^n}}$ is a primitive N^{th}-root of 1 in $\mathbb{H}(\mathbf{R})$, where $N=2^n$) and for this reason quaternion DFT have the same fast algorithms as ordinary DFT:

$$\mathcal{QF}\left(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,...,\boldsymbol{\varphi}_n;\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,...,\boldsymbol{\beta}_n;\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,...,\boldsymbol{\gamma}_n;\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,...\boldsymbol{\theta}_n\right)=$$
$$=\prod_{r=1}^{n}\left(\prod_{s_r=0}^{2^{n-r}-1}\prod_{k_r=0}^{2^{r-1}-1}\left[\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left({}_{LS}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^p}\mid_{LS}e^{{}^3\boldsymbol{\mu}(\gamma_r^q,\theta_r^q)\beta_r^q}\right)\cdot\mathbf{J}_{2^n}^{p(k_r,s_r),q(k_r,s_r)}\left(\varphi_r^{(k_r,s_r)}\right)\right]\right),$$
$$\mathcal{FQ}\left(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,...,\boldsymbol{\varphi}_n;\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,...,\boldsymbol{\beta}_n;\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,...,\boldsymbol{\gamma}_n;\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,...\boldsymbol{\theta}_n\right)=$$
$$=\prod_{r=1}^{n}\left(\prod_{s_r=0}^{2^{n-r}-1}\prod_{k_r=0}^{2^{r-1}-1}\left[\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left({}_{RS}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^p}\mid{}_{RS}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^q}\right)\cdot\mathbf{J}_{2^n}^{p(k_r,s_r),q(k_r,s_r)}\left(\varphi_r^{(k_r,s_r)}\right)\right]\right), \tag{17}$$

where labels LS and RS at ${}_{LS}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^p}$ and ${}_{RS}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^p}$ indicate about the left side or the right side multiplications, respectively. They are $4n\cdot 2^{n-1}$- parameter quaternion Fourier-like transforms $\mathcal{QF}(\boldsymbol{\omega})$ and $\mathcal{FQ}(\boldsymbol{\omega})$ with parameters $\boldsymbol{\omega}=\left(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,...,\boldsymbol{\varphi}_n;\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,...,\boldsymbol{\beta}_n;\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,...,\boldsymbol{\gamma}_n;\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,...\boldsymbol{\theta}_n\right)$.

Let us introduce a bunch of binary N -D crypto-keys $\mathfrak{b}=\{\mathbf{b}_r\}_{r=1}^{n}=\{(b_r(0),...,b_r(p),...,b_r(N-1))\}_{r=1}^{n}$ and define

$$ {}_{b_r(p)}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^p} := \begin{cases} {}_{LS}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^p}, & \text{if } b_r(p)=0, \\ {}_{RS}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^p}, & \text{if } b_r(p)=1. \end{cases} $$

Then quaternion Fourier-like transform with the branch of binary crypto-keys

$$ \mathcal{F}_{\mathcal{Q}}(\boldsymbol{\omega}|\mathfrak{b})=\prod_{r=1}^{n}\left(\prod_{s_r=0}^{2^{n-r}-1}\prod_{k_r=0}^{2^{r-1}-1}\left[\Delta_{2^n}^{(p(k_r,s_r),q(k_r,s_r))}\left({}_{b_r(p)}e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^p}\,|_{b_r(p)}\,e^{{}^3\boldsymbol{\mu}(\gamma_r^p,\theta_r^p)\beta_r^q}\right)\cdot\mathbf{J}_{2^n}^{p(k_r,s_r),q(k_r,s_r)}\left(\varphi_r^{(k_r,s_r)}\right)\right]\right) $$

generalizes both $\mathcal{QF}(\boldsymbol{\omega})$ and $\mathcal{FQ}(\boldsymbol{\omega})$.

3. Quaternion all-pass filters

In this section we introduce special classes of many-parametric all-pass discrete cyclic filters. The output/input relation of the discrete cyclic filter is described by the discrete cyclic convolution:

$$ y(n)=\mathbf{Filt}_{\mathcal{F}}\{x(n)\}=\sum_{m=0}^{N-1}h(n\underset{N}{\ominus}m)x(m)=(h*x)(n)=\left(\mathcal{F}^{\dagger}\cdot\mathbf{Diag}\left\{|H(k)|e^{i\varphi(k)}\right\}\cdot\mathcal{F}\right)\cdot x(n), $$

where $x(n), y(n)$ are input and output signals, respectively, $h(n)$ is the impulse response, $H(k)=|H(k)|e^{i\varphi(k)}=\mathcal{F}\{h(n)\}$ is the frequency response, $\underset{N}{\ominus}$ is difference modulo N and $*$ is the symbol of cyclic convolution, $\mathbf{Filt}_{\mathcal{F}}=\left[h(n\underset{N}{\ominus}m)\right]_{n,m=0}^{N-1}$ is the cyclic $(N\times N)-$matrix with the kernel $h(n)$

We will concentrate our analysis on all-pass filters whose frequency response can be expressed in the form $H(k)=|H(k)|e^{i\varphi(k)}$, where frequency response magnitude is constant for all frequencies, for example, $|H(k)|\equiv 1,\ k=0,1,2,...,N-1$. So, for all-pass filter $\mathbf{Filt}_{\mathcal{F}}$ has the following complex-valued impulse $|h(n)\rangle=\mathcal{F}\cdot|e^{i\varphi(k)}\rangle$ and frequency responses $|H(k)\rangle=|e^{i\varphi(k)}\rangle$. Hence,

$$ y(n)=\mathbf{Filt}_{\mathcal{F}}\{x(n)\}=\left(\mathcal{F}^{\dagger}\cdot\mathbf{Diag}\left\{e^{i\varphi(k)}\right\}\cdot\mathcal{F}\right)\{x(n)\}. $$

We are going to consider this filter as a parametric filter

$$ \mathbf{Filt}_{\mathcal{F}}^{(\boldsymbol{\varphi})}=\mathbf{Filt}_{\mathcal{F}}^{(\varphi_0,\varphi_1,...,\varphi_{N-1})}=\mathcal{F}^{\dagger}\cdot\mathbf{Diag}\left\{e^{i\varphi(k)}\right\}\cdot\mathcal{F}=\mathcal{F}^{\dagger}\cdot\mathbf{Diag}\left\{e^{i\varphi_0},e^{i\varphi_1},...,e^{i\varphi_{N-1}}\right\}\cdot\mathcal{F} \tag{18} $$

with N free parameters $\boldsymbol{\varphi}=(\varphi_0,\varphi_1,...,\varphi_{N-1})$. Obviously, all-pass filter $\mathbf{Filt}_{\mathcal{F}}^{(\boldsymbol{\varphi})}$ (as linear transform) is many-parameter unitary cyclic $(N\times N)-$matrix, since

$$ \mathbf{Filt}_{\mathcal{F}}^{(\boldsymbol{\varphi})}\cdot\mathbf{Filt}_{\mathcal{F}}^{\dagger(\boldsymbol{\varphi})}=\left(\mathcal{F}^{\dagger}\cdot\mathbf{Diag}\left\{e^{i\varphi(k)}\right\}\cdot\mathcal{F}\right)\cdot\left(\mathcal{F}^{\dagger}\cdot\mathbf{Diag}\left\{e^{-i\varphi(k)}\right\}\cdot\mathcal{F}\right)=\mathcal{F}^{\dagger}\cdot\mathbf{Diag}\left\{e^{i\varphi(k)}\cdot e^{-i\varphi(k)}\right\}\mathcal{F}=I. $$

Our the first natural generalization of (18) is based on an arbitrary unitary transform $\mathcal{U}$ instead of Fourier transform $\mathcal{F}$:

$$ \mathbf{Filt}_{\mathcal{U}}^{(\boldsymbol{\varphi})}=\mathbf{Filt}_{\mathcal{U}}^{(\varphi_0,\varphi_1,...,\varphi_{N-1})}=\mathcal{U}^{\dagger}\cdot\mathbf{Diag}\left\{e^{i\varphi(k)}\right\}\cdot\mathcal{U}=\mathcal{U}^{\dagger}\cdot\mathbf{Diag}\left\{e^{i\varphi_0},e^{i\varphi_1},...,e^{i\varphi_{N-1}}\right\}\cdot\mathcal{U}. \tag{19} $$

The second generalized is based on quaternion-valued exponents ${}_{b_0(p)}e^{{}^3\boldsymbol{\mu}(\gamma_0^p,\theta_0^p)\alpha_0^p}$ $(p=0,1,...,N-1)$ and quaternion Fourier transforms $\mathcal{F}_{\mathcal{Q}}(\boldsymbol{\omega}|\mathfrak{b})$ in (19):

$$ \mathbf{Filt}_{\mathcal{F}_{\mathcal{Q}}}\left(\boldsymbol{\alpha}_0,\boldsymbol{\gamma}_0,\boldsymbol{\theta}_0;\boldsymbol{\omega}\,|\,\mathbf{b}_0,\mathfrak{b}\right)=\mathcal{F}_{\mathcal{Q}}^{\dagger}(\boldsymbol{\omega}|\mathfrak{b})\cdot\mathbf{Diag}\left\{{}_{b_0(0)}e^{{}^3\boldsymbol{\mu}(\gamma_0^0,\theta_0^0)\alpha_0^0},\ {}_{b_0(1)}e^{{}^3\boldsymbol{\mu}(\gamma_0^1,\theta_0^1)\alpha_0^1},...,{}_{b_0(1)}e^{{}^3\boldsymbol{\mu}(\gamma_0^{N-1},\theta_0^{N-1})\alpha_0^{N-1}}\right\}\cdot\mathcal{F}_{\mathcal{Q}}^{\dagger}(\boldsymbol{\omega}|\mathfrak{b}). \tag{20} $$

where $\boldsymbol{\alpha}_0=(\alpha_0^0,\alpha_0^1,...,\alpha_0^{N-1})$, $\boldsymbol{\gamma}_0=(\gamma_0^0,\gamma_0^1,...,\gamma_0^{N-1})$, $\boldsymbol{\theta}_0=(\theta_0^0,\theta_0^1,...,\theta_0^{N-1})$, $\mathbf{b}_0=(b_0(0),b_0(1),...,,b_r(N-1))$ and $\boldsymbol{\omega}=(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2,...,\boldsymbol{\varphi}_n;\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,...,\boldsymbol{\beta}_n;\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,...,\boldsymbol{\gamma}_n;\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,...\boldsymbol{\theta}_n)$. Let $\tilde{\mathfrak{b}}=\{\mathbf{b}_0,\mathfrak{b}\}=\left\{\mathbf{b}_0,\{\mathbf{b}_r\}_{r=1}^{n}\right\}=\{\mathbf{b}_r\}_{r=0}^{n}$ and

$$\tilde{\boldsymbol{\omega}} = \left(\boldsymbol{\alpha}_0, \boldsymbol{\gamma}_0, \boldsymbol{\theta}_0; \boldsymbol{\omega}\right) = \left(\underbrace{\overbrace{\boldsymbol{\alpha}_0, \boldsymbol{\gamma}_0, \boldsymbol{\theta}_0}^{3\cdot 2^n \text{ parameters}}}_{\text{All-pass}}; \underbrace{\overbrace{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, ..., \boldsymbol{\beta}_n}^{n2^n \text{ parameters}}}_{\text{Angles of phasors}}; \underbrace{\overbrace{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, ..., \boldsymbol{\varphi}_n}^{n2^{n-1} \text{ parameters}}}_{\text{Jacobi angles}}; \underbrace{\overbrace{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, ..., \boldsymbol{\gamma}_n}^{n2^n \text{ parameters}}; \overbrace{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ...\boldsymbol{\theta}_n}^{n2^n \text{ parameters}}}_{\text{Imaginary units}} \right)$$

Then we can write

$$\mathbf{Filt}_{\mathcal{F}_{\mathcal{Q}}}\left(\tilde{\boldsymbol{\omega}} \mid \tilde{\mathfrak{b}}\right) = \mathcal{F}_{\mathcal{Q}}^{\dagger}\left(\boldsymbol{\omega}\middle|\mathfrak{b}\right) \cdot \mathbf{Diag}\left\{ {}_{b_0(0)}e^{{}^3\boldsymbol{\mu}(\gamma_0^0, \theta_0^0)\alpha_0^0}, \ {}_{b_0(1)}e^{{}^3\boldsymbol{\mu}(\gamma_0^1, \theta_0^1)\alpha_0^1}, ..., {}_{b_0(1)}e^{{}^3\boldsymbol{\mu}(\gamma_0^{N-1}, \theta_0^{N-1})\alpha_0^{N-1}} \right\} \cdot \mathcal{F}_{\mathcal{Q}}^{\dagger}\left(\boldsymbol{\omega}\middle|\mathfrak{b}\right) \qquad (21)$$

instead of (21). It is $(7n+6)\cdot 2^{n-1}$ - parameter quaternion cyclic transform with parameters $\tilde{\boldsymbol{\omega}}$ and with bunch of binary crypto-keys $\tilde{\mathfrak{b}}$.

4. Conclusion

In this paper, we have shown a new unified approach to the many-parametric representation of complex and quaternion Fourier transforms. This form is the product of sparse rotation matrixes and it describes fast algorithms for introduced many-parameter transforms. Defined representation of many-parameter transforms (MPT) depend on finite set of free parameters, which could be changed independently of one another. For each set of values of parameter we get the unique orthogonal transform. We are going to use these MPTs for constructing of novel Intelligent OFDM-telecommunication systems The new systems will use inverse MPT (or inverse MPT) for modulation at the transmitter and direct MPT (or direct MPT) for demodulation at the receiver.

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