Uniqueness and existence of viscosity solutions under a degenerate dynamic boundary condition

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Abstract

We consider the initial boundary value problem for a fully-nonlinear parabolic equation in a half space. The boundary condition we study is a degenerate one in the sense that it does not depend on the normal derivative on the boundary. A typical example is a stationary boundary condition prescribing the value of the time derivative of the unknown function. Our setting also covers the classical Dirichlet boundary condition. We establish a comparison principle for a viscosity sub- and supersolution under a weak continuity assumption on the solutions on the boundary. We also prove existence of solutions and give some examples of solutions under several boundary conditions. We show among other things that, in the sense of viscosity solutions, the stationary boundary condition can be different from the Dirichlet boundary condition which is obtained by integrating the stationary condition.

Key words: degenerate dynamic boundary condition; comparison principle; viscosity solutions
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1 Introduction

Equation and Goals. We consider the initial boundary value problem for a second order parabolic equation of the form

$$\text{(IBV)} \begin{cases} u_t(x,t) + F(x,t,u(x,t),\nabla u(x,t),\nabla^2 u(x,t)) = 0 & \text{in } \Omega\times(0,T), \quad (1.1)\\ B(x',t,u(x,t),\nabla' u(x,t),u_t(x,t)) = 0 & \text{on } \partial\Omega\times(0,T), \quad (1.2)\\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}. \quad (1.3)\end{cases}$$

Here $u = u(x,t) : \overline{\Omega}\times[0,T) \to \mathbf{R}$ is an unknown function and $u_t = \partial_t u$, $\nabla u = (\partial_{x_i} u)_{i=1}^n$, $\nabla^2 u = (\partial_{x_i x_j} u)_{i,j=1}^n$ are its derivatives. Besides, $T > 0$ and $\Omega := \{x = (x', x_n) \in \mathbf{R}^{n-1}\times\mathbf{R} \mid x_n > 0\}$ is the half space of $\mathbf{R}^n$. Throughout this paper, the prime $'$ and the index n denote the first $(n-1)$ components and the last component of a vector in $\mathbf{R}^n$, respectively. We write $\nabla' u = (\partial_{x_i} u)_{i=1}^{n-1}$.

The goal of this paper is to establish a unique existence of viscosity solutions to (IBV), especially a comparison principle for a viscosity subsolution and a viscosity supersolution to (IBV). A typical boundary condition in our mind is a *degenerate dynamic boundary condition*:

$$u_t(x,t) = g(x',t) \quad \text{on } \partial\Omega\times(0,T), \tag{1.4}$$

and its special case

$$u_t(x,t) = c \quad \text{on } \partial\Omega\times(0,T), \tag{1.5}$$

where $c \in \mathbf{R}$ is a constant. In this paper we call (1.5) a *stationary boundary condition*. If B is strictly monotone with respect to the normal derivative $-u_{x_n}$ of u on the boundary (see (1.12) below), the comparison principle is well-known in the literature ([2, 3]). Our boundary condition (1.2) does not satisfy this

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monotonicity. In this sense we say that (1.2) (and hence (1.4)) is degenerate. The known comparison results cannot be applied to (IBV).

When the boundary condition is degenerate, the comparison principle does not hold for semicontinuous viscosity solutions. This is known when the boundary condition is of the Dirichlet type:

$$u(x,t) = h(x',t) \quad \text{on } \partial\Omega \times (0,T), \tag{1.6}$$

which our boundary condition (1.2) covers. Our comparison result is thus established under a suitable (weak) continuity of solutions on the boundary.

Background. 1. *Asymptotic problem for a non-degenerate boundary condition.* A typical non-degenerate dynamic boundary condition is a linear one

$$u_t(x,t) - \beta u_{x_n}(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{1.7}$$

with $\beta > 0$. When $\beta \to 0$, the boundary condition

$$u_t(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{1.8}$$

naturally appears. This is the stationary boundary condition (1.5) with $c = 0$. Thus the solutions for (1.7) are expected to converge to the solution for (1.8). To make this convergence assertion rigorous, we need a uniqueness result under (1.8). The comparison principle for (1.8) therefore plays an important role to study the asymptotic analysis for a non-degenerate boundary condition. See Sections 4.4–4.6 for the details.

2. *Crystal growth problem:* The boundary condition of the type (1.4) is considered to describe the *two-dimensional nucleation* in crystal growth ([5, Part III], [20, Section 2], [21]). This crystal growth is started by external supply of crystal molecules at a *step source*. Let $u(x,t)$ be the height of the crystal surface at a position x and a time t. Suppose now that the step source is located on the boundary of Ω. For example, in one-dimensional case where $\Omega = (0,\infty)$, the step source is $x = 0$. If crystal molecules are supplied at a rate of $c > 0$ on $\partial\Omega$, then the height function u should satisfy (1.5). The height function u is thus obtained by solving the initial value problem of an evolution equation in Ω under (1.5). In [22], instead of (1.5) the Dirichlet boundary condition

$$u(x,t) = ct + u_0(x) \quad \text{in } \partial\Omega \times (0,T) \tag{1.9}$$

is imposed when $\Omega = (0,\infty)$.

In [12] another approach is presented to understand the two-dimensional nucleation. There, instead of imposing the boundary condition, a source term is added into the equation. More precisely, the equation studied in [12] is

$$u_t(x,t) + H(x, \nabla u(x,t)) = cI(x) \quad \text{in } \mathbf{R}^n \times (0,T) \tag{1.10}$$

when the step source is located at $x = 0$ and the supplying rate is c. Here $I(x)$ is defined by

$$I(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases} \tag{1.11}$$

Since (1.10) has a discontinuous source term, the classical viscosity solution theory cannot be applied; in fact, solutions are not unique. We thus introduced a new notion of viscosity solutions called *envelope solutions*, and proved the unique existence of envelope solutions when the Hamiltonian H is coercive. In Section 5, we will give some examples of the solution u to (IBV) in the one-dimensional case and compare it with the envelope solution $\hat{u}$ of (1.10). It turns out that they are the same solution in the sense that $u(x,t) = \hat{u}(|x|,t)$. See Remarks 5.8 and 5.11. A natural question is whether these two solutions are the same for more general setting, but the author does not know any nice answer so far.

Literature overview. We below give some known results in the viscosity solution theory related to dynamic boundary value conditions such as (1.4) and (1.7). The list is not exhaustive at all.

1. Non-degenerate case: A unique existence result for a fully nonlinear parabolic equation with a general dynamic boundary condition is established by Barles ([2, 3]). The boundary condition is of the form

$$u_t(x,t) + B(x,t,u(x,t),\nabla u(x,t)) = 0 \quad \text{on } \partial U \times (0,T),$$

where U is a bounded domain in $\mathbf{R}^n$ with smooth boundary. The function B is assumed to be non-degenerate, that is, for any $R > 0$ there exists $\mu_R > 0$ such that

$$B(x,t,r,p+\lambda\nu(x)) - B(x,t,r,p) \geqq \mu_R \lambda \tag{1.12}$$

for all $x \in \partial U$, $t \in (0,T)$, $r \in \mathbf{R}$ with $|r| \leqq R$, $p \in \mathbf{R}^n$ and $\lambda > 0$. Here $\nu(x)$ denotes the unit outward normal vector to ∂U at x. The comparison principle is established for a upper semicontinuous subsolution and a lower semicontinuous supersolution. Note that our boundary condition (1.2) does not satisfy (1.12) since B is independent of u_{x_n}.

In [1] the asymptotic behavior of solutions to the initial value problem of the eikonal equation

$$\varepsilon u_t(x,t) + |\nabla u(x,t)| = 1 \quad \text{in } U \times (0,\infty) \quad (\varepsilon > 0)$$

is studied under a linear boundary condition

$$u_t(x,t) + \langle \nu(x), \nabla u(x,t)\rangle = 0 \quad \text{on } \partial U \times (0,\infty). \tag{1.13}$$

Here $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the usual Euclidean norm and inner product, respectively. The comparison results are presented not only for (1.13) but also for the stationary boundary condition

$$u_t(x,t) = 0 \quad \text{on } \partial U \times (0,\infty) \tag{1.14}$$

provided that both a subsolution and a supersolution are Lipschitz continuous ([1, Lemma 3.2]). The idea of the proof for (1.14) is to reduce it to (1.13) with a small coefficient on $\langle \nu, \nabla u\rangle$. Since the same idea works for our (IBV), we later give the comparison result for Lipschitz continuous solutions of (IBV) (Theorem 2.9).

Unfortunately, Lipschitz continuous solutions of (IBV) may not exist under (1.5) even if the initial datum is zero. We give such an example in Section 5.3 for some first order equation with a non-coercive Hamiltonian. For this problem, a continuous solution u exists but it is not Lipschitz continuous. Thus Theorem 2.9 does not imply the uniqueness of u. We need our main comparison result, Theorem 3.2, to deduce the uniqueness of such u.

Recently, in [13] the authors established a uniqueness and existence of viscosity solutions to possibly singular geometric evolution equations such as the level-set mean curvature flow equation. There the linear dynamic boundary condition (1.7) is imposed on the boundary of the half space Ω. Some convergence results as $\beta \to +0$ are also obtained.

In [14] the authors proposed deterministic discrete game-theoretic interpretations for fully nonlinear equations with nonlinear dynamic boundary conditions. To study fast evolution asymptotics of solutions to parabolic equations, a comparison principle for elliptic equations with dynamic boundary conditions is established ([14, Theorems 2.5 and 5.5]). See also [15] for a game-theoretic approach to the level-set curvature flow equation under dynamic boundary conditions.

2. Degenerate case: A typical boundary condition of degenerate type is the Dirichlet boundary condition. For a stationary Hamilton-Jacobi equation

$$H(x,u(x),\nabla u(x)) = 0 \quad \text{in } U, \quad u(x) = h(x) \quad \text{on } \partial U,$$

the comparison principle is established in [16] for a bounded U and [6] for a unbounded U. The continuity of solutions is assumed in both the results.

In [9] the initial value problem for a Hamilton-Jacobi equation

$$u_t(x,t) - F(x,t)\sqrt{\{u_x(x,t)\}^2 + \gamma^2} = 0 \quad \text{in } (0,L) \times (0,T)$$

is studied under the degenerate boundary condition

$$u_t(x,t) - F(x,t)\alpha(x,t) = 0 \quad \text{on } \{0, L\} \times (0,T).$$

The comparison principle is established with the aid of an equivalent notion of viscosity solutions. To prove the equivalence, continuity of solutions on the boundary is used. More precisely, the continuity guarantees that $(x_\varepsilon, t_\varepsilon) \to (L, \hat{t})$ as $\varepsilon \to 0$ in the proof of [9, Lemma 2.5]. Although the continuity of solutions on the boundary is not explicitly assumed in the statement of the comparison result [9, Theorem 3.1], it is implicitly assumed due to this lemma.

The theory of viscosity solutions for junction (network) spaces was recently developed under possibly degenerate dynamic boundary conditions. A typical junction problem is given by

$$\begin{cases} u_t(x,t) + H_i(\partial_i u(x,t)) = 0 & \text{in } J \times (0,T), \\ u_t(x,t) + F(\partial_1 u(x,t), \ldots, \partial_N u(x,t)) = 0 & \text{on } \{0\} \times (0,T). \end{cases}$$

Here J is a junction space defined as the union of the half-line $J_i = [0,\infty)$ $(i = 1, \ldots, N)$ whose boundaries $\{0\}$ are identified as a junction point. In [17] Hamiltonians H_i are assumed to be quasi-convex and coercive, and F is non-increasing with respect to all variables. The comparison principle ([17, Theorem 1.5]) is established when a subsolution u satisfies a weak continuity condition at the junction point given by

$$u(t,0) = \limsup_{(s,y)\to(t,0),\, y\in J_i\setminus\{0\}} u(s,y) \quad \text{for all } i = 1, \ldots, N.$$

The reader is also referred to [18] for multi-dimensional junction spaces and [10] for the case where H is non-convex.

In [19] junction problems with Kirchhoff-type boundary conditions are studied without convexity assumptions of Hamiltonians. It is shown that there is a unique viscosity solution in the class of bounded and uniformly continuous functions ([19, Theorem 2.1]), and as stated in [19, Section 2], the arguments apply to more general boundary conditions of the form $G(\partial_1 u, \ldots, \partial_N u, u)$ which are strictly increasing with respect to all the variables. In the proof of the theorem, solutions are regularized by the sup- and inf-convolutions. The same regularization procedure also plays an important role in the proof of our main comparison theorem.

Relation to the Dirichlet boundary condition. One may think that the stationary boundary condition (1.5) is equivalent to the Dirichlet boundary condition (1.9). However, they are not equivalent in the sense of viscosity solutions. Namely, the unique solution of (IBV) with (1.5) can be different from that with (1.9). We will give such an example in Section 5.2. See also [9, Section 5] and [12, Section 5.3] for such non-equivalence.

For the example presented in Section 5.2, from physical point of view, the solution u_1 under (1.5) seems to be properer than the solution u_2 under (1.9) in the sense that solutions for (1.9) do not possess a semi-group property. It also turns out that the solution u^β under (1.7) converges not to u_2 but to u_1 as $\beta \to 0$.

Results. *1. Comparison principle:* The main result of this paper is a comparison principle for a viscosity subsolution u and a viscosity supersolution v to (IBV) being continuous on the boundary $\partial\Omega \times (0,T)$ in the sense of (3.1) and (3.2). The idea of the proof is as follows:

Let us explain the idea when the boundary condition is the simplest one (1.5). First, we regularize a subsolution u and a supersolution v by the sup- and inf-convolutions with respect to x' and t. As is well known, the regularized functions u^ε and v_ε solve (IBV) with a small error of ε, which we neglect here for simplicity. Moreover u^ε and $-v_\varepsilon$ are semiconvex. As in the standard proof of the comparison principle, we suppose by contradiction that the function $u^\varepsilon(x,t) - v_\varepsilon(x,t)$ (with additional terms) attains a positive maximum at (x_*, t_*). Our main concern lies in the case where x_* belongs to the boundary $\partial\Omega$.

From the semiconvexity it follows that both u^ε and v_ε are differentiable at (x_*, t_*) with respect to t. By maximality we then have $u^\varepsilon_t(x_*, t_*) - (v_\varepsilon)_t(x_*, t_*) > 0$. This in particular implies that either u^ε or v_ε does not satisfy the boundary condition. In fact, otherwise we would have $0 < u^\varepsilon_t(x_*, t_*) - (v_\varepsilon)_t(x_*, t_*) \leqq c - c = 0$, a contradiction. Let us suppose that u^ε does not satisfy the boundary condition, i.e., $u^\varepsilon_t(x_*, t_*) > c$. We now double the variables to set $\Phi = u^\varepsilon(x,t) - v_\varepsilon(y,s) - \phi(x,t,y,s)$ and let $\hat{Z}_\alpha = (\hat{x}_\alpha, \hat{t}_\alpha, \hat{y}_\alpha, \hat{s}_\alpha)$ be a maximum point of Φ, where α is a coefficient of a penalty term of ϕ. The smooth function ϕ is chosen so that $\hat{y}_\alpha$

belongs to Ω. This is possible in a way similar to the case of the Dirichlet boundary condition. Since $\hat{y}_\alpha \in \Omega$, we have a viscosity superinequality $-\phi_s(\hat{Z}_\alpha) + F \geqq 0$ for (1.1).

Using the continuity of the solutions on the boundary, we see that $\hat{Z}_\alpha$ converges to (x_*, t_*, x_*, t_*) as $\alpha \to \infty$. The point $\hat{x}_\alpha$ may lie on the boundary $\partial\Omega$, but when there are infinitely many such α, it turns out that the semiconvexity of u^ε implies that $\phi_t(\hat{Z}_\alpha)$ converges to $u^\varepsilon_t(x_*, t_*)$ as $\alpha \to \infty$. This comes from the fact that derivatives of semiconvex functions converge (Proposition 2.15 (3)). We thus have $\phi_t(\hat{Z}_\alpha) \approx u^\varepsilon_t(x_*, t_*) > c$, i.e., the boundary condition is violated at $(\hat{x}_\alpha, \hat{t}_\alpha)$. Accordingly, a viscosity subinequality $\phi_t(\hat{Z}_\alpha) + F \leqq 0$ holds. Finally, subtracting the two viscosity inequalities yields a contradiction.

2. Existence results: We also give some existence results of solutions. By the standard Perron's method ([8, Section 4]), we are able to construct viscosity solutions to

$$\text{(IBV.n)} \begin{cases} (1.1),\ (1.3), \\ B(x', t, u(x,t), \nabla' u(x,t), u_t(x,t)) - \beta u_{x_n}(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T) \end{cases} \tag{1.15}$$

with $\beta \geqq 0$. Thanks to the term $-\beta u_{x_n}(x,t)$, the boundary condition (1.15) is not degenerate if $\beta > 0$, so that (IBV.n) admits a unique, continuous viscosity solution. When $\beta = 0$, to obtain the uniqueness and continuity of solutions, we need to construct solutions being continuous on the boundary in order to apply our comparison principle. To do this, we consider the limit of solutions u^β to (IBV.n) as $\beta \to +0$ and prove that the limit function possesses the desired continuity if there exist suitable barrier functions or if F is a first order coercive operator. Barrier assumptions similar to the ones in this paper can be found in [13, Section 6] and [14, Sections 4, 5], where the asymptotic behavior of solutions is studied.

All of the results in this paper are presented for the half space Ω, and almost the same methods apply to a domain of layer type $\{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid 0 < x_n < L\}$ with $L > 0$. Extension of the results to a domain with non-flat boundary is left for future work.

Organization. This paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3 we establish our main comparison result, while we prove existence of solutions in Section 4. Several examples of solutions are given in Section 5.

2 Preliminaries

2.1 Assumptions

We list assumptions on the functions $F = F(x,t,r,p,X) : \Omega \times (0,T) \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$ in (1.1) and $B = B(x', t, r, p', \tau) : \mathbf{R}^{n-1} \times (0,T) \times \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} \to \mathbf{R}$ in (1.2). Here $\mathbf{S}^n$ stands for the space of real $n \times n$ symmetric matrices with the usual ordering. Namely, for $X, Y \in \mathbf{S}^n$ we write $X \leqq Y$ if $\langle (Y - X)\xi, \xi \rangle \geqq 0$ for all $\xi \in \mathbf{R}^n$. Also, set $\|X\| := \sup_{p \in \mathbf{R}^n,\, |p| \leqq 1} |Xp|$ for $X \in \mathbf{S}^n$. We define

$$\mathcal{M} := \{\omega \in C([0,\infty)) \mid \omega(0) = 0,\ \omega > 0 \text{ in } (0,\infty) \text{ and } \omega \text{ is non-decreasing in } [0,\infty)\}.$$

Our assumptions on F are as follows:

(F1) (**Continuity**) F is continuous in $\Omega \times (0,T) \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{S}^n$.

(F2) (**Uniform continuity in** (x,t,p,X)) For every $R > 0$ there exists $\omega_R \in \mathcal{M}$ such that

$$|F(x,t,r,p,X) - F(y,s,r,q,Y)| \leqq \omega_R(|x-y| + |t-s| + |p-q| + \|X - Y\|)$$

for all $x, y \in \Omega$, $t, s \in (0,T)$, $r \in \mathbf{R}$ with $|r| \leqq R$, $p, q \in \mathbf{R}^n$ and $X, Y \in \mathbf{S}^n$.

(F3) (**Monotonicity in** r)

$$F(x,t,r,p,X) \leqq F(x,t,s,p,X)$$

for all $x \in \Omega$, $t \in (0,T)$, $r, s \in \mathbf{R}$ with $r < s$, $p \in \mathbf{R}^n$ and $X \in \mathbf{S}^n$.

(F4) (**Degenerate ellipticity**)
$$F(x,t,r,p,X) \geqq F(x,t,r,p,Y)$$
for all $x \in \Omega$, $t \in (0,T)$, $r \in \mathbf{R}$, $p \in \mathbf{R}^n$ and $X, Y \in \mathbf{S}^n$ with $X \leqq Y$.

We assume that B satisfies

(B1) (**Continuity**) B is continuous in $\mathbf{R}^{n-1} \times (0,T) \times \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R}$.

(B2) (**Uniform continuity in** (x', t, p')) For every $R > 0$ there exists $\rho_R \in \mathcal{M}$ such that
$$|B(x',t,r,p',\tau) - B(y',s,r,p',\tau)| \leqq \rho_R(|x'-y'| + |t-s| + |p'-q'|)$$
for all $x', y' \in \mathbf{R}^{n-1}$, $t, s \in (0,T)$, $r \in \mathbf{R}$ with $|r| \leqq R$, $p', q' \in \mathbf{R}^{n-1}$ and $\tau \in \mathbf{R}$.

(B3) (**Strict monotonicity in** r **or** τ) There exists some $\zeta \in \mathcal{M}$ such that
$$\zeta(\min\{s-r,\, \sigma-\tau\}) \leqq B(x',t,s,p',\sigma) - B(x',t,r,p',\tau)$$
for all $x' \in \mathbf{R}^{n-1}$, $t \in (0,T)$, $r, s \in \mathbf{R}$ with $r < s$, $p' \in \mathbf{R}^{n-1}$ and $\tau, \sigma \in \mathbf{R}$ with $\tau < \sigma$.

Throughout this paper we assume all of the above conditions (F1)–(F4) and (B1)–(B3).

Remark 2.1. It is easy to see that the degenerate dynamic boundary condition (1.4) and the Dirichlet boundary condition (1.6) satisfy (B3).

Remark 2.2. By (B1), sending $\sigma \to \tau + 0$ in (B3) implies

(B3)$'$ (**Monotonicity in** r)
$$B(x',t,r,p',\tau) \leqq B(x',t,s,p',\tau)$$
for all $x' \in \mathbf{R}^{n-1}$, $t \in (0,T)$, $r, s \in \mathbf{R}$ with $r < s$, $p' \in \mathbf{R}^{n-1}$ and $\tau \in \mathbf{R}$.

2.2 Definition of viscosity solutions

Let $K \subset \mathbf{R}^N$ be a subset. For a function $h : K \to \mathbf{R}$, we define the *upper semicontinuous envelope* h^* and the *lower semicontinuous envelope* h_* by
$$h^*(x) := \lim_{r \to +0} \sup_{y \in K \cap B_r(x)} h(y), \quad h_*(x) := \lim_{r \to +0} \inf_{y \in K \cap B_r(x)} h(y) \quad (x \in \overline{K}).$$
Here $B_r(x)$ denotes the open ball of radius r centered at x.

We next introduce a notion of the *parabolic semi-jets* ([8, Section 8], [11, Chapter 3.2.1]). For $u : \overline{\Omega} \times (0,T) \to \mathbf{R}$ and $(x_0, t_0) \in \overline{\Omega} \times (0,T)$, we define
$$\begin{aligned}\mathcal{P}^{2,+}u(x_0,t_0) := \{&(\nabla\phi(x_0,t_0), \phi_t(x_0,t_0), \nabla^2\phi(x_0,t_0)) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{S}^n \mid \\ &\phi \in C^{2,1}(\overline{\Omega} \times (0,T)),\ u - \phi \text{ attains a maximum at } (x_0,t_0) \text{ over } \overline{\Omega} \times (0,T)\},\end{aligned}$$
where $C^{2,1}(\overline{\Omega} \times (0,T))$ is the set of functions of C^2-class in x and C^1-class in t over $\overline{\Omega} \times (0,T)$. Similarly, $\mathcal{P}^{2,-}u(x_0,t_0)$ is defined by replacing "maximum" by "minimum" in the above. The following *extended parabolic semi-jets* will also be used:
$$\begin{aligned}\overline{\mathcal{P}}^{2,\pm}u(x_0,t_0) := \{&(p,\tau,X) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{S}^n \mid \text{there exist } \{(x_m,t_m)\}_{m=1}^{\infty} \subset \overline{\Omega} \times (0,T) \text{ and} \\ &\{(p_m,\tau_m,X_m)\}_{m=1}^{\infty} \subset \mathbf{R}^n \times \mathbf{R} \times \mathbf{S}^n \text{ such that } (p_m,\tau_m,X_m) \in \mathcal{P}^{2,\pm}u(x_m,t_m) \text{ and} \\ &(x_m,t_m) \to (x_0,t_0),\ (p_m,\tau_m,X_m) \to (p,\tau,X) \text{ and } u(x_m,t_m) \to u(x_0,t_0) \text{ as } m \to \infty\}.\end{aligned}$$

Definition 2.3 (Viscosity solution)**.** Let $u : \overline{\Omega} \times [0, T) \to \mathbf{R}$ be bounded from above (resp. from below). We say that u is a *viscosity subsolution* (resp. *viscosity supersolution*) *of* (1.1) *and* (1.2) if for all $(x_0, t_0) \in \overline{\Omega} \times (0, T)$ and $(p, \tau, X) \in \mathcal{P}^{2,+}u^*(x_0, t_0)$ (resp. $(p, \tau, X) \in \mathcal{P}^{2,-}u_*(x_0, t_0)$), we have

$$(*) \begin{cases} \tau + F(x_0, t_0, u^*(x_0, t_0), p, X) \leqq 0 & & (2.1) \\ (\text{resp.}\, \tau + F(x_0, t_0, u_*(x_0, t_0), p, X) \geqq 0) & \text{if } x_0 \in \Omega, & (2.2) \\ \tau + F(x_0, t_0, u^*(x_0, t_0), p, X) \leqq 0 \ \text{ or } \ B(x_0, t_0, u^*(x_0, t_0), p', \tau) \leqq 0 & & (2.3) \\ (\text{resp.}\, \tau + F(x_0, t_0, u_*(x_0, t_0), p, X) \geqq 0 \ \text{ or } \ B(x_0, t_0, u_*(x_0, t_0), p', \tau) \geqq 0) & \text{if } x_0 \in \partial\Omega. & (2.4) \end{cases}$$

If u further satisfies $u^*(\cdot, 0) \leqq u_0$ (resp. $u_*(\cdot, 0) \geqq u_0$) in $\overline{\Omega}$, then we say that u is a *viscosity subsolution* (resp. *viscosity supersolution*) *of* (IBV). A *viscosity solution* is a function which is both a viscosity subsolution and a viscosity supersolution.

We often call the inequality in (2.1) (resp. (2.2)) a viscosity subinequality (resp. viscosity superinequality) for the equation (1.1). Similarly, the second inequality in (2.3) (resp. (2.4)) is called a viscosity subinequality (resp. viscosity superinequality) for the boundary condition (1.2).

Remark 2.4. By definition, a viscosity solution is always supposed to be bounded in $\overline{\Omega} \times [0, T)$ in this paper. In Section 3 we establish a comparison principle for a bounded viscosity sub- and supersolution, and hence it follows that bounded viscosity solutions are unique. In the literature, however, comparison results in unbounded domain often apply to possibly unbounded solutions such as solutions with linear growth. It is a possible extension of our result for (IBV), but we do not pursue the issue of unbounded solutions in this paper.

Remark 2.5. From the continuity assumptions (F1), (B1) and the definitions of $\overline{\mathcal{P}}^{2,\pm}$, the following fact immediately follows: Assume that u is a viscosity subsolution (resp. viscosity supersolution) of (1.1) and (1.2). Then $(*)$ holds for any $(p, \tau, X) \in \overline{\mathcal{P}}^{2,+}u^*(x_0, t_0)$ (resp. $(p, \tau, X) \in \overline{\mathcal{P}}^{2,-}u_*(x_0, t_0)$) with $(x_0, t_0) \in \overline{\Omega} \times (0, T)$.

Example 2.6. Viscosity solutions can be discontinuous and may not be unique under the stationary boundary condition and the Dirichlet boundary condition. Let us give such an example. In Section 5, we again give examples of discontinuous solutions and non-uniqueness, but we now provide a very simple one.

Let $n = 1$ and consider the equation

$$u_t(x, t) + F(u_x(x, t)) = 0 \quad \text{in } \Omega \times (0, T) \tag{2.5}$$

with the initial condition

$$u(x, 0) = 0 \quad \text{in } \overline{\Omega}. \tag{2.6}$$

Here we assume that F is uniformly continuous in $\mathbf{R}$, $F(0) = 0$ and $F \geqq 0$ in $(-\infty, 0]$. Let $c > 0$ be a constant, and we consider the stationary boundary value problem

$$(2.5), \quad u_t(0, t) = c \ \text{ on } \partial\Omega \times (0, T), \quad (2.6), \tag{St0}$$

and the Dirichlet boundary value problem

$$(2.5), \quad u(0, t) = ct \ \text{ on } \partial\Omega \times (0, T), \quad (2.6). \tag{Di0}$$

Let us define

$$u(x, t) := \begin{cases} 0 & (x > 0), \\ f(t) & (x = 0), \end{cases}$$

where $f \in C([0, T)) \cap C^1((0, T))$ satisfies $f(0) = 0$ and

$$0 \leqq f(t) \leqq ct, \quad f'(t) \leqq c \quad \text{for all } t \in (0, T). \tag{2.7}$$

Then u is a solution to both (St0) and (Di0) whatever f satisfying (2.7) is chosen. To check this, we first note that $u^* = u$ and $u_* = 0$. Since $u^* = u_* = 0$ in $\Omega \times (0, T)$ and $F(0) = 0$, we see that u is a solution of (2.5). At the initial time, we have $u^*(x, 0) = u_*(x, 0) = 0$ in $\overline{\Omega}$, and so (2.6) is satisfied. On the boundary

$\partial\Omega \times (0,T)$, we have $u^*(0,t) = f(t) \leqq ct$ and $u_*(0,t) = 0$. If $u^* - \phi$ attains its maximum at $(0,t)$ for $\phi \in C^1(\overline{\Omega} \times (0,T))$, then $\phi_t(0,t) = f'(t) \leqq c$ and $u^*(0,t) \leqq ct$. Namely, the viscosity subinequality for the boundary condition is satisfied for both (St0) and (Di0). Assume next that $u_* - \phi$ attains its minimum at $(0,t)$ for $\phi \in C^1(\overline{\Omega} \times (0,T))$. Then $\phi_t(0,t) = 0$ and $\phi_x(0,t) \leqq 0$, and thus we have $\phi_t(0,t) + F(\phi_x(0,t)) \geqq 0$.

We therefore conclude that solutions are not unique for both (St0) and (Di0). By Corollary 3.3 established later, we see that $u \equiv 0$ is a unique continuous solution of (St0) and (Di0). However, the solution does not satisfy the boundary condition in the classical sense. Namely, $u_t(0,t) \neq c$ and $u(0,t) \neq ct$ for $t \in (0,T)$. When F takes negative values in $(-\infty, 0)$, there is a chance that the unique continuous solution satisfies $u(0,t) = ct + u_0(0)$ for $t \in (0,T)$. See Section 5.1 for the details.

2.3 Some relevant comparison results

We first state a comparison and uniqueness result for (IBV.n) with $\beta > 0$. The unique, continuous solution u^β of (IBV.n) is used in Section 4 to construct solutions of (IBV).

Theorem 2.7 (Comparison principle for (IBV.n))**.** *Let $\beta > 0$. Assume that u and v are respectively a viscosity subsolution of* (IBV.n) *and a viscosity subsolution of* (IBV.n)*. If $u^*(\cdot,0) \leqq v_*(\cdot,0)$ in $\overline{\Omega}$, then $u^* \leqq v_*$ in $\overline{\Omega} \times [0,T)$.*

When Ω is a bounded domain, this comparison result is established in [2, Theorem II.1], whose proof is essentially given in [2, Proof of Theorem I.2]. Here we only state how to modify the argument in [2] so that it applies to a unbounded domain Ω.

Sketch of proof. We argue by contradiction. The argument of [2, Proof of Theorem I.2] (adapted for the case of [2, Theorem II.1]), where Ω is a bounded set, begins with taking a maximum point (x_0, t_0) of

$$M = \max_{\overline{\Omega} \times [0,T]} (u^*(x,t) - v_*(x,t)) > 0.$$

Due to the classical comparison principle ([8, Theorem 3.3, Theorem 8.2]), we may assume that the maximum is achieved only at a point on the boundary. Namely, $(x_0,t_0) \in \partial\Omega \times (0,T)$. We then introduce a suitable test function $\phi(x,t,y,s)$ with the doubled variables and study a maximum point of $u^*(x,t) - v_*(y,s) - \phi(x,t,y,s)$. The maximum point approaches to (x_0,t_0,x_0,t_0), and so the proof reduces to the one in a bounded set around (x_0,t_0). We obtain a contradiction by subtracting two viscosity inequalities for the equation (1.1).

When Ω is the half-space, instead of the above M, we consider

$$M = \max_{\overline{\Omega} \times [0,T]} \left(u^*(x,t) - v_*(x,t) - \frac{\sigma}{T-t} - \gamma f(x) \right),$$

where $\sigma, \gamma > 0$ are small constants and $f(x) = \sqrt{1+|x|^2}$. This maximum is attained, and again we may suppose $(x_0,t_0) \in \partial\Omega \times (0,T)$. Since the argument is local near (x_0,t_0), we can discuss as if Ω were bounded. After subtracting two viscosity inequalities for (1.1), we get a contradiction for σ, γ small enough. □

Theorem 2.7 immediately gives the following uniqueness result:

Corollary 2.8 (Uniqueness of solutions for (IBV.n))**.** *Let $\beta > 0$. Then* (IBV.n) *admits at most one viscosity solution. If u is a viscosity solution of* (IBV.n)*, then it is continuous in $\overline{\Omega} \times [0,T)$.*

Proof. Assume that u and v are viscosity solutions of (IBV.n). Then Theorem 2.7 implies that $u^* \leqq v_*$ and $v^* \leqq u_*$ in $\overline{\Omega} \times [0,T)$. Combining these, we obtain $u^* \leqq v_* \leqq v^* \leqq u_* \leqq u^*$ in $\overline{\Omega} \times [0,T)$, and so every inequality should be an equality. We thus conclude that u and v are continuous and $u = v$. □

Following the same argument as [1, Lemma 3.2], we next prove that Lipschitz continuous solutions of (IBV) are unique. Our main result, Theorem 3.2, generalizes the following theorem except that Theorem 3.2 requires extra conditions (3.3) and (3.4) near the initial time. We assume (B5) given in Section 4.1.

Theorem 2.9 (Comparison principle for Lipschitz continuous solutions)**.** *Assume* (B5)*. Let $u, v : \overline{\Omega} \times [0,T) \to \mathbf{R}$ be Lipschitz continuous functions in $\overline{\Omega} \times [0,T)$. Assume that u is a viscosity subsolution of* (IBV) *and v is a viscosity supersolution of* (IBV)*. If $u(\cdot,0) \leqq v(\cdot,0)$ in $\overline{\Omega}$, then $u \leqq v$ in $\overline{\Omega} \times [0,T)$.*

Proof. Fix $\beta > 0$. Let M be the maximum of the Lipschitz constants of u and v, and define $u^\beta(x,t) = u(x,t) - (\beta M t)/k$ and $v^\beta(x,t) = v(x,t) + (\beta M t)/k$. It is then easy to see that u^β and v^β are respectively a subsolution and a supersolution of (IBV.n). Here we only check the boundary condition for u^β. Assume that $(p,\tau,X) \in \mathcal{P}^{2,+}u^\beta(x,t)$ for $(x,t) \in \partial\Omega \times (0,T)$. Then $(p, \tau + (\beta M/k), X) \in \mathcal{P}^{2,+}u(x,t)$. If $\tau + (\beta M/k) + F(x,t,u(x,t),p,X) \leqq 0$ holds, it is clear that $\tau + F(x,t,u(x,t),p,X) \leqq 0$. We consider the other case. Note that the Lipschitz continuity of u implies $|p_n| \leqq M$. Then, by (B3)$'$ and (B5) we have

$$\begin{aligned} B(x',t,u^\beta(x,t),p',\tau) - \beta p_n &\leqq B(x',t,u(x,t),p',\tau) + \beta M \\ &\leqq -\beta M + B(x',t,u(x,t),p',\tau + (\beta M/k)) + \beta M \leqq 0. \end{aligned}$$

Thus u^β satisfies the boundary condition (1.15).

Theorem 2.7 implies that $u^\beta \leqq v^\beta$ in $\overline{\Omega} \times [0,T)$. The proof is complete if we send $\beta \to +0$. □

We therefore conclude that Lipschitz continuous solutions of (IBV) are unique. Unfortunately, Lipschitz continuous solutions may not exist even if the initial datum is Lipschitz continuous. We give such an example in Section 5.3. In order to deduce the uniqueness of solutions to such problems, we need our comparison result, Theorem 3.2.

Remark 2.10. The Lipschitz continuity of solutions assumed in Theorem 2.9 can be relaxed. In fact, it is enough to assume that there is $M > 0$ such that, for every $t \in (0,T)$,

$$(p,\tau,X) \in \mathcal{P}^{2,+}u^*(0,t) \text{ implies } p_n \geqq -M, \quad (p,\tau,X) \in \mathcal{P}^{2,-}v_*(0,t) \text{ implies } p_n \leqq M.$$

2.4 Tangential sup-/inf-convolution

We next introduce a sup-convolution and inf-convolution ([4, Chapter II, Section 4.2], [7, Section 3.5]), which will be used to regularize solutions to (IBV). Usually, the convolution is taken with respect to all the variables of a function, but we here exclude the normal variable x_n and define the convolutions as follows. For a function $u : \overline{\Omega} \times [0,T] \to \mathbf{R}$ and $\varepsilon > 0$, we define a *sup-convolution* u^ε and a *inf-convolution* u_ε as

$$u^\varepsilon(x',x_n,t) := \sup_{y' \in \mathbf{R}^{n-1}, s \in [0,T]} \left\{ u(y',x_n,s) - \frac{1}{2\varepsilon}|x'-y'|^2 - \frac{1}{2\varepsilon}(t-s)^2 \right\}, \tag{2.8}$$

$$u_\varepsilon(x',x_n,t) := \inf_{y' \in \mathbf{R}^{n-1}, s \in [0,T]} \left\{ u(y',x_n,s) + \frac{1}{2\varepsilon}|x'-y'|^2 + \frac{1}{2\varepsilon}(t-s)^2 \right\} \tag{2.9}$$

for $(x,t) \in \overline{\Omega} \times [0,T]$. If u is bounded, i.e., $\|u\| := \sup_{\overline{\Omega} \times [0,T]} |u| < \infty$, then u^ε and u_ε are real-valued and

$$-\|u\| \leqq u_\varepsilon \leqq u \leqq u^\varepsilon \leqq \|u\| \quad \text{in } \overline{\Omega} \times [0,T].$$

We will give some properties of these convolutions. In what follows, we only consider u^ε since statements for u_ε are parallel. First, the following facts are well-known:

Proposition 2.11. *Let $\varepsilon > 0$ and $u : \overline{\Omega} \times [0,T] \to \mathbf{R}$ be bounded and upper semicontinuous. Set $C_0 := 2\sqrt{\|u\|}$.*

(1) *For every $x_n \geqq 0$, the function $(x',t) \mapsto u^\varepsilon(x',x_n,t)$ is semiconvex and Lipschitz continuous in $\mathbf{R}^{n-1} \times [0,T]$; more precisely,*

$$(x',t) \mapsto u^\varepsilon(x',x_n,t) + \frac{1}{2\varepsilon}|x'|^2 + \frac{1}{2\varepsilon}t^2 \quad \textit{is convex in } \mathbf{R}^{n-1} \times [0,T]$$

and

$$|u^\varepsilon(x',x_n,t) - u^\varepsilon(y',x_n,s)| \leqq \frac{C_0}{\sqrt{\varepsilon}}(|x'-y'| + |t-s|) \qquad \textit{for all } (x',t),(y',s) \in \mathbf{R}^{n-1} \times [0,T].$$

(2) *For every* $(x,t) \in \overline{\Omega} \times [0,T]$, *there exists some* $(y_0', s_0) \in \mathbf{R}^{n-1} \times [0,T]$ *such that* $|x' - y_0'| \leqq C_0\sqrt{\varepsilon}$, $|t - s_0| \leqq C_0\sqrt{\varepsilon}$ *and*

$$u^\varepsilon(x', x_n, t) = u(y_0', x_n, s_0) - \frac{1}{2\varepsilon}|x' - y_0'|^2 - \frac{1}{2\varepsilon}(t - s_0)^2.$$

For the proofs, see, e.g., [4, Lemmas II.4.11, II.4.12]. The same proofs as in [4] work even if the x_n-variable is fixed.

We next show that the convolutions keep viscosity properties. Although the assertions of this type are known ([4, Proposition II.4.13]), we provide the proofs in order to show that fixing the variable does not cause any problems for the proofs.

Proposition 2.12. *Let* $\varepsilon > 0$ *and* $u : \overline{\Omega} \times [0,T] \to \mathbf{R}$ *be bounded and upper semicontinuous. Set* $C_0 := 2\sqrt{\|u\|}$, $R := \|u\|$ *and* $I_\varepsilon := (C_0\sqrt{\varepsilon}, T - C_0\sqrt{\varepsilon})$. *We suppose that* ε *satisfies* $2C_0\sqrt{\varepsilon} < T$ *so that* $I_\varepsilon \neq \emptyset$.

(1) *Assume that*

$$\max_{\overline{\Omega}\times(0,T)} (u^\varepsilon - \phi) = (u^\varepsilon - \phi)(\hat{x}', \hat{x}_n, \hat{t})$$

for some $(\hat{x}, \hat{t}) \in \overline{\Omega} \times I_\varepsilon$ *and* $\phi \in C^{2,1}(\overline{\Omega} \times (0,T))$. *Then*

$$\max_{\overline{\Omega}\times I_\varepsilon} (u - \tilde{\phi}) = (u - \tilde{\phi})(\hat{x}' + \varepsilon p', \hat{x}_n, \hat{t} + \varepsilon\tau), \tag{2.10}$$

where $(p', \tau) := (\nabla'\phi, \phi_t)(\hat{x}, \hat{t})$ *and* $\tilde{\phi}(x', x_n, t) := \phi(x' - \varepsilon p', x_n, t - \varepsilon\tau)$. *Moreover,*

$$u^\varepsilon(\hat{x}', \hat{x}_n, \hat{t}) = u(\hat{x}' + \varepsilon p', \hat{x}_n, \hat{t} + \varepsilon\tau) - \frac{\varepsilon}{2}|p'|^2 - \frac{\varepsilon}{2}\tau^2. \tag{2.11}$$

(2) *If* u *is a viscosity subsolution of* (1.1) *and* (1.2), *then* u^ε *is a viscosity subsolution of*

$$u_t(x,t) + F(x,t,u(x,t),\nabla u(x,t),\nabla^2 u(x,t)) = \omega_R(2C_0\sqrt{\varepsilon}) \qquad \text{in } \Omega \times I_\varepsilon, \tag{2.12}$$
$$B(x',t,u(x,t),\nabla' u(x,t),u_t(x,t)) = \rho_R(2C_0\sqrt{\varepsilon}) \qquad \text{on } \partial\Omega \times I_\varepsilon, \tag{2.13}$$

where ω_R *and* ρ_R *are respectively the functions in* (F2) *and* (B2).

Proof. (1) Set $m := (u^\varepsilon - \phi)(\hat{x}', \hat{x}_n, \hat{t})$. By assumption we have

$$u(y', x_n, s) - \frac{1}{2\varepsilon}|x' - y'|^2 - \frac{1}{2\varepsilon}(t - s)^2 - \phi(x', x_n, t) \leqq m \tag{2.14}$$

for all $(x,t) \in \overline{\Omega}\times(0,T)$ and $(y', s) \in \mathbf{R}^{n-1}\times[0,T]$. By Proposition 2.11 (2) there exists $(y_0', s_0) \in \mathbf{R}^{n-1}\times[0,T]$ such that

$$u^\varepsilon(\hat{x}', \hat{x}_n, \hat{t}) = u(y_0', \hat{x}_n, s_0) - \frac{1}{2\varepsilon}|\hat{x}' - y_0'|^2 - \frac{1}{2\varepsilon}(\hat{t} - s_0)^2. \tag{2.15}$$

Since the maximum of the left-hand side on (2.14) is attained at $(\hat{x}, \hat{t}, y_0', s_0)$, we see that the map

$$(x', t) \mapsto u(y_0', \hat{x}_n, s_0) - \frac{1}{2\varepsilon}|x' - y_0'|^2 - \frac{1}{2\varepsilon}(t - s_0)^2 - \phi(x', \hat{x}_n, t) \tag{2.16}$$

attains a maximum at $(\hat{x}', \hat{t})$. This shows that

$$-\frac{\hat{x}' - y_0'}{\varepsilon} - \nabla'\phi(\hat{x}', \hat{x}_n, \hat{t}) = 0, \quad -\frac{\hat{t} - s_0}{\varepsilon} - \phi_t(\hat{x}', \hat{x}_n, \hat{t}) = 0,$$

that is, $y_0' = \hat{x}' + \varepsilon p'$ and $s_0 = \hat{t} + \varepsilon\tau$. Substituting these for (2.15) implies (2.11). Also, since the maximum value of (2.16) is m, we have

$$\begin{aligned} m &= u(y_0', \hat{x}_n, s_0) - \frac{1}{2\varepsilon}|\hat{x}' - y_0'|^2 - \frac{1}{2\varepsilon}(\hat{t} - s_0)^2 - \phi(\hat{x}', \hat{x}_n, \hat{t}) \\ &= u(y_0', \hat{x}_n, s_0) - \frac{\varepsilon}{2}|p'|^2 - \frac{\varepsilon}{2}\tau^2 - \phi(\hat{x}', \hat{x}_n, \hat{t}). \end{aligned}$$

Using this equality, we see

$$\begin{aligned}(u-\tilde{\phi})(\hat{x}'+\varepsilon p',\hat{x}_n,\hat{t}+\varepsilon\tau)&=u(\hat{x}'+\varepsilon p',\hat{x}_n,\hat{t}+\varepsilon\tau)-\phi(\hat{x}',\hat{x}_n,\hat{t})\\&=u(y_0',\hat{x}_n,s_0)-\phi(\hat{x}',\hat{x}_n,\hat{t})\\&=m+\frac{\varepsilon}{2}|p'|^2+\frac{\varepsilon}{2}\tau^2.\end{aligned}\tag{2.17}$$

We also note that the result of Proposition 2.11 (1) implies that

$$|p'|\leqq\frac{C_0}{\sqrt{\varepsilon}},\quad |\tau|\leqq\frac{C_0}{\sqrt{\varepsilon}}.\tag{2.18}$$

Let us fix $(z,r)\in\overline{\Omega}\times I_\varepsilon$, and then choose $x'=z'-\varepsilon p'$, $x_n=z_n$, $y'=z'$, $t=r-\varepsilon\tau$ and $s=r$ in (2.14). Here, since $|\varepsilon\tau|\leqq C_0\sqrt{\varepsilon}$ by (2.18), we have $t=r-\varepsilon\tau\in(0,T)$. Then

$$\begin{aligned}m&\geqq u(z',z_n,r)-\frac{\varepsilon}{2}|p'|^2-\frac{\varepsilon}{2}\tau^2-\phi(z'-\varepsilon p',z_n,r-\varepsilon\tau)\\&=(u-\tilde{\phi})(z',z_n,r)-\frac{\varepsilon}{2}|p'|^2-\frac{\varepsilon}{2}\tau^2.\end{aligned}$$

Hence, we conclude by (2.17) that $(u-\tilde{\phi})(z',z_n,r)\leqq(u-\tilde{\phi})(\hat{x}'+\varepsilon p',\hat{x}_n,\hat{t}+\varepsilon\tau)$.

(2) Assume that $\max_{\overline{\Omega}\times(0,T)}(u^\varepsilon-\phi)=(u^\varepsilon-\phi)(\hat{x},\hat{t})$ for some $(\hat{x},\hat{t})\in\overline{\Omega}\times I_\varepsilon$ and $\phi\in C^{2,1}(\overline{\Omega}\times(0,T))$. Set $(p',\tau):=(\nabla'\phi,\phi_t)(\hat{x},\hat{t})$. Since u is a viscosity subsolution of (1.1) and (1.2), it follows from (2.10) that, if $\hat{x}_n>0$,

$$\tau+F(\hat{x}'+\varepsilon p',\hat{x}_n,\hat{t}+\varepsilon\tau,u(\hat{x}'+\varepsilon p',\hat{x}_n,\hat{t}+\varepsilon\tau),\nabla\phi(\hat{x},\hat{t}),\nabla^2\phi(\hat{x},\hat{t}))\leqq 0,\tag{2.19}$$

while, if $\hat{x}_n=0$, then we have (2.19) or

$$B(\hat{x}'+\varepsilon p',\hat{t}+\varepsilon\tau,u(\hat{x}'+\varepsilon p',\hat{x}_n,\hat{t}+\varepsilon\tau),p',\tau)\leqq 0.\tag{2.20}$$

By (2.11) and the monotonicities (F3) and (B3)′, we see that (2.19) and (2.20) respectively yield

$$\begin{aligned}\tau+F(\hat{x}'+\varepsilon p',\hat{x}_n,\hat{t}+\varepsilon\tau,u^\varepsilon(\hat{x},\hat{t}),\nabla\phi(\hat{x},\hat{t}),\nabla^2\phi(\hat{x},\hat{t}))\leqq 0,\\B(\hat{x}'+\varepsilon p',\hat{t}+\varepsilon\tau,u^\varepsilon(\hat{x},\hat{t}),p',\tau)\leqq 0.\end{aligned}$$

Since $|u^\varepsilon(\hat{x},\hat{t})|\leqq\|u\|=R$ and $|\varepsilon p'|+|\varepsilon\tau|\leqq 2C_0\sqrt{\varepsilon}$ by (2.18), we conclude the proof. □

We next study (semi)continuity with respect to x_n. Since the definitions of our convolutions exclude x_n-variable, such a continuity is not a direct consequence of well-known results in the literature. For this reason, we give the proofs again.

Proposition 2.13. *Let $\varepsilon>0$ and $u:\overline{\Omega}\times[0,T]\to\mathbf{R}$ be bounded and upper semicontinuous.*

(1) *u^ε is upper semicontinuous in $\overline{\Omega}\times[0,T]$.*

(2) *Let $(x',t)\in\mathbf{R}^{n-1}\times[0,T]$ and assume that*

$$u(x',0,t)\leqq\liminf_{x_n\to+0}u(x',x_n,t).\tag{2.21}$$

Then

$$u^\varepsilon(x',0,t)\leqq\liminf_{x_n\to+0}u^\varepsilon(x',x_n,t),\tag{2.22}$$

and hence

$$u^\varepsilon(x',0,t)=\lim_{x_n\to+0}u^\varepsilon(x',x_n,t).$$

Proof. (1) Fix $(x,t) \in \overline{\Omega} \times [0,T]$, and let $\{(x_j,t_j)\}_{j=1}^{\infty} \subset \overline{\Omega} \times [0,T]$ be a sequence such that $(x_j,t_j) \to (x,t)$, $(x_j,t_j) \neq (x,t)$ and $u^\varepsilon(x_j,t_j) \to \alpha$ as $j \to \infty$ for some $\alpha \in \mathbf{R}$. We have to show that $\alpha \leqq u^\varepsilon(x,t)$.

For every j we see by Proposition 2.11 (2) that there is $(y_j', s_j) \in \mathbf{R}^{n-1} \times [0,T]$ such that $|x_j' - y_j'| \leqq C_0\sqrt{\varepsilon}$ with $C_0 = 2\sqrt{\|u\|}$ and

$$u^\varepsilon(x_j',(x_j)_n,t_j) = u(y_j',(x_j)_n,s_j) - \frac{1}{2\varepsilon}|x_j' - y_j'|^2 - \frac{1}{2\varepsilon}(t_j - s_j)^2. \tag{2.23}$$

Since $\{y_j'\}_j$ is bounded, it contains a convergent subsequence. We let $y_{j(k)}' \to \bar{y}'$ and $s_{j(k)} \to \bar{s}$ as $k \to \infty$. Then, taking lim sup in (2.23) along this subsequence, we get

$$\limsup_{k\to\infty} u^\varepsilon(x_{j(k)}',(x_{j(k)})_n,t_{j(k)}) \leqq u(\bar{y}',x_n,\bar{s}) - \frac{1}{2\varepsilon}|x' - \bar{y}'|^2 - \frac{1}{2\varepsilon}(t-\bar{s})^2,$$

where the upper semicontinuity of u has been applied. The left-hand side equals α while the right-hand side is dominated by $u^\varepsilon(x',x_n,t)$ by its definition. Thus we obtain $\alpha \leqq u^\varepsilon(x',x_n,t)$.

(2) We only need to prove (2.22) since u^ε is upper semicontinuous. Fix $(x',t) \in \mathbf{R}^{n-1} \times [0,T]$ and let $x_n > 0$. Again, by Proposition 2.11 (2) there is $(y_0',s_0) \in \mathbf{R}^{n-1} \times [0,T]$ such that

$$u^\varepsilon(x',0,t) = u(y_0',0,s_0) - \frac{1}{2\varepsilon}|x' - y_0'|^2 - \frac{1}{2\varepsilon}(t-s_0)^2. \tag{2.24}$$

By the definition of $u^\varepsilon(x',x_n,t)$, we have

$$u^\varepsilon(x',x_n,t) \geqq u(y_0',x_n,s_0) - \frac{1}{2\varepsilon}|x' - y_0'|^2 - \frac{1}{2\varepsilon}(t-s_0)^2.$$

Taking $\liminf_{x_n\to+0}$, we see by (2.21) that

$$\liminf_{x_n\to+0} u^\varepsilon(x',\eta,t) \geqq u(y_0',0,s_0) - \frac{1}{2\varepsilon}|x' - y_0'|^2 - \frac{1}{2\varepsilon}(t-s_0)^2.$$

Since the right-hand side is equal to $u^\varepsilon(x',0,t)$ by (2.24), we conclude the proof. □

The next proposition shows the behavior of $u^\varepsilon(x,t)$ near $t=0$. In what follows, $BUC(K)$ denotes the set of bounded and uniformly continuous functions in $K \subset \mathbf{R}^N$.

Proposition 2.14. *Let $u : \overline{\Omega} \times [0,T] \to \mathbf{R}$ be bounded and upper semicontinuous. Let $\varepsilon > 0$, $u_0 \in BUC(\overline{\Omega})$, $a \in \mathcal{M}$ and define*

$$u_0^\varepsilon(x) := \sup_{y'\in\mathbf{R}^{n-1}} \left\{ u_0(y',x_n) - \frac{1}{2\varepsilon}|x'-y'|^2 \right\}.$$

(1) *u_0^ε converges to u_0 uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$.*

(2) *Assume that, for some $T' \in (0,T]$,*

$$u(x,t) \leqq u_0(x) + a(t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,T']. \tag{2.25}$$

Then, for all $\theta > 0$, there exist $\varepsilon_0 > 0$ and $\kappa > 0$ such that

$$u^\varepsilon(x,t) \leqq u_0(x) + \theta \quad \text{for all } \varepsilon \in (0,\varepsilon_0) \text{ and } (x,t) \in \overline{\Omega} \times [0,\kappa].$$

Proof. (1) Let $\omega \in \mathcal{M}$ be a modulus of continuity of u_0, i.e, $|u_0(x) - u_0(y)| \leqq \omega(|x-y|)$ $(x,y \in \overline{\Omega})$. Set $C_0 := 2\sqrt{\|u_0\|}$. By Proposition 2.11 (2), for all $x \in \overline{\Omega}$,

$$\begin{aligned} 0 \leqq u_0^\varepsilon(x) - u_0(x) &= \sup_{\substack{y'\in\mathbf{R}^{n-1} \\ |x'-y'|\leqq C_0\sqrt{\varepsilon}}} \left\{ u_0(y',x_n) - u_0(x',x_n) - \frac{1}{2\varepsilon}|x'-y'|^2 \right\} \\ &\leqq \sup_{\substack{y'\in\mathbf{R}^{n-1} \\ |x'-y'|\leqq C_0\sqrt{\varepsilon}}} \{\omega(|x'-y'|) - 0\} = \omega\left(C_0\sqrt{\varepsilon}\right). \end{aligned}$$

This shows the uniform convergence.

(2) By changing a in an appropriate way, we may assume that $T' = T$. Taking the sup-convolution of both the sides of (2.25), we have

$$u^\varepsilon(x,t) \leqq u_0^\varepsilon(x) + a^\varepsilon(t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,T], \tag{2.26}$$

where

$$a^\varepsilon(t) = \sup_{s\in[0,T]} \left\{ a(s) - \frac{1}{2\varepsilon}(t-s)^2 \right\}.$$

Fix $\theta > 0$. By (1) there is $\varepsilon_1 > 0$ such that $u_0^\varepsilon - u_0 \leqq \theta/3$ in $\overline{\Omega}$ for all $\varepsilon \in (0, \varepsilon_1)$. Similarly, for some $\varepsilon_2 > 0$, we have $a^\varepsilon - a \leqq \theta/3$ in $[0,T]$ whenever $\varepsilon \in (0, \varepsilon_2)$. Thus (2.26) implies that

$$u^\varepsilon(x,t) \leqq u_0(x) + a(t) + \frac{2\theta}{3} \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,T].$$

Since $a(0) = 0$ and a is continuous at 0, the above inequality shows the desired result. □

Finally, we recall some properties of sub-/superdifferentials of semiconvex functions. For $K \subset \mathbf{R}^N$, $f : K \to \mathbf{R}$ and $x_0 \in K$, we define

$$D^+f(x_0) := \{\nabla\phi(x_0) \in \mathbf{R}^N \mid \phi \in C^1(K),\ f - \phi \text{ attains a maximum at } x_0 \text{ over } K\},$$
$$D^-f(x_0) := \{\nabla\phi(x_0) \in \mathbf{R}^N \mid \phi \in C^1(K),\ f - \phi \text{ attains a minimum at } x_0 \text{ over } K\}.$$

Proposition 2.15. *Let $U \subset \mathbf{R}^N$ be an open set and $f : U \to \mathbf{R}$ be a semiconvex function. Let $x \in U$ and $p \in \mathbf{R}^N$. Then*

1. *$D^-f(x) \neq \emptyset$.*
2. *Either $D^+f(x) = \emptyset$ or f is differentiable at x.*
3. *Let $\{(x_n, p_n)\}_{n=1}^\infty \subset U \times \mathbf{R}^N$. If $p_n \in D^+f(x_n)$, $(x_n, p_n) \to (x, p)$ as $n \to \infty$ and f is differentiable at x, then $p = \nabla f(x)$.*

The proofs of (1) and (2) are given in [4, Proposition II.4.7 (a), (b)] while (3) is found in the argument in [4, Proof of Proposition 4.8]. Obviously, similar assertions hold for a semiconcave function f when $D^\pm$ are replaced by $D^\mp$.

3 Comparison principle

For the comparison principle, we will assume that a viscosity subsolution u and a viscosity supersolution v of (IBV) are continuous in x_n-direction on the boundary. More precisely,

$$u^*(x', 0, t) \leqq \liminf_{x_n \to +0} u^*(x', x_n, t) \quad \text{for all } (x', t) \in \mathbf{R}^{n-1} \times (0,T), \tag{3.1}$$
$$v_*(x', 0, t) \geqq \limsup_{x_n \to +0} v_*(x', x_n, t) \quad \text{for all } (x', t) \in \mathbf{R}^{n-1} \times (0,T). \tag{3.2}$$

These conditions are weaker than the usual continuity. We also impose conditions which control the behaviors of u and v near the initial time:

$$\text{There exist } a \in \mathcal{M} \text{ and } T' \in (0,T] \text{ such that } u^*(x,t) \leqq u_0(x) + a(t) \text{ for all } (x,t) \in \overline{\Omega} \times [0,T'], \tag{3.3}$$
$$\text{There exist } a \in \mathcal{M} \text{ and } T' \in (0,T] \text{ such that } v_*(x,t) \geqq u_0(x) - a(t) \text{ for all } (x,t) \in \overline{\Omega} \times [0,T']. \tag{3.4}$$

These are not restrictive conditions in the sense that, when $u_0 \in BUC(\overline{\Omega})$, there always exist barrier functions satisfying (3.3) and (3.4). Thus we are able to construct viscosity solutions satisfying (3.3) and (3.4) by Perron's method; see Section 4 for the details.

We define

$$X^- := \left\{u \mid u : \overline{\Omega} \times [0,T) \to \mathbf{R} \text{ is a viscosity subsolution of (IBV) satisfying (3.1) and (3.3)}\right\},$$
$$X^+ := \left\{v \mid v : \overline{\Omega} \times [0,T) \to \mathbf{R} \text{ is a viscosity supersolution of (IBV) satisfying (3.2) and (3.4)}\right\},$$

and $X := X^- \cap X^+$.

Remark 3.1. One of typical classes of functions satisfying (3.1)–(3.4) is $BUC(\mathbf{R}^n \times [0,T))$.

Theorem 3.2 (Comparison principle)**.** *Let $u, v : \overline{\Omega} \times [0,T) \to \mathbf{R}$ be bounded. If $u \in X^-$ and $v \in X^+$, then $u^* \leqq v_*$ in $\overline{\Omega} \times [0,T)$.*

Proof. **1. Preliminary arguments.** To simplify notation we write u for u^* and v for v_*. Also, we define $R := \max\{\|u\|, \|v\|\}$, $f(x) := \sqrt{1+|x|^2}$ and $l := \sup_{x\in\mathbf{R}^n} \|\nabla^2 f(x)\| < \infty$.

Suppose by contradiction that $M_0 := u(x_0,t_0) - v(x_0,t_0) > 0$ for some $(x_0,t_0) \in \overline{\Omega} \times (0,T)$. Let $\sigma > 0$ be a constant satisfying

$$0 < \sigma < \frac{M_0(T-t_0)}{4},$$

and then choose $\gamma \in (0,1)$ such that

$$0 < \gamma < \frac{M_0}{4f(x_0)}, \quad \rho_R(\gamma) < \zeta\left(\frac{\sigma}{T^2} \cdot \min\{T,1\}\right), \quad \omega_R(\gamma(1+l+l^2)) < \frac{\sigma}{T^2},$$

where ρ_R, ζ and ω_R are the functions in (B2), (B3) and (F2), respectively. By (3.3), (3.4) and Proposition 2.14 (2), there are $\varepsilon_0 > 0$ and $\kappa > 0$ such that

$$u^\varepsilon(x,t) \leqq u_0(x) + \frac{\sigma}{2T}, \; v_\varepsilon(x,t) \geqq u_0(x) - \frac{\sigma}{2T} \quad \text{for all } \varepsilon \in (0,\varepsilon_0) \text{ and } (x,t) \in \overline{\Omega} \times [0,\kappa].$$

Here u^ε and v_ε are the convolutions defined by (2.8) and (2.9), respectively. In particular, for such ε and (x,t), we have

$$u^\varepsilon(x,t) - v_\varepsilon(x,t) \leqq \frac{\sigma}{T}. \tag{3.5}$$

Hereafter we take $\varepsilon \in (0,\varepsilon_0)$ small so that

$$C_1\sqrt{\varepsilon} < \min\left\{\frac{T}{2}, \kappa, \frac{\sigma}{\|u\|+\|v\|}\right\},$$
$$2\rho_R(2C_1\sqrt{\varepsilon}) < \zeta\left(\frac{\sigma}{T^2} \cdot \min\{T,1\}\right) - \rho_R(\gamma), \quad 2\omega_R(2C_1\sqrt{\varepsilon}) < \frac{\sigma}{T^2} - \omega_R(\gamma(1+l+l^2)),$$

where $C_1 := 2\sqrt{R}$.

Let us recall Proposition 2.12 (2), which guarantees that u^ε is a viscosity subsolution of (2.12), (2.13) with C_1 instead of C_0 and $I_\varepsilon = (C_1\sqrt{\varepsilon}, T - C_1\sqrt{\varepsilon}) \neq \emptyset$. Similarly, v_ε is a viscosity supersolution of

$$\begin{aligned} &v_t(x,t) + F(x,t,v(x,t),\nabla v(x,t),\nabla^2 v(x,t)) = -\omega_R(2C_1\sqrt{\varepsilon}) && \text{in } \Omega \times I_\varepsilon, \\ &B(x',t,v(x,t),\nabla' v(x,t),v_t(x,t)) = -\rho_R(2C_1\sqrt{\varepsilon}) && \text{on } \partial\Omega \times I_\varepsilon. \end{aligned} \tag{3.6}$$

We now set

$$M := \max_{\overline{\Omega}\times[0,T]} \left(u^\varepsilon(x,t) - v_\varepsilon(x,t) - \frac{\sigma}{T-t} - \gamma f(x)\right). \tag{3.7}$$

Here we interpret $\sigma/(T-t) = \infty$ for $t = T$. By the boundedness of u^ε and v_ε, the maximum is attained and M is finite. Also, the choices of σ and γ implies that

$$u^\varepsilon(x_0,t_0) - v_\varepsilon(x_0,t_0) - \frac{\sigma}{T-t_0} - \gamma f(x_0) > M_0 - \frac{M_0}{4} - \frac{M_0}{4} = \frac{M_0}{2},$$

and so $M > 0$. From this positivity of M it follows that the maximum in (3.7) is attained over $\overline{\Omega} \times [\kappa, T^-]$, where

$$T^- := T - \frac{\sigma}{\|u\|+\|v\|}.$$

Indeed, if $0 \leqq t \leqq \kappa$, then (3.5) is valid and

$$u^\varepsilon(x,t) - v_\varepsilon(x,t) - \frac{\sigma}{T-t} - \gamma f(x) \leqq \frac{\sigma}{T} - \frac{\sigma}{T} - 0 = 0.$$

For $T^- \leqq t < T$, we see that

$$u^\varepsilon(x,t) - v_\varepsilon(x,t) - \frac{\sigma}{T-t} - \gamma f(x) \leqq \|u\| + \|v\| - \frac{\sigma}{T-T^-} - 0 = 0.$$

Thus the maximizer of (3.7) lies in $\overline{\Omega} \times [\kappa, T^-]$. Recall that the convolutions u^ε and v_ε possess viscosity properties in $\overline{\Omega} \times (C_1\sqrt{\varepsilon}, T - C_1\sqrt{\varepsilon})$ in view of Proposition 2.12 (2), and by the choice of ε, we now have $[\kappa, T^-] \subset (C_1\sqrt{\varepsilon}, T - C_1\sqrt{\varepsilon})$. This guarantees that we are able to apply the definitions of viscosity sub- and supersolutions to u^ε and v_ε near the maximum point of (3.7).

If

$$\max_{\partial\Omega \times [0,T]} \left(u^\varepsilon(x,t) - v_\varepsilon(x,t) - \frac{\sigma}{T-t} - \gamma f(x) \right) < M, \tag{3.8}$$

the proof is classical. In fact, in this case we use a test function ϕ in (3.16) with $\delta = 0$. It then follows that a maximum point $\hat{Z}_\alpha = (\hat{x}_\alpha, \hat{t}_\alpha, \hat{y}_\alpha, \hat{s}_\alpha)$ of Φ converges to some $(\bar{x}, \bar{t}, \bar{x}, \bar{t})$ with $(\bar{x}, \bar{t}) \in \Omega \times [\kappa, T^-]$. The fact $\bar{x} \in \Omega$ is a consequence of (3.8), and by this we do not need to take the boundary condition (1.2) into account when we apply the definitions of viscosity sub- and supersolutions after penalization.

For this reason, we have to consider the case where the maximum in (3.8) is equal to M. We let $(x_*, t_*) = (x'_*, 0, t_*) \in \partial\Omega \times [\kappa, T^-]$ be a point attaining the maximum, i.e.,

$$\begin{aligned} M &= \max_{\partial\Omega \times [0,T]} \left(u^\varepsilon(x,t) - v_\varepsilon(x,t) - \frac{\sigma}{T-t} - \gamma f(x) \right) \\ &= u^\varepsilon(x_*, t_*) - v_\varepsilon(x_*, t_*) - \frac{\sigma}{T-t_*} - \gamma f(x_*). \end{aligned} \tag{3.9}$$

Note that

$$u^\varepsilon(x_*, t_*) - v_\varepsilon(x_*, t_*) > \frac{\sigma}{T-t_*} + \gamma f(x_*). \tag{3.10}$$

We also remark that, by Proposition 2.13 (2) and (3.2),

$$v_\varepsilon(x', 0, t) = \lim_{x_n \to +0} v_\varepsilon(x', x_n, t). \tag{3.11}$$

2. Differentiablity of u^ε and v_ε at (x_*, t_*). Let us define

$$U(x', t) := u^\varepsilon(x', 0, t), \quad \tilde{U}(x', t) := U(x', t) - \frac{\sigma}{T-t} - \gamma f(x', 0) - M, \quad V(x', t) := v_\varepsilon(x', 0, t).$$

Then, by Proposition 2.11 (1), U and $\tilde{U}$ are semiconvex and V is semiconcave in $\mathbf{R}^{n-1} \times [0,T]$. Also,

$$\max_{\mathbf{R}^{n-1} \times [0,T]} (\tilde{U} - V) = (\tilde{U} - V)(x'_*, t_*) = 0. \tag{3.12}$$

We claim that $\tilde{U}$ and V are differentiable at (x'_*, t_*). First, it follows from (3.12) that $0 \in D^+(\tilde{U} - V)(x'_*, t_*)$. Since $\tilde{U} - V$ is semiconvex, Proposition 2.15 (2) guarantees that $\tilde{U} - V$ is differentiable at (x'_*, t_*). By (3.12) again, we see that V touches $\tilde{U}$ from above at (x'_*, t_*), which shows that $D^-\tilde{U}(x'_*, t_*) \subset D^-V(x'_*, t_*)$. Recall $D^-\tilde{U}(x'_*, t_*) \neq \emptyset$ by Proposition 2.15 (1), and therefore $D^-V(x'_*, t_*) \neq \emptyset$. Proposition 2.15 (2) thus implies that V is differentiable at (x'_*, t_*), and so the sum $(\tilde{U} - V) + V = \tilde{U}$ is differentiable at (x'_*, t_*) too. The proof of the claim is now complete.

The claim implies that u^ε and v_ε are differentiable at (x_*, t_*) in the directions of x' and t. Since $(\nabla', \partial_t)(\tilde{U} - V)(x'_*, t_*) = (0, 0)$, we have

$$\nabla' u^\varepsilon(x_*, t_*) - \nabla' v_\varepsilon(x_*, t_*) = \gamma \nabla' f(x_*), \tag{3.13}$$

$$u^\varepsilon_t(x_*, t_*) - (v_\varepsilon)_t(x_*, t_*) = \frac{\sigma}{(T-t_*)^2}. \tag{3.14}$$

Since $|\nabla f| \leqq 1$, we deduce from (3.13) that

$$|\nabla' u^\varepsilon(x_*, t_*) - \nabla' v_\varepsilon(x_*, t_*)| \leqq \gamma. \tag{3.15}$$

We now divide the situation into the following three cases:

(i) $B(x_*', t_*, u^\varepsilon(x_*, t_*), \nabla' u^\varepsilon(x_*, t_*), u_t^\varepsilon(x_*, t_*)) > \rho_R(2C_1\sqrt{\varepsilon})$;

(ii) $B(x_*', t_*, v_\varepsilon(x_*, t_*), \nabla' v_\varepsilon(x_*, t_*), (v_\varepsilon)_t(x_*, t_*)) < -\rho_R(2C_1\sqrt{\varepsilon})$;

(iii) Neither (i) nor (ii) holds.

Among these three possibilities, the case (iii) is easy to derive a contradiction. In fact, if (iii) were true, we would have

$$\begin{aligned} L := B(x_*', t_*, u^\varepsilon(x_*, t_*), \nabla' u^\varepsilon(x_*, t_*), u_t^\varepsilon(x_*, t_*)) - B(x_*', t_*, v_\varepsilon(x_*, t_*), \nabla' v_\varepsilon(x_*, t_*), (v_\varepsilon)_t(x_*, t_*)) \\ \leqq 2\rho_R(2C_1\sqrt{\varepsilon}). \end{aligned}$$

Let us estimate L. First, by (B2) and (3.15) we have

$$L \geqq B(x_*', t_*, u^\varepsilon(x_*, t_*), \nabla' u^\varepsilon(x_*, t_*), u_t^\varepsilon(x_*, t_*)) - B(x_*', t_*, v_\varepsilon(x_*, t_*), \nabla' u^\varepsilon(x_*, t_*), (v_\varepsilon)_t(x_*, t_*)) - \rho_R(\gamma).$$

Next, from (B3), (3.10) and (3.14) it follows that

$$L \geqq \zeta\left(\min\left\{\frac{\sigma}{T-t_*} + \gamma f(x_*),\ \frac{\sigma}{(T-t_*)^2}\right\}\right) - \rho_R(\gamma) \geqq \zeta\left(\min\left\{\frac{\sigma}{T},\ \frac{\sigma}{T^2}\right\}\right) - \rho_R(\gamma).$$

We therefore have

$$0 < \zeta\left(\frac{\sigma}{T^2}\cdot\min\{T,\ 1\}\right) - \rho_R(\gamma) \leqq 2\rho_R(2C_1\sqrt{\varepsilon}),$$

which contradicts the choice of ε.

Hereafter we discuss the case (i); the proof for the case (ii) is parallel.

3. Doubling the variables. Assume (i). Let us define a function $\Phi : (\overline{\Omega} \times [0, T])^2 \to \mathbf{R} \cup \{-\infty\}$ by

$$\Phi(x, t, y, s) := u^\varepsilon(x, t) - v_\varepsilon(y, s) - \phi(x, t, y, s)$$

with

$$\begin{aligned} \phi(x, t, y, s) := \alpha\left\{\left|x - y + \frac{\delta}{\sqrt{\alpha}}e_n\right|^2 + (t-s)^2\right\} \\ + \delta\left\{|x - x_*|^2 + \left|y - x_* - \frac{\delta}{\sqrt{\alpha}}e_n\right|^2 + (t - t_*)^2\right\} + \frac{\sigma}{T-t} + \gamma f(x), \end{aligned} \tag{3.16}$$

where $\alpha > 1$, $0 < \delta < \min\{\sqrt{M}/2,\ 1\}$ are constants and $e_n = (0, \ldots, 0, 1) \in \mathbf{R}^n$. This choice of δ guarantees that

$$\Phi(x_*, t_*, x_*, t_*) \geqq \frac{M}{2}. \tag{3.17}$$

Indeed, by the definition of Φ

$$\begin{aligned} \Phi(x_*, t_*, x_*, t_*) &= u^\varepsilon(x_*, t_*) - v_\varepsilon(x_*, t_*) - \phi(x_*, t_*, x_*, t_*) \\ &= M - \delta^2 - \frac{\delta^3}{\alpha} > M - \delta^2 - \frac{\delta^2}{1} = M - 2\delta^2 > M - \frac{M}{2} = \frac{M}{2}, \end{aligned}$$

which gives (3.17).

Let $\hat{Z}^\alpha = (\hat{x}_\alpha, \hat{t}_\alpha, \hat{y}_\alpha, \hat{s}_\alpha)$ be a maximum point of Φ over $(\overline{\Omega} \times [0,T])^2$. Since u^ε and v_ε are bounded, the maximum is attained on a compact set $(\overline{B_d(0)} \times [0,T])^2$ with some large $d > 0$ independent of α. We thus have $\hat{x}_\alpha, \hat{y}_\alpha \in \overline{B_d(0)}$ and $\hat{t}_\alpha, \hat{s}_\alpha \in [0,T)$. Note also that (3.17) gives

$$\Phi(\hat{Z}_\alpha) \geqq \Phi(x_*, t_*, x_*, t_*) \geqq \frac{M}{2}. \tag{3.18}$$

For later use we compute the derivatives of ϕ:

$$\nabla_x \phi(x,t,y,s) = 2\alpha(x-y) + 2\sqrt{\alpha}\delta e_n + 2\delta(x - x_*) + \gamma \nabla f(x), \tag{3.19}$$

$$\nabla_y \phi(x,t,y,s) = -2\alpha(x-y) - 2\sqrt{\alpha}\delta e_n + 2\delta\left(y - x_* - \frac{\delta}{\sqrt{\alpha}} e_n\right), \tag{3.20}$$

$$\phi_t(x,t,y,s) = 2\alpha(t-s) + 2\delta(t-t_*) + \frac{\sigma}{(T-t)^2}, \tag{3.21}$$

$$\phi_s(x,t,y,s) = -2\alpha(t-s), \tag{3.22}$$

$$\nabla^2_{(x,y)}\phi(x,t,y,s) = \begin{pmatrix} 2\alpha I + 2\delta I + \gamma\nabla^2 f(x) & -2\alpha I \\ -2\alpha I & 2\alpha I + 2\delta I \end{pmatrix}. \tag{3.23}$$

In particular, by (3.21) and (3.22),

$$\phi_t(x,t,y,s) + \phi_s(x,t,y,s) = 2\delta(t-t_*) + \frac{\sigma}{(T-t)^2} \geqq 2\delta(t - t_*) + \frac{\sigma}{T^2}. \tag{3.24}$$

4. Behavior of the maximum point $\hat{Z}_\alpha$. Let us prove that

$$\hat{Z}_\alpha = (\hat{x}_\alpha, \hat{t}_\alpha, \hat{y}_\alpha, \hat{s}_\alpha) \to (x_*, t_*, x_*, t_*) \quad \text{as } \alpha \to \infty. \tag{3.25}$$

First, since $\hat{x}_\alpha, \hat{y}_\alpha \in \overline{B_d(0)}$ and $\hat{t}_\alpha, \hat{s}_\alpha \in [0,T)$, it follows that, up to a subsequence

$$(\hat{x}_\alpha, \hat{t}_\alpha, \hat{y}_\alpha, \hat{s}_\alpha) \to (\bar{x}, \bar{t}, \bar{y}, \bar{s}) \quad \text{as } \alpha \to \infty$$

for some $\bar{x}, \bar{y} \in \overline{B_d(0)}$ and $\bar{t}, \bar{s} \in [0,T]$. Next, rearranging the inequality $\Phi(\hat{Z}_\alpha) \geqq \Phi(x_*, t_*, x_* + (\delta/\sqrt{\alpha})e_n, t_*)$, we have

$$\begin{aligned}
&\alpha\left\{\left|\hat{x}_\alpha - \hat{y}_\alpha + \frac{\delta}{\sqrt{\alpha}}e_n\right|^2 + (\hat{t}_\alpha - \hat{s}_\alpha)^2\right\} \\
&\leqq u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) - v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha) - \delta\left\{|\hat{x}_\alpha - x_*|^2 + \left|\hat{y}_\alpha - x_* - \frac{\delta}{\sqrt{\alpha}}e_n\right|^2 + (\hat{t}_\alpha - t_*)^2\right\} \\
&\quad - \frac{\sigma}{T - \hat{t}_\alpha} - \gamma f(\hat{x}_\alpha) - u^\varepsilon(x_*, t_*) + v_\varepsilon\left(x_* + \frac{\delta}{\sqrt{\alpha}}e_n, t_*\right) + \frac{\sigma}{T - t_*} + \gamma f(x_*).
\end{aligned} \tag{3.26}$$

Since the right-hand side is bounded from above as $\alpha \to \infty$, we see that $\bar{x} = \bar{y}$ and $\bar{t} = \bar{s}$. Then we take $\limsup_{\alpha\to\infty}$ in (3.26) to see that, by (3.11), (3.7) and (3.9)

$$\begin{aligned}
0 &\leqq \limsup_{\alpha\to\infty} \alpha\left\{\left|\hat{x}_\alpha - \hat{y}_\alpha + \frac{\delta}{\sqrt{\alpha}}e_n\right|^2 + (\hat{t}_\alpha - \hat{s}_\alpha)^2\right\} \\
&\leqq u^\varepsilon(\bar{x}, \bar{t}) - v_\varepsilon(\bar{x}, \bar{t}) - \delta\left\{2|\bar{x} - x_*|^2 + (\bar{t} - t_*)^2\right\} \\
&\quad - \frac{\sigma}{T - \bar{t}} - \gamma f(\bar{x}) - u^\varepsilon(x_*, t_*) + v_\varepsilon(x_*, t_*) + \frac{\sigma}{T - t_*} + \gamma f(x_*) \\
&\leqq -\delta\left\{2|\bar{x} - x_*|^2 + (\bar{t} - t_*)^2\right\}.
\end{aligned}$$

This shows that $\bar{x} = x_*$, $\bar{t} = t_*$ and

$$\alpha\left|\hat{x}_\alpha - \hat{y}_\alpha + \frac{\delta}{\sqrt{\alpha}}e_n\right|^2 \to 0, \quad \alpha(\hat{t}_\alpha - \hat{s}_\alpha)^2 \to 0 \quad \text{as } \alpha \to \infty. \tag{3.27}$$

(3.25) has therefore been proved.

By the first fact in (3.27), we see that $\sqrt{\alpha}((\hat{x}_\alpha)_n - (\hat{y}_\alpha)_n) + \delta \to 0$ as $\alpha \to \infty$. Accordingly, $(\hat{x}_\alpha)_n < (\hat{y}_\alpha)_n$ for α large enough. Since $(\hat{x}_\alpha)_n \geqq 0$, we have $(\hat{y}_\alpha)_n > 0$, i.e.,

$$\hat{y}_\alpha \in \Omega. \tag{3.28}$$

We furthermore claim that

$$u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) \to u^\varepsilon(x_*, t_*), \quad v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha) \to v_\varepsilon(x_*, t_*) \quad \text{as } \alpha \to \infty. \tag{3.29}$$

By (3.26) we have

$$\begin{aligned}
&u^\varepsilon(x_*, t_*) - v_\varepsilon\left(x_* + \frac{\delta}{\sqrt{\alpha}} e_n, t_*\right) \\
&\leqq u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) - v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha) - \delta\left\{|\hat{x}_\alpha - x_*|^2 + \left|\hat{y}_\alpha - x_* - \frac{\delta}{\sqrt{\alpha}} e_n\right|^2 + (\hat{t}_\alpha - t_*)^2\right\} \\
&\quad - \frac{\sigma}{T - \hat{t}_\alpha} - \gamma f(\hat{x}_\alpha) + \frac{\sigma}{T - t_*} + \gamma f(x_*).
\end{aligned}$$

Taking $\liminf_{\alpha\to\infty}$ implies that $u^\varepsilon(x_*, t_*) - v_\varepsilon(x_*, t_*) \leqq \liminf_{\alpha\to\infty}\{u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) - v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha)\}$. Since $u^\varepsilon - v_\varepsilon$ is upper semicontinuous, this shows that

$$\lim_{\alpha\to\infty}\{u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) - v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha)\} = u^\varepsilon(x_*, t_*) - v_\varepsilon(x_*, t_*). \tag{3.30}$$

Then,

$$\begin{aligned}
\liminf_{\alpha\to\infty} u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) &= \liminf_{\alpha\to\infty}[\{u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) - v_\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha)\} + v_\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha)] \\
&\geqq \{u^\varepsilon(x_*, t_*) - v_\varepsilon(x_*, t_*)\} + v_\varepsilon(x_*, t_*) = u^\varepsilon(x_*, t_*).
\end{aligned}$$

Thus $\lim_{\alpha\to\infty} u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) = u^\varepsilon(x_*, t_*)$. This and (3.30) give (3.29). Also, by (3.10) and (3.29), we have

$$u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha) > v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha) \quad \text{for } \alpha > 0 \text{ large enough.} \tag{3.31}$$

5. Dividing the situation about $\hat{x}_\alpha$. Either (a) or (b) below or both occurs:

There is a sequence $\{\alpha_j\}_{j\in\mathbf{N}}$ such that $\alpha_j \to \infty$ as $j \to \infty$ and

(a) $\hat{x}_{\alpha_j} \in \Omega$ for all $j \in \mathbf{N}$;

(b) $\hat{x}_{\alpha_j} \in \partial\Omega$ for all $j \in \mathbf{N}$.

In what follows we write α for α_j to simplify notation. The case (a) is an easier case since we always have a viscosity subinequality for the equation (2.12) at $\hat{x}_\alpha \in \Omega$.

Let us discuss the case (b). We want to prove that

$$B(\hat{x}'_\alpha, \hat{t}_\alpha, u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha), \nabla_{x'}\phi(\hat{Z}_\alpha), \phi_t(\hat{Z}_\alpha)) > \rho_R(2C_1\sqrt{\varepsilon}) \quad \text{for } \alpha > 0 \text{ large enough,} \tag{3.32}$$

that is, the boundary condition (2.13) is violated. Recall the function $U(x', t) = u^\varepsilon(x', 0, t)$ introduced in Step 2. Considering a map

$$(x', t) \mapsto \Phi(x', 0, t, \hat{y}_\alpha, \hat{s}_\alpha) = U(x', t) - v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha) - \phi(x', 0, t, \hat{y}_\alpha, \hat{s}_\alpha),$$

we see that

$$(\nabla_{x'}\phi(\hat{Z}_\alpha), \phi_t(\hat{Z}_\alpha)) \in D^+U(\hat{x}'_\alpha, \hat{t}_\alpha).$$

We also recall that $U(x',t)$ is Lipschitz continuous. Thus, taking a subsequence of α if necessary, we may assume that $(\nabla_{x'}\phi(\hat{Z}_\alpha), \phi_t(\hat{Z}_\alpha)) \to (\bar{p}', \bar{\tau})$ as $\alpha \to \infty$ for some $(\bar{p}', \bar{\tau}) \in \mathbf{R}^{n-1} \times \mathbf{R}$. Since $(\hat{x}'_\alpha, \hat{t}_\alpha) \to (x'_*, t_*)$ as $\alpha \to \infty$ and U is differentiable at (x'_*, t_*), Proposition 2.15 (3) implies that

$$(\bar{p}', \bar{\tau}) = (\nabla_{x'}U(x'_*, t_*), U_t(x'_*, t_*)) = (\nabla' u^\varepsilon(x'_*, t_*), u^\varepsilon_t(x'_*, t_*)).$$

By this and (3.29), we see that, as $\alpha \to \infty$,

$$B(\hat{x}'_\alpha, \hat{t}_\alpha, u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha), \nabla_{x'}\phi(\hat{Z}_\alpha), \phi_t(\hat{Z}_\alpha)) \to B(x'_*, t_*, u^\varepsilon(x_*, t_*), \nabla' u^\varepsilon(x'_*, t_*), u^\varepsilon_t(x'_*, t_*)).$$

Since we now discuss the case (i) in Step 2, the above fact shows (3.32).

6. Use of Crandall-Ishii lemma. We now apply Crandall-Ishii lemma ([8, Theorem 3.2, Theorem 8.3]) to the function Φ at $\hat{Z}_\alpha$. Then there exist $X_\alpha, Y_\alpha \in \mathbf{S}^n$ such that

$$(\nabla_x\phi(\hat{Z}_\alpha), \phi_t(\hat{Z}_\alpha), X_\alpha) \in \overline{\mathcal{P}}^{2,+} u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha), \tag{3.33}$$

$$(-\nabla_y\phi(\hat{Z}_\alpha), -\phi_s(\hat{Z}_\alpha), -Y_\alpha) \in \overline{\mathcal{P}}^{2,-} v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha), \tag{3.34}$$

$$\begin{pmatrix} X_\alpha & O \\ O & Y_\alpha \end{pmatrix} \leqq A + A^2 \tag{3.35}$$

with $A = \nabla^2_{(x,y)}\phi(\hat{Z}_\alpha)$. Since A is of the form (3.23), operating $(\xi, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$ to (3.35) implies that

$$X_\alpha + Y_\alpha \leqq (4\delta + 8\delta^2)I + \{\gamma(1+4\delta)\nabla^2 f(\hat{x}_\alpha) + \gamma^2(\nabla^2 f(\hat{x}_\alpha))^2\} =: \delta' I + G_\delta(\hat{x}_\alpha). \tag{3.36}$$

Note that, since $\|\nabla^2 f(\hat{x}_\alpha)\| \leqq l$, we have

$$\|G_\delta(\hat{x}_\alpha)\| \leqq \gamma(1+4\delta)l + \gamma^2 l^2. \tag{3.37}$$

Set $p_\alpha := 2\alpha(\hat{x}_\alpha - \hat{y}_\alpha) + 2\sqrt{\alpha}\delta e_n$, so that (3.19) and (3.20) at $\hat{Z}_\alpha$ are represented as

$$\nabla_x\phi(\hat{Z}_\alpha) = p_\alpha + 2\delta(\hat{x}_\alpha - x_*) + \gamma\nabla f(\hat{x}_\alpha), \quad \nabla_y\phi(\hat{Z}_\alpha) = -p_\alpha + 2\delta\left(\hat{y}_\alpha - x_* - \frac{\delta}{\sqrt{\alpha}}e_n\right).$$

In Step 5, we have proved that the viscosity subinequality for (2.12) holds even if $\hat{x}_\alpha \in \partial\Omega$ since the boundary condition (2.13) breaks for ϕ. Also, $\hat{y}_\alpha$ never lies on the boundary by (3.28). Therefore we deduce from (3.33) and (3.34) that

$$\phi_t(\hat{Z}_\alpha) + F(\hat{x}_\alpha, \hat{t}_\alpha, u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha), p_\alpha + 2\delta(\hat{x}_\alpha - x_*) + \gamma\nabla f(\hat{x}_\alpha), X_\alpha) \leqq \omega_R(2C_1\sqrt{\varepsilon}),$$
$$-\phi_s(\hat{Z}_\alpha) + F\left(\hat{y}_\alpha, \hat{s}_\alpha, v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha), p_\alpha - 2\delta\left(\hat{y}_\alpha - x_* - \frac{\delta}{\sqrt{\alpha}}e_n\right), -Y_\alpha\right) \geqq -\omega_R(2C_1\sqrt{\varepsilon}).$$

Due to (3.31) and (F3), we are able to replace $v_\varepsilon(\hat{y}_\alpha, \hat{s}_\alpha)$ in the second inequality by $u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha)$. Then, subtracting the two inequalities, we see that, by (3.24), (3.36) and (F4),

$$\begin{aligned}
2\omega_R(2C_1\sqrt{\varepsilon}) &\geqq \phi_t(\hat{Z}_\alpha) + \phi_s(\hat{Z}_\alpha) + F(\hat{x}_\alpha, \hat{t}_\alpha, u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha), p_\alpha + 2\delta(\hat{x}_\alpha - x_*) + \gamma\nabla f(\hat{x}_\alpha), X_\alpha) \\
&\quad - F\left(\hat{y}_\alpha, \hat{s}_\alpha, u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha), p_\alpha - 2\delta\left(\hat{y}_\alpha - x_* - \frac{\delta}{\sqrt{\alpha}}e_n\right), -Y_\alpha\right) \\
&\geqq 2\delta(\hat{t}_\alpha - t_*) + \frac{\sigma}{T^2} + F(\hat{x}_\alpha, \hat{t}_\alpha, u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha), p_\alpha + 2\delta(\hat{x}_\alpha - x_*) + \gamma\nabla f(\hat{x}_\alpha), X_\alpha) \\
&\quad - F\left(\hat{y}_\alpha, \hat{s}_\alpha, u^\varepsilon(\hat{x}_\alpha, \hat{t}_\alpha), p_\alpha - 2\delta\left(\hat{y}_\alpha - x_* - \frac{\delta}{\sqrt{\alpha}}e_n\right), X_\alpha - \delta' I - G_\delta(\hat{x}_\alpha)\right).
\end{aligned}$$

We next apply (F2) to obtain

$$\begin{aligned}
&2\omega_R(2C_1\sqrt{\varepsilon}) \\
&\geqq 2\delta(\hat{t}_\alpha - t_*) + \frac{\sigma}{T^2} \\
&\quad - \omega_R\left(|\hat{x}_\alpha - \hat{y}_\alpha| + |\hat{t}_\alpha - \hat{s}_\alpha| + 2\delta|\hat{x}_\alpha - x_*| + \gamma + 2\delta\left|\hat{y}_\alpha - x_* - \frac{\delta}{\sqrt{\alpha}}e_n\right| + \delta'\|I\| + \gamma(1+4\delta)l + \gamma^2 l^2\right),
\end{aligned}$$

where we have used the fact that $|\nabla f| \leqq 1$ and (3.37). Sending $\alpha \to \infty$, we obtain

$$2\omega_R(2C_1\sqrt{\varepsilon}) \geqq \frac{\sigma}{T^2} - \omega_R(\gamma + \delta'\|I\| + \gamma(1+4\delta)l + \gamma^2 l^2),$$

and then letting $\delta \to 0$ yields

$$2\omega_R(2C_1\sqrt{\varepsilon}) \geqq \frac{\sigma}{T^2} - \omega_R(\gamma + \gamma l + \gamma^2 l^2) \geqq \frac{\sigma}{T^2} - \omega_R(\gamma(1 + l + l^2)).$$

This is a contradiction to the choice of ε. □

In the same manner as the proof of Corollary 2.8, we obtain

Corollary 3.3 (Uniqueness of solutions for (IBV))**.** (IBV) *admits at most one viscosity solution belonging to X. If $u \in X$ is a viscosity solution of* (IBV)*, then it is continuous in $\overline{\Omega} \times [0, T)$.*

4 Existence of continuous solutions

We construct the unique solution $u \in X$ of (IBV), especially as the limit of the solution u^β to (IBV.n). Application and examples will be given in Section 5.

4.1 Assumptions

In addition to (F1)–(F4) and (B1)–(B3), we further impose the following conditions on F and B:

(F5) (**Boundedness in** (x, t, p))

$$\mathcal{F}_L(r, \mu) := \sup\{|F(x, t, r, p, \mu I)| \mid x \in \overline{\Omega},\ t \in (0, T),\ p \in \mathbf{R}^n \text{ with } |p| \leqq L\} < \infty$$

for all $L > 0$, $r \in \mathbf{R}$ and $\mu \in \mathbf{R}$.

(B4) (**Boundedness in** (x', t, p'))

$$\mathcal{B}_L(r) := \sup\{|B(x', t, r, p', 0)| \mid x' \in \mathbf{R}^{n-1},\ t \in (0, T),\ p' \in \mathbf{R}^{n-1} \text{ with } |p'| \leqq L\} < \infty$$

for all $L > 0$ and $r \in \mathbf{R}$.

(B5) (**Strict monotonicity in** τ) There exists some $k > 0$ such that

$$k(\sigma - \tau) \leqq B(x', t, r, p', \sigma) - B(x', t, r, p', \tau)$$

for all $x' \in \mathbf{R}^{n-1}$, $t \in (0, T)$, $r \in \mathbf{R}$, $p' \in \mathbf{R}^{n-1}$ and $\tau, \sigma \in \mathbf{R}$ with $\tau < \sigma$.

Throughout this section we assume (F5), (B4) and (B5). Furthermore we fix a constant $\beta_0 > 0$ as an upper bound of β in (IBV.n). Also, assume that $u_0 \in BUC(\overline{\Omega})$.

We prepare stability results for viscosity sub- and supersolutions. See, e.g., [8, Lemma 4.2] and [11, Lemma 2.4.1] for the proof.

Proposition 4.1 (Stability)**.** *Let $\mathcal{S}$ be a family of viscosity subsolutions (resp. viscosity supersolutions) of* (1.1) *and* (1.2) *which are uniformly bounded from above (resp. below) in $\overline{\Omega} \times (0, T)$. Set $u(x, t) := \sup\{w(x, t) \mid w \in \mathcal{S}\}$ (resp. $u(x, t) := \inf\{w(x, t) \mid w \in \mathcal{S}\}$). Then u is a viscosity subsolution (resp. viscosity supersolution) of* (1.1) *and* (1.2)*.*

4.2 Initial barriers

To carry out Perron's method, we first construct barrier functions satisfying the initial condition. To do this, we regularize the initial datum $u_0 \in BUC(\overline{\Omega})$ by the standard sup-convolution u_0^ε and the inf-convolution $u_{0\varepsilon}$, which are defined as

$$u_0^\varepsilon(x) := \sup_{y\in\overline{\Omega}} \left\{ u_0(y) - \frac{1}{2\varepsilon}|x-y|^2 \right\}, \quad u_{0\varepsilon}(x) := \inf_{y\in\overline{\Omega}} \left\{ u_0(y) + \frac{1}{2\varepsilon}|x-y|^2 \right\}. \tag{4.1}$$

They are Lipschitz continuous in $\overline{\Omega}$. Besides u_0^ε and $u_{0\varepsilon}$ are respectively semiconvex and semiconcave in $\overline{\Omega}$ (Proposition 2.11). Namely, for $C_0 = 2\sqrt{\|u_0\|}$,

$$(p,X) \in J^{2,+}u_0^\varepsilon(x) \text{ implies } \begin{cases} |p| \leqq C_0/\varepsilon \text{ and } X \geqq -(1/\varepsilon)I & \text{if } x \in \Omega, \\ |p'| \leqq C_0/\varepsilon \text{ and } p_n \geqq -C_0/\varepsilon & \text{if } x \in \partial\Omega, \end{cases} \tag{4.2}$$

$$(p,X) \in J^{2,-}u_{0\varepsilon}(x) \text{ implies } \begin{cases} |p| \leqq C_0/\varepsilon \text{ and } X \leqq (1/\varepsilon)I & \text{if } x \in \Omega, \\ |p'| \leqq C_0/\varepsilon \text{ and } p_n \leqq C_0/\varepsilon & \text{if } x \in \partial\Omega. \end{cases} \tag{4.3}$$

Here, for $v_0 : \overline{\Omega} \to \mathbf{R}$, we denote by $J^{2,+}v_0(x)$ (resp. $J^{2,-}v_0(x)$) the set of $(\nabla\phi(x), \nabla^2\phi(x)) \in \mathbf{R}^n \times \mathbf{S}^n$ such that $v_0 - \phi$ attains a maximum (resp. minimum) at x over $\overline{\Omega}$ for $\phi \in C^2(\overline{\Omega})$.

Proposition 4.2 (Initial barriers)**.** *Let $\varepsilon > 0$ and define*

$$M_\varepsilon^\pm := \max \left\{ \mathcal{F}_{C_0/\sqrt{\varepsilon}}(\mp\|u_0\|, \pm 1/\varepsilon),\ \frac{1}{k}\left\{ \mathcal{B}_{C_0/\sqrt{\varepsilon}}(\mp\|u_0\|) + (\beta_0 C_0/\sqrt{\varepsilon}) \right\} \right\},$$

where $C_0 = 2\sqrt{\|u_0\|}$ and k is the constant in (B5)*.*

(1) *Define*

$$w_\varepsilon^-(x,t) = -\|u_0^\varepsilon - u_0\| - M_\varepsilon^- t + u_0^\varepsilon(x), \quad w_\varepsilon^+(x,t) = \|u_{0\varepsilon} - u_0\| + M_\varepsilon^+ t + u_{0\varepsilon}(x) \quad \text{for } (x,t) \in \overline{\Omega} \times [0,T).$$

Then w_ε^- is a viscosity subsolution and w_ε^+ is a viscosity supersolution of (IBV.n) *for every $\beta \in [0, \beta_0]$.*

(2) *Define*

$$v^-(x,t) := \sup_{\varepsilon>0} w_\varepsilon^-(x,t), \quad v^+(x,t) := \inf_{\varepsilon>0} w_\varepsilon^+(x,t) \quad \text{for } (x,t) \in \overline{\Omega} \times [0,T),$$

and

$$\alpha^-(r) := \inf_{\varepsilon>0}(M_\varepsilon^- r + \|u_0^\varepsilon - u_0\|), \quad \alpha^+(r) := \inf_{\varepsilon>0}(M_\varepsilon^+ r + \|u_{0\varepsilon} - u_0\|) \quad \text{for } r > 0.$$

Then v^- is a viscosity subsolution and v^+ is a viscosity supersolution of (IBV.n) *for every $\beta \in [0,\beta_0]$. Moreover, $\alpha^\pm \in \mathcal{M}$ and*

$$u_0(x) - \alpha^-(t) \leqq v^-(x,t) \leqq u_0(x) \leqq v^+(x,t) \leqq u_0(x) + \alpha^+(t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,T). \tag{4.4}$$

Proof. (1) We only prove that w_ε^- is a subsolution since the proof for w_ε^+ is parallel. First, observe that

$$w_\varepsilon^-(x,t) \leqq -\|u_0^\varepsilon - u_0\| + u_0^\varepsilon(x) \leqq -\{u_0^\varepsilon(x) - u_0(x)\} + u_0^\varepsilon(x) = u_0(x). \tag{4.5}$$

In particular, $w_\varepsilon^-(\cdot,0) \leqq u_0$ in $\overline{\Omega}$. In what follows, we prove that w_ε^- is a subsolution of (1.1) and (1.15).

Let $(p,\tau,X) \in \mathcal{P}^{2,+}w_\varepsilon^-(x,t)$ for $(x,t) \in \overline{\Omega} \times (0,T)$. Then $\tau = -M_\varepsilon^-$ and $(p,X) \in J^{2,+}u_0^\varepsilon(x)$. If $x \in \Omega$, then we have $|p| \leqq C_0/\sqrt{\varepsilon}$ and $X \geqq -(1/\varepsilon)I$ by (4.2). Applying (F3) and (F4), we see

$$F(x,t,w_\varepsilon^-(x,t),p,X) \leqq F(x,t,\|u_0\|,p,-(1/\varepsilon)I) \leqq \mathcal{F}_{C_0/\sqrt{\varepsilon}}(\|u_0\|, -1/\varepsilon).$$

Therefore

$$\tau + F(x,t,w_\varepsilon^-(x,t),p,X) \leqq -M_\varepsilon^- + \mathcal{F}_{C_0/\sqrt{\varepsilon}}(\|u_0\|, -1/\varepsilon) \leqq 0.$$

Next, assume that $x \in \partial\Omega$. We then have $|p'| \leqq C_0/\sqrt{\varepsilon}$ and $p_n \geqq -C_0/\sqrt{\varepsilon}$ by (4.2). From (B3)$'$ and (B5), we deduce

$$B(x', t, w_\varepsilon^-(x,t), p', \tau) \leqq B(x', t, \|u_0\|, p', -M_\varepsilon^-) \leqq -kM_\varepsilon^- + B(x', t, \|u_0\|, p', 0) \leqq -kM_\varepsilon^- + \mathcal{B}_{C_0/\sqrt{\varepsilon}}(\|u_0\|).$$

This shows

$$B(x', t, w_\varepsilon^-(x,t), p', \tau) - \beta p_n \leqq -kM_\varepsilon^- + \mathcal{B}_{C_0/\sqrt{\varepsilon}}(\|u_0\|) + \beta_0 \cdot (C_0/\sqrt{\varepsilon}) \leqq 0.$$

We thus conclude that w_ε^- is a viscosity subsolution of (IBV.n).

(2) The viscosity properties for $v^\pm$ follow from the stability (Proposition 4.1). Since $\|u_0^\varepsilon - u_0\|, \|u_{0\varepsilon} - u_0\| \to 0$ as $\varepsilon \to 0$, we see that $\alpha^\pm \in \mathcal{M}$. Taking $\sup_{\varepsilon>0}$ in the inequality $w_\varepsilon^-(x,t) \geqq -\|u_0^\varepsilon - u_0\| - M_\varepsilon^- t + u_0(x)$ implies that $v^-(x,t) \geqq -\alpha^-(t) + u_0(x)$. The inequality $v^-(x,t) \leqq u_0(x)$ is a consequence of (4.5). The remaining inequalities in (4.4) for v^+ can be obtained in the same manner. □

Remark 4.3. The functions $\alpha^\pm$ depend on β_0, but they are independent of β.

We are now in a position to give the existence result of viscosity solutions to (IBV.n) by Perron's method. Set

$$\mathcal{S}^\beta := \left\{ w \;\middle|\; \begin{array}{c} w : \overline{\Omega} \times [0,T) \to \mathbf{R} \text{ is a viscosity subsolution of (IBV.n)} \\ \text{such that } v^- \leqq w \leqq v^+ \text{ in } \overline{\Omega} \times [0,T) \end{array} \right\},$$

where $v^\pm$ are the functions in Proposition 4.2. Note that $\mathcal{S}^\beta$ is not empty since $v^- \in \mathcal{S}^\beta$.

Theorem 4.4 (Existence of solutions to (IBV.n))**.** *Define $u^\beta(x,t) = \sup\{w(x,t) \mid w \in \mathcal{S}^\beta\}$ for $\beta \in [0, \beta_0]$. Then u^β is a viscosity solution of* (IBV.n) *satisfying*

$$u_0(x) - \alpha^-(t) \leqq u^\beta(x,t) \leqq u_0(x) + \alpha^+(t) \quad \textit{for all } (x,t) \in \overline{\Omega} \times [0,T), \tag{4.6}$$

where $\alpha^\pm \in \mathcal{M}$ are the functions in Proposition 4.2. If $\beta \in (0, \beta_0]$, then $u^\beta \in C(\overline{\Omega} \times [0,T))$ and it is the unique viscosity solution of (IBV.n)*.*

Proof. First, the estimate (4.6) immediately follows from (4.4). In particular, the initial condition (1.3) is satisfied. Next, the stability result (Proposition 4.1) implies that u^β is a viscosity subsolution of (1.1) and (1.2). If u^β were not a supersolution of (1.1) and (1.2), there would exist $w \in \mathcal{S}^\beta$ such that $u^\beta(x_0, t_0) < w(x_0, t_0)$ for some $(x_0, t_0) \in \overline{\Omega} \times (0,T)$, which is a contradiction to the definition of u^β. For the details of this argument, see [8, Lemma 4.4] and [11, Lemma 2.4.2]. The continuity and uniqueness of u^β with $\beta \in (0, \beta_0]$ are consequences of Corollary 2.8. □

4.3 Boundary barriers (I)

If we know a subsolution and a supersolution which are continuous and share the same boundary value on $\partial\Omega \times (0,T)$, then Perron's method directly gives a solution of (IBV) in X. Our assumption is

(Br1) There exist $\phi^- \in X^-$, $\phi^+ \in X^+$ and $\psi \in C(\partial\Omega \times (0,T))$ such that

- (i) $\phi^\pm$ are continuous at every $(x,t) \in \partial\Omega \times (0,T)$;
- (ii) $\phi^\pm(x', 0, t) = \psi(x', t)$ for all $(x', t) \in \mathbf{R}^{n-1} \times (0,T)$.

We first prepare

Proposition 4.5 (Initial-boundary barriers)**.** *Assume* (Br1)*. Let $v^\pm$ be the functions in Proposition 4.2. Define*

$$\tilde{v}^-(x,t) = \max\{v^-(x,t),\, \phi^-(x,t)\}, \quad \tilde{v}^+(x,t) = \min\{v^+(x,t),\, \phi^+(x,t)\} \quad \textit{for } (x,t) \in \overline{\Omega} \times [0,T).$$

Then $\tilde{v}^-$ is a viscosity subsolution and $\tilde{v}^+$ is a viscosity supersolution of (IBV)*, and $\tilde{v}^- \leqq \tilde{v}^+$ in $\overline{\Omega} \times [0,T)$.*

Proof. Proposition 4.1 guarantees that $\tilde{v}^-$ and $\tilde{v}^+$ are respectively a subsolution and a supersolution of (IBV). Let us prove that $\tilde{v}^- \leqq \tilde{v}^+$. Since $v^- \leqq v^+$ by (4.4), it is enough to prove

$$v^- \leqq \phi^+, \quad \phi^- \leqq v^+ \quad \text{in } \overline{\Omega} \times [0, T). \tag{4.7}$$

To show the first one, let $\varepsilon > 0$ and w_ε^- be the function in Proposition 4.2. Since $w_\varepsilon^- \in X^-$ and $\phi^+ \in X^+$, the comparison principle (Theorem 3.2) implies that $w_\varepsilon^- \leqq (\phi^+)_* \leqq \phi^+$ in $\overline{\Omega} \times [0, T)$. Taking $\sup_{\varepsilon > 0}$, we obtain $v^- \leqq \phi^+$ in $\overline{\Omega} \times [0, T)$. Similarly, one can prove the second inequality in (4.7). ☐

Proposition 4.5 enables us to define a non-empty set

$$\tilde{\mathcal{S}} := \left\{ w \;\middle|\; \begin{array}{c} w : \overline{\Omega} \times [0, T) \to \mathbf{R} \text{ is a viscosity subsolution of (IBV)} \\ \text{such that } \tilde{v}^- \leqq w \leqq \tilde{v}^+ \text{ in } \overline{\Omega} \times [0, T) \end{array} \right\}.$$

Theorem 4.6 (Existence of solutions to (IBV) in X under barriers (I))**.** *Assume* (Br1)*. Define* $u(x, t) = \sup\{w(x, t) \mid w \in \tilde{\mathcal{S}}\}$*. Then* u *is the unique viscosity solution of* (IBV) *in* X*, and* $u \in C(\overline{\Omega} \times [0, T))$*.*

Proof. For the same reason as in the proof of Theorem 4.4, u is a viscosity solution of (IBV). We check that u belongs to X. Note that, by the definition of u,

$$\phi^- \leqq \tilde{v}^- \leqq u \leqq \tilde{v}^+ \leqq \phi^+ \quad \text{in } \overline{\Omega} \times [0, T).$$

Since ϕ^- satisfies (3.4), so does u_*. Similarly, u^* fulfills (3.3). Next, taking the semicontinuous envelopes in the above inequalities, we obtain $(\phi^-)_* \leqq u_* \leqq u^* \leqq (\phi^+)^*$ in $\overline{\Omega} \times [0, T)$. On the boundary $\partial\Omega \times (0, T)$, by (Br1) we have $\psi = \phi^- = (\phi^-)_* \leqq u_* \leqq u^* \leqq (\phi^+)^* = \phi^+ = \psi$. This shows that $u^* = u_* = \psi$ on $\partial\Omega \times (0, T)$; in particular, u is continuous on $\partial\Omega \times (0, T)$. From this we deduce that, for any $(x, t) \in \partial\Omega \times (0, T)$,

$$u^*(x, t) = u(x, t) = \liminf_{(y,s)\to(x,t)} u(y, s) \leqq \liminf_{(y,s)\to(x,t)} u^*(y, s),$$

which implies that u^* is continuous on $\partial\Omega \times (0, T)$. For the same reason, u_* is continuous on $\partial\Omega \times (0, T)$, and hence $u \in X$. The continuity of u follows from Corollary 3.3. ☐

4.4 Half-relaxed limits

We next study the asymptotic behavior of solutions u^β to (IBV.n) as $\beta \to +0$ in order to construct a solution u of (IBV) in X. For this purpose, we consider the *upper half-relaxed limit* $\overline{u}$ and the *lower half-relaxed limit* $\underline{u}$ of u^β. The definitions are as follows: for $(x, t) \in \overline{\Omega} \times [0, T)$

$$\overline{u}(x, t) = \limsup_{\beta\to 0}{}^* u^\beta(x, t) := \lim_{\delta\to 0} \sup\{u^\beta(y, s) \mid (y, s) \in \overline{\Omega} \times [0, T),\ |x - y| < \delta,\ |t - s| < \delta,\ 0 < \beta < \delta\},$$
$$\underline{u}(x, t) = \liminf_{\beta\to 0}{}_* u^\beta(x, t) := \lim_{\delta\to 0} \inf\{u^\beta(y, s) \mid (y, s) \in \overline{\Omega} \times [0, T),\ |x - y| < \delta,\ |t - s| < \delta,\ 0 < \beta < \delta\}.$$

From definitions it follows that $\overline{u}$ is upper semicontinuous and $\underline{u}$ is lower semicontinuous in $\overline{\Omega} \times [0, T)$. By (4.6) we have

$$u_0(x) - \alpha^-(t) \leqq \underline{u}(x, t) \leqq \overline{u}(x, t) \leqq u_0(x) + \alpha^+(t) \quad \text{for all } (x, t) \in \overline{\Omega} \times [0, T). \tag{4.8}$$

In particular, $\overline{u}$ and $\underline{u}$ are continuous on $\overline{\Omega} \times \{0\}$ and satisfy $\overline{u}(\cdot, 0) = \underline{u}(\cdot, 0) = u_0$ in $\overline{\Omega}$. Thus, from stability results under the half-relaxed limits ([8, Lemma 6.1, Remarks 6.2 and 6.3]), it follows that $\overline{u}$ and $\underline{u}$ are respectively a viscosity subsolution and a viscosity supersolution of (IBV). If $\overline{u} = \underline{u} =: u$ in $\overline{\Omega} \times [0, T)$, then we conclude that u^β converges to u locally uniformly in $\overline{\Omega} \times [0, T)$; see [8, Remark 6.4]. Under appropriate assumptions, we will prove that $\overline{u} = \underline{u}$ in $\overline{\Omega} \times [0, T)$ by applying the comparison principle to them.

4.5 Boundary barriers (II)

Assuming the existence of sub- and supersolutions approximating a common boundary value, we prove the convergence of u^β to the unique solution u of (IBV) in X. We assume

(Br2) There exist a family of upper semicontinuous viscosity subsolutions $\{\phi_\eta^-\}_{\eta>0}$ of (IBV), a family of lower semicontinuous viscosity supersolutions $\{\phi_\eta^+\}_{\eta>0}$ of (IBV) and $\psi \in C(\partial\Omega \times (0,T))$ such that, for every $\eta > 0$,

(i) $\phi_\eta^\pm$ are continuous at every $(x,t) \in \partial\Omega \times (0,T)$;

(ii) $\lim_{\eta\to+0} \phi_\eta^\pm(x',0,t) = \psi(x',t)$ for all $(x',t) \in \mathbf{R}^{n-1} \times (0,T)$;

(iii) there exist some $M_\eta > 0$ such that, for every $(x,t) \in \partial\Omega \times (0,T)$,

$$(p,\tau,X) \in \mathcal{P}^{2,+}\phi_\eta^-(x,t) \text{ implies } p_n \geqq -M_\eta, \quad (p,\tau,X) \in \mathcal{P}^{2,-}\phi_\eta^+(x,t) \text{ implies } p_n \leqq M_\eta.$$

Theorem 4.7 (Existence of solutions to (IBV) in X under barriers (II))**.** *Assume* (Br2)*. Let u^β be the unique viscosity solution of* (IBV.n) *with $\beta \in (0,\beta_0]$. Then u^β converges to the unique viscosity solution u of* (IBV) *in X locally uniformly in $\overline{\Omega} \times [0,T)$ as $\beta \to +0$, and $u \in C(\overline{\Omega} \times [0,T))$.*

Proof. 1. Let us fix $\eta > 0$ and take the functions $\phi_\eta^\pm$ in (Br2). For $\varepsilon > 0$ we define $V^\pm(x,t) := \pm\varepsilon t + \phi_\eta^\pm(x,t)$. We claim that V^- and V^+ are respectively a viscosity subsolution and a viscosity supersolution of (IBV.n) when $0 < \beta \leqq \min\{k\varepsilon/M_\eta,\, \beta_0\}$. Here k is the constant in (B5) and M_η is the constant in (Br2)-(iii).

At the initial time, we have $V^-(\cdot,0) = \phi_\eta^-(\cdot,0) \leqq u_0 \leqq \phi_\eta^+(\cdot,0) = V^+(\cdot,0)$ in $\overline{\Omega}$. Next, let $(p,\tau,X) \in \mathcal{P}^{2,+}V^-(x,t)$ for $(x,t) \in \overline{\Omega} \times (0,T)$. Then $(p,\tau+\varepsilon,X) \in \mathcal{P}^{2,+}\phi_\eta^-(x,t)$. If $x \in \Omega$, we have

$$\tau + \varepsilon + F(x,t,\phi_\eta^-(x,t),p,X) \leqq 0. \tag{4.9}$$

Since $V^-(x,t) \leqq \phi_\eta^-(x,t)$, the monotonicity (F3) yields the desired viscosity subinequality:

$$\tau + F(x,t,V^-(x,t),p,X) \leqq 0. \tag{4.10}$$

We next consider the case $x \in \partial\Omega$. If (4.9) holds, then (4.10) is derived in the same manner. Assume that

$$B(x',t,\phi_\eta^-(x,t),p',\tau+\varepsilon) \leqq 0.$$

Applying (B3)$'$, (B5) and (Br2)-(iii), we observe

$$\begin{aligned} B(x',t,V^-(x,t),p',\tau) - \beta p_n &\leqq B(x',t,\phi_\eta^-(x,t),p',\tau) - \beta p_n \\ &\leqq -k\varepsilon + B(x',t,\phi_\eta^-(x,t),p',\tau+\varepsilon) + \beta M_\eta \\ &\leqq -k\varepsilon + \beta M_\eta. \end{aligned}$$

By the choice of β, the right-hand side is non-positive. We thus conclude that V^- is a subsolution of (IBV.n). The proof for V^+ is similar.

2. The comparison principle for (IBV.n) (Theorem 2.7) implies that $V^- \leqq u^\beta \leqq V^+$ in $\overline{\Omega} \times (0,T)$. Taking $\limsup^*_{\beta\to 0}$, we get $V^- \leqq \overline{u} \leqq (V^+)^* = \varepsilon t + (\phi_\eta^+)^*$ in $\overline{\Omega} \times (0,T)$. Then, sending $\varepsilon \to 0$ gives $\phi_\eta^- \leqq \overline{u} \leqq (\phi_\eta^+)^*$, and so $(\phi_\eta^-)_* \leqq (\overline{u})_* \leqq \overline{u} \leqq (\phi_\eta^+)^*$ in $\overline{\Omega} \times (0,T)$. In particular, from the continuity (Br2)-(i) of $\phi_\eta^\pm$ on the boundary, we deduce that $\phi_\eta^- \leqq (\overline{u})_* \leqq \overline{u} \leqq \phi_\eta^+$ on $\partial\Omega \times (0,T)$. By (Br2)-(ii), letting $\eta \to +0$ yields $\overline{u} = (\overline{u})_* = \psi$ on $\partial\Omega \times (0,T)$. Similarly, one can prove that $\underline{u} = (\underline{u})^* = \psi$ on $\partial\Omega \times (0,T)$. Thus both $\overline{u}$ and $\underline{u}$ are continuous on $\partial\Omega \times (0,T)$.

Since (4.8) holds, we see that $\overline{u} \in X^-$ and $\underline{u} \in X^+$, and so $\overline{u} \leqq \underline{u}$ in $\overline{\Omega} \times [0,T)$ by the comparison principle (Theorem 3.2). Hence u^β converges to $u := \overline{u} = \underline{u} \in C(\overline{\Omega} \times [0,T)) \cap X$ as $\beta \to +0$ locally uniformly in $\overline{\Omega} \times [0,T)$. The proof is complete. □

4.6 Coercive equations

We next consider first order equations with coercive F, instead of assuming the existence of barrier functions from the boundary. The coercivity (F6) below guarantees that continuities of solutions u^β to (IBV.n) are uniform in β. Accordingly, a continuous solution of (IBV) is obtained as $\beta \to +0$.

In this subsection, we assume

(FB1) F is independent of (t, X), and B is independent of t.

(F6) For every $R > 0$

$$\lim_{L\to\infty} \inf\{F(x,r,p) \mid x \in \overline{\Omega},\, r \in [-R,R],\, p \in \overline{B_L(0)}\} = \infty \tag{4.11}$$

or

$$\lim_{L\to\infty} \sup\{F(x,r,p) \mid x \in \overline{\Omega},\, r \in [-R,R],\, p \in \overline{B_L(0)}\} = -\infty. \tag{4.12}$$

Let us denote by $BLip(K)$ the set of bounded and Lipschitz continuous functions in $K \subset \mathbf{R}^N$. Our result is

Theorem 4.8 (Existence in X under coercivity). *Let u^β be the unique viscosity solution of* (IBV.n) *with $\beta \in (0, \beta_0]$. Then u^β converges to the unique viscosity solution u of* (IBV) *in X locally uniformly in $\overline{\Omega} \times [0,T)$ as $\beta \to +0$, and $u \in BUC(\overline{\Omega} \times [0,T))$. Moreover, if $u_0 \in BLip(\overline{\Omega})$, then $u \in BLip(\overline{\Omega} \times [0,T))$.*

To prove Theorem 4.8 we prepare some notations and facts. For $R > 0$ and $m > 0$ we define

$$\begin{aligned}\mathcal{C}_R^+(m) &:= \sup\{|p| \mid F(x,r,p) \leqq m \text{ for some } (x,r) \in \overline{\Omega} \times [-R,R]\},\\ \mathcal{C}_R^-(m) &:= \sup\{|p| \mid F(x,r,p) \geqq -m \text{ for some } (x,r) \in \overline{\Omega} \times [-R,R]\}.\end{aligned}$$

Then

$$(4.11) \iff \mathcal{C}_R^+(m) < \infty \text{ for all } m > 0, \quad (4.12) \iff \mathcal{C}_R^-(m) < \infty \text{ for all } m > 0.$$

For $L > 0$ let us define

$$M^\pm[L] := \max\left\{\mathcal{F}_L(\mp\|u_0\|),\, \frac{1}{k}\left\{\mathcal{B}_L(\mp\|u_0\|) + \beta_0 L\right\}\right\}.$$

Here we denote $\mathcal{F}_L(r,\mu)$ in (F5) simply by $\mathcal{F}_L(r)$ since F is independent of X.

The following lemma is given in [12, Lemma A.2].

Lemma 4.9. *Let $u : \Omega \times (0,T) \to \mathbf{R}$ be a bounded and continuous function.*

(1) *For any $t_0 \in (0,T)$,*

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x,t_0) - u(y,t_0)|}{|x-y|} = \sup_{\substack{x_0\in\Omega\\ p\in D^+(u|_{t=t_0})(x_0)}} |p|.$$

(2) *Assume that u satisfies*

$$\sup_{x\in\Omega} \sup_{\substack{t,s\in(0,T)\\ t\neq s}} \frac{|u(x,t)-u(x,s)|}{|t-s|} < \infty.$$

Then

$$\sup_{\substack{(x_0,t_0)\in\Omega\times(0,T)\\ (p,\tau)\in D^+u(x_0,t_0)}} |p| = \sup_{t_0\in(0,T)} \sup_{\substack{x_0\in\Omega\\ p\in D^+(u|_{t=t_0})(x_0)}} |p|.$$

Thanks to the estimate (4.6), the solutions u^β of (IBV.n) are bounded uniformly in $\beta \in (0, \beta_0]$. Namely,

$$R := \sup_{\beta\in(0,\beta_0]} \|u^\beta\| \tag{4.13}$$

is finite.

Proposition 4.10 (Regularity results for solutions to (IBV.n))**.** *Let u^β be the unique viscosity solution of* (IBV.n) *with $\beta \in (0, \beta_0]$.*

(1) *Assume that $u_0 \in BLip(\overline{\Omega})$. Let L_0 be the Lipschitz constant of u_0, and define*

$$L_{\mathrm{t}} = \max\{M^+[L_0],\, M^-[L_0]\}, \quad L_{\mathrm{x}} = \begin{cases} \mathcal{C}_R^+(L_{\mathrm{t}}) & \textit{if } (4.11) \textit{ holds,} \\ \mathcal{C}_R^-(L_{\mathrm{t}}) & \textit{if } (4.12) \textit{ holds,} \end{cases}$$

where R is the constant in (4.13)*. Then,*

$$|u^\beta(x,t) - u^\beta(x,s)| \leqq L_{\mathrm{t}}|t-s| \quad \textit{for all } x \in \overline{\Omega} \textit{ and } t,s \in [0,T), \tag{4.14}$$

$$|u^\beta(x,t) - u^\beta(y,t)| \leqq L_{\mathrm{x}}|x-y| \quad \textit{for all } x,y \in \overline{\Omega} \textit{ and } t \in [0,T). \tag{4.15}$$

(2) *Assume that $u_0 \in BUC(\overline{\Omega})$. Let $\varepsilon > 0$ and u_0^ε be the sup-convolution of u_0 define by* (4.1)*. Define*

$$L_{\mathrm{t}}^\varepsilon = \max\{M^+[2\sqrt{\|u_0\|/\varepsilon}],\, M^-[2\sqrt{\|u_0\|/\varepsilon}]\}, \quad L_{\mathrm{x}}^\varepsilon = \begin{cases} \mathcal{C}_R^+(L_{\mathrm{t}}^\varepsilon) & \textit{if } (4.11) \textit{ holds,} \\ \mathcal{C}_R^-(L_{\mathrm{t}}^\varepsilon) & \textit{if } (4.12) \textit{ holds} \end{cases}$$

and

$$\alpha_{\mathrm{t}}(r) = \inf_{\varepsilon>0}\{L_{\mathrm{t}}^\varepsilon r + \|u_0^\varepsilon - u_0\|\}, \quad \alpha_{\mathrm{x}}(r) = \inf_{\varepsilon>0}\{L_{\mathrm{x}}^\varepsilon r + \|u_0^\varepsilon - u_0\|\} \quad \textit{for } r \geqq 0.$$

Then, $\alpha_{\mathrm{t}}, \alpha_{\mathrm{x}} \in \mathcal{M}$ and

$$|u^\beta(x,t) - u^\beta(x,s)| \leqq \alpha_{\mathrm{t}}(|t-s|) \quad \textit{for all } x \in \overline{\Omega} \textit{ and } t,s \in [0,T), \tag{4.16}$$

$$|u^\beta(x,t) - u^\beta(y,t)| \leqq \alpha_{\mathrm{x}}(|x-y|) \quad \textit{for all } x,y \in \overline{\Omega} \textit{ and } t \in [0,T). \tag{4.17}$$

Proof. (1) 1. Let us define $w^\pm : \overline{\Omega} \times [0,T) \to \mathbf{R}$ by $w^\pm(x,t) = \pm L_{\mathrm{t}} t + u_0(x)$. Then, by the same argument as in the proof of Proposition 4.2, we see that w^- and w^+ are respectively a viscosity subsolution and a viscosity supersolution of (IBV.n). Thus, the comparison principle for (IBV.n) (Theorem 2.7) implies that $w^- \leqq u^\beta \leqq w^+$ in $\overline{\Omega} \times [0,T)$.

Let $h \in (0,T)$, and define

$$\tilde{w}^\pm(x,t) = \begin{cases} w^\pm(x,t) & \text{if } (x,t) \in \overline{\Omega} \times [0,h], \\ u^\beta(x,t-h) \pm L_{\mathrm{t}} h & \text{if } (x,t) \in \overline{\Omega} \times [h,T), \end{cases}$$

which is continuously connected at $t = h$. Note that, for $(x,t) \in \overline{\Omega} \times [h,T)$, we have

$$\tilde{w}^-(x,t) - w^-(x,t) = u^\beta(x,t-h) - L_{\mathrm{t}} h - (w^-(x,t-h) - L_{\mathrm{t}} h) = u^\beta(x,t-h) - w^-(x,t-h) \geqq 0.$$

We now claim that $\tilde{w}^-$ is a viscosity subsolution of (IBV.n). We only need to check the viscosity subinequalities at the time $t = h$ since both $w^-(x,t)$ and $u^\beta(x,t-h) - L_{\mathrm{t}} h$ are subsolutions. Let $(p,\tau) \in D^+\tilde{w}^-(x,h)$. Then, since $\tilde{w}^- = w^-$ in $\overline{\Omega} \times [0,h]$ and $\tilde{w}^- \geqq w^-$ in $\overline{\Omega} \times [h,T)$, we have $(p,\tau) \in D^+w^-(x,h)$. Thus the claim follows since w^- is a subsolution. For the same reason, $\tilde{w}^+$ is a viscosity supersolution of (IBV.n).

By comparison we have $\tilde{w}^- \leqq u^\beta \leqq \tilde{w}^+$ in $\overline{\Omega} \times [0,T)$. In particular, for $t \in [0,T-h)$

$$u^\beta(x,t) - L_{\mathrm{t}} h = \tilde{w}^-(x,t+h) \leqq u^\beta(x,t+h) \leqq \tilde{w}^+(x,t+h) = u^\beta(x,t) + L_{\mathrm{t}} h,$$

which shows (4.14).

2. Let us next prove (4.15). We assume that (4.11) holds in (F6). Let $(x_0,t_0) \in \Omega \times (0,T)$ and take any $(p,\tau) \in D^+u^\beta(x_0,t_0)$. By (4.14) we have $|\tau| \leqq L_{\mathrm{t}}$, and so

$$F(x_0, u^\beta(x_0,t_0), p) \leqq -\tau \leqq L_{\mathrm{t}}.$$

The definition of $\mathcal{C}_R^+$ thus implies that $|p| \leqq \mathcal{C}_R^+(L_{\mathrm{t}})$. Therefore

$$\sup_{\substack{(x_0,t_0)\in\Omega\times(0,T)\\(p,\tau)\in D^+u^\beta(x_0,t_0)}} |p| \leqq \mathcal{C}_R^+(L_{\mathrm{t}}).$$

By Lemma 4.9 we conclude (4.15) for $x, y \in \Omega$ and $t \in (0,T)$, and it is extended for $x, y \in \overline{\Omega}$ and $t \in [0,T)$ since u^β is continuous in $\overline{\Omega} \times [0,T)$. The same proof works for the case of (4.12).

(2) Let $u^{\beta,\varepsilon}$ be the solution of (IBV.n) with the initial datum u_0^ε. Since u_0^ε is Lipschitz continuous with the Lipschitz constant less than or equal to $2\sqrt{\|u_0\|/\varepsilon}$, the result of (1) guarantees that $u^{\beta,\varepsilon}$ satisfies the estimates (4.14) and (4.15) with $L_{\mathrm{t}}^\varepsilon$ and $L_{\mathrm{x}}^\varepsilon$ instead of L_{t} and L_{x}, respectively. We also note that

$$0 \leqq u^{\beta,\varepsilon} - u^\beta \leqq \|u_0^\varepsilon - u_0\| \quad \text{in } \overline{\Omega} \times [0,T),$$

which is due to the fact that $u^{\beta,\varepsilon} - \|u_0^\varepsilon - u_0\|$ and u^β are respectively a viscosity subsolution and a viscosity supersolution of (IBV.n).

Let us prove (4.16). Fix $x \in \overline{\Omega}$, $t, s \in [0,T)$ and $\varepsilon > 0$. We may assume that $u^\beta(x,t) \geqq u^\beta(x,s)$. Then

$$\begin{aligned}|u^\beta(x,t) - u^\beta(x,s)| &= u^\beta(x,t) - u^\beta(x,s) \leqq u^{\beta,\varepsilon}(x,t) - u^\beta(x,s)\\ &= \{u^{\beta,\varepsilon}(x,t) - u^{\beta,\varepsilon}(x,s)\} + \{u^{\beta,\varepsilon}(x,s) - u^\beta(x,s)\}\\ &\leqq L_{\mathrm{t}}^\varepsilon|t-s| + \|u_0^\varepsilon - u_0\| \leqq \alpha_{\mathrm{t}}(|t-s|).\end{aligned}$$

Similarly, one can prove (4.17). □

Remark 4.11. The constants L_{t}, L_{x} and the functions α_{t}, α_{x} do not depend on β.

Proof of Theorem 4.8. Assume that $u_0 \in BUC(\overline{\Omega})$. Since α_{t} and α_{x} are independent of β, the half-relaxed limits $\overline{u}$ and $\underline{u}$ also satisfy (4.16) and (4.17). Accordingly, $\overline{u} \in X^- \cap BUC(\overline{\Omega}\times[0,T))$ and $\underline{u} \in X^+ \cap BUC(\overline{\Omega}\times[0,T))$. The comparison principle (Theorem 3.2) thus implies that $\overline{u} \leqq \underline{u}$ in $\overline{\Omega} \times [0,T)$, and so u^β converges to $u := \overline{u} = \underline{u} \in X \cap BUC(\overline{\Omega} \times [0,T))$ as $\beta \to +0$ locally uniformly in $\overline{\Omega} \times [0,T)$. If $u_0 \in BLip(\overline{\Omega})$, the limits $\overline{u}$ and $\underline{u}$ satisfy (4.14) and (4.15). Therefore $u := \overline{u} = \underline{u} \in BLip(\overline{\Omega} \times [0,T))$. The proof is complete. □

5 Examples

In this section we let the spatial dimension n be one, so that $\Omega = (0,\infty)$, and give several examples of solutions to (IBV). We mainly study the stationary boundary condition

$$u_t(0,t) = c \quad \text{on } \partial\Omega \times (0,T) \tag{5.1}$$

and the Dirichlet boundary condition

$$u(0,t) = ct + u_0(0) \quad \text{on } \partial\Omega \times (0,T). \tag{5.2}$$

We are especially interested in whether the solution u satisfies

$$u(0,t) = ct + u_0(0) \quad \text{for all } t \in [0,T), \tag{5.3}$$

which is the Dirichlet condition in the classical sense. Throughout this section we assume that $c > 0$.

5.1 Hamilton-Jacobi equations

We apply the existence result, Theorem 4.6, to simple Hamilton-Jacobi equations. We consider

$$u_t(x,t) - h(|u_x(x,t)|) = 0 \quad \text{in } \Omega \times (0,T). \tag{5.4}$$

Here $h : [0,\infty) \to \mathbf{R}$ satisfies (h1) h is uniformly continuous in $[0,\infty)$, $h(0)=0$ and $h_\infty := \liminf_{r\to\infty} h(r) \in (0,\infty]$.

We remark that $H(p) := -h(|p|)$ is uniformly continuous in $\mathbf{R}^n$ under (h1), and thus the comparison principle (Theorem 3.2) can be applied to (5.4). For $d \in [0,h_\infty)$ we define $\pi(d) := \max\{r \geqq 0 \mid h(r) = d\} \in [0,\infty)$. Note that $h(\pi(d)) = d$ and $h(r) > d$ for all $r > \pi(d)$. We consider the stationary boundary value problem

$$(5.4), \quad (5.1), \quad u(x,0) = u_0(x) \ \text{ in } \overline{\Omega}. \tag{HJ}$$

Theorem 5.1. *Assume* (h1).

(1) *Assume that $c < h_\infty \leqq \infty$. Then, for any $u_0 \in BUC(\overline{\Omega})$ satisfying*

$$u_0(x) \leqq u_0(0) + \pi(c)x \quad \textit{for all } x \in \overline{\Omega}, \tag{5.5}$$

there exists a unique solution $u \in C(\overline{\Omega} \times [0,T))$ of (HJ) *in X. Moreover, u satisfies* (5.3).

(2) *Assume that $c = h_\infty = \lim_{r\to\infty} h(r) < \infty$. Then, for any $u_0 \in BUC(\overline{\Omega})$, there exists a unique solution $u \in C(\overline{\Omega} \times [0,T))$ of* (HJ) *in X. Moreover, u satisfies* (5.3).

Proof. (1) We construct barrier functions $\phi^\pm$ in (Br1). First, by the definition of π and (5.5), we see that $\phi_1^+(x,t) := ct + u_0(0) + \pi(c)x$ is a supersolution of (HJ). Next, define

$$\omega(r) := \sup\{|u_0(x) - u_0(y)| \mid x, y \in \overline{\Omega},\ |x-y| \leqq r\},$$

and choose $\omega_\infty \in \mathcal{M}$ such that

$$\omega_\infty \in C^\infty((0,\infty)),\ \omega_\infty \geqq \omega_0 \text{ in } [0,\infty).$$

See, e.g., [11, Lemma 2.1.9 (i)] for the existence of such ω_∞. We then set $\tilde{\omega}_\infty(r) = \omega_\infty(r) + \pi(c)r$ for $r \geqq 0$, so that $\tilde{\omega}'_\infty \geqq \pi(c)$ in $(0,\infty)$. From this we deduce that $\phi_1^-(x,t) := ct + u_0(0) - \tilde{\omega}_\infty(x)$ is a subsolution of (HJ). In fact, for $(x,t) \in \Omega \times (0,T)$, we have

$$(\phi_1^-)_t(x,t) - h(|(\phi_1^-)_x(x,t)|) = c - h(\tilde{\omega}'_\infty(x)) \leqq c - c = 0,$$

and $(\phi_1^-)_t(0,t) = c$ on the boundary. When $t=0$, we have

$$\phi_1^-(x,0) = u_0(0) - \tilde{\omega}_\infty(x) \leqq u_0(0) - \omega_\infty(x) \leqq u_0(0) - \omega_0(x) \leqq u_0(x).$$

Accordingly, ϕ_1^- is a subsolution of (HJ).

We further prepare bounded functions $\phi_2^\pm(x,t) = ct \pm \|u_0\|$. It is easily seen that ϕ_2^- is a subsolution and ϕ_2^+ is a supersolution of (HJ). Thus $\phi^- = \max\{\phi_1^-, \phi_2^-\}$ and $\phi^+ = \min\{\phi_1^+, \phi_2^+\}$ are respectively a bounded subsolution and a bounded supersolution of (HJ). The functions $\phi^\pm$ above satisfies the conditions in (Br1) with $\psi(t) = ct + u_0(0)$, and therefore Theorem 4.6 gives the desired conclusion.

(2) We modify the functions $\phi_1^\pm$ in (1). Fix $\varepsilon \in (0,c)$ and choose $R_\varepsilon > 0$ such that $|h(r) - h_\infty| < \varepsilon$ for all $r \geqq R_\varepsilon$. Using this constant R_ε, we define $\tilde{\omega}_{\infty,\varepsilon}(r) = \omega_\infty(r) + R_\varepsilon r$ and $\phi_{1,\varepsilon}^\pm(x,t) = (c \pm \varepsilon)t + u_0(0) \pm \tilde{\omega}_{\infty,\varepsilon}(x)$, where ω_∞ is the function in the proof of (1). Then, as in the proof of (1), we see that $\phi_{1,\varepsilon}^-$ and $\phi_{1,\varepsilon}^+$ are respectively a subsolution and a supersolution of (HJ).

We now define $\phi_1^- = \sup_{\varepsilon\in(0,c)} \phi_{1,\varepsilon}^-$ and $\phi_1^+ = \inf_{\varepsilon\in(0,c)} \phi_{1,\varepsilon}^+$. Then $\phi_1^\pm(0,t) = ct + u_0(0)$, and it follows from stability that ϕ_1^- and ϕ_1^+ are respectively a subsolution and a supersolution of (HJ). Moreover, they are continuous on $\partial\Omega \times (0,T)$. In fact, for a fixed $\varepsilon_0 \in (0,c)$, we have

$$ct + u_0(0) + \omega_\infty(x) \leqq \phi_1^+(x,t) \leqq (c+\varepsilon_0)t + u_0(0) + \tilde{\omega}_{\infty,\varepsilon_0}(x).$$

Thus

$$ct + u_0(0) \leqq \liminf_{x\to+0} \phi_1^+(x,t) \leqq \limsup_{x\to+0} \phi_1^+(x,t) \leqq (c+\varepsilon_0)t + u_0(0).$$

Since $\varepsilon_0 \in (0,c)$ is arbitrary, this shows the continuity. Similarly, ϕ_1^- is continuous on $\partial\Omega \times (0,T)$. The rest of the proof is the same as (1). □

Remark 5.2. In both (1) and (2), the solutions satisfy (5.3). Thus they solve the Dirichlet problem

$$(5.4), \quad (5.2), \quad u(x,0) = u_0(x) \text{ in } \overline{\Omega}$$

and they are the unique solutions in X. If the assumptions in the theorem does not hold, then the solution of (HJ) may not satisfy (5.3) and it can be different from the solution of the above Dirichlet problem. We give such an example in Section 5.2 for the initial datum u_0 not satisfying (5.5) in the case $c < h_\infty$. Also, the case $c > h_\infty$ is discussed in Section 5.3, where the solution does not satisfy (5.3).

Eikonal equation. As an example, let us consider the eikonal equation

$$u_t(x,t) - |u_x(x,t)| = 0 \quad \text{in } \Omega \times (0,T), \tag{5.6}$$

for which $h(r) = r$, $h_\infty = \infty$ and $\pi(d) = d$. Since $u_t/|u_x|$ represents the normal velocity of the level set of $u(\cdot, t)$, the equation (5.6) describes a growth model where the height function u spreads at a uniform speed one in the horizontal direction.

We assume that the initial datum u_0 is zero. Namely, our problem is

$$(5.6), \quad (5.1), \quad u(x,0) = 0 \text{ in } \overline{\Omega}. \tag{St1}$$

Since the initial datum satisfies (5.5) for any $c > 0$, Theorem 5.1 (1) can be applied. Therefore, by Remark 5.2, the unique solution of (St1) in X is also the unique solution in X of the Dirichlet problem

$$(5.6), \quad (5.2), \quad u(x,0) = 0 \text{ in } \overline{\Omega}.$$

We here prove that the unique solution of (St1) is

$$U_0(x,t) = c(t-x)_+, \tag{5.7}$$

where $a_+ = \max\{a, 0\}$ denotes the positive part of $a \in \mathbf{R}$. The proof is straightforward, but we give it since the solution U_0 also plays an important role in an example of the next subsection, where the difference between (5.1) and (5.2) is shown.

Theorem 5.3. *U_0 is the unique solution of* (St1) *in X.*

To prove this we prepare

Lemma 5.4. *Let $f_1, f_2 \in C^1(U)$ for $U \subset \mathbf{R}^N$ open, and define $g := \min\{f_1, f_2\}$. Assume that $f_1(z) = f_2(z)$ at $z \in U$.*

1. *Let $\phi \in C^1(U)$. If $g - \phi$ attains a local maximum at z, then $\nabla\phi(z) = \lambda\nabla f_1(z) + (1-\lambda)\nabla f_2(z)$ for some $\lambda \in [0,1]$.*
2. *If $\nabla f_1(z) \neq \nabla f_2(z)$, then there does not exist $\phi \in C^1(U)$ such that $g - \phi$ attains a local minimum at z.*

Proof. Set $Y(z) := \{\nabla f_1(z), \nabla f_2(z)\}$. By [4, Proposition II.2.13], the superdifferential $D^+g(z)$ of g at z is equal to the convex hull of $Y(z)$, which proves (1). If $\nabla f_1(z) \neq \nabla f_2(z)$, the same proposition implies that the subdifferential $D^-g(z)$ of g at z is empty since $Y(z)$ is not a singleton. Accordingly, (2) holds. □

Proof of Theorem 5.3. It is easy to see that U_0 fulfills (3.1)–(3.4). At the initial time, we have $U_0(x,0) = 0$ in $\overline{\Omega}$. Moreover, since $U_0(0,t) = ct$, the boundary condition (5.1) is satisfied.

We check that U_0 is a solution of (5.6) in $\Omega \times (0,T)$. In $\{(x,t) \in \Omega \times (0,T) \mid t \neq x\}$, U_0 is differentiable and

$$\begin{cases} (U_0)_t(x,t) = c, \ (U_0)_x(x,t) = -c & (t < x), \\ (U_0)_t(x,t) = 0, \ (U_0)_x(x,t) = 0 & (t > x). \end{cases}$$

Thus $(U_0)_t(x,t) - |(U_0)_x(x,t)| = 0$ if $t \neq x$.

Let $(\hat{x}, \hat{x}) \in \Omega \times (0,T)$ and $\phi \in C^1(\overline{\Omega} \times (0,T))$. We assume that $U_0 - \phi$ attains its maximum at $(\hat{x}, \hat{x})$. Then, since $U_0(x,t) = \max\{c(t-x), 0\}$ near $(\hat{x}, \hat{x})$, it follows from Lemma 5.4 (2) that there is no such ϕ. Next, suppose that $U_0 - \phi$ attains its minimum at $(\hat{x}, \hat{x})$. In this case, Lemma 5.4 (1) implies that

$$(\phi_t(\hat{x}, \hat{t}), \phi_x(\hat{x}, \hat{t})) = \lambda(c, -c) + (1-\lambda)(0,0) = \lambda(c, -c)$$

for some $\lambda \in [0,1]$. Therefore $\phi_t(\hat{x}, \hat{t}) - |\phi_x(\hat{x}, \hat{t})| = 0$. □

Remark 5.5. Let us compare the solution U_0 with envelope solutions ([12]) of

$$u_t(x,t) - |u_x(x,t)| = cI(x) \quad \text{in } \mathbf{R}\times(0,T), \quad u(x,0) = u_0(x) \quad \text{in } \mathbf{R}. \tag{5.8}$$

Here $I(x)$ is the function given in (1.11). Define $\hat{u}(x,t) = U_0(|x|,t) = c(t-|x|)_+$ for $(x,t)\in\mathbf{R}\times[0,T)$, which is obtained by reflecting U_0 through $\{x=0\}$. By [12, Example 3.16], $\hat{u}$ is the unique envelope solution of (5.8) with $u_0 = 0$.

5.2 Stationary condition v.s. Dirichlet condition

We continue to study the eikonal equation (5.6). Now, impose the initial condition

$$u(x,t) = u_0(x) := 2c\min\{x,\, 1\} = \begin{cases} 2cx & (0 \leqq x \leqq 1), \\ 2c & (x \geqq 1) \end{cases} \quad \text{in } \overline{\Omega}. \tag{5.9}$$

This u_0 does not satisfy (5.5). We show that the unique solution under (5.1) is different from the unique solution under (5.2).

Let $T > 2$. In addition to (5.1) and (5.2), we study the Neumann boundary condition

$$-u_x(0,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{5.10}$$

and the dynamic boundary condition

$$u_t(0,t) - \beta u_x(0,t) = c \quad \text{on } \partial\Omega\times(0,T) \quad (\beta > 0). \tag{5.11}$$

In particular, we are interested in the asymptotic behavior of the solution for (5.11) with respect to β. Set

$$(5.6),\quad (5.1),\quad (5.9), \tag{St2}$$
$$(5.6),\quad (5.2),\quad (5.9), \tag{Di2}$$
$$(5.6),\quad (5.10),\quad (5.9), \tag{Ne2}$$
$$(5.6),\quad (5.11),\quad (5.9). \tag{Dy2}$$

Let us define $u_i : \overline{\Omega}\times[0,T)\to\mathbf{R}$ $(i=1,2,3)$ and $u_4 = u_4^\beta : \overline{\Omega}\times[0,T)\to\mathbf{R}$ by

$$u_1(x,t) := \begin{cases} 2c\min\{x+t,\, 1\} & (0\leqq t\leqq 1), \\ U_0(x,t-1)+2c & (t\geqq 1), \end{cases} \qquad u_2(x,t) := \begin{cases} 2c\min\{x+t,\, 1\} & (0\leqq t\leqq 1), \\ 2c & (1\leqq t\leqq 2), \\ U_0(x,t-2)+2c & (t\geqq 2), \end{cases}$$

$$u_3(x,t) := \begin{cases} 2c\min\{x+t,\, 1\} & (0\leqq t\leqq 1), \\ 2c & (t\geqq 1), \end{cases} \qquad u_4(x,t) := \begin{cases} 2c\min\{x+t,\, 1\} & (0\leqq t\leqq 1), \\ \dfrac{1}{1+\beta}U_0(x,t-1)+2c & (t\geqq 1). \end{cases}$$

Here U_0 is the function give by (5.7). Note that $u_1 = u_4^0$ if we let $\beta = 0$. Moreover, $u_4^\beta \to u_1$ uniformly in $\overline{\Omega}\times[0,T)$ as $\beta\to+0$ and $u_4^\beta\to u_3$ uniformly in $\overline{\Omega}\times[0,T)$ as $\beta\to\infty$. We prove that u_1,u_2,u_3,u_4 are, respectively, the unique solution of (St2), (Di2), (Ne2), (Dy2). See Figure 1 for the graphs of these solutions.

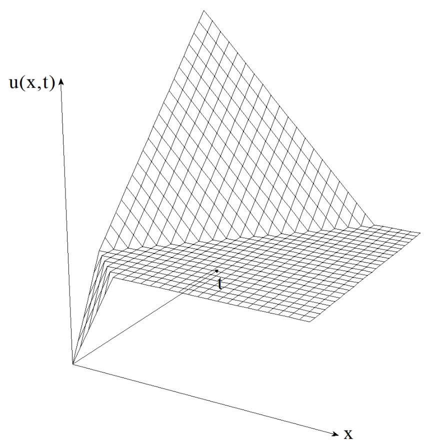
Theorem 5.6. (1) u_1 *is the unique solution of* (St2) *in* X.

(2) u_2 *is the unique solution of* (Di2) *in* X.

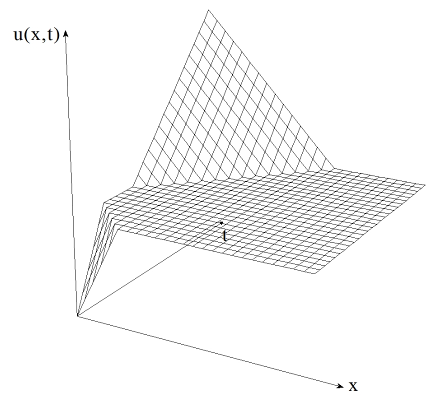
(3) u_3 *is the unique solution of* (Ne2).

(4) u_4^β *is the unique solution of* (Dy2) *for any* $\beta > 0$.

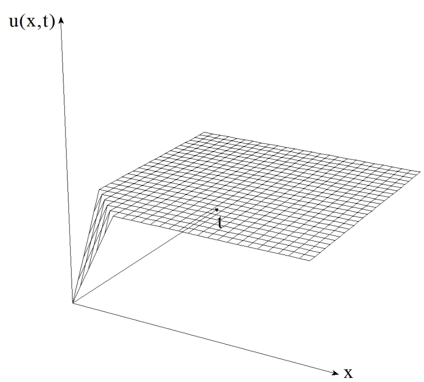
In particular, we see by this theorem that the solution u_1 of (St2) is different from the solution u_2 of (Di2). Moreover, $u_1(0,t)\neq ct$ for $t\in(0,T)$ and $u_2(0,t)\neq ct$ for $t\in(0,2)$. Also, the limit of the solution u_4^β of (Dy2) as $\beta\to+0$ is characterized not by (Di2) but by (St2).



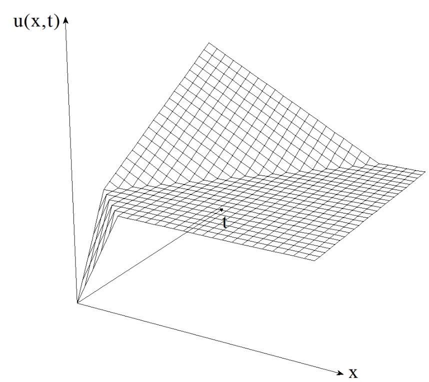
(a) The solution u_1 of (St2)



(b) The solution u_2 of (Di2)



(c) The solution u_3 of (Ne2)



(d) The solution u_4 of (Dy2)

Figure 1: Solutions of (5.6) with (5.9).

Proof. By definitions we have $u_i(x,0) = u_0(x)$ in $\overline{\Omega}$ for all $i = 1,2,3,4$. Besides, it is easily seen that u_1 and u_2 satisfy (3.1)–(3.4). and so the uniqueness of them is a consequence of Corollary 3.3. The uniqueness of u_3 and u_4 follows from the classical comparison result, Corollary 2.8. Let us check that u_i is a solution for every $i = 1,2,3,4$. Since the assertion for u_1 is included in (4) by allowing $\beta = 0$, we give the proofs of (2), (3) and (4) with $\beta \geqq 0$. Moreover, in a similar way to the proof of Theorem 5.3, one can check that u_i is a solution of (5.6) in $\Omega \times (0,T)$. Thus we only check the boundary condition.

1. u_i **is a subsolution on** $\partial\Omega \times (0,T)$**.** Assume that $u_i - \phi$ attains its maximum at $(0,\hat{t}) \in \partial\Omega \times (0,T)$ for $\phi \in C^1(\overline{\Omega} \times (0,T))$. Set $\tau = \phi_t(0,\hat{t})$ and $p = \phi_x(0,\hat{t})$.

Case 1: $\hat{t} \in (0,1]$**.** We prove that the equation (5.6) is satisfied in this case. Suppose first that $\hat{t} \in (0,1)$. Since $u_i(x,t) = 2c(x+t)$ near $(0,\hat{t})$ for every $i = 2,3,4$, we have $\tau = 2c$ and $p \geqq 2c$. This implies that

$$\tau - |p| \leqq 2c - 2c = 0.$$

Next, let $\hat{t} = 1$. For $h > 0$ small enough, we have

$$\frac{\phi(h,1-h) - \phi(0,1)}{\sqrt{2}h} \geqq \frac{u_i(h,1-h) - u_i(0,1)}{\sqrt{2}h} = \frac{2c-2c}{\sqrt{2}h} = 0 \quad (i = 2,3,4).$$

Sending $h \to +0$ implies that $p - \tau \geqq 0$. Now, as $u_i(x,1) = 2c$ for $x \geqq 0$, we have $p \geqq 0$. Consequently, $\tau - |p| = \tau - p \leqq 0$.

Case 2: $\hat{t} \in (1,T)$**.** By the definitions of u_2, u_3, u_4 we have

$$(2)\ \begin{cases} \tau = 0,\ p \geqq 0 & \text{if } 1 < \hat{t} < 2, \\ \text{there is no such } \phi & \text{if } \hat{t} = 2, \\ u_2(0,\hat{t}) = c\hat{t} & \text{if } \hat{t} > 2 \end{cases} \qquad (3)\ \tau = 0,\ p \geqq 0 \qquad (4)\ \tau = \frac{c}{1+\beta},\ p \geqq -\frac{c}{1+\beta}.$$

Therefore

$$(2)\ \tau - |p| \leqq 0 \quad \text{if } 1 < \hat{t} < 2 \qquad (3)\ \tau - |p| \leqq 0 \qquad (4)\ \tau - \beta p \leqq \frac{c}{1+\beta} - \beta \cdot \frac{-c}{1+\beta} = c.$$

2. u_i **is a supersolution on** $\partial\Omega \times (0,T)$**.** Assume that $u_i - \phi$ attains its minimum at $(0,\hat{t}) \in \partial\Omega \times (0,T)$ for $\phi \in C^1(\overline{\Omega} \times (0,T))$. Set $\tau = \phi_t(0,\hat{t})$ and $p = \phi_x(0,\hat{t})$. Then, for $i = 2$, the boundary condition (5.2) is satisfied wherever $\hat{t}$ lies since we have $u_2(0,\hat{t}) \geqq c\hat{t}$. Let us discuss the cases of $i = 3,4$.

Case 1: $\hat{t} \in (0,1]$**.** It is easily seen that there is no such ϕ when $\hat{t} = 1$. Thus we may let $\hat{t} \in (0,1)$. For both $i = 3,4$ we have $u_i(x,t) = 2c(x+t)$ near $(0,\hat{t})$. This implies that $\tau = 2c$ and $p \leqq 2c$, and so

$$(3)\ \begin{cases} \tau - |p| \geqq 2c - 2c = 0 & \text{if } 0 \leqq p \leqq 2c, \\ -p \geqq 0 & \text{if } p \leqq 0 \end{cases} \qquad (4)\ \begin{cases} \tau - |p| \geqq 2c - 2c = 0 & \text{if } 0 \leqq p \leqq 2c, \\ \tau - \beta p \geqq 2c + 0 \geqq c & \text{if } p \leqq 0. \end{cases}$$

Case 2: $\hat{t} \in (1,T)$**.** In this case

$$(3)\ \tau = 0,\ p \leqq 0 \qquad (4)\ \tau = \frac{c}{1+\beta},\ p \leqq -\frac{c}{1+\beta},$$

and thus

$$(3)\ -p \geqq 0 \qquad (4)\ \tau - \beta p \geqq \frac{c}{1+\beta} - \beta \cdot \frac{-c}{1+\beta} = c.$$

Namely, the boundary condition is satisfied. The proof is complete. □

Remark 5.7. The behaviors of u_1 and u_2 and their difference can be understood as follows: Since the equation (5.6) requires that the horizontal growth speed of the solution should be one, both u_1 and u_2 grow in the horizontal direction until they become flat at $t = 1$. During the period $0 \leqq t \leqq 1$, the value of them at $x = 0$

is $2ct$ and the vertical growth speed is $2c$. They are beyond the prescribed boundary conditions. After they get flat, the behaviors are different. Since the value of the solution u_2 is larger than the Dirichlet boundary value at $x = 0$, the solution waits until it recovers the boundary condition. Once it is recovered at $t = 2$, the solution starts to grow again with the shape U_0 as in the previous example (Theorem 5.3). Different from u_2, the solution u_1 of the stationary problem does not need to wait since the prescribed condition is not the value but the speed. Thus u_1 immediately starts to grow after $t = 1$.

This difference is caused by the steep slope of the initial datum near $x = 0$. As we have already proven in Theorem 5.1 (1), if the initial slope is small compared with c, then the stationary condition and the Dirichlet condition give the same solution.

This example also shows that solutions for the Dirichlet boundary problem do not possess a semi-group property. In fact, as we observed in the previous example, if the initial datum is constant, the solution immediately grows. However, the solution u_2 does not grow during $1 \leqq t \leqq 2$ in spite that it is flat at $t = 1$.

Remark 5.8. Let us define $\hat{u}(x,t) = u_1(|x|,t)$ for $(x,t) \in \mathbf{R} \times [0,T)$, which is the reflection of the solution u_1 of (St2). As shown in [12, Example 5.7 (2)], $\hat{u}$ is the unique envelope solution of (5.8) with u_0 given by (5.9).

In [12, Section 5.3], we also studied the problem

$$u_t(x,t) - |u_x(x,t)| = 0 \quad \text{in } (\mathbf{R} \setminus \{0\}) \times (0,T), \quad u(x,0) = u_0(x) \quad \text{in } \mathbf{R}, \quad u_t(0,t) = c. \tag{5.12}$$

The above $\hat{u}$ is a viscosity solution of (5.12) but it is not a unique solution. The problem (5.12) has infinitely many continuous viscosity solutions.

Non-uniqueness of discontinuous solutions. There are infinitely many discontinuous solutions of (St2) and (Di2). This shows that solutions may lose continuity even if a Hamiltonian is coercive. Such discontinuous solutions are obtained by modifying u_1 and u_2 on $\partial\Omega \times (0,T)$. Let us define

$$\tilde{u}_1(x,t) := \begin{cases} u_1(x,t) & (x > 0), \\ f_1(t) & (x = 0), \end{cases} \qquad \tilde{u}_2(x,t) := \begin{cases} u_2(x,t) & (x > 0), \\ f_2(t) & (x = 0) \end{cases}$$

with functions $f_1 \in C([0,T)) \cap C^1((0,T))$ and $f_2 \in C([0,T))$ satisfying $f_1(0) = f_2(0) = 0$ and

$$\begin{aligned} 0 \leqq f_1(t) \leqq u_1(0,t), \quad f_1'(t) \geqq c \quad &\text{for all } t \in (0,T), \\ ct \leqq f_2(t) \leqq u_2(0,t) \quad &\text{for all } t \in (0,T). \end{aligned}$$

There are infinitely many choices of such f_1 and f_2, and examples include $f_1(t) = f_2(t) = ct$. Unless $f_i(t) = u_i(0,t)$ for all $t \in [0,T)$, the function $\tilde{u}_i$ has discontinuity on $\partial\Omega \times (0,T)$ for $i = 1,2$. Also, it is easily seen that

$$(\tilde{u}_i)^* = u_i \quad \text{in } \overline{\Omega} \times [0,T), \quad (\tilde{u}_i)_* = \begin{cases} u_i & \text{in } \Omega \times [0,T), \\ f_i & \text{on } \partial\Omega \times [0,T) \end{cases} \quad (i = 1,2). \tag{5.13}$$

Proposition 5.9. *$\tilde{u}_1$ is a solution of* (St2) *and $\tilde{u}_2$ is a solution of* (Di2)*. Therefore, solutions are not unique for these problems.*

Proof. By Theorem 5.6 and (5.13), we only need to check that $(\tilde{u}_i)_*$ is a supersolution on the boundary for $i = 1,2$. Assume first that $u_1 - \phi$ attains its minimum at $(0,\hat{t})$ for $\phi \in C^1(\overline{\Omega} \times (0,T))$ and $\hat{t} \in (0,T)$. Then, $\phi_t(0,\hat{t}) = f_1'(0,\hat{t}) \geqq c$. Let us next assume that $u_2 - \phi$ attains its minimum at $(0,\hat{t})$ for $\phi \in C^1(\overline{\Omega} \times (0,T))$ and $\hat{t} \in (0,T)$. Then we have $(\tilde{u}_2)_*(0,\hat{t}) = f_2(\hat{t}) \geqq c\hat{t}$. □

5.3 Non-existence of Lipschitz continuous solutions

Even if the initial datum is zero, the unique continuous solution may immediately lose the Lipschitz continuity when the equation is not coercive. We below give such an example. For such problems we cannot apply Theorem 2.9 to show the uniqueness.

We consider a problem similar to [12, Example 5.15]. We study the equation

$$u_t(x,t) - \frac{|u_x(x,t)|}{1+|u_x(x,t)|} = 0 \quad \text{in } \Omega \times (0,T). \tag{5.14}$$

Here $h(r) = r/(1+r)$ and $h_\infty = 1$. Also, note that the corresponding $F(p) = -h(|p|) = -|p|/(1+|p|)$ is not coercive. For $c > 0$ we consider the stationary boundary problem

$$(5.14), \quad (5.1), \quad u(x,0) = 0 \ \text{ in } \overline{\Omega}. \tag{St3}$$

Let us define

$$U(x,t) = \left\{ \left(\sqrt{t} - \sqrt{x} \right)_+ \right\}^2,$$

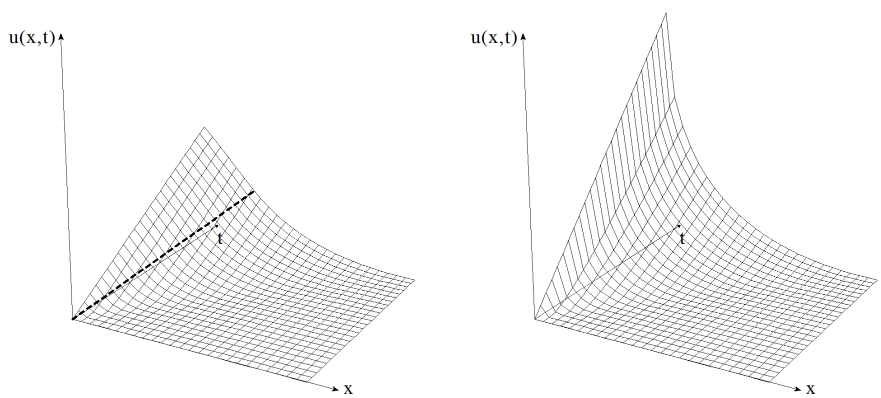
and

$$u_c(x,t) = \begin{cases} ct - \dfrac{c}{1-c}x & (x \leqq (1-c)^2 t), \\ U(x,t) & (x \geqq (1-c)^2 t) \end{cases} \quad \text{if } 0 < c < 1, \quad u_c(x,t) = U(x,t) \ \text{ if } c \geqq 1.$$

In $\overline{\Omega} \times [0,T)$, the above u_c is Lipschitz continuous if $0 < c < 1$, while it is not Lipschitz continuous if $c \geqq 1$. The latter fact follows from

$$\lim_{h \to +0} \frac{U(h,t) - U(0,t)}{h} = -\infty \quad \text{for } t \in (0,T). \tag{5.15}$$

See Figure 2 for the graphs of u_c.



(a) The solution u_c with $0 < c < 1$ (b) The solution u_c with $c \geqq 1$

Figure 2: Solutions of (5.14) with $u(x,0) = 0$.

When $c > 1$, let us define

$$\tilde{u}_c(x,t) := \begin{cases} U(x,t) & (x > 0), \\ f(t) & (x = 0) \end{cases}$$

with a function $f \in C([0,T)) \cap C^1((0,T))$ satisfying $f(0) = 0$ and

$$t \leqq f(t), \quad f'(t) \leqq c \quad \text{for all } t \in (0,T). \tag{5.16}$$

Note that $\tilde{u}_c$ is continuous in $\overline{\Omega} \times [0,\infty)$ if and only if $f(t) = t$.

Theorem 5.10. (1) *u_c is the unique solution of* (St3) *in X for every $c > 0$.*

(2) *Let $c > 1$. Then $\tilde{u}_c$ is a solution of* (St3).

Thus the unique solution u_c in X is not Lipschitz continuous when $c \geqq 1$. If $c > 1 = h_\infty$ (i.e, the assumptions of Theorem 5.1 are not fulfilled), the solution u_c does not satisfy (5.3) since $u_c(0,t) = t$ for $t \in (0,T)$. Theorem 5.10 (2) asserts that there are infinitely many discontinuous solutions.

Proof. (1) It is easily seen that u_c satisfies (3.1)–(3.4) and $u_c(x,0) = 0$ in $\overline{\Omega}$ for every $c > 0$.

Case 1: $0 < c < 1$**.** Let $a(x,t) := ct - cx/(1-c)$. Differentiating a and U, we find

$$a_t(x,t) = c, \quad a_x(x,t) = -\frac{c}{1-c}, \quad U_t(x,t) = \frac{(\sqrt{t}-\sqrt{x})_+}{\sqrt{t}}, \quad U_x(x,t) = -\frac{(\sqrt{t}-\sqrt{x})_+}{\sqrt{x}}.$$

This shows that both a and U solve (5.14) in $\Omega \times (0,T)$. Moreover, they are smoothly connected on the line $\{x = (1-c)^2 t\}$. Namely, $u_c \in C^1(\overline{\Omega} \times (0,T))$, and thus u_c is a classical (and hence viscosity) solution of (5.14). Since $u_c(0,t) = ct$, the boundary condition is also satisfied.

Case 2: $c \geqq 1$**.** We have already checked that $u_c = U$ solves (5.14) in $\Omega \times (0,T)$. On the boundary $\partial\Omega \times (0,T)$, we have $u_c(0,t) = U(0,t) = t \leqq ct$, and so u_c is a subsolution of (St3). Furthermore, due to (5.15), there is no test functions ϕ touching u_c from below at $(0,t)$. Thus u_c is a supersolution of (St3).

(2) Since $t \leqq f(t)$ for $t \in [0,T)$, we have

$$(\tilde{u}_c)^* = \begin{cases} U & \text{in } \Omega \times [0,T), \\ f & \text{on } \partial\Omega \times [0,T), \end{cases} \qquad (\tilde{u}_c)_* = U \quad \text{in } \overline{\Omega} \times [0,T).$$

These and the second assumption $f'(\hat{t}) \leqq c$ in (5.16) give the desired conclusion. ☐

Remark 5.11. Set $\hat{u}_c(x,t) = u_c(|x|,t)$ for $(x,t) \in \mathbf{R} \times [0,T)$. In [12, Example 5.15] we studied

$$u_t(x,t) - \frac{|u_x(x,t)|}{1+|u_x(x,t)|} = cI(x) \quad \text{in } \mathbf{R} \times (0,T), \quad u(x,0) = 0 \quad \text{in } \mathbf{R} \tag{5.17}$$

and proved that $\hat{u}_c$ is the unique envelope solution of (5.17) when $0 < c \leqq 1$. If $c > 1$, envelope solutions are not unique.

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