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NUMERICAL SOLUTION OF PERFECT PLASTIC PROBLEMS WITH CONTACT: PART I - THEORY AND NUMERICAL METHODS

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Abstract. The contribution deals with a static case of discretized elasto-perfectly plastic problems obeying Hencky's law in combination with frictionless contact boundary conditions. The main interest is focused on the analysis of the formulation in terms of displacements, limit load analysis and related numerical methods. This covers the study of: *i*) the dependence of the solution set on the loading parameter ζ, *ii*) relation between ζ and the parameter α representing the work of external forces, *iii*) loading process controlled by ζ and by α, *iv*) numerical methods for solving problems with prescribed value of ζ and α.

1 INTRODUCTION

This contribution deals with a static case of discretized elasto-perfectly plastic problems obeying Hencky's law. In particular, we confine ourselves to the von Mises yield criterion and a standard (conventional) finite element discretization of the problem. We study the problem mainly in terms of displacements, however we also introduce the formulation in terms of stresses since its knowledge is very useful from both, the theoretical and numerical point of view. The related continuous problem is analyzed in many papers, we refer to e.g. [5, 12, 13, 19]. A discretization of the problem using discontinuous finite elements can be found e.g. in [14].

Since the used discretization of the problem is not fully compatible with the continuous setting of the problem which is formulated on nonseparable BD-spaces, we will pay our attention to analysis of the discretized problem. We slightly extend known existence results in dependence on the load parameter ζ. Furthermore, we establish a relation between ζ and the parameter α representing the work of external forces. This is useful for the evaluation and controlling the loading process and consequently for the limit load analysis.

The limit load analysis is very important from the mechanical point of view in perfect plasticity. We refer e.g. [2, 5, 12, 19] for various approaches. We propose the approach based on the mutual relation between ζ and α. In particular, if α tends to $+\infty$ then ζ tends to its limit value ζ_{lim}. Therefore α seems to be more sensitive than ζ for controlling the monotone loading process up to the limit load.

We introduce several numerical methods for solving this type of problems. In particular, we consider a modified semi-smooth Newton method with damping and an augmented Lagrangian method. Newton-like methods in plasticity are studied e.g. in [1, 9, 16, 17]. The augmented Lagrangian method can be found in [8].

Moreover, we consider the elasto-perfectly plastic material in combination with frictionless contact boundary conditions following [10, 11]. For the sake of simplicity, we confine ourselves to bounded contact zones. The presented results and methods can be used for both, classical and contact, boundary conditions. In the case of the contact problem, these methods can also be interpreted as methods of sequential quadratic programming. Thus we solve a contact problem with a linearized material in each iteration.

For the sake of brevity, this contribution does not contain any proofs. For more details, we refer to [18] and [4] (in preparation). Numerical realization of these methods in combination with the T-FETI domain decomposition method and numerical examples can be found in PART II ([3]) of our contribution.

This contribution is organized as follows. In Section 2, the discretized problem is formulated and analyzed mainly in terms of displacements. In Section 3 and 4, numerical methods are presented. Concluding remarks can be found in Section 5.

2 SETTING OF THE DISCRETIZED PROBLEM

Let us consider two 3D bodies in mutual contact which are made of elasto - perfectly plastic materials. The bodies are represented by bounded domains $\Omega^1, \Omega^2 \subseteq \mathbb{R}^3$ whose Lipschitz boundaries are decomposed as follows: $\partial\Omega^j = \overline{\Gamma}^j_u \cup \overline{\Gamma}^j_f \cup \overline{\Gamma}^j_c$, $j = 1, 2$, where Γ^j_u, Γ^j_f, Γ^j_c are open and mutually disjoint. On $\Gamma^j_u \neq \emptyset$, the structure is fixed, while surface tractions are applied on Γ^j_f, $j = 1, 2$. Further, $\Gamma_c := \Gamma^1_c = \Gamma^2_c$ is a bounded contact zone, where the frictionless contact boundary conditions are prescribed.

For the sake of simplicity we shall suppose that Ω^1, Ω^2 are polyhedric domains. Let $\mathcal{T}^j_h$

be regular partitions of $\overline{\Omega}^j$, $j = 1, 2$ into tetrahedrons denoted by $\triangle$ with a discretization parameter $h > 0$. Next we shall suppose that $\mathcal{T}_h^1|_{\Gamma_c} = \mathcal{T}_h^2|_{\Gamma_c}$, i.e. the nodes of both partitions on Γ_c coincide. Let us denote $\mathcal{T}_h := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ and introduce the following finite-dimensional spaces:

$$\mathbb{V}_h = \{v = (v_1, v_2) \in (C(\overline{\Omega}^1))^3 \times (C(\overline{\Omega}^2))^3 \mid \; v|_\triangle \in (P_1(\triangle))^3 \;\; \forall \triangle \in \mathcal{T}_h, \;\; v = 0 \text{ on } \Gamma_u^1 \cup \Gamma_u^2\},$$

$$S_h = \{\tau \in (L^2(\Omega^1))^{3\times 3}_{sym} \times (L^2(\Omega^2))^{3\times 3}_{sym} \;\mid\; \tau|_\triangle \in (P_0(\triangle))^{3\times 3}_{sym} \quad \forall \triangle \in \mathcal{T}_h\},$$

where $P_k(\triangle),\; k \geq 0$ integer, stands for the space of all polynomials of degree less or equal k defined in $\triangle \in \mathcal{T}_h$. The spaces $\mathbb{V}_h$ and S_h are the simplest finite element approximations of displacements and stresses (or strains), respectively. We shall consider the following scalar product and norm on S_h:

$$\langle \sigma, \tau \rangle \;:=\; \int_{\Omega^1 \cup \Omega^2} \sigma : \tau \, \mathrm{d}x = \sum_{\triangle \in \mathcal{T}_h} |\triangle| \; \sigma|_\triangle : \tau|_\triangle \quad \forall \sigma, \tau \in S_h,$$

$$\|\tau\|_E \;:=\; \sqrt{\langle C^{-1}\tau, \tau \rangle} = \left(\sum_{\triangle \in \mathcal{T}_h} |\triangle| \; C^{-1}\tau|_\triangle : \tau|_\triangle \right)^{1/2} \quad \forall \tau \in S_h.$$

Here $C = (c_{ijkl})^3_{i,j,k,l=1}$ is the fourth order symmetric elasticity tensor of generalized Hooke's law for an isotropic homogeneous material, i.e.

$$\tau = Ce \;\Leftrightarrow\; \tau = \lambda \, \mathrm{tr}(e) \, \delta + 2\mu e = (3\lambda + 2\mu) \mathrm{tr}(e) \, \delta + 2\mu e^D, \quad e, \tau \in \mathbb{R}^{3\times 3}_{sym},$$

where δ is the (3×3) identity matrix, $\mathrm{tr}(e)$, e^D denote the trace and deviatoric part of a tensor e, respectively, and $\lambda, \mu > 0$ are Lame's coefficients. On $\mathbb{V}_h$, we shall consider the energy norm related to the small strain assumption:

$$|\!|\!| v |\!|\!| := \|C\varepsilon(v)\|_E \qquad \forall v \in \mathbb{V}_h.$$

We define the convex sets of kinematically admissible displacement fields, plastically and statically admissible stress fields, respectively, as follows:

$$\begin{aligned} K_h &:= \{v \in \mathbb{V}_h \,|\, [v]_n \leq 0 \text{ on } \Gamma_c\}, \\ P_h &:= \left\{\tau \in S_h \,|\, \|\tau^D|_\triangle\|_F \leq \gamma \;\; \forall \triangle \in \mathcal{T}_h \right\}, \\ \Lambda^h_{\zeta L} &= \{\tau \in S \,|\, \langle \tau, \varepsilon(v) \rangle \geq \zeta L(v) \;\; \forall v \in K_h\}. \end{aligned}$$

Here $[v]_n := (v_1 - v_2).n \leq 0$, where n is the outward unit normal vector to $\partial\Omega^1$, represents the non-penetration condition on Γ_c. Further $\|.\|_F$, $\gamma > 0$ denote the Frobenius norm, and the initial yield stress, respectively. The linear functional $L : \mathbb{V}_h \to \mathbb{R}$ represents an applied (fixed) load including volume and traction forces and $\zeta \geq 0$ is the load parameter. We shall also assume that L satisfies the following condition:

$$\exists z \in K_h : \quad L(z) > 0. \tag{1}$$

The dual and primal formulation of the discretized problem depending on $\zeta \geq 0$ reads as follows:

$$\begin{array}{lll} (\mathcal{P})^*_{h,\zeta} & \text{minimize} & \mathcal{S}(\tau) = \frac{1}{2}\|\tau\|^2_E \quad \text{on } \Lambda^h_{\zeta L} \cap P_h, \\ (\mathcal{P})_{h,\zeta} & \text{minimize} & \mathcal{J}_\zeta(v) = \Psi(\varepsilon(v)) - \zeta L(v) \quad \text{on } K_h, \end{array}$$

respectively. The functional $\mathcal{S}$ is quadratic and strictly convex. Therefore a unique solution to $(\mathcal{P})^*_{h,\zeta}$ exists if and only if

$$\Lambda^h_{\zeta L} \cap P_h \neq \emptyset. \tag{2}$$

It is known that there exists the so-called limit load parameter $\zeta_{lim} > 0$ (possibly $\zeta_{lim} = +\infty$) such that (2) holds if and only if $\zeta \in [0, \zeta_{lim}]$.

The functional $\Psi: \ S_h \to \mathbb{R}$ in the primal problem $(\mathcal{P})_{h,\zeta}$ has the following form:

$$\Psi(e) = -\frac{1}{2}\|\Sigma(e)\|^2_E + \langle e, \Sigma(e)\rangle \quad \forall e \in S_h,$$

where

$$\Sigma(e)|_\triangle = \begin{cases} Ce|_\triangle, & \|(Ce|_\triangle)^D\|_F \leq \gamma, \\ \frac{1}{3}(3\lambda + 2\mu)\mathrm{tr}(e|_\triangle)\delta + \gamma\frac{e^D}{\|e^D\|_F}, & \|(Ce|_\triangle)^D\|_F \geq \gamma, \end{cases}$$

Notice that Σ is the Fréchet derivative of Ψ, i.e. $\Sigma(e) := \mathbb{D}\Psi(e)\ \forall e \in S_h$.

The functional Ψ is convex. However, it has only a linear growth at infinity and it is not strictly convex, in general. It means that the coercivity of $\mathcal{J}_\zeta$ and uniqueness of a solution to $(\mathcal{P})_{h,\zeta}$ are not generally guaranteed. It is known that for $\zeta < \zeta_{lim}$ the functional $\mathcal{J}_\zeta$ is coercive and the solution set to $(\mathcal{P})_{h,\zeta}$, denoted by $\mathcal{K}_{h,\zeta}$ in what follows, is non-empty and bounded. If $\zeta = \zeta_{lim}$, then $\mathcal{K}_{h,\zeta}$ is either empty or unbounded. For $\zeta > \zeta_{lim}$, it holds that $\mathcal{K}_{h,\zeta} = \emptyset$. For sufficiently small ζ, the problem $(\mathcal{P})_{h,\zeta}$ has a unique solution which also solves the corresponding contact problem for elastic bodies.

The following relationship between the dual and primal problem [19] holds:

$$\inf_{v \in K_h} \mathcal{J}_\zeta(v) = \sup_{\tau \in \Lambda^h_{\zeta L} \cap P_h} \{-\mathcal{S}(\tau)\} \quad \forall \zeta \geq 0,$$

where we set $\sup\{-\mathcal{S}(\tau)\} = -\infty$ if $\Lambda^h_{\zeta L} \cap P_h = \emptyset$, i.e. if $\zeta > \zeta_{lim}$. If there exists a solution $u_\zeta \in \mathbb{V}_h$ of $(\mathcal{P})_{h,\zeta}$, then $\sigma_\zeta = \Sigma(\varepsilon(u_\zeta))$ solves $(\mathcal{P})^*_{h,\zeta}$, i.e. Σ represents the stress-strain relation.

In perfect plasticity, it is typical to investigate a loading process up to the limit load represented by ζ_{lim}, which is not known a priori. So increasing ζ, we would like to know how far we are from ζ_{lim}. To this end, it will be useful to know how the parameter $\alpha \geq 0$ representing the work of external forces $L(u_\zeta)$, $u_\zeta \in \mathcal{K}_{h,\zeta}$, depends on ζ.

The following assertions hold provided that (1) is satisfied:

i) Let $0 \leq \zeta_1 < \zeta_2 \leq \zeta_{lim}$ and $(\mathcal{P})_{\zeta_2}$ has a solution. Then
$$L(u_{\zeta_1}) < L(u_{\zeta_2}) \quad \forall u_{\zeta_i} \in \mathcal{K}_{h,\zeta_i},\ i = 1, 2.$$

ii) Let $\alpha \geq 0$ be given. Then
$$\exists!\ \zeta := \zeta(\alpha) \leq \zeta_{lim},\ \ \exists u_\zeta \in \mathcal{K}_{h,\zeta} : \quad L(u_\zeta) = \alpha.$$

iii) If $\alpha \to +\infty$ then $\zeta(\alpha) \to \zeta_{lim}$.

iv) The function $\alpha \mapsto \zeta(\alpha)$ is linear for sufficiently small α (elastic branch).

v) The function $\alpha \mapsto \zeta(\alpha)$ is continuous and nondecreasing on $\mathbb{R}_+$.

Thus the parameter α is more sensitive for controlling the loading process than ζ. If the curve representing the relation between α and ζ is far from to be linear, one can expect that ζ is close to ζ_{lim}.

To control the loading process through the parameter α, we define the following problem: given $\alpha \geq 0$,

$$(\mathcal{S})_h^\alpha \qquad \text{find } (\zeta, u) \in \mathbb{R}^+ \times K_h : \quad u \in \mathcal{K}_{h,\zeta},\ \ L(u) = \alpha.$$

From the previous results, we know that problem $(\mathcal{S})_h^\alpha$ has a solution for any $\alpha \geq 0$, whose first component is unique. Moreover, $(\mathcal{S})_h^\alpha$ can be equivalently expressed as the following saddle-point problem:

$$\text{find } (\zeta, u) \in \mathbb{R}^+ \times K_h : \quad \mathcal{L}_\alpha(\omega, u) \leq \mathcal{L}_\alpha(\zeta, u) \leq \mathcal{L}_\alpha(\zeta, v) \quad \forall (\omega, v) \in \mathbb{R}^+ \times K_h.$$

with
$$\mathcal{L}_\alpha(\omega, v) := \Psi(\varepsilon(v)) + \omega(\alpha - L(v)) = \mathcal{J}_\omega(v) + \alpha\omega, \quad \omega \in \mathbb{R}^+,\ v \in K_h.$$

The existence of a saddle-point of $\mathcal{L}_\alpha$ yields

$$\min_{v \in K_h} \sup_{\omega \in \mathbb{R}^+} \mathcal{L}_\alpha(\omega, v) = \max_{\omega \in \mathbb{R}^+} \inf_{v \in K_h} \mathcal{L}_\alpha(\omega, v) = \mathcal{L}_\alpha(\zeta, u) = \Psi(\varepsilon(u)). \tag{3}$$

One can easily find that the min-sup problem in (3) can be written into the form

$$(\mathcal{P})_h^\alpha \qquad \begin{cases} \text{find } u \in K_h^\alpha : \quad \Psi(\varepsilon(u)) \leq \Psi(\varepsilon(v)) \quad \forall v \in K_h^\alpha, \text{ where} \\ \qquad\qquad K_h^\alpha := \{v \in K_h \mid\ L(v) = \alpha\}. \end{cases}$$

Problem $(\mathcal{P})_h^\alpha$ is related to $(\mathcal{S})_h^\alpha$ in the following sense: If $\alpha \geq 0$ and (ζ, u) is a solution to $(\mathcal{S})_h^\alpha$, then u is a solution to $(\mathcal{P})_h^\alpha$. On the contrary, if u is a solution to $(\mathcal{P})_h^\alpha$ for some $\alpha > 0$, then (ζ, u) is a solution to $(\mathcal{S})_h^\alpha$, where $\zeta = \frac{1}{\alpha}\langle \Sigma(\varepsilon(u)), \varepsilon(u)\rangle$.

3 NUMERICAL METHODS FOR SOLVING $(\mathcal{P})_{h,\zeta}$

Since the solution set $\mathcal{K}_{h,\zeta}$ to the primal problem $(\mathcal{P})_{h,\zeta}$ need not be singleton in general, we consider two different numerical methods: the modified semi-smooth Newton method and the augmented Lagrangian method. A comparison of the methods and numerical experiments can be found in PART II ([3]).

3.1 Modified semi-smooth Newton method

In Section 2, the function Σ representing the non-linear stress-strain relation has been introduced. Notice that Σ is not smooth everywhere. Since Σ is Lipschitz continuous and S_h is finite-dimensional, Σ is almost everywhere differentiable. Thus one can define the generalized derivative $\partial\Sigma(e)$ of Σ at any $e \in S_h$ in the sense of Clark (see [6]). Clearly $\partial\Sigma(e) = \{\mathbb{D}\Sigma(e)\}$ for any $e \in S_h$ satisfying $\|(Ce|_\triangle)^D\|_F \neq \gamma \;\; \forall \triangle \in \mathcal{T}_h$. Let us define a function $\Sigma^o : \; S_h \to \mathcal{L}(S_h, S_h)$ such that $\Sigma^o(e) \in \partial\Sigma(e)$ for any $e \in S_h$.

Further, the function Σ is strongly semi-smooth ([15]) on S_h as follows from e.g. [9, 16, 17]. This yields the following estimate:

$$\begin{array}{l} \forall v \in \mathbb{V}_h, \; \exists c, r_1 > 0, \; \forall \delta v \in \mathbb{V}_h, \; \|\!|\delta v|\!\| \leq r_1 : \\ \qquad \|\Sigma(\varepsilon(v+\delta v)) - \Sigma(\varepsilon(v)) - \Sigma^o(\varepsilon(v+\delta v))\varepsilon(\delta v)\|_E \leq c\|\!|\delta v|\!\|^2. \end{array}$$

Notice that this estimate depends on the discretization parameter h. The dependence of c on h has been investigated in [16]. Since the matrix representation of $\Sigma^o(\varepsilon(v))$ need not be positive definite on $\mathbb{V}_h$, we introduce the regularization of Σ^o as follows:

$$\Sigma^{o,\rho} := \rho C + (1-\rho)\Sigma^o, \quad \rho \in [0,1].$$

Clearly $\Sigma^{o,0} = \Sigma^o$ and

$$\langle \Sigma^{o,\rho}(\varepsilon(v))\varepsilon(w), \varepsilon(w)\rangle \geq \rho\|\!|w|\!\|^2 \quad \forall v, w \in \mathbb{V}_h, \; \forall \rho \in [0,1].$$

Newton-like methods introduced in this and the next section are based on the approximation of the non-quadratic term Ψ by the quadratic one:

$$\begin{array}{rcl} \Psi(\varepsilon(\overline{u})) & \approx & \Psi(\varepsilon(u^k)) + \langle \Sigma(\varepsilon(u^k)), \varepsilon(\overline{u} - u^k)\rangle + \\ & & + \frac{1}{2}\langle \Sigma^{o,\rho}(\varepsilon(u^k))\varepsilon(\overline{u} - u^k), \varepsilon(\overline{u} - u^k)\rangle, \end{array} \qquad (4)$$

for some $\overline{u}, u^k \in K_h$. To represent the admissible set of functions for differences $\overline{u} - u^k$, we define the convex set

$$K_{h,k} := \{\delta v \in \mathbb{V}_h \mid \delta v + u^k \in K_h\}.$$

The k-th step of the modified Newton method reads as follows: If $u^k \in K_h$ is known then

$$u^{k+1} = u^k + \beta_k \delta u^k \in K_h,$$

where $\delta u^k \in K_{h,k}$ is a solution to the minimization problem

$$\min\left\{\frac{1}{2}\langle\Sigma^{o,\rho}(\varepsilon(u^k))\varepsilon(\delta v),\varepsilon(\delta v)\rangle+\langle\Sigma(\varepsilon(u^k)),\varepsilon(\delta v)\rangle-\zeta L(\delta v),\quad \delta v\in K_{h,k}\right\} \tag{5}$$

and the damping parameter β_k is defined as follows:

$$\beta_k=\arg\min_{\beta\in(0,1]}\mathcal{J}_\zeta(u^k+\beta\delta u^k).$$

This algorithm will be denoted ALG1 in what follows. As the initial iteration we choose some $u^0\in K_h$ and set

$$e^k:=\varepsilon(u^k),\;\;\delta e^k:=\varepsilon(\delta u^k),\;\;\sigma^k:=\Sigma(e^k),\quad k=0,1,2,\ldots.$$

Let us note (5) has a unique solution for any $\rho>0$. For $\rho=0$, (5) need not have a unique solution. If $\delta u^k=0$, then one can prove that u^k solves problem $(\mathcal{P})_{h,\zeta}$. In such a case, we formally set $\beta_k=1$ for the sake of completeness of the algorithm.

If $\rho>0$, then $\{u^k\}$ is the minimizing sequence of the functional $\mathcal{J}_\zeta$ for any $\zeta\geq 0$. In particular, if $\zeta\leq\zeta_{lim}$, then one can prove convergence of $\{\sigma^k\}$ to the solution of the dual problem $(\mathcal{P})^*_{h,\zeta}$. In addition, if $\zeta<\zeta_{lim}$, then the sequence $\{u^k\}$ is bounded in $\mathbb{V}_h$ and its accumulation points solve problem $(\mathcal{P})_{h,\zeta}$.

In addition, if we assume that

$$\exists r_2>0,\;\exists\tilde{\epsilon}:=\tilde{\epsilon}(\zeta,r_2)>0:\langle\Sigma^o(\varepsilon(w))\varepsilon(v),\varepsilon(v)\rangle\geq\tilde{\epsilon}|\!|\!|v|\!|\!|^2\quad\forall v,w\in\mathbb{V}_h,\,|\!|\!|u-w|\!|\!|\leq r_2,$$

where u is a solution to $(\mathcal{P})_{h,\zeta}$, then this solution is unique and local convergence properties of the method in terms of displacements can be proven for any $\rho\in[0,1]$. In particular, for $\rho=0$, one can prove local quadratic convergence.

Remark 1 One can choose $u^0=0$ for the initialization of ALG1. On the other hand, if we are interested in the loading process, it is reasonable to choose $u^0=u_{\tilde{\zeta}}$, where $\tilde{\zeta}<\zeta$ and $u_{\tilde{\zeta}}$ is a numerical solution to $(\mathcal{P})_{h,\tilde{\zeta}}$. Further in the next part (see [3]), we shall consider and compare different stopping criteria, such as

$$\frac{|\!|\!|\delta u^k|\!|\!|}{|\!|\!|u^k+\delta u^k|\!|\!|+|\!|\!|u^k|\!|\!|}<\epsilon_u,\;\frac{\|\Sigma(e^k+\delta e^k)-\Sigma(e^k)\|_E}{\|\Sigma(e^k+\delta e^k)\|_E+\|\Sigma(e^k)\|_E}<\epsilon_\sigma\text{ or }\frac{|\mathcal{J}_\zeta(u^{k+1})-\mathcal{J}_\zeta(u^k)|}{|\mathcal{J}_\zeta(u^{k+1})|+|\mathcal{J}_\zeta(u^k)|}<\epsilon_J.$$

3.2 Augmented Lagrangian method

This method has been used and analyzed by Fortin and Glowinski for numerical solution of a large class of problems with classical boundary conditions (see [8]). The method is based on releasing the constraint $e=\varepsilon(v)$ by using the augmented Lagrangian

$$\mathcal{H}_{\zeta,r}(v,e,\tau)=\Psi(e)-\zeta L(v)+\langle\tau,\varepsilon(v)-e\rangle+\frac{r}{2}\langle C(\varepsilon(v)-e),\varepsilon(v)-e\rangle,$$

where $r \geq 0$ is a penalty parameter. The related saddle-point problem $(\mathcal{S})_{h,\zeta,r}$ reads as follows: find a triplet $(u, \overline{e}, \sigma) \in K_h \times S_h \times S_h$ such that

$$\mathcal{H}_{\zeta,r}(u, e, \tau) \leq \mathcal{H}_{\zeta,r}(u, \overline{e}, \sigma) \leq \mathcal{H}_{\zeta,r}(v, e, \sigma) \quad \forall (v, e) \in K_h \times S_h \quad \forall \tau \in S_h. \tag{6}$$

It is known that any triplet $(u, \varepsilon(u), \sigma)$, where $u,\ \sigma$ solves $(\mathcal{P})_{h,\zeta}$, and $(\mathcal{P})^*_{h,\zeta}$, respectively, is a solution of (6), where $\zeta \leq \zeta_{lim}$.

We now introduce the algorithm ALG2 (in terminology of [8]). It is an Uzawa type algorithm with a separate minimization of $\mathcal{H}_{\zeta,r}$ with respect to each of the first two variables:

ALG2: $(e^0, \sigma^0) \in S_h \times S_h$ given; set $k := 0$

Find $(u^{k+1}, e^{k+1}, \sigma^{k+1}) \in K_h \times S_h \times S_h$ as follows:

Step 1 $\quad \mathcal{H}_{\zeta,r}(u^{k+1}, e^k, \sigma^k) = \arg\min \left\{ \mathcal{H}_{\zeta,r}(v, e^k, \sigma^k),\ v \in K_h \right\}$

Step 2 $\quad \mathcal{H}_{\zeta,r}(u^{k+1}, e^{k+1}, \sigma^k) = \arg\min \left\{ \mathcal{H}_{\zeta,r}(u^{k+1}, e, \sigma^k),\ e \in S_h \right\}$

Step 3 $\quad \sigma^{k+1} = \Sigma(e^{k+1})$

set $k := k + 1$ and go to Step 1.

Step 1 is equivalent to the following variational inequality in K_h:

$$\left. \begin{array}{l} \text{Find } u^{k+1} \in K_h \text{ such that} \\ \langle C\varepsilon(u^{k+1}), \varepsilon(v - u^{k+1}) \rangle \geq L_{r,\zeta,k}(v - u^{k+1}) \quad \forall v \in K_h, \end{array} \right\} \tag{7}$$

where

$$L_{r,\zeta,k}(v) := \frac{\zeta}{r} L(v) - \frac{1}{r} \langle \sigma^k, \varepsilon(v) \rangle + \langle C e^{k+1}, \varepsilon(v) \rangle.$$

Problem (7) is nothing else than the weak formulation of a frictionless contact problem for elastic bodies in which only the right hand side $L_{r,\zeta,k}$ changes during the iteration process. Further, problems in Step 2 and 3 can be solved explicitely (see [8]). In particular,

$$e^{k+1} = \frac{1}{r} \left((1 + r)\omega^k - C^{-1}\Sigma(\omega^k) \right), \quad \omega^k = \frac{C^{-1}\sigma^k + r\varepsilon(u^{k+1})}{1 + r},$$

and

$$\sigma^{k+1} = \Sigma(e^{k+1}) = \sigma^k + rC(\varepsilon(u^{k+1}) - e^{k+1}).$$

It can be shown that if a saddle-point $(u,\ \varepsilon(u),\ \sigma)$ of $\mathcal{H}_{\zeta,r}$ on $K_h \times S_h \times S_h$ exists then the sequence $\left\{ \sigma^k \right\}$ tends to σ, i.e. $\sigma^k \to \sigma$ as $k \to \infty$. Moreover σ is the solution to $(\mathcal{P})^*_{h,\zeta}$.

4 NUMERICAL SOLUTION OF $(\mathcal{P})_h^\alpha$ AND $(\mathcal{S})_h^\alpha$

In this section, we briefly present Newton-like methods for solving $(\mathcal{P})_h^\alpha$ and $(\mathcal{S})_h^\alpha$. In the case of $(\mathcal{S})_h^\alpha$ we confine ourselves to the classical boundary conditions (without contact). We will use notation introduced in the previous sections.

4.1 Numerical solution of $(\mathcal{P})_h^\alpha$

Let us recall the definition of $(\mathcal{P})_h^\alpha$:

$$(\mathcal{P})_h^\alpha \qquad \begin{cases} \text{find } u \in K_h^\alpha : \quad \Psi(\varepsilon(u)) \leq \Psi(\varepsilon(v)) \quad \forall v \in K_h^\alpha, \\ K_h^\alpha := \{v \in K_h \mid \ L(v) = \alpha\}. \end{cases}$$

We describe the algorithm ALG3. Its k-th step reads as follows: If $u^k \in K_h$ is known then

$$u^{k+1} = u^k + \delta u^k \in K_h,$$

where $\delta u^k \in K_{h,k}^\alpha$ is a solution to the minimization problem

$$\min \left\{ \frac{1}{2} \langle \Sigma^{o,\rho}(\varepsilon(u^k))\varepsilon(\delta v), \varepsilon(\delta v)\rangle + \langle \Sigma(\varepsilon(u^k)), \varepsilon(\delta v)\rangle, \quad \delta v \in K_{h,k}^\alpha \right\},$$

where

$$K_{h,k}^\alpha := \{\delta v \in \mathbb{V}_h \mid u^k + \delta v \in K_h^\alpha, \ L(\delta v) = \alpha - L(u^k)\}.$$

We will also define

$$\zeta^k = \frac{1}{\alpha} \langle \Sigma(\varepsilon(u^k)), \varepsilon(u^k)\rangle.$$

The algorithm starts from some $u^0 \in K_h$. If we are interested in the loading process, then we choose u^0 as a solution to $(\mathcal{P})_h^{\tilde{\alpha}}$, where $\tilde{\alpha} < \alpha$. Notice that such an initialization does not belong to K_h^α. On the other hand, $u^k \in K_h^\alpha$ for any $k \geq 1$. One can also use damping for $k = 1, 2, \ldots$, similarly as in Subsection 3.1.

4.2 Numerical solution of $(\mathcal{S})_h^\alpha$

In this case, we restrict ourselves to the classical boundary conditions. Then $K_h = \mathbb{V}_h$, i.e. the whole space $\mathbb{V}_h$ is used. Problem $(\mathcal{S})_h^\alpha$ leads to the following system of non-linear equations:

$$(\mathcal{S})_h^\alpha \qquad \text{find } (\zeta, u) \in \mathbb{R} \times \mathbb{V}_h : \begin{cases} \langle \Sigma(\varepsilon(u)), \varepsilon(v)\rangle = \zeta L(v) \quad \forall v \in \mathbb{V}_h, \\ L(u) = \alpha. \end{cases}$$

This problem will be solved by ALG4, whose the k-th step reads as follows: given $(\zeta^k, u^k) \in \mathbb{R} \times \mathbb{V}_h$, find $(\delta\zeta^k, \delta u^k) \in \mathbb{R} \times \mathbb{V}_h$ such that

$$\begin{cases} \langle \Sigma^{o,\rho}(\varepsilon(u^k))\varepsilon(\delta u^k), \varepsilon(v)\rangle = (\zeta^k + \delta\zeta^k)L(v) - \langle \Sigma(\varepsilon(u^k)), \varepsilon(v)\rangle \quad \forall v \in \mathbb{V}_h, \\ L(u^k + \delta u^k) = \alpha \end{cases}$$

and set

$$u^{k+1} = u^k + \delta u^k, \quad \zeta^{k+1} = \zeta^k + \delta\zeta^k.$$

One can easily find $\delta u^k, \delta\zeta^k$. Indeed,

$$\delta u^k = v^k + \zeta^k w^k, \quad \zeta^k = \frac{\alpha - L(u^k + v^k)}{L(w^k)},$$

where $v^k, w^k \in \mathbb{V}_h$ solve the following systems of linear equations:

$$\begin{aligned} \langle \Sigma^{o,\rho}(\varepsilon(u^k))\varepsilon(v^k), \varepsilon(v)\rangle &= \zeta^k L(v) - \langle \Sigma(\varepsilon(u^k)), \varepsilon(v)\rangle \quad \forall v \in \mathbb{V}_h, \\ \langle \Sigma^{o,\rho}(\varepsilon(u^k))\varepsilon(w^k), \varepsilon(v)\rangle &= L(v) \quad \forall v \in \mathbb{V}_h. \end{aligned}$$

A similar algorithm is also typical for the arc-length methods, see e.g. [7]. A generalization of this algorithm to the contact boundary conditions seems to be more involved.

5 CONCLUSIONS

In this contribution the discretized contact problem for elasto-perfectly plastic bodies has been investigated. We analyzed this problem and proposed a way how to evaluate the loading path using the relation between the load parameter ζ and the parameter α representing the work of external forces. Finally, several numerical methods for solving this class of problems were mentioned.

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