ELASTOPLASTIC STRESS ANALYSIS OF FUNCTIONALLY GRADED DISC UNDER INTERNAL PRESSURE– COMPLAS XII

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Abstract. The study deals with elastoplastic stress analysis of a hollow disk made of functionally graded materials (FGMs) subjected to an internal pressure. The material properties of disc are assumed to vary radially according to power law function, but Poisson's ratio is taken constant. Small deformations and a state of plane stress are presumed, and the analysis of disk is based on Von-Mises yield criterion. The materials are assumed to be linear strain hardening, isotropic and not be affected by temperature. Variation of stresses and displacements according to gradient parameters are investigated by using analytical and finite element method. The results show that gradient parameters have an important role in determining the elastoplastic stress of functionally graded disc.

1 INTRODUCTION

Functionally graded materials (FGMs) are provided a spatial variation in composition and properties, as an alternative to homogenous a bi-material interface structures which are used in a broad array of applications that range from aerospace structures and cutting tools to electronics and biomedical engineering [1]. Here, we investigate a new application of FGM for designing pressured discs which are used in rotors, turbines, jet engines, flywheels, automobiles, pumps, compressors and many other applications [2].

The analysis of elastic stresses distribution has been investigated by many authors and researchers. However, this elastoplastic and residual stresses distribution does not exist for pressured disc. Hassani et al.[2, 3]obtained the analytical solutions of rotating annular disk with non-uniform thickness and material properties subjected to thermo-elasto-plastic loadings is solved using homotopy analysis method (HAM) as an analytic solution and also finite element method. Widjaja et al. [4] using finite element analysis (FEA) was performed to investigate the effects of different cooling rates and substrate preheating process on the residual stress distribution. The results show that lower cooling rate and substrate preheating process reduce stresses within duplex coating. Nie and Batra [5] use the Airy stress function to derive exact solutions for plane strain deformations of a functionally graded hollow cylinder with the inner and the outer surfaces subjected to different boundary conditions, and the cylinder composed of an isotropic and incompressible linear elastic material. They investigated tailoring material properties for producing the desired stress distribution in a given body and under prescribed boundary conditions. Kurşun [6] studied elastoplastic stress analysis of functionally graded disc subjected mechanical, thermal and thermomechanical loads. He obtained elastic, elastoplastic and residual stress distribution along the radius.

In this paper, annular disk with non-uniform material properties subjected to internal pressure load is solved using the infinitesimal deformation theory of elasticity as an analytic solution and also finite element method. Results obtained both analytical and numerical solutions are found very well consistent with each other.

2 ELASTIC AND PLASTIC SOLUTIONS

The governing differential equation of equilibrium for an internal pressure disc is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

Due to the rotational symmetry, the strain-displacement relations are given by

$$\varepsilon_r = \frac{du}{dr}, \qquad \varepsilon_\theta = \frac{u}{r} \tag{2}$$

where u is the displacement component in the radial direction. The strain compatibility equation is

$$\varepsilon_r = \frac{d}{dr}\left(r\,\varepsilon_\theta\right) \tag{3}$$

The total radial and hoop strains of the rotating disc are

$$\varepsilon_r^{tot} = \frac{1}{E(r)}(\sigma_r - \nu\sigma_\theta) + \varepsilon_r^p \tag{4}$$

$$\varepsilon_\theta^{tot} = \frac{1}{E(r)}(\sigma_\theta - \nu\sigma_r) + \varepsilon_\theta^p \tag{5}$$

where ε_r and ε_θ are the strains in radial and tangential directions, respectively. The equation equilibrium (1) is satisfied by the stress function F defined as

$$\sigma_r = \frac{F}{r}, \qquad \sigma_\theta = \frac{dF}{dr} \tag{6}$$

Substituting equation (5) into equation (4) and (5) after using equation (4) and (5) into compatibility equation (3) yields the following governing equation

$$r^2\frac{d^2F}{dr^2} + r\left(1 - r\frac{E'(r)}{E(r)}\right)\frac{dF}{dr} + \left(\nu r\frac{E'(r)}{E(r)} - 1\right)F = E(r)\left(\varepsilon_r^p - \varepsilon_\theta^p - r\frac{d\varepsilon_\theta^p}{dr}\right) \tag{7}$$

In the elastic region of an internal pressure disc in which $\varepsilon_r^p = \varepsilon_\theta^p = 0$, equation (7) reduces to

$$r^2\frac{d^2F}{dr^2}+r\left(1-r\frac{E'(r)}{E(r)}\right)\frac{dF}{dr}+\left(\nu r\frac{E'(r)}{E(r)}-1\right)F=0 \tag{8}$$

In relation to Hencky's deformation theory that the plastic strain tensor is related to the deviatoric part of stress tensor is [17]

$$\varepsilon_{ij}^{p}=\phi S_{ij} \tag{9}$$

in which S_{ij} is the deviatoric stress tensor and ϕ is a scalar valued function which are obtained as

$$S_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}S_{ij} \tag{10}$$

$$\phi=\frac{3\varepsilon_{eq}^{p}}{2\sigma_{eq}} \tag{11}$$

here ε_{eq}^{p} is the equivalent plastic strain which depends on the material model used. In this study, an elastic-linear strain hardening model is used for modeling the stress–strain curve of the disk material. By using this model strains are obtained as

$$\begin{aligned}&\varepsilon=\frac{\sigma}{E(r)} && \text{for } \sigma\le\sigma(r)_y\\ &\varepsilon=\frac{\sigma(r)_y}{E(r)}+\frac{1}{Et(r)}\left(\sigma-\sigma(r)_y\right) && \text{for } \sigma>\sigma(r)_y\end{aligned} \tag{12}$$

where $E_{(r)}$, $\sigma_{(r)y}$, and $E_{t(r)}$ are the elasticity modules, yield strength and tangent modulus of the material, respectively. By using equation (9), the radial and hoop plastic strains of the disc can be calculated as

$$\varepsilon_{r}^{p}=\frac{\varepsilon_{eq}^{p}}{\sigma_{eq}}\left(\sigma_r-\frac{1}{2}\sigma_\theta\right) \tag{13}$$

$$\varepsilon_{\theta}^{p}=\frac{\varepsilon_{eq}^{p}}{\sigma_{eq}}\left(\sigma_\theta-\frac{1}{2}\sigma_r\right) \tag{14}$$

the equivalent plastic strain ε_{eq}^{p} is

$$\begin{aligned}&\varepsilon_{eq}^{e}=\frac{\sigma}{E(r)}\;\varepsilon_{eq}^{p}=0 && \text{for } \sigma\le\sigma(r)_y\\ &\varepsilon_{eq}^{e}=\frac{\sigma}{E(r)}\;\varepsilon_{eq}^{p}=\left(\frac{1}{Et(r)}-\frac{1}{E(r)}\right)+\left(\sigma-\sigma(r)_y\right) && \text{for } \sigma>\sigma(r)_y\end{aligned} \tag{15}$$

In which ε_{eq}^{e} is the equivalent elastic strain. In the plastic zone of the disk, ε_{eq}^{p} derived from equation (15) and substituting into the equations (13) and (14) results in

$$\varepsilon_r^p = \frac{\sigma_{eq} - \sigma(r)_y}{\sigma_{eq}}\left(\frac{1}{Et(r)} - \frac{1}{E(r)}\right)\left(\sigma_r - \frac{1}{2}\sigma_\theta\right) \tag{16}$$

$$\varepsilon_\theta^p = \frac{\sigma_{eq} - \sigma(r)_y}{\sigma_{eq}}\left(\frac{1}{Et(r)} - \frac{1}{E(r)}\right)\left(\sigma_\theta - \frac{1}{2}\sigma_r\right) \tag{17}$$

when equations (16) and (17) are substituted into equation (7), differential equation of the elastic-linear strain hardening disc in the plastic region.

$$r^2\frac{d^2F}{dr^2} + r\left(1 - r\frac{E'(r)}{E(r)}\right)\frac{dF}{dr} + \left(\nu r\frac{E'(r)}{E(r)} - 1\right)F = \tag{18}$$

$$+E(r)\begin{pmatrix} \dfrac{\sigma_{eq} - \sigma(r)_y}{\sigma_{eq}}\left(\dfrac{1}{Et(r)} - \dfrac{1}{E(r)}\right)\left(\sigma_r - \dfrac{1}{2}\sigma_\theta\right) \\ -\dfrac{\sigma_{eq} - \sigma(r)_y}{\sigma_{eq}}\left(\dfrac{1}{Et(r)} - \dfrac{1}{E(r)}\right)\left(\sigma_\theta - \dfrac{1}{2}\sigma_r\right) \\ -r\left(\dfrac{\sigma_{eq} - \sigma(r)_y}{\sigma_{eq}}\left(\dfrac{1}{Et(r)} - \dfrac{1}{E(r)}\right)\left(\sigma_\theta - \dfrac{1}{2}\sigma_r\right)\right)\dfrac{d}{dr} \end{pmatrix}$$

Now suppose that

$$E_{(r)} = E_0\left(\frac{r}{r_o}\right)^n, \quad E_{t(r)} = E_{t0}\left(\frac{r}{r_o}\right)^\psi, \quad \sigma_{(r)y} = \sigma_o\left(\frac{r}{r_o}\right)^\xi \tag{19}$$

here n, ψ and ξ are gradient parameters, and E_0, E_{t0} and σ_o are nominal modulus of elasticity, tangent modulus and yield strength of the material, respectively. If gradient parameters are zero, materials will be homogeneous. When substituting equation (19) into equation (8), Airy stress function in elastic region can be obtained as

$$F = C_1 r^{\frac{n+k}{2}} + C_2 r^{\frac{n-k}{2}} \tag{20}$$

C_1 and C_2 integration constants can be obtained from boundary conditions which its inner surface subject to an internal pressure and outer surface is free.

$$C_1 = \frac{-Pr_o^{\frac{n-k-2}{2}}}{r_i^{\frac{n+k-2}{2}} r_o^{\frac{n-k-2}{2}} - r_o^{\frac{n+k-2}{2}} r_i^{\frac{n-k-2}{2}}} \tag{21}$$

$$C_2 = \frac{P r_o^{\frac{n+k-2}{2}}}{r_i^{\frac{n+k-2}{2}} r_o^{\frac{n-k-2}{2}} - r_o^{\frac{n+k-2}{2}} r_i^{\frac{n-k-2}{2}}} \tag{22}$$

Here, P is an internal pressure of disc. A closed-form solution of nonlinear equation (18) with variable coefficients seems to be difficult if not impossible to obtain. Thus, in this study a code is written in ANSYS Parametric Design Language (APDL) for the solution of nonlinear equation (21) is attempted.

3 RESULTS AND DISCUSSION

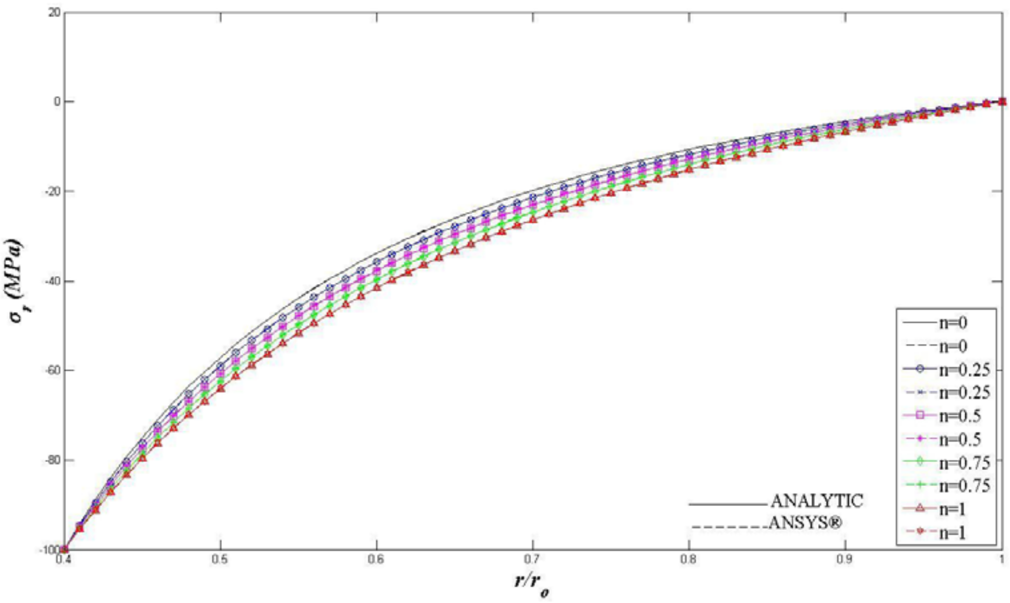
In this study, elastic and elastoplastic stress analysis are carried out on a circular disc made of functionally graded materials (FGM) by using an analytical and numerical solution. For numerical method, the disc is modeled and meshed by an axisymmetric element (Plane42 2D Structural Solid) in ANSYS®, which is a commercial finite element program [7]. A code is written in ANSYS Parametric Design Language (APDL) regarding to power law functions. In this code, it is considered that elasticity, tangent modules and yield strength of the materials vary according to equation (19). Inner and outer radii of disc are r_i=*40mm* and r_o=*100mm* with plane stress assumption, respectively. Mechanical properties of the disc such as elasticity modulus, tangential modulus and yield strength are given in Table 1.

Table 1: Mechanical properties of the disc

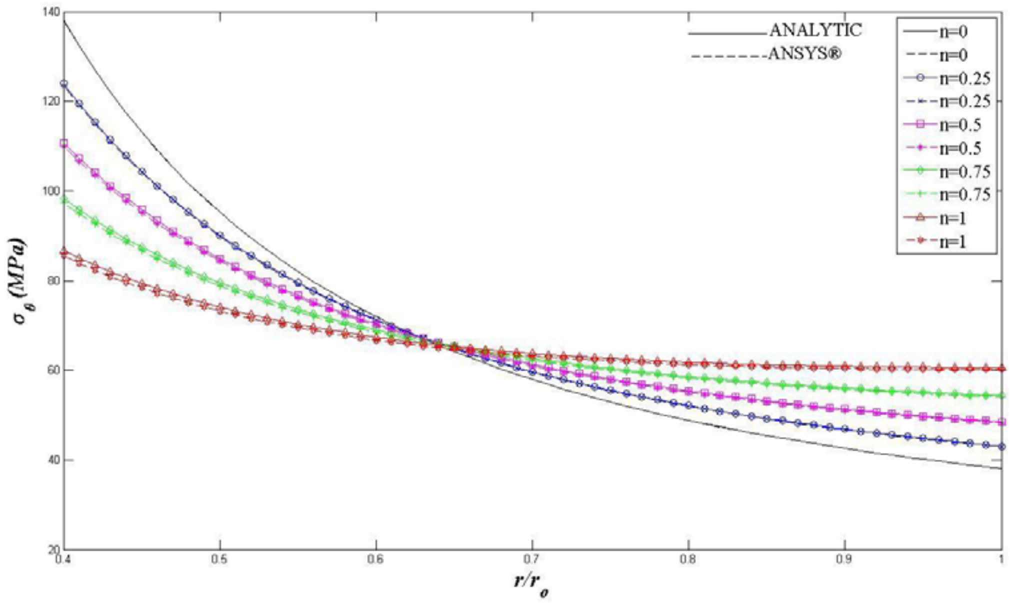
$E_o(GPa)$	$E_{to}(GPa)$	$\sigma_o(MPa)$
200	50	300

In Figure 2, the variations of radial and circumferential stresses for FGM disc subjected to an internal pressure of disc are presented. It is clear from this figure that, the analytical and numerical results are consistent very well with each other. In Figure 2 (a), Radial stress is compressive along the radius, and equal to internal pressure at inner and equal to zero at outer surface and it is the highest value for the gradient parameter equal to 1, the lowest value for homogenous material.

The circumferential stresses decrease gradually from inner to outer surface, as seen in Figure 2 (b). They take maximum values at inner surface and minimum values at outer surface as tensile. The figure 2 (b) suggests that the circumferential stresses for an internal pressure discs are smaller compared homogeneous material. For gradient parameters, zero values represent homogeneous material, but others as 0.25, 0.5, 0.75 and 1, represent FGMs. It is also seen that increment of gradient parameters uniform the circumferential stresses.



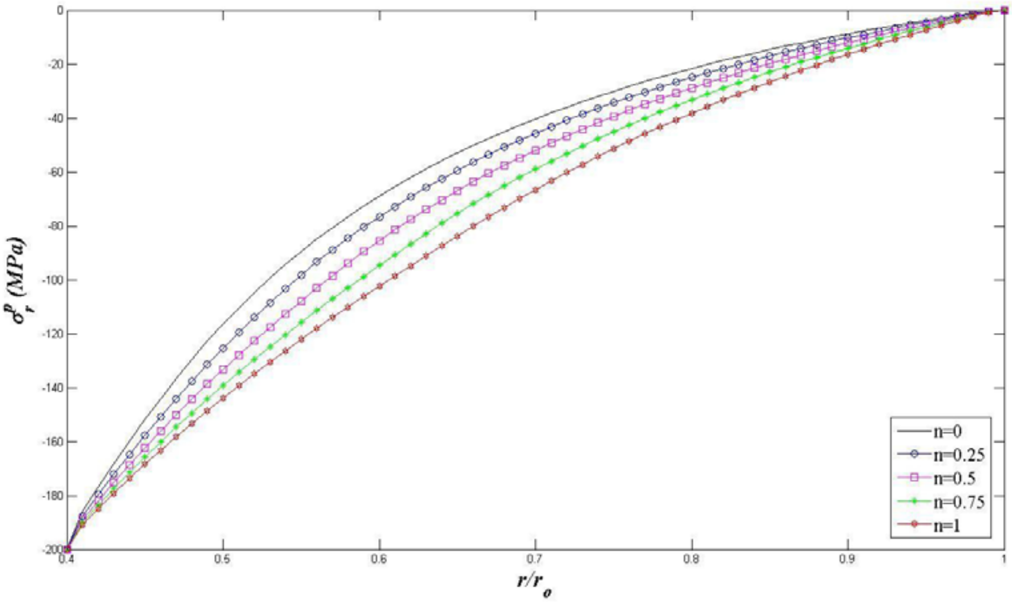
(a)



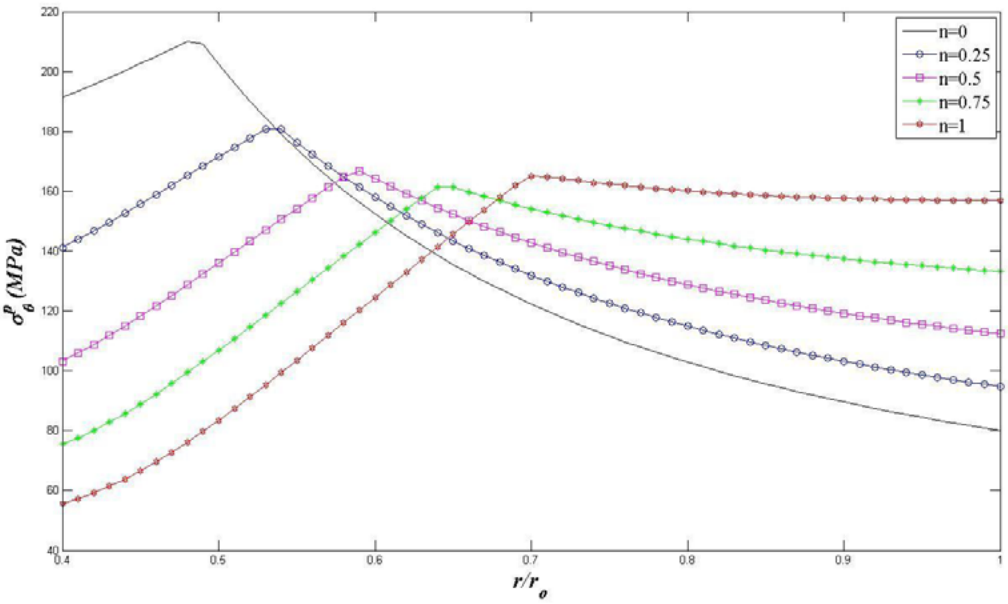
(b)

Figure 1: Elastic stress distribution along the radius for different gradient parameters, (a) radial stress and (b) circumferential stress, *P=100 MPa*.

Elastoplastic solution is obtained using by finite element method, ANSYS®. The materials are assumed to be linear strain hardening. Radial and circumferential elastoplastic stresses distribution for different gradient parameters at the disc subjected to an internal pressure, *P=200 MPa* are shown in Figure 2 (a) and (b). In Figure 4 (a), the maximum radial elastoplastic stresses are found for FGMs *n=1*, and the minimum is found for homogenous material *n=0*, as seen Figure 4 (a).



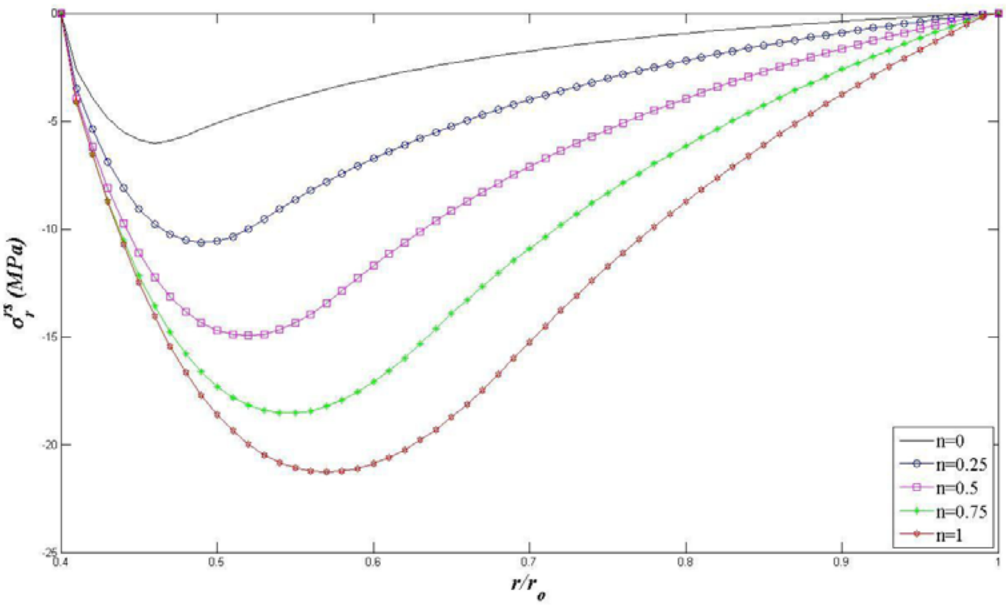
(a)



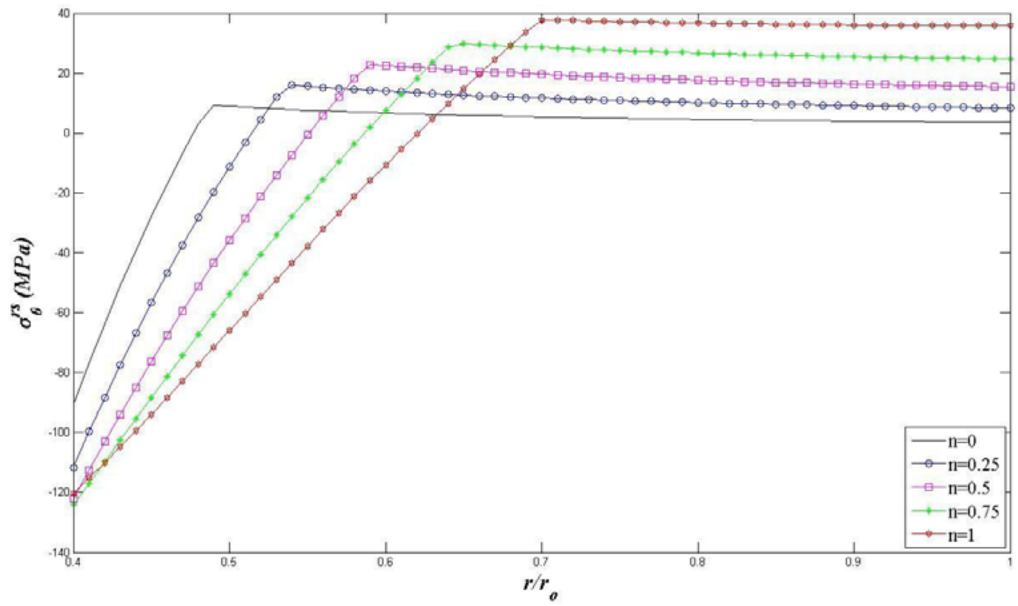
(b)

Figure 2: Elastoplastic stress distribution along the radius for different gradient parameters, (a) radial stress and (b) circumferential stress, *P=200 MPa*.

As it can be seen from Figure 4 (b), yield starts at inner surface of the disc for all gradient parameters. Plastic region is found maximum for maximum gradient parameter, and it is decreased at what time the gradient parameter is decreased as well. The circumferential elastoplastic stresses are found maximum at inner surface by means of tensile.



(a)



(b)

Figure 3: Residual stress distribution along the radius for different gradient parameters, (a) radial stress and (b) circumferential stress, *P=200 MPa*.

Residual stresses are determined by superposing a completely elastic system on the stress system obtained from the elastoplastic solution. As seen Figure 3, disc can be divided in two regions as first plastic region and second elastic region from inner to outer surface. Residual radial stresses are zero at inner and outer radius and compressive at left behind of the disc, as seen Figure 3 (a). It is clear from this figure that radial and circumferential stresses are the highest value for the highest gradient parameter.

4 CONCLUSIONS

- Gradient parameters effect the variation of stresses along the radius of the disc.
- Plastic yielding starts first at the inner surface where σ_θ is the greatest.
- The magnitude of the plastic flow is found to be highest at the inner surface for the highest gradient parameter *n=1*.

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