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COMBINED CREEP AND PLASTIC ANALYSIS WITH NUMERICAL METHODS

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Abstract. The combination of plastic and creep analysis formulation are developed in this paper. The boundary element method and the finite element method are applied in plates in order to do the numerical analysis. This new approach is developed to combine the constitutive equation for time hardening creep and the constitutive equation for plasticity, which is based on the von Mises criterion and the Prandtl-Reuss flow. The implementation of creep strain in the formulation is achieved through domain integrals. The creep phenomenon takes place in the domain which is discretized into quadratic quadrilateral continuous and discontinuous cells. The creep analysis is applied to metals with a power law creep for the secondary creep stage. Results obtained for three models studied are compared to those published in the literature. The obtained results are in good agreement and evinced that the Boundary Element Method could be a suitable tool to deal with combined nonlinear problems.

1 INTRODUCTION

For many years the Finite Element Method (FEM) has been used as the main tool to solve problems in engineering [1]. The domain of the body is divided into several small subdomains, of quite simple shape, called finite elements. Any continuous parameter such as pressure or displacement can be approximated to the actual behavior of the solution with trial functions, usually polynomials. These functions are uniquely defined in terms of the approximated values of the solution at some nodal points, inside or on the boundary of each element. A weighted residual technique is the most popular tool to assess this approximation, leading to a symmetric system of equations which involves the unknown values of the approximated solution at nodal points. Without doubt, this method is computationally efficient and during many years has reached such popularity that a very wide range of linear and non-linear engineering problems have been solved with this powerful numerical method [2].

In many branches of science and engineering, the Boundary Element Method (BEM) has become a powerful tool for the solution of boundary value problems. The integral formulation is the foundation of the method and has been used to solve linear and nonlinear problems in finite and infinite regions. One of the first successful applications of the BEM to nonlinear problems in solid mechanics was focused to elasto-plastic flow for work- hardening materials, for both anisotropic and compressible behavior, by Swedlow and Cruse [3]. This was followed by the numerical implementation of the boundary-integral technique for planar problems of elasticity and elasto-plasticity by Riccardella [4]. Kumar and Mukherjee [5] who presented the boundary integral equation analysis of time-dependent inelastic deformation of arbitrarily shaped three-dimensional metallic bodies subjected to arbitrary mechanical and thermal loading histories. Examples of creep of thick-walled spheres, long thick-walled cylinders and rotating discs were also discussed. Another formulation for plasticity based on initial stress is due to Banerjee and Mustoe [6]. Mukherjee [7] showed an indepth treatment of problems in nonlinear solid mechanics together with several interesting fracture mechanics applications using the Boundary Element Method. Also formulations for three-dimensional, two dimensional, axi-symmetric elastoplasticity as well as viscoplasticity and bending of plates, were covered. Telles [8] showed an extensive research on the elastoplastics, viscoplasticity and creep of structural components and civil engineering structures using BEM. His work represented a wide range of benchmark problems, where important comparisons between BEM and FEM were done. Time-dependent solution was obtained by the Euler step procedure highlighting for the selection of the time step length. Brebbia et al. [9] also included elastoplasticy, viscoplasticity and wave propagation problems using BEM formulations.

On the other hand, nonlinear problems of creep bending of plates by means of the FEM were solved [10], where an initial strain approach was adapted to solve the resulting nonlinear simultaneous equations, the algorithm developed lead to a time incremental set of equations. FEM formulations for plasticity, viscoplasticity, and creep of solids under multiaxial loading conditions applied to mechanical components are due to [11, 12]. The solution of axisymmetrical thin shells considering the elasto-plastic and creep behavior using FEM is due to Xu [13], where the geometric non-linear analysis was also discussed. While elastic-plastic creep buckling analysis of circular cylindrical shells subjected to axial compression was developed by Hagihara et al. [14]. Three-dimensional elasto-plastic and creep analysis of slabs was carried out by [15], where the effects of narrow faces of slabs were emphasized. Finite element algorithm for elasto-plastic creep applied to continuum damage that allows large time increments is due to Chulya and Walker [16]. Sato et al. [17] studied the stress relaxation in an inclusion bearing material at high temperatures using FEM, here the dependence of the steady-state creep rate on inclusion aspect ratio and volume fraction has been also highlighted. Several lead-free solder alloys represented by constitutive plasticity-creep models were incorporated to FEM analyses by [18], including the thermal deformation occurred during the process. Additionally, bolt joint behavior of cast aluminum alloy by coupling creep and plasticity in finite element analysis was done by Chang and Wang [19]. Recently, plastic and creep models are been focusing to damage analyses using FEM, some of the most important technical contributions are applied to pipes subjected to super-heat and high-pressure conditions [20-22].

Moreover, BEM formulations have been extended to play an important role in the field of science and engineering, the elastoplastic boundary value problem in terms of rates was used to solve a linear complementary problem endowed with a symmetric matrix, in contrast to the traditional boundary element formulation [23]. Non linear fracture mechanics considering an improve scheme of the boundary element formulation was developed by Leitao [24]. Furthermore, works of Atluri contributed enormously to the comprehension on boundary element formulation mainly on the non linear problems [e.g. 25,26]. An important review for the application of boundary element method applied to non linear solid mechanics was done by Kane [27]. A broad range of time-dependent material non-linearity problems including creep and plasticity are due to Chandenduang [28] and Aliabadi [29]. Some researches on creep continuum damage problems with plastic effects and crack propagation damage problems using BEM are by [30,31]. Creeping analysis with variable temperature in plates applying the boundary element method was recently presented by Pineda et al. [32], here BEM showed good agreement between BEM results and experimental ones.

2 COMBINED PLASTICITY AND CREEP

For the combined plasticity and creep analysis, the creep strain will be included to the total strain modeled above for plasticity only. Now the total strain rate consists of the elastic, plastic and creep strain rates as follow:

$$\dot{\varepsilon} = \dot{\varepsilon}^{e} + \dot{\varepsilon}^{p} + \dot{\varepsilon}^{c}, \tag{1}$$

where $\dot{\varepsilon}$ is the total strain rate and $\dot{\varepsilon}^{e}$, $\dot{\varepsilon}^{p}$ and $\dot{\varepsilon}^{c}$ are the elastic, plastic and creep strain rate, respectively. The constitutive equation for time hardening creep analysis can be presented as follows:

$$\dot{\varepsilon}^{c} = \frac{3}{2} mB \; \sigma_{eq}^{(n-1)} S_{ij} t^{(m-1)}, \tag{2}$$

where *B*, *m* and *n* are the material constants which dependent on the temperature. σ_{eq} is the equivalent stress, S_{ij} is the deviatory stress and *t* is the time.

The constitutive equation for plasticity based on von Mises yield criterion and Prandtl-Reuss flow rule is

$$\dot{\varepsilon}^{p} = \frac{3}{2} \left[\frac{S_{ij} \dot{\varepsilon}_{ij}}{1 + \frac{H'}{3\mu}} \right] \sigma_{eq}^{(-2)} S_{ij}, \tag{3}$$

where H´ is the plastic hardening modulus and μ is the shear modulus.

It is assumed that the plasticity and creep analysis are separable since elasto-plasticity is a time-independent process and creep is a time-dependent process. Therefore to combine the plastic and creep analysis superposition is used.

3 DISPLACEMENT BOUNDARY INTEGRAL FORMULATION

The Somigliana's equation rate can be obtained by neglecting the body forces, substituting the Dirac delta function property and the fundamental fields (displacements, tractions and stresses) into the equilibrium equations and traction definition to give

$$\dot{u}_i = \int_\Gamma u'_{ij} \dot{t}_j \, d\Gamma - \int_\Gamma t'_{ij} \dot{u}_j \, d\Gamma + \int_\Omega \sigma'_{ij} \dot{\varepsilon}^a_{ij} \, d\Omega \quad (4)$$

The above equation computes the displacement in any internal point of the domain Ω once that we know the values of the boundary displacements and tractions as well as the anelastic deformation $\dot{\varepsilon}^a_{ij}$. For linear elastic problems $\dot{\varepsilon}^a_{ij} = 0$, but in this work this deformation will be considered.

The equation (4) is for any internal point within the domain Ω'. In order to obtain a solution for the points on the boundary it is necessary to apply the definition of the limit to Somigliana's equation when $x \rightarrow x'$ like in elasticity, see Aliabadi [29]. Here x' is any point on the boundary Γ and x represent any point in the domain Ω' This leads to the following boundary Integral representation of the *boundary displacements* when the *initial strain approach* for the solution of elastoplastic problems is used

$$c_{ij} \dot{u}_j + \int_\Gamma t'_{ij} \dot{u}_j \, d\Gamma = \int_\Gamma u'_{ij} \dot{t}_j \, d\Gamma + \int_\Omega \sigma'_{ijk} \dot{\varepsilon}^a_{jk} \, d\Omega \quad (5)$$

On the left hand side of the equation (5), the integral $\int_\Gamma$ stands for the Cauchy principal value integral. In this equation c_{ij} is called the jump term which depends on the geometry

4 NUMERICAL IMPLEMENTATION

The numerical expression for the displacement boundary equation (5) can be written as follows:

$$c\dot{u} + \sum_{n=1}^{N_{el}} \left(\int_\Gamma T\phi d\Gamma \right) \dot{u}^n = \sum_{n=1}^{N_{el}} \left(\int_\Gamma U\phi d\Gamma \right) \dot{t}^n + \sum_{n=1}^{N_{el}} \left(\int_{\Omega_N} \sigma \Psi d\Omega \dot{\varepsilon}^{a,n} \right) \quad (6)$$

The terms T, U and σ in this equation are sub matrices containing the fundamental solution. N_{el} is the number of integration elements, ϕ and Ψ are the shape functions for the boundary and the domain respectively. It is possible to represent in matrices every term of the equation (36) like:

$$H = \sum_{n=1}^{N_{el}} \left(\int_\Gamma T \phi \, d\Gamma \right) \dot{u}^n$$

$$G = \sum_{n=1}^{N_{el}} \left(\int_{\Gamma} U \phi \, d\Gamma \right) t^{n}$$

$$W = \sum_{n=1}^{N_{el}} \left(\int_{\Omega_n} \sigma \Psi \, d\Omega \, \varepsilon^{a} \right) \quad (7)$$

These matrices are used to represent a system of equations conformed as follows:

$$H \dot{u} = G \dot{t} + W \dot{\varepsilon}^{a} \quad (8)$$

The matrices *H* and *G* are the generalized displacement and tractions elastic contribution and the matrix *W* is the inelastic strain influence over the discretized domain.
After applying boundary conditions we rearrange the system and collecting all the unknowns in the left-hand side of the equation (8) the following expression is obtained:

$$A \dot{Y} = \dot{F} + W \dot{\varepsilon}^{a} \quad (9)$$

Where the matrix *A* corresponds to the system of matrices derived from *G* and *H*, $\dot{Y}$ represents the vector of the unknowns. $\dot{F}$ is the known values vector on the boundary. By taking the *A* matrix to the right-hand side of the equation (9) and considering the initial strain approach, this can be rearranged to give

$$\dot{Y} = A^{-1}\dot{F} + A^{-1}W\,(\varepsilon^{a} + \Delta\varepsilon^{a}) \quad (10)$$

The solution of the equation (10) represents the displacements on the boundary. The same procedure, above described, is applied for the internal stress equation.

5 BENCHMARK PROBLEMS FOR COMBINED PLASTICITY AND CREEP

7.1 Square Plate

This example corresponds to a square plate (Figure 1), subjected to a uniaxial tensile stress of 300 MPa in *x*-direction the square plate has the dimension of 100 mm. It is assumed linear hardening in this case. An initial time step of 10^{-3} is used together with an automatic time step control with maximum and minimum creep strain tolerances of 10^{-4} and 10^{-5} respectively. This test is secondary creep and plane stress problem which is performed for the total time of 1 hour for a full load approach. The material properties are as follows: E= 207000 MPa, $\nu = 0.3$, applied stress $\sigma_a = 300$ MPa, yield stress $\sigma_y = 250$ MPa, hardening coefficient $H' = 4223.8267$ MPa, while creep parameters are B=3.124x10^{-14} MPa/h, m=1 (for secondary creep) and n=5.

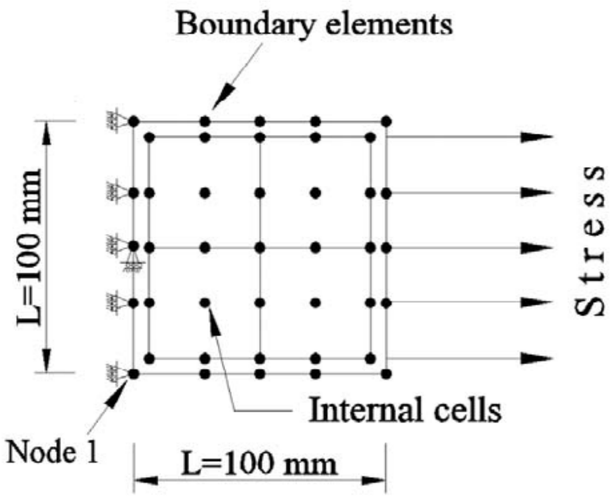

Figure 1. Square Plate with internal 9 node cells.

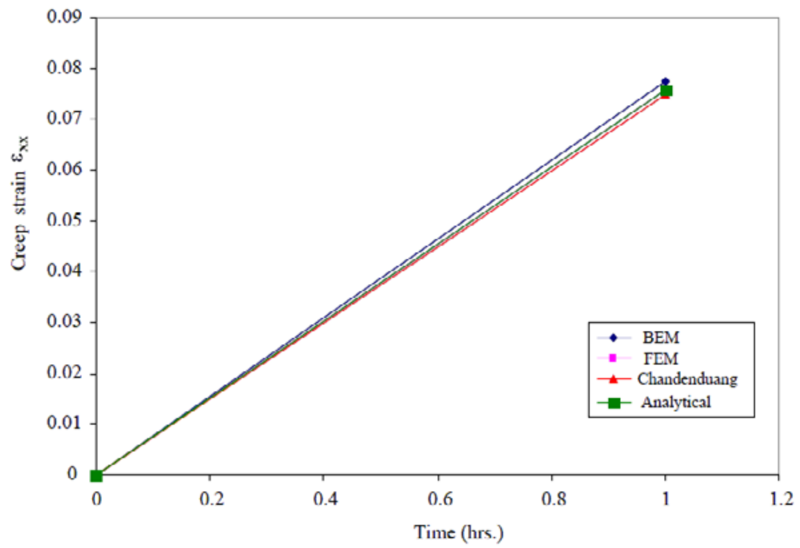

Figure 2. Comparison for creep strain in x direction, with different methods.

The results for creep strains in *x* direction is shown in Figure 2. The creep strains are calculated in the first node of the plate. These results are compared to Chandenduang [28], which includes the analytical solutions and the corresponding finite element solutions. The results are in good agreement with both the analytical solutions and the finite element solutions. The error compared to the finite element solution and analytical solutions is less than 2%. It is found that the finite element solutions are very close to the analytical solutions.

7.2 Plate with a Circular Hole

A tensile stress of 12 N/mm2 in the y-direction is applied in a plate with a circular hole. Because of the symmetry half of the plate is modeled to do the analysis (Figure 3). The material properties are E= 7000 kg/mm2, $\nu = 0.2$, yield stress $\sigma_y = 24.3$ kg/mm2, hardening coefficient $H' = 224.2$ kg/mm2, while creep parameters are B=3.125x10^{-10} MPa/h, m=1 (for secondary creep) and n=5. An initial time step of 10^{-3} is used together with an automatic time step control with maximum and minimum creep strain tolerances of 10^{-3} and 10^{-4} respectively. This test is secondary creep and plane stress problem which is performed for the total time of 1 hour for a full load approach.

The results for the stress distribution in y-direction are presented in Figure 4. Here x/r is the ratio of the distance along the root in the x-direction to the hole radius. These results agree well with the finite element solutions from [28].

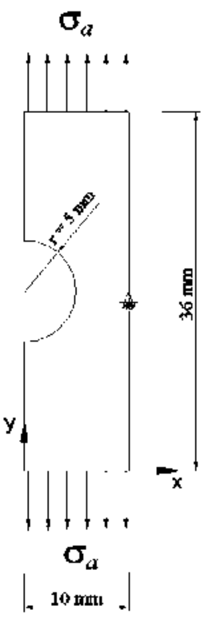

Figure 3. Plate with a circular hole for tensile stress.

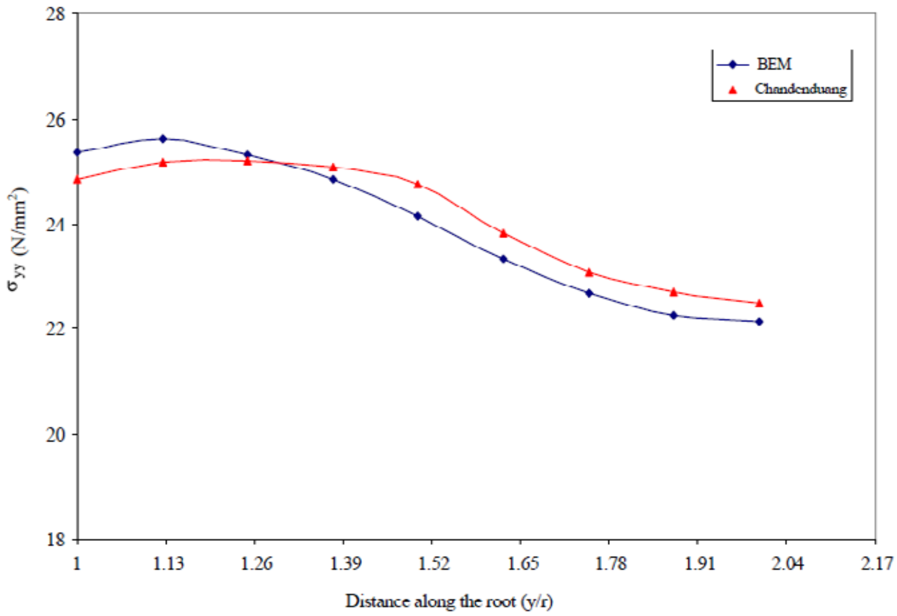

Figure 4. Results for stress distribution in y direction of the circular plate.

8. CONCLUSION

This work presented a formulation to performance a combined plastic and creep analyses in 2 D plates using the Boundary Elements Method. This new approach is develop to combine the constitutive equation for time hardening creep and the constitutive equation for plasticity, which is based on the von Mises criterion and the Prandtl-Reuss flow. It has been shown that plasticity and creep analysis can be combined and, then, it is possible to state the total strain rate as a function of the elastic, plastic and creep strain rates. Results obtained by this approach were verified against those published by Chandenduang [28] and finite element results. For the case of a square plate and a plate with a circular hole, results match very well. On the other hand, for the case of a plate with a semi-circular notch, results show slight differences having an error nearly to 3 %.

These results evince the possibility of the BEM´s methods to deal properly with non-linear problems, including those that represent combined effects of two rheological behaviors.

9. REFERENCES

[1] Zienkiewicz OC. The Finite Element Method. New York: McGraw-Hill; 1971.
[2] Oden JT. Finite Elements of Nonlinear Continua. New York: McGraw-Hill; 1972.
[3] Swedlow JL, Cruse TA. Formulation of the boundary integral equation for three-dimensional elastoplastic flow. International Journal of Solids and Structures 1971;7:1673-1681.

[4] Riccardella P. An Implementation of the Boundary Integral Technique for plane problems of Elasticity and Elastoplasticity. PhD Thesis, Carnegie Mellon University, Pitsburg, PA 1973.
[5] Kumar V, Mukherjee S. A boundary-integral equation formulation for time-dependent inelastic deformation in metals. International Journal of Mechanical Sciences 1977; 19:713-724.
[6] Banerjee PK, Mustoe GCW. The boundary element method for two-dimensional problems of elasto-plasticity. Recent Advances in Boundary Element Methods, C. A.Brebbia(ed.). Plymouth, Devon, UK: Pentech Press; 283-300; 1978.
[7] Mukherjee SA. Boundary Element Methods in Creep and Fracture. Barking Essex, England; Applied Science Publishers LTD 1982.
[8] Telles JCF. The Boundary Element Method Applied to Inelastic Problems, Springer-Verlag, Berlin 1983.
[9] Brebbia CA, Telles JCF, Wrobel LC. Boundary Element Techniques, theory and application in engineering, Springer-Verlag, Berlin 1984.
[10] Hamza H, Muggeridge DB. Plastic creep bending analysis of plates. *Journal of the Engineering Mechanics Division* 1982, 108: 1252-1261.
[11] Pian THH. Plasticity, viscoplasticity, and creep of solids by mechanical subelement models. *Numer Methods in Coupled Syst* 1984, 119-126.
[12] Kaufman A. Simplified method for elastic-plastic-creep structural analyses. NASA Technical Memorandum,1984.

[13] Xu J. Elasto-plastic and creep analysis of axisymmetrical thin shells by the finite element method, *Journal of Northwestern Polytechnical University*, 1987, 5: 49-60.
[14] Hagihara S, Miyazaki N, Munakata T. Elastic-plastic creep buckling analysis of circular cylindrical shells subjected to axial compression, *American Society of Mechanical Engineers, Pressure Vessels and Piping Division* 1989, 175: 117-122.
[15] Okamura K, Kawashima H. Three-dimensional elasto-plastic and creep analysis of bulging in continuously cast slabs, *ISIJ International*, 1989, 29: 666-672.
[16] Chulya A, Walker KP. New uniformly valid asymptotic integration algorithm for elasto-plastic creep and unified viscoplastic theories including continuum damage, *International Journal for Numerical Methods in Engineering*, 1991, 32: 385-418.
[17] Sato E, Ookawara T, Kuribayashi K, Kodama S. Steady-state creep in an inclusion bearing material by plastic accommodation, *Acta Materialia* 1998, 46: 4153-4159.
[18] Ohguchi K, Sasaki K, Ishibashi M, Hoshino T. Plasticity-creep separation method for viscoplastic deformation of lead -free solders, *JSME International Journal*, Series A: Solid Mechanics and Material Engineering 2004, 47: 371-379.
[19] Chang CC, Wang QG. Modeling of bolt joint behavior of cast aluminum alloy (A380-T5) by coupling creep and plasticity in finite element analysis, *Metallurgical and Materials Transactions B*: Process Metallurgy and Materials Processing Science 2007, 38: 607-613.
[20] Saï K, Laiarinandrasana L, Naceur IB, Besson J, Jeridi M, Cailletaud G. Multi-mechanism damage-plasticity model for semi-crystalline polymer: Creep damage of notched specimen of PA6, *Materials Science and Engineering A* 2011, 528: 1087-1093.
[21] Hossain S, Truman CE, Smith DJ. Generation of residual stress and plastic strain in a fracture mechanics specimen to study the formation of creep damage in type 316 stainless steel, *Fatigue and Fracture of Engineering Materials and Structures* 2011, 34: 654-666.

[22] Guo X, Lu H, Xu W. Finite element analysis of Thermo-elastic-plastic creep of super-heat and High-pressure steam pipe, *Key Engineering Materials* 2011, 458:119-124.
[23] Maier G, Polizzotto C. Galerkin approach to boundary element elastoplastic analysis, *Computer Methods in Applied Mechanics and Engineering* 1987, 60: 175-194.
[24] Leitao V. An Improved Boundary Element Formulation for Nonlinear Fracture Mechanics I, PhD Thesis, Wessex Institute of Technology, University of Porsmouth, 1993.
[25]Okada H, Rajiyah H, Atluri SN. A full tangent stiffness field boundary element formulation for geometric and material non linear problems in solid mechanics, *International Journal for Numerical Methods in Engineering* 1990, 29: 15-35.
[26] Okada H, Atluri S N. Recent developments in the field boundary element method for finite/small strain elastoplasticity, *Int. J. Solids Structures* 1994, 31: 1737-1775.
[27] Kane JH. Boundary Element Analysis in Engineering Continuum Mechanics, Prentice Hall, New Jersey, 1994.
[28] Chandenduang C. Boundary Element Analysis of Time-Dependent Material Non-Linearity, University of Nottingham (Ed), 1999.
[29] Aliabadi MH. The Boundary Element Method. Applications in Solids and Structures. Vol. 2. West Sussex, England: JohnWiley & Sons, Ltd 2002.
[30] Kebir H. Dual boundary element method modelling of aircraft structural joints with multiple site damage, *Engineering Fracture Mechanics* 2006, 73: 418-434.

[31] Gun H. Two-dimensional boundary element analysis of creep continuum damage problems with plastic effects, *Computational Materials Science* 2007, 41: 322-329.
[32] Pineda LE, Samayoa OD, Rodríguez-Castellanos A. Aliabadi MH, Avalos GE, Olivera-Villaseñor E. Creeping analysis with variable temperature applying the boundary element method, *Engineering Analysis with Boundary Elements* 2012, 36: 1715-1720.
[33] Telles JCF, Brebbia C. On the application of the boundary element method to plasticity, *Appl. Math. Modelling* 1979, 3: 466-470).
[34] Leitao V, Aliabadi MH, Rooke DP. The Dual Boundary Element Method for Elasto-plastic Fracture Mechanics, Int. J. Num. Meth. Engng 1994, 38: 315–333.
[35] Yoshimura S, Chen KL, Atluri SN. Study of two alternate tangent modulus formulations and attendant implicit algorithms for creep as well as high-strain-rate plasticity, *International journal of plasticity* 1987, 3: 391-413.
[36] Cao J, Lin J, Dean TA. An implicit unitless error and step-size control method in integrating unified viscoplastic/creep ODE-type constitutive equations, *International Journal for Numerical Methods in Engineering* 2008, 73: 1094-1112.
[37] Gernay T, Franssen JM. A comparison between explicit and implicit modelling of transient creep strain in concrete uniaxial constitutive relationships, Conference Proceedings - Fire and Materials 2011, 12th International Conference and Exhibition, Sponsor: Fire Testing Technology (FTT); FM Global; Rockwool; Publisher: Interscience Communications Ltd, 2011.
[38] Chen GG, Hsu TR. Mixed explicit-implicit (ei) algorithm for creep stress analysis. *International Journal for Numerical Methods in Engineering* 1988, 26: 511-524.