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# Distributed Plasticity Analysis of Frame Structures in Rate Form

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**Abstract.** Distributed plasticity beam column elements are able to efficiently track hysteretic nonlinear behavior of structures under static or dynamic loading. This is accomplished by a refined discretization of the element in control sections along its length, each one being represented by a set of longitudinal fibers. The global response of the element results from a two level integration. In the first the non-linear stress of every fiber is integrated across the cross-sectional area to derive the constitutive relation of the control section and then integration along the element's length is proved sufficient to yield the current state of the element.

This work focuses on the formulation of both displacement and force based beam-column elements where the internal variables that describe the element's state, namely fiber stresses or strains are expressed in rate form, herein using Bouc-Wen hysteretic models. Both formulations are derived from a unified approach based on the two field Hellinger-Reissner potential which highlights their differences. For simplicity reasons the methodology is applied on plane frame elements based on Euler–Bernoulli kinematics.

The main advantage of expressing the evolution of each internal variable through a differential equation offers the ability to solve the entire set simultaneously with the global structure's equations of motion in state space form. Accurate solutions are derived from proper implementation of an efficient numerical ODE solver.

## 1 INTRODUCTION

Distributed plasticity models monitor plasticity in multiple sections along the element's length. These control sections are described by constitutive relations of classical plasticity in terms of stress resultants, or they are subdivided in longitudinal fibers representing a uniaxial stress-strain law [1]. The first elements of this category were based on the classical finite elements method (Bathe [2]) where the displacement field along the element is expressed with cubic polynomials. This methodology describes only constant axial force and linear curvature, which is not accurate in the plastic region where curvature is distributed non-linearly along the element. To address this problem, force based models have been proposed that interpolate nodal forces inside the element maintaining equilibrium. The thorough investigation of these models, in the general framework of the direct stiffness method following a standard non-linear finite element procedure, was performed by Spacone et al. [1]. They suggested an iterative procedure under constant displacements for the element state determination establishing compatibility. Later, Neuenhofer and Filippou [3] showed that elemental iterations are not necessary as the element stresses gradually converge while the whole structure is in equilibrium. Although these formulations have been proved very efficient algorithmically, they were not established variationally as notified by Hjelmstad and Taciroglu [4]. In their paper force based formulations are proposed based on mixed variational approaches. In the same context Alemdar and White [5] review classical displacement and force based formulation along with the mixed ones and highlight the advantages of the later. Hence, mixed methods seem to dominate recent research for the formulation of numerical strategies for the nonlinear beam problem as they are proved more efficient considering also the works of Taylor et al. [6] and Saritas and Soydas [7].

In all previous works plasticity is incorporated in its classical form by linearization of the resulting equations. On the other hand Simeonov et al. [8] have developed a force based element where material constitutive relations are considered in rate form and they are solved simultaneously in state space form with the global differential equations of motion. Jafari et al. [9] have extended this formulation in large displacement analysis. In addition Triantafyllou and Koumousis [10] proposed a generic finite element procedure where material nonlinearity is treated at the elemental level through proper implementation of the Bouc-Wen hysteretic rule. Herein, distributed plasticity beam formulations for both displacement and force based approach in state space form are developed where constitutive equations are expressed in rate form.

## 2 EULER-BERNOULLI BEAM THEORY

The basic degrees of freedom of a beam after excluding rigid body motions are obvious in Figure 1 and consist of deformations $\{q\}=\{u,\theta_1,\theta_2\}^T$ and their energy counterpart forces $\{Q\}=\{N,M_1,M_2\}^T$.

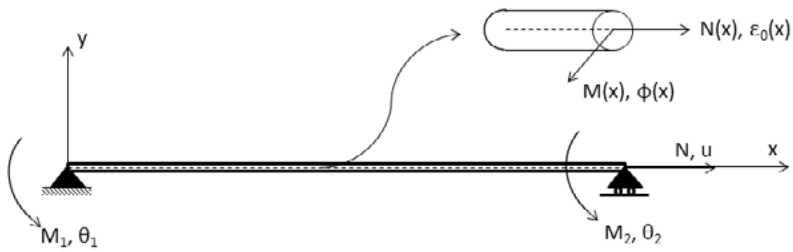


Figure 1: Local degrees of freedom of the beam element

In order to fully describe the current state of a deformable beam, in general knowledge of three fields is required, namely the displacement field *u(x,y)*, the deformation field *ε(x,y)* and the stress field *σ(x,y)*.

From compatibility and equilibrium relations that connect displacements and deformations the following relations are obtained:

$$\{q\} = T \cdot \{\bar{q}\}, \quad \{Q\} = T \cdot \{\bar{Q}\}, \qquad T = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/L & 1 & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 0 & -1/L & 1 \end{bmatrix} \tag{1}$$

where $\{\bar{q}\}$, $\{\bar{Q}\}$ are the displacements and forces at the original system and $T$ is the equilibrium transformation matrix. According to Euler-Bernoulli beam theory kinematics, the displacement field is defined by the assumption that plane sections before deformation remain plane and normal to the elastic line after deformation, thus for a beam problem we have:

$$u_1(x,y) = u(x) - y \cdot \theta(x) = u(x) - y \cdot \frac{\partial v(x)}{\partial x} \quad , \quad u_2(x,y) = v(x) \tag{2}$$

The same assumption is also valid for the deformation field. Hence, the axial strains at any point of the beam are given by the relation:

$$\varepsilon_1(x,y) = \varepsilon_0(x) - y \cdot \varphi(x) \tag{3}$$

where, the deformation field $d(x) = \{\varepsilon_0(x), \varphi(x)\}^T$ consists of the axial centerline deformation $\varepsilon_0(x)$ and curvature $\varphi(x)$. Stress field can be fully derived from the deformation field through constitutive relations $\sigma(x,y)=f(\varepsilon(x,y))$. However, the opposite is not unique, as the deformation field can be expressed both by the stress or displacement field. Euler-Bernoulli beam theory considers only the effects of axial stress $\sigma_1$, thus by applying equilibrium conditions in a cross section, the stress resultants are calculated as follows:

$$N(x) = \int_A \sigma_1(x,y)dA \quad , \quad M(x) = -\int_A y \cdot \sigma_1(x,y)dA \tag{4}$$

The same relations can be casted in matrix form as:

$$P(x) = \int_A \begin{bmatrix} 1 & -y \end{bmatrix}^T \cdot \sigma_1(x,y)dA \tag{5}$$

where $P(x) = \{N(x), M(x)\}^T$

## 4 CONSTITUTIVE BEHAVIOUR

Based on the endochronic theory of plasticity [11] the Bouc-Wen model [12] provides a robust method to describe inelastic hysteretic behavior of materials. The advantage of the model is based on its ability to incorporate the whole inelastic loading path, namely elastic loading, yielding, kinematic hardening and unloading, in a single differential equation.

The uniaxial state of every nonlinear material can be defined when internal variables expressed by evolution equations are considered along with the generalized state variables of stress and strain. Considering the Bouc-Wen model, stress can be decomposed in two parts, a reduced elastic part and a hysteretic one.

$$\sigma = \sigma_{el} + \sigma_h = \alpha \cdot E \cdot \varepsilon + (1-\alpha) \cdot E \cdot z \tag{6}$$

where $\alpha$ is the post yield stiffness to elastic stiffness ratio and $z$ is a hysteretic deformation parameter which serves as an internal variable, whose evolution with time follows the nonlinear differential equation:

$$\dot{z}(z,\dot{\varepsilon}) = (1 - h_1 \cdot h_2) \cdot \dot{\varepsilon}, \qquad h_1 = \left| \frac{z}{\varepsilon_y} \right|^n, \quad h_2 = \beta + \gamma \cdot \mathrm{sgn}(z \cdot \dot{\varepsilon}) \tag{7}$$

where $h_1, h_2$ are Heaviside type of functions that control the hysteretic behavior of the material and dot $(\cdot)$ denotes differentiation with respect of time. More specifically, $h_1$ can be regarded as the uniaxial flow rule that controls yielding, while $h_2$ controls loading-unloading situations. Parameter $n$ controls the smoothness of the transition from the elastic to the inelastic regime, while the terms $\beta$ and $\gamma$ are factors that affect the shape of the hysteresis loop.

An equivalent model was proposed by Sivaselvan and Reinhorn [13] by considering both deformation and stress as independent variables of the model. By substituting equation (7) in the rate form of equation (6) we can obtain:

$$\dot{\sigma} = \left[1 + (\alpha - 1) \cdot h_1 \cdot h_2\right] \cdot E \cdot \dot{\varepsilon} = E_{\tan} \cdot \dot{\varepsilon}, \qquad h_1 = \left| \frac{\sigma^h}{\sigma_y{}^h} \right|^n, \quad h_2 = \beta + \gamma \cdot \mathrm{sgn}(\sigma^h \cdot \dot{\varepsilon}) \tag{8}$$

where, $\sigma^h = \sigma - \alpha \cdot E \cdot \varepsilon$ and $\sigma_y{}^h = (1 - a) \cdot \sigma_y$ are the hysteretic part of the total stress and the hysteretic part of the yield stress respectively.

Also from equation (8) the tangent material modulus is given by the equation:

$$E_{\tan} = \left[1 + (\alpha - 1) \cdot h_1 \cdot h_2\right] \cdot E \tag{9}$$

By combining equations (3), (4) and (8), cross sectional tangent stiffness can be derived as:

$$k(x) = \int_A \left[1 \ \ -y\right]^T \cdot E_{\tan} \cdot \left[1 \ \ -y\right] dA \tag{10}$$

Therefore, by eliminating rates from equation (8) the standard constitutive material law results ($\sigma_1 = E_{tan} \cdot \varepsilon_0$) and substituting this equation along with equations (3) and (10) in (5) results in the cross sectional constitutive equation:

$$P(x) = k(x) \cdot d(x) \tag{11}$$

Also, the cross sectional flexibility matrix can be calculated from the inverse of the stiffness matrix:

$$f(x) = k(x)^{-1} \tag{12}$$

It turns out that the dependence of the classical Bouc-Wen model only on the deformation field makes it appropriate for a displacement based beam formulation, while the dependence of the Sivaselvan-Reinhorn model on both deformation and stress fields renders it appropriate for a force-based beam formulation.

# 5 VARIATIONAL FORMULATION

## 5.1 Two field approach

Apart from the two classical principles of virtual displacements and virtual forces more generalized ones exist that use more than one fields as independent. Herein, the mixed Hellinger-Reissner variational principle [14] which takes into account both the displacement and the stress fields as independent is employed. This hybrid method is used to derive in a unified way the element's state for the case of displacement and force based methods.

For an elastic material with stress $\sigma_1$ and strain $\varepsilon_1$ the Hellinger-Reissner functional can be stated in terms of the two independent variables of stress $\sigma_1$ and displacement $u_1$ as follows:

$$\Pi_{HR}(\sigma_1, u_1) = \int_V \left\{ \frac{\partial u_1}{\partial x_1} \cdot \sigma_1 - \chi(\sigma_1) \right\} dV - \Pi_{ext}(u) \tag{13}$$

Where $\chi(\sigma_1)$ is the complementary energy density function, $\Pi_{ext}(u)$ is the functional of external loading and integration is performed in the undeformed volume $V$ of the element. The above functional can be written also in terms of stress resultants $\{P\}$ as described in [15].

$$\Pi_{HR}(P,u) = \int_L \left\{ \begin{bmatrix} N(x) & M(x) \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial u(x)}{\partial x} \\ \dfrac{\partial^2 v(x)}{\partial x^2} \end{bmatrix} - \chi(P) \right\} dx - \{Q\}^T \cdot \{q\} \tag{14}$$

where $L$ is the undeformed length of the element and $N(x)$, $M(x)$ are given from equations (4).

In order to calculate the state of the element, stationarity of the Hellinger-Reissner functional is imposed by setting its first variation with respect to the two independent fields equal to zero.

$$\delta\Pi_{HR} = \delta_u \Pi_{HR} + \delta_P \Pi_{HR} = 0 \tag{15}$$

where $\delta_u$ is the variation with respect to the displacement field and $\delta_P$ is the variation with respect to the stress field expressed in stress resultant terms. Each of the two added terms in equation (15) is calculated as follows:

$$\delta_u \Pi_{HR} = \int_L \left\{ \{P(x)\}^T \cdot \begin{bmatrix} \dfrac{\partial \delta u(x)}{\partial x} \\ \dfrac{\partial^2 \delta v(x)}{\partial x^2} \end{bmatrix} \right\} dx - \{Q\}^T \cdot \delta\{q\} = 0 \tag{16}$$

$$\delta_P \Pi_{HR} = \int_L \delta\{P(x)\}^T \cdot \left\{ \begin{bmatrix} \dfrac{\partial u(x)}{\partial x} \\ \dfrac{\partial^2 v(x)}{\partial x^2} \end{bmatrix} - \frac{\partial \chi(P(x))}{\partial P(x)} \right\} dx \;=\; \int_L \delta\{P(x)\}^T \cdot \left\{ \begin{bmatrix} \dfrac{\partial u(x)}{\partial x} \\ \dfrac{\partial^2 v(x)}{\partial x^2} \end{bmatrix} - \{d(x)\} \right\} dx = 0 \tag{17}$$

where in the last equation the derivative of the complementary energy with respect to the stress resultants is equal to the strains $d = \{\varepsilon_0(x), \varphi(x)\}^T$ namely the axial deformation and curvature at the cross sectional centroid. Equations (16) and (17) fully describe the state of an elastic element, while with the additional consideration of proper constitutive equations and evolution equations of internal variables the state of the nonlinear element can also be defined. In particular, equation (16) corresponds to the principle of virtual work or principle of virtual displacements and is the weak form of equilibrium equations, while equation (17) corresponds to the principle of virtual complementary work or principle of virtual forces which is the weak form of the compatibility equations.

### 5.2 Displacement based derivation

In displacement based or stiffness based formulation the only independent field is the displacement field. Deformations are defined with proper differentiations of displacements and stresses are calculated using the constitutive laws. Therefore the quantity inside the bracket in equation (17) is always equal to zero. This means that in displacement based formulation compatibility equations are valid in their strong form as they are satisfied pointwise and not in an average sense.

To proceed with the formulation, the displacement field must be defined in advance. This is accomplished in classical FEM with the implementation of cubic polynomial shape functions *[N]* that relates internal with nodal displacements (*u(x)=N(x)·q*). This is the source of inaccuracies of the formulation as the outcome displacement field loses accuracy in the nonlinear case, although it is exact in the linear one. Deformations are defined as derivatives of displacements which leads to the following relation:

$$\{d(x)\} = \begin{Bmatrix} \varepsilon_0(\mathrm{x}) \\ \varphi(\mathrm{x}) \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u(x)}{\partial x} \\ \dfrac{\partial^2 v(x)}{\partial x^2} \end{Bmatrix} = \partial N(x) \cdot \{q\} = a(x) \cdot \{q\} \tag{18}$$

where *α(x)* are shape functions that relate cross sectional deformations with nodal displacements. These shape functions account for constant distribution of axial deformation along the element's length and linear distribution of curvature. Equation (16) when considering equation (18) can be written as follows:

$$\int_L \{P(x)\}^T \cdot \delta\{d(x)\} dx = \{Q\}^T \cdot \delta\{q\} \tag{19}$$

By substituting equations (11) and (18) in the above relation and after performing some algebraic calculations, we end to the basic equations that determine the state of the element, namely its current stiffness and nodal forces. The local stiffness matrix of the element is determined through integration of cross sectional stiffness over its length as:

$$K = \int_L \alpha(x)^T \cdot k(x) \cdot a(x) dx \tag{20}$$

Also, the vector of element's nodal forces is derived through integration of cross sectional forces over its length as:

$$\{Q\} = \int_L \alpha(x)^T \cdot \{P(x)\} dx \tag{21}$$

It should be noted that equilibrium equation (19) is not satisfied for every admissible displacement field that satisfies essential boundary conditions, but is only satisfied for the applied displacement field described by the cubic polynomials. For this reason equilibrium equations are not satisfied in a strong form but only in an average sense.

### 5.3 Force based derivation

In order to satisfy equilibrium in strong form a force based or flexibility method is required. Equation (19) is valid for all admissible variations of $\delta u$ that satisfy the essential boundary conditions $\delta u(0) = \delta v(0) = \delta v(L) = 0$. Integrating by parts and considering the essential boundary conditions, the classical equilibrium equations, in the absence of body forces, are derived:

$$\frac{dN(x)}{dx} = 0, \quad \frac{d^2 M(x)}{dx^2} = 0 \tag{22}$$

By integrating these equations and applying the natural boundary conditions $N(L) = N$, $M(0) = -M_1$, $M(L) = M_2$ we end to the basic equation that interpolates nodal forces to cross sectional forces:

$$P(x) = b(x) \cdot Q(x) \;\; = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{x}{L} - 1 & \frac{x}{L} \end{bmatrix} \cdot Q(x) \tag{23}$$

The displacement field in this formulation is arbitrary as no interpolation functions were adopted, which establishes the strong form formulation of the equilibrium equations.

Equation (23) is then substituted into the weak form of the compatibility equation (17) and the following relation is derived:

$$\int_L \delta \{P(x)\}^T \cdot \{d(x)\} \, dx = \delta \{Q\} \cdot \{q\} \tag{24}$$

Substituting again equation (23) in the above equation implies the following result:

$$\{q\} = \int_L b(x)^T \cdot \{d(x)\} \, dx \tag{25}$$

which means that nodal displacements can be calculated from cross sectional deformations. If the basic relation $\{d(x)\} = f(x) \cdot \{P(x)\}$ (where *f(x)* is the cross sectional flexibility) is considered, element's flexibility matrix is derived as:

$$F(x) = \int_L b(x)^T \cdot f(x) \cdot b(x) dx \tag{26}$$

The last two equations are sufficient to determine the state of the element in the elastoplastic case if they are combined with the nonlinear evolution equations of the cross sectional stresses and flexibility.

# 6 NUMERICAL IMPLEMENTATION

## 6.1 Discretization aspects

All the presented integrals should be calculated numerically, which means that the integrants should be evaluated at discrete points. To calculate the integrals over the length of the element Gauss-Lobatto integration is chosen as the more appropriate one, as it includes both element ends, where plastic deformations concentrate, as quadrature points. On the other hand integrals over the cross-section are evaluated by discretizing cross sections in longitudinal fibers, each one representing the respective material properties and by adding each fiber's contribution.

## 6.2 Displacement based formulation

In the displacement based formulation the primary unknowns are the nodal displacements which also define the internal displacement field through shape function interpolation. In the nonlinear case the internal Bouc-Wen hysteretic variables $z$ should be evaluated at the discretization points. Hence, the total unknowns of the element are ($6+N_{GL}\cdot N_{fib}$), where $N_{GL}$ are the Gauss-Lobatto quadrature points and $N_{fib}$ are the number of cross-sectional fibers.

By substituting equation (6) in equation (5) the expression of the cross sectional forces is derived:

$$\{P\} = [l]^T \cdot [C_{el}] \cdot [l] \cdot \{d\} + [l]^T \cdot [C_{hys}] \cdot \{z\} = [k_{el}] \cdot \{d\} + [k_{hys}] \cdot \{z\} \tag{27}$$

where $[l]$ is the matrix with fibers coordinates, $[C_{el}]$ and $[C_{hys}]$ are diagonal matrices of ($N_{fib}$, $N_{fib}$) dimension that include $\alpha_i \cdot E_i \cdot A_i$ and $(1-\alpha_i) \cdot E_i \cdot A_i$ terms respectively and $\{z\}$ is hysteretic variables vector of ($N_{fib}$,1) dimension. In the above equation the first part is the reduced elastic contribution and the second part is the hysteretic one. The elastic part is treated normally which ends to the derivation of the reduced elastic matrix of the element $K_{el}$.

The nodal hysteretic forces are calculated using equation (21) in matrix form using Gauss-Lobatto integration as follows.

$$\{Q_{hys}\} = \begin{bmatrix} [GL]_1 & \cdots & [GL]_{N_{GL}} \end{bmatrix} \cdot \begin{bmatrix} [k_{hys}]_1 & & \\ & \ddots & \\ & & [k_{hys}]_{N_{GL}} \end{bmatrix} \cdot \begin{bmatrix} \{z\}_1 \\ \vdots \\ \{z\}_{N_{GL}} \end{bmatrix} \tag{28}$$

where $[GL]_i = L/2 \cdot w_i \cdot \left[ a\left( L/2 \cdot (1+\xi_i) \right) \right]^T$ is the Gauss-Lobatto addition term concerning i cross section with $w_i$ and $\xi_i$ being quadrature's weight and position respectively. After incorporating rigid body motion and transforming to global system, the global element's hysteretic forces are derived. Assembling every element's contribution forms the global equation of the hysteretic nodal forces of a structure with $N_{el}$ number of elements and $N_{dof}$ degrees of freedom.

$$\{F_H\}_S = [K_H]_S \cdot \{z\}_S \tag{29}$$

where $\left[K_H\right]_S$ of dimension ($N_{dof}$, $N_{el}\cdot N_{GL}\cdot N_{fib}$) is the global hysteretic matrix of the structure and $\{z\}_S$ is the total vector of hysteretic variables with dimension ($N_{el}\cdot N_{GL}\cdot N_{fib}$, 1)

Global dynamic equilibrium leads to the global dynamic equations of motion in the following form:

$$[M]_S\cdot\{\ddot{u}\}_S+[C]_S\cdot\{\dot{u}\}_S+\left[K_{el}\right]_S\cdot\{u\}_S+\left[K_H\right]_S\cdot\{z\}_S=P(t) \quad (30)$$

These equations along with the Bouc-Wen evolution equations are solved in state space form by adding the velocities as additional unknown vector. Consequently, the following 1st order ODE system is formed.

$$\{\dot{x}\}=\begin{Bmatrix}\{\dot{u}\}_S\\ \{\ddot{u}\}_S\\ \{\dot{z}\}_S\end{Bmatrix}=\begin{Bmatrix}\{\dot{u}\}_S\\ -[M]_S^{-1}\cdot\left([C]_S\cdot\{\dot{u}\}_S+\left[K_{el}\right]_S\cdot\{u\}_S+\left[K_H\right]_S\cdot\{z\}_S-P(t)\right)\\ \left(1-\left|\dfrac{z}{\varepsilon_y}\right|^n\cdot\left(\beta+\gamma\cdot\operatorname{sgn}(z_i\cdot\dot{\varepsilon}_i)\right)\right)\cdot\dot{\varepsilon}_i,\ i=1\ldots N_{el}\cdot N_{GL}\cdot N_{fib}\end{Bmatrix} \quad (31)$$

The main advantage of the displacement based formulation is that all matrices are formed at the beginning and remain constant throughout the numerical procedure. In every time step only fiber rate strains have to be calculated through nodal velocities that are inserted into the Bouc-Wen equations.

### 6.3 Force based numerical formulation

In force based approach the internal stress field of the element is considered unknown and should be evaluated through the solution procedure. Although cross sectional forces are calculated from nodal forces through equilibrium considerations, cross sectional stresses at fiber points are difficult to evaluate as there isn't a unique stress field that satisfies cross sectional equilibrium. This fact leads to the introduction of all fiber stresses in the beam element's unknown vector. Also, introducing cross sectional deformations in the unknown vector resolves the lack of interpolation of the deformation field through nodal displacements. Fiber uniaxial strains are then easily calculated from cross sectional deformations as linear strain distribution is considered always valid. Hence, the total unknowns of the element are ($6+N_{GL}\cdot N_{fib}+N_{GL}\cdot 2$) and the global equation of motion is written in the form:

$$[M]_S\cdot\{\ddot{u}\}_S+[C]_S\cdot\{\dot{u}\}_S+\left[K_{\tan}\right]_S\cdot\{u\}_S=P(t) \quad (32)$$

where $[K_{tan}]_S$ is the whole structure's tangent global stiffness matrix. The ODE system is:

$$\{\dot{x}\}=\begin{Bmatrix}\{\dot{u}\}_S\\ \{\ddot{u}\}_S\\ \{\dot{d}\}_S\\ \{\dot{\sigma}\}_S\end{Bmatrix}=\begin{Bmatrix}\{\dot{u}\}_S\\ -[M]_S^{-1}\cdot\left([C]_S\cdot\{\dot{u}\}_S+\left[K_{\tan}\right]_S\cdot\{u\}_S-P(t)\right)\\ [f]_S\cdot[b]_S\cdot[F]_S^{-1}\cdot\{\dot{q}\}_S\\ \left[E_{\tan}\right]_S\cdot[l]_S\cdot\{\dot{d}\}_S\end{Bmatrix} \quad (33)$$

where matrix $[E_{tan}]_S$ is a block diagonal matrix ($N_{el}{\cdot}N_{GL}{\cdot}N_{Nfib}$, $N_{el}{\cdot}N_{GL}{\cdot}N_{Nfib}$) that contains all fiber's material tangent modulus. Every fiber's tangent modulus can be calculated from equation (9). The cross sectional flexibility matrix is the inverse of the tangent stiffness matrix derived in equation (10). Then the stiffness of the element calculated by inverting the flexibility stiffness matrix (equation (26)) which finally is assembled to form the global stiffness matrix $[K_{tan}]_S$ of the structure.

Vector $\{\dot{d}\}_S$ is a block vector ($2{\cdot}N_{el}{\cdot}N_{GL}$, 1) that contains all cross sectional deformations of the structure. Vector $\{\dot{q}\}_S$ is a block vector with dimension ($3{\cdot}N_{el}$, 1) that includes every element's nodal velocities as they are extracted from the global structure's nodal velocities vector $\{\dot{u}\}_S$. Matrix $[f]_S$ is a block diagonal matrix ($2{\cdot}N_{el}{\cdot}N_{GL}$, $2{\cdot}N_{el}{\cdot}N_{GL}$) that contains current flexibility matrices of all control sections. Matrix $[\ b\ ]_S$ is a block diagonal matrix ($2{\cdot}N_{el}{\cdot}N_{GL}$, $3{\cdot}N_{el}$) that contains equilibrium interpolation matrices evaluated at every cross section of all elements. Finally, Matrix $[F]_S^{-1}$ is a block diagonal matrix ($3{\cdot}N_{el}$, $3{\cdot}N_{el}$) containing the stiffness matrices of all structure's elements.

Vector $\{\dot{\sigma}\}_S$ is a block vector ($N_{el}{\cdot}N_{GL}{\cdot}N_{Nfib}$, 1) that contains all fiber stresses of the structure and matrix $[\ l]_S$ is a block diagonal matrix ($N_{el}{\cdot}N_{GL}{\cdot}N_{Nfib}$, $2{\cdot}N_{el}{\cdot}N_{GL}$) with fiber cross sectional coordinates.

## 7 NUMERICAL EXAMPLE

To highlight the efficiency of the proposed formulations, a simple, one bay steel frame is analyzed under static and dynamic loading. The span and height of the frame is *6m* and *3m* respectively and both columns and beam have a standard *IPE300* cross section. The constitutive law of steel is considered bilinear with *2%* post-yield to pre-yield stiffness ratio and yield stress $\sigma_y=235\ MPa$ .To simulate this behavior with the Bouc-Wen model the following parameters were selected: $n=25$, $\beta=\gamma=0.5$.

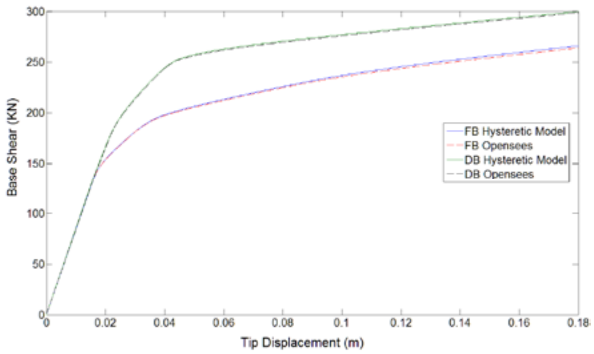


Figure 2: Comparison of both displacement and force based formulations.

First, a static nonlinear analysis is carried out for both formulations and nonlinear base shear- tip displacement curves are presented in Figure 2. The results are compared against those of Opensees Software [16] using the classical displacement and force based formulations with the iterative Newton –Raphson procedures. As it can be observed the proposed formulations are able to track the exact nonlinear behavior of the steel frame as they yield nearly identical results with the well documented Opensees software. Difference between force and displacement based formulation is due to the inefficiency of the latter to satisfy equilibrium in its strong form. This error which originates from the enforcement of the displacement field can be diminished with h-refinement of the elements by adding multiple internal nodes.

The same frame is also analyzed dynamically under a scaled seismic excitation.

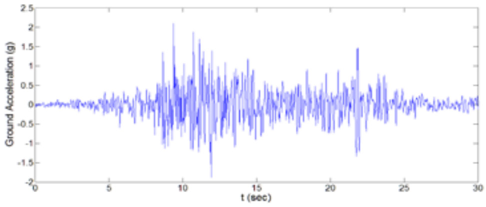


Figure 3: Input acceleration time history.

Seismostruct software [17] is utilized for the comparison. After performing the analysis the following tip displacement time histories are derived for both formulations. The results of the proposed model are in full agreement with the Seismostruct software.

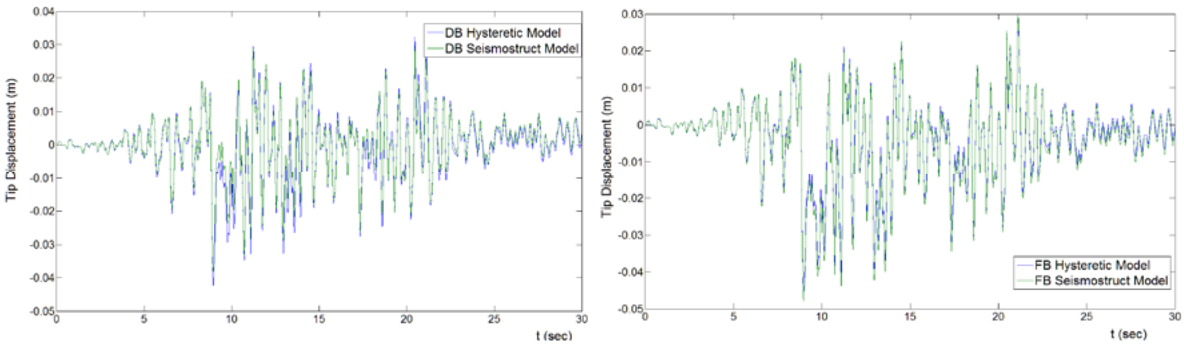


Figure 4: Tip displacement time histories of the displacement and force based models.

The fact that the force based models present greater displacements proves the results of Figure 2 as their ultimate strength seems to be smaller comparing to the displacement based models.

## 8 CONCLUSIONS

- Distributed plasticity beam models have been proposed where constitutive behavior is incorporated in rate form by using hysteretic models of Bouc-Wen Type.
- Displacement and force based formulations were derived from the Hellinger-Reissner variational principle and their pros and cons were highlighted.
- For both formulations a system of equations for solving the nonlinear problem simultaneously were developed in state space form.
- The main advantage of the proposed methodology is that avoids the need to linearize equations following a predictor- corrector scheme, as the system of equations are solved in differential form.
- The accuracy and efficiency of the proposed methodology is verified numerically in both static and transient analysis.

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