XV International Conference on Computational Plasticity. Fundamentals and Applications
COMPLAS 2019
E. Oñate, D.R.J. Owen, D. Peric, M. Chiumenti & Eduardo de Souza Neto (Eds)

# THE BREZIS-EKELAND-NAYROLES MINIMIZATION PRINCIPLE WITH MIXED FINITE ELEMENT METHOD FOR ELASTOPLASTIC DYNAMIC PROBLEMS

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**Key words:** Numerical simulations, Elastic plastic materials, Variational principal, Brezis-Ekeland-Nayroles principle, Constrained optimization

**Abstract.** We propose a modification of the Hamiltonian formalism which can be used for dissipative systems, the Brezis-Ekeland-Nayroles principle. The formalism is specialized to the standard plasticity in small strains and dynamics. We apply it to solve the classical problem of a thin tube in plane strain subjected to an internal pressure. The continuum is discretized with mixed finite elements.

## 1 INTRODUCTION

There are two types energy loss, one originated from external actions, another derived from the internal effect. If the cause is internal, such as plasticity etc., we call them dissipative system. Hamilton's variational principle does not work in this case. The proposition, the Brezis-Ekeland-Nayroles principle (in short, BEN principle) is presented in the paper. For more detail of the BEN principle, see [1, 2, 3, 4, 5].

The symplectic version of the Brezis-Ekeland-Nayroles variational principle is proposed in [4]: *The natural evolution curve* $z : [t_0, t_1] \to X \times Y$ *minimizes the functional:*

$$\Pi(z) := \int_{t_0}^{t_1} \left[\phi(\dot{z}) + \phi^{*\omega}(\dot{z} - X H) - \omega(\dot{z} - X H, \dot{z})\right] \, \mathrm{dt} \tag{1}$$

*among all the curves verifying the initial conditions* $z(t_0) = z_0$ *and, remarkably, the minimum is zero.*

With $\phi$ dissipation potential, $\phi^{*\omega}$ its Fenchel transform, and $\omega$ a natural symplectic form. Observing that $\omega(\dot{z}, \dot{z})$ vanishes and integrating by part, we have also the variant:

$$\Pi(z) = \int_{t_0}^{t_1} [\phi(\dot{z}) + \phi^{*\omega}(\dot{z} - XH) - \frac{\partial H}{\partial t}(t, z)] \, \mathrm{dt} + H(t_1, z(t_1)) - H(t_0, z_0) \tag{2}$$

One advantage of the BEN principle is its ability in solving a mechanical problem simultaneously for all time steps. It can avoid the usual computation failures in the step-by-step method. Objective of this paper is to numerically test the feasibility of the BEN principle by modeling a simple mechanical problem, the tube subjected to an internal pressure in dynamics.

## 2 APPLICATION TO THE STANDARD PLASTICITY AND VISCOPLASTICITY IN DYNAMICS

Application of the BEN principle in statics case is discussed in [4, 5]. The small difference for the BEN principle between the case of statics and dynamics in the mechanical sense is the consideration of the inertia term $\dot{\boldsymbol{p}}$. In dynamics, the equilibrium equations

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \dot{\boldsymbol{p}} \quad \text{on} \quad \Omega, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{f}} \quad \text{on} \quad \partial\Omega_1 \tag{3}$$

are satisfied. This expression can be transformed as follows. For sake of easiness, let us put:

$$\langle \boldsymbol{l}(t), u \rangle = \int_{\Omega} \boldsymbol{f}(t) \cdot \boldsymbol{u} + \int_{\partial\Omega_1} \bar{\boldsymbol{f}}(t) \cdot \boldsymbol{u}$$

Then,

$$\frac{\partial H}{\partial t}(t, z) = -\langle \dot{\boldsymbol{l}}(t), \boldsymbol{u} \rangle$$

On the other hand

$$\frac{d}{dt}\left[H(t, z(t))\right] = \langle \frac{\boldsymbol{p}}{\rho}, \dot{\boldsymbol{p}} \rangle + \langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} - \dot{\boldsymbol{\varepsilon}}_I \rangle - \langle \boldsymbol{l}(t), \dot{\boldsymbol{u}} \rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{u} \rangle$$

For the minimizer, the kinematical conditions on $\partial\Omega_0$ and the balance of linear momentum (3) are satisfied and using Green's formula:

$$\langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} \rangle = \langle \boldsymbol{l}(t), \dot{\boldsymbol{u}} \rangle - \langle \dot{\boldsymbol{p}}, \dot{\boldsymbol{u}} \rangle \tag{4}$$

that leads to

$$\frac{d}{dt}\left[H(t, z(t))\right] - \frac{\partial H}{\partial t}(t, z) = -\langle \boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}_I \rangle \tag{5}$$

Putting (5) in (2) and time-integrating, we have:

$$\Pi(\boldsymbol{\sigma}, \dot{\boldsymbol{u}}) = \int_{t_0}^{t_1} \left\{ \varphi(\boldsymbol{\sigma}) + \varphi^*(\nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}}) - \langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}} \rangle \right\} \, \mathrm{dt} \tag{6}$$

among all curves $\boldsymbol{u} : [t_0, t_1] \to U$ satisfying the kinematical conditions on $\partial\Omega_0$ and all curves $\boldsymbol{\sigma} : [t_0, t_1] \to E$ such that $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ and the balance of linear momentum (3) are satisfied.

## 3 THE TUBE PROBLEM

We consider a tube of internal radius $a$ and external one $b$ within the plane strain hypothesis. Its internal wall is subjected to an internal pressure $\tilde{p} > 0$ monotonic increasing from zero. The material is elastic perfectly plastic and isotropic with von Mises model and yield stress $\sigma_Y$. The initial stresses, displacements and velocities are null. The problem is assumed to be axisymmetric, the stress tensor is diagonal in the local basis of the cylindrical coordinates. The axial tensile stress being supposed as usual to be the intermediate principal stress. We consider the elastic domain:

$$K = \{\boldsymbol{\sigma} \quad \text{such that} \quad f(\boldsymbol{\sigma}) - \sigma_Y \leq 0\}$$

The dissipation potential is:

$$\varphi(\boldsymbol{\sigma}) = \int_\Omega \chi_K(\boldsymbol{\sigma})$$

In the sequel, the inelastic strain $\dot{\boldsymbol{\varepsilon}}_I$ is plastic and denoted $\dot{\boldsymbol{\varepsilon}}^p$. The Fenchel conjugate is obtained conbining this rule with the expression of the dissipation power by unit volume and the yield condition:

$$D = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p = \sigma_Y \dot{\varepsilon}^p_{\theta\theta} \tag{7}$$

As $\dot{\varepsilon}^p_{\theta\theta}$ must be non negative, the Fenchel conjugate function is:

$$\varphi^*(\dot{\boldsymbol{\varepsilon}}^p) = \int_\Omega \left\{\sigma_Y \dot{\varepsilon}^p_{\theta\theta} + \chi_{\mathbb{R}_+}(\dot{\varepsilon}^p_{\theta\theta})\right\}$$

In plane strain and axisymmetry, the displacement is radial. The only non vanishing components of the strain rate tensor are:

$$\dot{\varepsilon}_{rr} = \frac{d\dot{u}_r}{dr}, \qquad \dot{\varepsilon}_{\theta\theta} = \frac{\dot{u}_r}{r} \tag{8}$$

In plane strain, Hooke's law reads:

$$\varepsilon_{rr} = \frac{1}{\bar{E}}(\sigma_{rr} - \bar{\nu}\,\sigma_{\theta\theta}), \qquad \varepsilon_{\theta\theta} = \frac{1}{\bar{E}}(\sigma_{\theta\theta} - \bar{\nu}\,\sigma_{rr})$$

with: $\bar{E} = \frac{E}{1-\nu^2}, \bar{\nu} = \frac{\nu}{1-\nu}$. Hence the compliance operator reads:

$$\boldsymbol{S} = \frac{1}{\bar{E}}\begin{bmatrix} 1 & -\bar{\nu} \\ -\bar{\nu} & 1 \end{bmatrix}$$

In this problem, there is no supports ($\partial\Omega_0 = \emptyset$). As the minimum is certainly finite, the functional (6) becomes:

$$\bar{\Pi}(\boldsymbol{\sigma}, \boldsymbol{u}) = \int_{t_0}^{t_1} \left\{ (\int_\Omega \sigma_Y \dot{\varepsilon}^p_{\theta\theta}) - \langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}} \rangle \right\} \,\mathrm{dt} \tag{9}$$

where $\dot{\varepsilon}^p_{\theta\theta}$ is given by normality rule, among all the curves among all curves $(\boldsymbol{\sigma}, \boldsymbol{u}) : [t_0, t_1] \to U \times E$ such that $\boldsymbol{\sigma}(t_0) = \mathbf{0}$, $\boldsymbol{u}(t_0) = \mathbf{0}$, satisfying the yield condition $f(\boldsymbol{\sigma}) - \sigma_Y \leq 0$ and the normality rule and the balance of linear momentum:

$$\frac{d}{dr}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = \rho\,\ddot{u}_r \quad \text{for} \quad a < r < b,$$
$$\sigma_{rr}(a, t) = -\tilde{p}(t), \qquad \sigma_{rr}(b, t) = 0 \tag{10}$$

## 4 MIXED FINITE ELEMENT OF TUBE PROBLEM

The continuum is discretized with mixed finite elements. There are two methods to model the tube problem: (i) Method A: balance of linear momentum satisfied exactly, (ii) Method B: balance of linear momentum satisfied in Gauss points.

### 4.1 Method A

**Displacement field.** We choose the displacement field:

$$u_r = v_1 + v_2 r + v_3 r^2 + v_4 r^3 \tag{11}$$

which provides the strain field:

$$\varepsilon_{rr} = \frac{du_r}{dr} = v_2 + 2\,v_3 r + 3\,v_4 r^2, \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} = \frac{v_1}{r} + v_2 + v_3 r + v_4 r^2$$

There is two connectors:

$$q_1 = u_r\,|_{r=\alpha}, \qquad q_2 = u_r\,|_{r=\beta}, \tag{12}$$

Considering two intermediate equidistant nodes of position, $\gamma = \frac{2\,\alpha+\beta}{3}, \delta = \frac{\alpha+2\,\beta}{3}$, we introduce two extra degrees of freedom internal to the element (not connected with the other ones):

$$q_3 = u_r\,|_{r=\gamma}, \qquad q_4 = u_r\,|_{r=\delta}$$

that defines a cubic Lagrange interpolation:

$$u_r(r) =$$

$$\frac{1}{16}\left[-(1-\bar{r})\,(1-9\,\bar{r}^2),\ -(1+\bar{r})\,(1-9\,\bar{r}^2),\ 9\,(1-\bar{r}^2)\,(1-3\,\bar{r}),\ 9\,(1-\bar{r}^2)\,(1+3\,\bar{r})\,\right]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{13}$$

where $\bar{r} = \frac{2\,r-(\beta+\alpha)}{\beta-\alpha}$. In short, we have:

$$u_r(r) = \boldsymbol{N}_e(r)\,\boldsymbol{q}_e$$

The corresponding strain field can be expressed in term of the nodal displacement:

$$\boldsymbol{\varepsilon}(r) = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{d\boldsymbol{N_e}}{d\bar{r}}\dfrac{d\bar{r}}{dr} \\ \dfrac{\boldsymbol{N_e}}{r} \end{bmatrix} \boldsymbol{q}_e = \boldsymbol{B}_e(r)\,\boldsymbol{q}_e$$

After calculation, one has:

$$\boldsymbol{B}_e(r) =$$

$$\frac{1}{16}\begin{bmatrix} J\,(1+18\,\bar{r}-27\,\bar{r}^2) & J\,(-1+18\,\bar{r}+27\,\bar{r}^2) & J\,(-27-18\,\bar{r}+81\,\bar{r}^2) & J\,(27-18\,\bar{r}-81\,\bar{r}^2) \\ -\dfrac{1}{r}(1-\bar{r})\,(1-9\,\bar{r}^2) & -\dfrac{1}{r}(1+\bar{r})\,(1-9\,\bar{r}^2) & \dfrac{9}{r}\,(1-\bar{r}^2)\,(1-3\,\bar{r}) & \dfrac{9}{r}\,(1-\bar{r}^2)\,(1+3\,\bar{r}) \end{bmatrix}$$

with $J = \dfrac{d\bar{r}}{dr} = \frac{2}{\beta-\alpha}$.

**Stress field.** We consider an axisymmetric element occupying a volume $\alpha < r < \beta$ with four stress connectors:

$$g_1 = \sigma_{rr} \,|_{r=\alpha}, \qquad g_2 = \sigma_{\theta\theta} \,|_{r=\alpha}, \qquad g_3 = \sigma_{rr} \,|_{r=\beta}, \qquad g_4 = \sigma_{\theta\theta} \,|_{r=\beta} \tag{14}$$

The choice of a polynomial statically admissible stress field is guided by the aim to avoid the global (or structural) equations of linear momentum balance in the constrained minimization problem. Only remains the local yield condition. The general solution of:

$$\nabla \cdot \boldsymbol{\sigma} = \dot{\boldsymbol{p}}$$

is the sum of the general solution $\boldsymbol{\sigma}_h$ of the homogeneous equation and a particular solution $\boldsymbol{\sigma}_d$ of the non homogeneous equation. Following a method due to Schaefer ([6, 7]), this last one is of the form:

$$\boldsymbol{\sigma}_d = 2\,\nabla \boldsymbol{w} - (\nabla \cdot \boldsymbol{w})\,\boldsymbol{I} \ , \tag{15}$$

where the vector potential $\boldsymbol{w}$ is solution of $\nabla^2 \boldsymbol{w} = \dot{\boldsymbol{p}}$. For the displacement field, we seek a radial vector potential. The previous equation reduces to:

$$\frac{d^2 w_r}{dr^2} + \frac{1}{r}\,\frac{d w_r}{dr} - \frac{w_r}{r^2} = \rho\,(\dot{v}_1 + \dot{v}_2 r + \dot{v}_3 r^2 + \dot{v}_4 r^3)$$

Clearly, a solution is given by a homogeneous polynomial in $r$ of degree five. Introducing it in the previous equation, we obtain by identification:

$$w_r = \rho\,\left(\frac{\dot{v}_1}{3}\,r^2 + \frac{\dot{v}_2}{8}\,r^3 + \frac{\dot{v}_3}{15}\,r^4 + \frac{\dot{v}_4}{24}\,r^5\right)$$

condition (15) reads in polar coordinates:

$$\sigma_{rr} = 2\,\frac{d w_r}{dr} - \frac{1}{r}\,\frac{d}{dr}(r\,w_r), \quad \sigma_{\theta\theta} = 2\,\frac{w_r}{r} - \frac{1}{r}\,\frac{d}{dr}(r\,w_r)$$

leads to the expression of $\boldsymbol{\sigma}_d$:

$$\sigma_{rr} = -\sigma_{\theta\theta} = \rho\,\left(\frac{\dot{v}_1}{3}\,r + \frac{\dot{v}_2}{4}\,r^2 + \frac{\dot{v}_3}{5}\,r^3 + \frac{\dot{v}_4}{6}\,r^4\right)$$

Besides, the stress field being defined by four connectors, we choose for $\boldsymbol{\sigma}_h$:

$$\sigma_{rr} = h_1 + h_2 r + h_3 r^2 + h_4 r^3$$

Using the internal equilibrium equation in (10), the hoop stress is:

$$\sigma_{\theta\theta} = h_1 + 2\,h_2 r + 3\,h_3 r^2 + 4\,h_4 r^3$$

In matrix form, the total stress field in terms of stress and displacement parameters reads:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{bmatrix} = \boldsymbol{\sigma}_e(r) = \boldsymbol{R}_e(r)\,\boldsymbol{h}_e + \boldsymbol{S}_e(r)\,\dot{\boldsymbol{v}}_e$$

$$= \begin{bmatrix} 1 & r & r^2 & r^3 \\ 1 & 2\,r & 3\,r^2 & 4\,r^3 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} + \rho \begin{bmatrix} \frac{r}{3} & \frac{r^2}{4} & \frac{r^3}{5} & \frac{r^4}{6} \\ -\frac{r}{3} & -\frac{r^2}{4} & -\frac{r^3}{5} & -\frac{r^4}{6} \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{bmatrix}$$

stress connectors (14) are linearly depending on the stress and displacement parameters:

$$\boldsymbol{g}_e = \boldsymbol{C}_e \boldsymbol{h}_e + \boldsymbol{D}_e \dot{\boldsymbol{v}}_e$$

with the connection matrix:

$$\boldsymbol{C}_e = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 1 & 2\,\alpha & 3\,\alpha^2 & 4\,\alpha^3 \\ 1 & \beta & \beta^2 & \beta^3 \\ 1 & 2\,\beta & 3\,\beta^2 & 4\,\beta^3 \end{bmatrix} \qquad \boldsymbol{D}_e = \rho \begin{bmatrix} \frac{\alpha}{3} & \frac{\alpha^2}{4} & \frac{\alpha^3}{5} & \frac{\alpha^4}{6} \\ -\frac{\alpha}{3} & -\frac{\alpha^2}{4} & -\frac{\alpha^3}{5} & -\frac{\alpha^4}{6} \\ \frac{\beta}{3} & \frac{\beta^2}{4} & \frac{\beta^3}{5} & \frac{\beta^4}{6} \\ -\frac{\beta}{3} & -\frac{\beta^2}{4} & -\frac{\beta^3}{5} & -\frac{4}{6} \end{bmatrix}$$

Hence, one has: $\boldsymbol{h}_e = \boldsymbol{C}_e^{-1}(\boldsymbol{g}_e - \boldsymbol{D}_e \dot{\boldsymbol{v}}_e)$. By identification of (11) with (13), we obtain the relation between displacement parameters and connectors:

$$\boldsymbol{v}_e = \boldsymbol{A}_e \dot{\boldsymbol{q}}_e$$

$$\boldsymbol{A}_e^T =$$

$$\begin{bmatrix} \frac{1}{16}\left(9[\frac{\hat\alpha}{\hat\beta}]^3 + 9[\frac{\hat\alpha}{\hat\beta}]^2 - \frac{\hat\alpha}{\hat\beta} - 1\right) & \frac{1}{8}\left(-27\frac{\hat\alpha^2}{\hat\beta^3} - 18\frac{\hat\alpha}{\hat\beta^2} + \frac{1}{\hat\beta}\right) & \frac{1}{4}\left(27\frac{\hat\alpha}{\hat\beta^3} + 9\frac{1}{\hat\beta^2}\right) & \frac{-9}{2\hat\beta^3} \\ \frac{1}{16}\left(-9[\frac{\hat\alpha}{\hat\beta}]^3 + 9[\frac{\hat\alpha}{\hat\beta}]^2 + \frac{\hat\alpha}{\hat\beta} - 1\right) & \frac{1}{8}\left(27\frac{\hat\alpha^2}{\hat\beta^3} - 18\frac{\hat\alpha}{\hat\beta^2} - \frac{1}{\hat\beta}\right) & \frac{1}{4}\left(-27\frac{\hat\alpha}{\hat\beta^3} + 9\frac{1}{\hat\beta^2}\right) & \frac{9}{2\hat\beta^3} \\ \frac{1}{16}\left(-27[\frac{\hat\alpha}{\hat\beta}]^3 - 9[\frac{\hat\alpha}{\hat\beta}]^2 + 27\frac{\hat\alpha}{\hat\beta} + 9\right) & \frac{1}{8}\left(81\frac{\hat\alpha^2}{\hat\beta^3} + 18\frac{\hat\alpha}{\hat\beta^2} - \frac{27}{\hat\beta}\right) & \frac{1}{4}\left(-81\frac{\hat\alpha}{\hat\beta^3} - 9\frac{1}{\hat\beta^2}\right) & \frac{27}{2\hat\beta^3} \\ \frac{1}{16}\left(27[\frac{\hat\alpha}{\hat\beta}]^3 - 9[\frac{\hat\alpha}{\hat\beta}]^2 - 27\frac{\hat\alpha}{\hat\beta} + 9\right) & \frac{1}{8}\left(-81\frac{\hat\alpha^2}{\hat\beta^3} + 18\frac{\hat\alpha}{\hat\beta^2} + \frac{27}{\hat\beta}\right) & \frac{1}{4}\left(81\frac{\hat\alpha}{\hat\beta^3} - 9\frac{1}{\hat\beta^2}\right) & \frac{-27}{2\hat\beta^3} \end{bmatrix}$$

where: $\hat\alpha = \alpha + \beta$, $\hat\beta = \beta - \alpha$. Eliminating the stress parameters provides the stress field in terms of stress and displacement connectors:

$$\boldsymbol{\sigma}_e(r) = \boldsymbol{T}_e(r)\boldsymbol{g}_e + \boldsymbol{U}_e(r)\ddot{\boldsymbol{q}}_e$$

where:

$$\boldsymbol{T}_e(r) = \boldsymbol{R}_e(r)\boldsymbol{C}_e^{-1}, \qquad \boldsymbol{U}_e(r) = (\boldsymbol{S}_e(r) - \boldsymbol{R}_e(r)\boldsymbol{C}_e^{-1}\boldsymbol{D}_e)\,\boldsymbol{A}_e$$

### 4.1.1 Space discretization of the principle

Introducing the plastic multiplier $\lambda \geq 0$, the yield rule reads:

$$\dot{\boldsymbol{\varepsilon}}^p = \lambda \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}$$

and the dissipation power (7) becomes:

$$D = \sigma_Y \lambda$$

As usual, the integral are approximated by numerical integration on every element:

$$\int_\alpha^\beta \boldsymbol{A}(r)\, 2\,\pi\, r\, dr \cong \sum_{g=1}^{n_e} w_g\, \boldsymbol{A}(r_g)\, 2\,\pi\, r_g$$

In particular, the total dissipation power in the element reads:

$$\int_\alpha^\beta D(r)\, 2\,\pi\, r\, dr = \Lambda_e^T \lambda_e$$

where:

$$\boldsymbol{\Lambda}_e = \sigma_Y \begin{bmatrix} w_1\, 2\,\pi\, r_1 \\ \cdots \\ w_{n_e}\, 2\,\pi\, r_{n_e} \end{bmatrix}, \qquad \boldsymbol{\lambda}_e = \begin{bmatrix} \dot{\lambda}_1 \\ \cdots \\ \dot{\lambda}_{n_e} \end{bmatrix}$$

Performing the assembling thanks to the localization matrices $\boldsymbol{L}_e, \boldsymbol{M}_e, \boldsymbol{P}_e$ such that:

$$\boldsymbol{g}_e = \boldsymbol{M}_e \boldsymbol{g}, \qquad \boldsymbol{q}_e = \boldsymbol{L}_e \boldsymbol{q}, \qquad \boldsymbol{\lambda}_e = \boldsymbol{P}_e \boldsymbol{\lambda}$$

the discretized form of the functional is:

$$\begin{aligned} \bar{\Pi}(\boldsymbol{g}, \boldsymbol{q}, \boldsymbol{\lambda}) = \int_{t_0}^{t_1} \Big[ & \boldsymbol{\Lambda}^T \boldsymbol{\lambda}(t) - \dot{\boldsymbol{q}}^T(t)\, (\boldsymbol{G}\, \boldsymbol{g}(t) + \tilde{\boldsymbol{G}}\, \ddot{\boldsymbol{q}}(t)) \\ & + \boldsymbol{g}^T(t)\, \boldsymbol{F_1}\, \dot{\boldsymbol{g}}(t) + \ddot{\boldsymbol{q}}^T(t)\, \boldsymbol{F_2}\, \dot{\boldsymbol{g}}(t) + \boldsymbol{g}^T(t)\, \boldsymbol{F_3}\, \dddot{\boldsymbol{q}}(t) + \ddot{\boldsymbol{q}}^T(t)\, \boldsymbol{F_4}\, \dddot{\boldsymbol{q}}(t) \Big]\, dt \end{aligned} \tag{16}$$

with:

$$\boldsymbol{\Lambda} = \sum_{e=1}^{n} \boldsymbol{P}_e^T \boldsymbol{\Lambda}_e,$$

$$\boldsymbol{G} = \sum_{e=1}^{n} \int_\alpha^\beta \boldsymbol{L}_e^T \boldsymbol{B}_e^T(r)\, \boldsymbol{T}_e(r)\, \boldsymbol{M}_e\, 2\,\pi\, r\, dr, \quad \tilde{\boldsymbol{G}} = \sum_{e=1}^{n} \int_\alpha^\beta \boldsymbol{L}_e^T \boldsymbol{B}_e^T(r)\, \boldsymbol{U}_e(r)\, \boldsymbol{L}_e\, 2\,\pi\, r\, dr,$$

$$\boldsymbol{F_1} = \sum_{e=1}^{n} \int_\alpha^\beta \boldsymbol{M}_e^T \boldsymbol{T}_e^T(r)\, \boldsymbol{S}\, \boldsymbol{T}_e(r)\, \boldsymbol{M}_e\, 2\,\pi\, r\, dr \quad \boldsymbol{F_2} = \sum_{e=1}^{n} \int_\alpha^\beta \boldsymbol{L}_e^T \boldsymbol{U}_e^T(r)\, \boldsymbol{S}\, \boldsymbol{T}_e(r)\, \boldsymbol{M}_e\, 2\,\pi\, r\, dr$$

$$\boldsymbol{F_3} = \sum_{e=1}^{n} \int_\alpha^\beta \boldsymbol{M}_e^T \boldsymbol{T}_e^T(r)\, \boldsymbol{S}\, \boldsymbol{U}_e(r)\, \boldsymbol{L}_e\, 2\,\pi\, r\, dr \quad \boldsymbol{F_4} = \sum_{e=1}^{n} \int_\alpha^\beta \boldsymbol{L}_e^T \boldsymbol{U}_e^T(r)\, \boldsymbol{S}\, \boldsymbol{U}_e(r)\, \boldsymbol{L}_e\, 2\,\pi\, r\, dr$$

The Brezis-Ekeland-Nayroles claims that we have to find the minimum of (16) with respect to the path $t \mapsto (\boldsymbol{g}(t), \boldsymbol{q}(t), \boldsymbol{\lambda}(t))$ under the constrains of:

- equilibrium (on the boundary, the internal equilibrium being satisfies *a priori*):

$$g_{r=a}(t) = -p(t), \quad g_{r=b}(t) = 0$$

- plasticity (at every integration point $g$ of every element $e$):

$$f_g(\boldsymbol{g}, \ddot{\boldsymbol{q}}) - \sigma_Y \leq 0, \qquad \lambda_g \geq 0 \qquad \boldsymbol{N}_Y \lambda_g = \boldsymbol{B}_e(r_g)\, \dot{\boldsymbol{q}}_e - \boldsymbol{S}\left(\boldsymbol{T}_e(r_g)\, \dot{\boldsymbol{g}}_e + \boldsymbol{U}_e(r) \dddot{\boldsymbol{q}}_e\right)$$

- initial conditions:

$$\begin{array}{llll} \boldsymbol{g}(t_0) = \boldsymbol{0}, & \boldsymbol{q}(t_0) = \boldsymbol{0}, & \boldsymbol{\lambda}(t_0) = \boldsymbol{0} & \\ \dot{\boldsymbol{g}}(t_0) = \boldsymbol{0}, & \dot{\boldsymbol{q}}(t_0) = \boldsymbol{0}, & \ddot{\boldsymbol{q}}(t_0) = \boldsymbol{0}, & \dddot{\boldsymbol{q}}(t_0) = \boldsymbol{0} \end{array}$$

### 4.1.2 Time discretization of the functional

For the time discretization of any physical quantity $a$, we put:

$$a_j = a(t_j), \quad \dot{a}_j = \dot{a}(t_j), \quad \cdots$$

On each step, we approximate the time rates at $t = t_j$ by:

$$\dot{a}_j = \frac{a_j - a_{j-1}}{t_j - t_{j-1}}, \qquad \ddot{a}_j = \frac{\dot{a}_j - \dot{a}_{j-1}}{t_j - t_{j-1}}, \qquad \dddot{a}_j = \frac{\ddot{a}_j - \ddot{a}_{j-1}}{t_j - t_{j-1}}$$

Considering $m$ time step from $t_0$ to $t_m$ and enforcing the yield condition only at the beginning and the end of the step, we have to minimize the objective function:

$$\bar{\Pi}(\boldsymbol{g}_0, \cdots, \boldsymbol{g}_m, \boldsymbol{q}_0, \cdots, \boldsymbol{q}_m, \boldsymbol{\lambda}_0, \cdots, \boldsymbol{\lambda}_m) = \sum_{j=1}^{j=m} \Big[ \boldsymbol{\Lambda}^T \boldsymbol{\lambda}_j - \dot{\boldsymbol{q}}_j^T \left( \boldsymbol{G}\, \boldsymbol{g}_j + \tilde{\boldsymbol{G}}\, \ddot{\boldsymbol{q}}_j \right) + \boldsymbol{g}_j{}^T(t)\, \boldsymbol{F_1}\, \dot{\boldsymbol{g}}_j(t)$$
$$+ \ddot{\boldsymbol{q}}_j{}^T(t)\, \boldsymbol{F_2}\, \dot{\boldsymbol{g}}_j(t) + \boldsymbol{g}_j{}^T(t)\, \boldsymbol{F_3}\, \dddot{\boldsymbol{q}}_j(t) + \ddot{\boldsymbol{q}}_j{}^T(t)\, \boldsymbol{F_4}\, \dddot{\boldsymbol{q}}_j(t) \Big] \left( t_j - t_{j-1} \right) \tag{17}$$

under the constrains of:

- equilibrium (on the boundary, at each time step):

$$g_{r=a,j}(t_j) = -p(t), \quad g_{r=b,j}(t_j) = 0$$

- plasticity (at every integration point $g$ of every element $e$ and at every time step):

$$f_{g,j}(\boldsymbol{g}, \ddot{\boldsymbol{q}}) - \sigma_Y \leq 0, \quad \lambda_{g,j} \geq 0, \quad \boldsymbol{N}_Y \lambda_{g,j} = \boldsymbol{B}_e(r_g)\, \boldsymbol{L}_e \dot{\boldsymbol{q}}_j^T - \boldsymbol{S}\, \left[\boldsymbol{T}_e(r_g)\, \boldsymbol{M}_e \dot{\boldsymbol{g}}_j + \boldsymbol{U}_e(r_g)\, \boldsymbol{L}_e \dddot{\boldsymbol{q}}_j\right]$$

- initial conditions:

$$\boldsymbol{g}_0 = \boldsymbol{0}, \qquad \boldsymbol{q}_0 = \boldsymbol{0}, \qquad \boldsymbol{\lambda}_0 = \boldsymbol{0}, \qquad \dot{\boldsymbol{g}}_0 = \boldsymbol{0}, \qquad \dot{\boldsymbol{q}}_0 = \boldsymbol{0}, \qquad \ddot{\boldsymbol{q}}_0 = \boldsymbol{0}, \qquad \dddot{\boldsymbol{q}}_0 = \boldsymbol{0}$$

## 4.2 Method B

The balance of linear momentum is satisfied in Gauss points.

**Displacement field.** Same as Method A.

**Stress field.** We choose the same position for the radial and hoop stress field.

$$\sigma_{rr} = h_1 + h_2\, r + h_3\, r^2 + h_4\, r^3 \qquad \sigma_{\theta\theta} = h_5 + h_6\, r + h_7\, r^2 + h_8\, r^3$$

There are four degrees of freedom for each stress:

$$g_1 = \sigma_{rr}\,|_{r=\alpha}, \qquad g_2 = \sigma_{rr}\,|_{r=\beta}, \qquad g_3 = \sigma_{rr}\,|_{r=\gamma}, \qquad g_4 = \sigma_{rr}\,|_{r=\delta} \tag{18}$$

$$s_1 = \sigma_{\theta\theta}\,|_{r=\alpha}, \qquad s_2 = \sigma_{\theta\theta}\,|_{r=\beta}, \qquad s_3 = \sigma_{\theta\theta}\,|_{r=\gamma}, \qquad s_4 = \sigma_{\theta\theta}\,|_{r=\delta} \tag{19}$$

In matrix form, we have:

$$\sigma_{rr}(r) = \boldsymbol{N}_e(r)\,\boldsymbol{g}_e \qquad \sigma_{\theta\theta}(r) = \boldsymbol{N}_e(r)\,\boldsymbol{s}_e$$

Thus:

$$\boldsymbol{\sigma}(r) = \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{N}_e(r) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{N}_e(r) \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_e \\ \boldsymbol{s}_e \end{bmatrix} = \boldsymbol{T}_e(r)\,\boldsymbol{t}_e$$

### 4.2.1 Space discretization of the principle

Same as Method A.

Performing the assembling thanks to the localization matrices $\boldsymbol{L}_e, \boldsymbol{M}_e, \boldsymbol{P}_e$ such that:

$$\boldsymbol{t}_e = \boldsymbol{M}_e \boldsymbol{t}, \qquad \boldsymbol{q}_e = \boldsymbol{L}_e \boldsymbol{q}, \qquad \boldsymbol{\lambda}_e = \boldsymbol{P}_e \boldsymbol{\lambda}$$

the discretized form of the functional is:

$$\bar{\Pi}(\boldsymbol{t}, \boldsymbol{q}, \boldsymbol{\lambda}) = \int_{t_0}^{t_1} (\boldsymbol{\Lambda}^T \boldsymbol{\lambda}(t) - \dot{\boldsymbol{q}}^T(t)\boldsymbol{G}\,\boldsymbol{t}(t) + \dot{\boldsymbol{t}}^T(t)\boldsymbol{F}\,\boldsymbol{t}(t))\ \mathrm{d}t \tag{20}$$

with:

$$\boldsymbol{\Lambda} = \sum_{e=1}^{n} \boldsymbol{P}_e^T \boldsymbol{\Lambda}_e,$$

$$\boldsymbol{G} = \sum_{e=1}^{n} \int_{\alpha}^{\beta} \boldsymbol{L}_e^T \boldsymbol{B}_e^T(r)\,\boldsymbol{T}_e(r)\,\boldsymbol{M}_e\, 2\,\pi\, r\, dr \quad \boldsymbol{F} = \sum_{e=1}^{n} \int_{\alpha}^{\beta} \boldsymbol{M}_e^T \boldsymbol{T}_e^T(r)\,\boldsymbol{S}\,\boldsymbol{T}_e(r)\,\boldsymbol{M}_e\, 2\,\pi\, r\, dr$$

The Brezis-Ekeland-Nayroles claims that we have to find the minimum of (20) with respect to the path $t \mapsto (\boldsymbol{t}(t), \boldsymbol{q}(t), \boldsymbol{\lambda}(t))$ under the constrains of:

- equilibrium (on the boundary, the internal equilibrium being satisfies *a priori*):

$$g_{r=a}(t) = -p(t), \quad g_{r=b}(t) = 0, \quad \frac{d}{dr}\sigma_r(r_g) + \frac{1}{r_g}\left[\sigma_r(r_g) - \sigma_\theta(r_g)\right] = \rho \ddot{u}_r(r_g)$$

- plasticity (at every integration point $g$ of every element $e$):

$$f_g(\boldsymbol{t}) - \sigma_Y \leq 0, \qquad \lambda_g \geq 0, \qquad \boldsymbol{N}_Y \lambda_g = \boldsymbol{B}_e(r_g)\,\dot{\boldsymbol{q}}_e - \boldsymbol{S}\,\boldsymbol{T}_e(r_g)\,\dot{\boldsymbol{t}}_e$$

- initial conditions:

$$\boldsymbol{g}(t_0) = \boldsymbol{0}, \qquad \boldsymbol{q}(t_0) = \boldsymbol{0}, \qquad \boldsymbol{\lambda}(t_0) = \boldsymbol{0}, \qquad \dot{\boldsymbol{g}}(t_0) = \boldsymbol{0}, \qquad \dot{\boldsymbol{q}}(t_0) = \boldsymbol{0}, \qquad \ddot{\boldsymbol{q}}(t_0) = \boldsymbol{0}$$

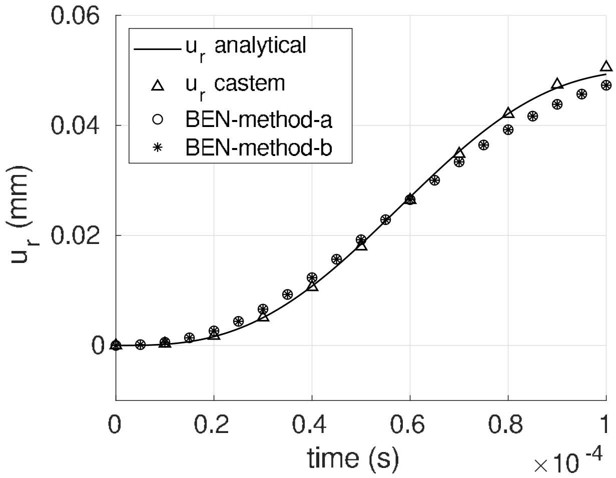


**Figure 1**: Comparison of radial displacement history of different data when $\tilde{p} = 1$ MPa in elastic case (i) analytical solution (ii) reference numerical solution (Cast3M software) with 10 time steps (iii) BEN method A solution with 20 time steps (iv) BEN method B solution with 20 time steps

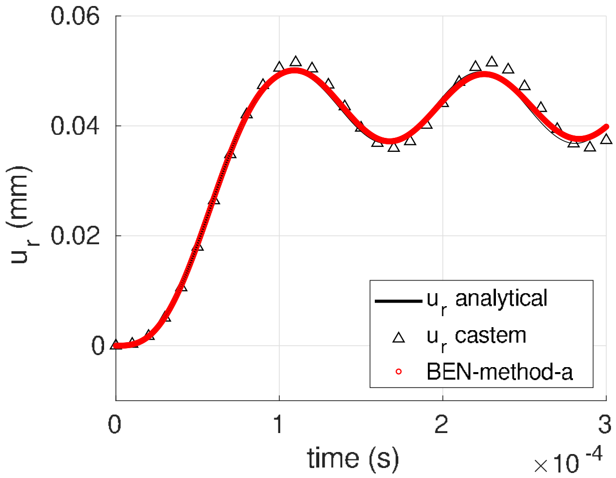


**Figure 2**: Comparison of radial displacement history of different data when $\tilde{p} = 1$ MPa in elastic case (i) analytical solution (ii) reference numerical solution (Cast3M software) with 30 time steps (iii) BEN method A solution with 600 time steps

### 4.2.2 Time discretization of the functional

Same as Method A.
Considering $m$ time step from $t_0$ to $t_m$ and enforcing the yield condition only at the beginning and the end of the step, we have to minimize the objective function:

$$\bar{\Pi}(\boldsymbol{t}_0, \cdots, \boldsymbol{t}_m, \boldsymbol{q}_0, \cdots, \boldsymbol{q}_m, \boldsymbol{\lambda}_0, \cdots, \boldsymbol{\lambda}_m) = \sum_{j=1}^{j=m} (\boldsymbol{\Lambda}^T \boldsymbol{\lambda}_j - \dot{\boldsymbol{q}}_j^T \boldsymbol{G}\, \boldsymbol{t}_j + \dot{\boldsymbol{t}}_j^T \boldsymbol{F}\, \boldsymbol{t}_j) \tag{21}$$

under the constrains of:

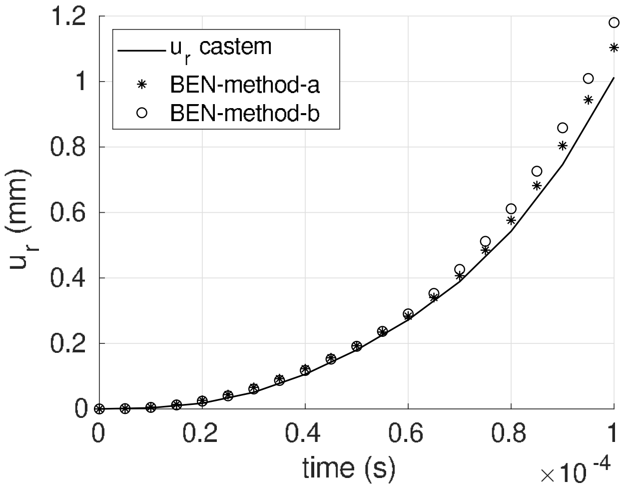


**Figure 3**: Comparison of radial displacement history of different data when $\tilde{p} = 10$ MPa in plastic case (i) analytical solution (ii) reference numerical solution (Cast3M software) (iii) BEN method A solution with 400 time steps

- equilibrium (on the boundary, at each time step):

$$g_{r=a,j} = -p(t_j), \quad g_{r=b,j} = 0, \quad \frac{d}{dr}\sigma_r(r_{g,j}) + \frac{1}{r_{g,j}}\left[\sigma_r(r_{g,j}) - \sigma_\theta(r_{g,j})\right] = \rho \ddot{u}_r(r_{g,j})$$

- plasticity (at every integration point $g$ of every element $e$ and at every time step):

$$f_{g,j}(\boldsymbol{t}) - \sigma_Y \leq 0, \qquad \lambda_{g,j} \geq 0, \qquad \boldsymbol{N}_Y(r_g)\lambda_{g,j} = \boldsymbol{B}_e(r_g)\,\boldsymbol{L}_e\dot{\boldsymbol{q}}_j - \boldsymbol{S}\,\boldsymbol{T}_e(r_g)\,\boldsymbol{M}_e\dot{\boldsymbol{t}}_j$$

- initial conditions:

$$\boldsymbol{g}_0 = \boldsymbol{0}, \qquad \boldsymbol{q}_0 = \boldsymbol{0}, \qquad \boldsymbol{\lambda}_0 = \boldsymbol{0}, \qquad \dot{\boldsymbol{g}}_0 = \boldsymbol{0}, \qquad \dot{\boldsymbol{q}}_0 = \boldsymbol{0}, \qquad \ddot{\boldsymbol{q}}_0 = \boldsymbol{0}$$

### 4.3 Simulation results

The program is coded in *Matlab*, the solver *fmincon* is applied to find the local minimum of the constrained functional (17, 21). Material parameters are, $E = 210$ GPa, $\nu = 0.3$, $\sigma_Y = 360$ MPa, $a = 100$ mm, $b = 101$ mm, $\rho = 7.8\,e^{-9}$ Kg/mm$^3$. Simulation results are displayed in figure (1), (2) and (3) for elastic and plastic cases. There is a good consistence between the BEN principle solution and the analytical or numerical solution. The BEN principle requires sufficient time steps to have a better precision than the one step-by-step. The method A and B does not change the simulation results.

## 5 CONCLUSIONS AND PERSPECTIVES

Thanks to the simulation results, the BEN method is numerically confirmed in dynamics. The BEN method allows to have a global view of all time steps which can avoid the usual difficulties in step-by-step method. As the computation time is significantly expansive, application of the Proper Generalized Decomposition (PGD) is the concept to obtain of a more effective program. Final objective of the present work is to apply the symplectic BEN principle in finite displacement to observe its ability in avoiding usual step-by-step method convergence problem.

## 6 ACKNOWLEDGMENT

This work was performed thanks to the international cooperation project *Dissipative Dynamical Systems by Geometrical and Variational Methods and Application to Viscoplastic Structures Subjected to Shock Waves* (DDGV) supported by the *Agence Nationale de la Recherche* (ANR) and the *Deutsche Forchungsgemeinschaft* (DFG).

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