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ON THE ACCURACY OF THE NUMERICAL INTEGRALS OF THE NEWMARK'S METHOD FOR COMPUTING INELASTIC SEISMIC RESPONSES

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Abstract. *The paper proposes an algorithm of the numerical integration with the modal analysis for computing inelastic seismic responses, and furthermore, the accuracy of the numerical integration with the Newmark's* β=1/4 *method that is most popular in the earthquake engineering is discussed by comparing with the response computed by the proposed method.*

1 INTRODUCTION

In nations exposed by great earthquakes, the seismic proofs of structures against those earthquakes permit that the state of a structure may excess the yield strength by bending moment, but it must not reach to the collapse. Therefore, the verification in computing inelastic seismic responses needs the maximum of the response after excessing the yield point and the residual deformation of the structure.

These inelastic analyses do not need only using a correct inelastic characteristics model but also adopting an accurate algorithm for numerically integrating the equation of the motion. In the numerical integrals for that, the Newmark's β=1/4 method is most popular and it has been using since before more than 30 years. The method is to obtain the incremental response in the incremental time from solving only the linear equation derived from the Newmark's β=1/4 method. However, verifying the validity of the method does not seem enough, because there is no comparison with the exact solution or more accurate solution.

The authors recently developed an algorithm with the modal analysis that enabled eliminating high frequency components certainly including errors from seismic response. The elimination of the high frequency components to advantage of the modal analysis makes the methods of numerical integrals giving accurate solutions usable. Those methods namely cannot directly solve the linear equation of the incremental response, because the linear equation always includes components with smaller period than the limitation of a time interval existing in those methods. The components with very small period do not contribute to the inelastic seismic response but breed divergence in those methods.

It has been published that there were the differences between the response computed by the modal analysis and that by the direct method with the Newmark's β=1/4 method. The paper is to show more amply the differences, the case of the agreement, comparing both responses of acceleration and so on.

2 THEORY OF NUMERICAL INTEGRALS WITH THE NEWMARK'S METHOD

The motion of an inelastic structure at the time t_m during an earthquake can be expressed by the following equation.

$$\mathbf{M}\ddot{\mathbf{u}}_m + \mathbf{C}_m\dot{\mathbf{u}}_m + \sum_e \boldsymbol{\alpha}_e \mathbf{S}_{e,m} = -\mathbf{M}\mathbf{J}\ddot{\phi}_m . \tag{1}$$

In the equation (1), $\mathbf{M}$ is the mass matrix of the structure, and $\mathbf{C}_m$ is the damping matrix possible to be arbitrarily composed though the paper uses the Rayleigh damping $\mathbf{C}_m = \nu\mathbf{M} + \mu\mathbf{K}_m$ where ν is a constant as well as μ and $\mathbf{K}_m$ is the tangent stiffness of the structure at the time t_m or the secant stiffness joining the two states at the times of t_m and $t_{m□1}$. Moreover, $\mathbf{S}_{e,m}$ is the end force vector of the element e, $\boldsymbol{\alpha}_e$ is the transforming matrix that changes the element end forces to the forces in the universal coordinates, $\mathbf{J}\ddot{\phi}_m$ is to give the input acceleration of the ground motion to all the nodes in the structure, and $\dot{\mathbf{u}}_m$ and $\ddot{\mathbf{u}}_m$ are the nodal velocity vector and the nodal acceleration vector respectively.

The method proposed in the paper applies the modal analysis with the Newmark's method to the equation (1) and then the existing methods directly apply the Newmark's method to the equation (1).

2.1 The method of modal analysis with the predictor-corrector method

Displacement modes and natural frequencies used in the modal analysis are derived from the eigenvalue analysis with the mass matrix $\mathbf{M}$ in the equation (1) and the tangent stiffness matrix $\mathbf{K}_{\mathrm{T}m}$ given by differentiating the third term in the equation (1) as follows.

$$\delta\left(\sum_e \boldsymbol{\alpha}_e \mathbf{S}_{e,m}\right) = \mathbf{K}_{\mathrm{T}m}\boldsymbol{\delta}\mathbf{u} , \tag{2}$$

where δ expresses the differential operator, and $\boldsymbol{\delta}\mathbf{u}$ is the infinitesimal displacement from the nodal position at the time t_m.

The generalized eigenvalue analysis using $\mathbf{M}$ and $\mathbf{K}_{\mathrm{T}m}$ gives the diagonal matrix $\boldsymbol{\Omega}_m^{(0)}$ composed by the squared natural circular frequencies and the matrix $\mathbf{X}_m^{(0)}$ composed by the displacement modes. The modes can transform the increment displacement $\boldsymbol{\Delta}\mathbf{u}^{(0)}$ in the time interval Δt between the times of t_m and t_{m+1} to the increment of the normal coordinates $\Delta\boldsymbol{\psi}^{(0)}$ as follows.

$$\Delta\mathbf{u}^{(0)} = \mathbf{X}_m^{(0)}\Delta\boldsymbol{\psi}^{(0)}, \tag{3}$$

where the upper suffix $^{(0)}$ attached to the variables means to relate with the predictor.

Applying the equation (3) and the Newmark's β method changes the equation (1) to the equation of $\Delta\boldsymbol{\psi}^{(0)}$, so that the equation of $\Delta\boldsymbol{\psi}^{(0)}$ is,

$$\left\{\frac{1}{\beta\Delta t}\left(\frac{1}{\Delta t}+\frac{\nu}{2}\right)\mathbf{I}+\left(\frac{\mu}{2\beta\Delta t}+1\right)\boldsymbol{\Omega}_m^{(0)}\right\}\Delta\boldsymbol{\psi}^{(0)} = -\boldsymbol{\zeta}_m^{(0)}\Delta\ddot{\boldsymbol{\phi}}+\frac{1}{\beta}\left\{\left(\frac{1}{\Delta t}+\frac{\nu}{2}\right)\mathbf{I}+\frac{\mu}{2}\boldsymbol{\Omega}_m^{(0)}\right\}\dot{\boldsymbol{\psi}}_m + \left\{\left(\frac{1}{2\beta}+\frac{\nu\Delta t}{4\beta}-\nu\Delta t\right)\mathbf{I}+\mu\Delta t\left(\frac{1}{4\beta}-1\right)\boldsymbol{\Omega}_m^{(0)}\right\}\ddot{\boldsymbol{\psi}}_m, \tag{4}$$

where $\boldsymbol{\zeta}_m^{(0)} = \mathbf{X}_m^{(0)\mathrm{T}}\mathbf{MJ}$ is the matrix composed by the participation factors, and $\mathbf{I}$ is the unit matrix.

Since all the matrices in the equation (4) are diagonal, the normal coordinates $\Delta\boldsymbol{\psi}^{(0)}$ are easily obtained. Then, the increment displacement $\Delta\mathbf{u}^{(0)}$ comes from the equation (3), and the displacement as the predictor at the time t_{m+1} is given by $\mathbf{u}_{m+1}^{(0)} = \mathbf{u}_m^{(0)} + \Delta\mathbf{u}^{(0)}$.

Those responses predicted, however, do not ordinarily satisfy the equation of the motion in changing the stiffness of the inelastic structure. Therefore, iterative computations are needed to correct them. In the computations of the iterative time number r, the variables of the response are provided with the upper suffix $^{(r)}$, e.g. the normal coordinates as corrector $\delta\boldsymbol{\psi}^{(r)}$, the matrix of the displacement modes $\mathbf{X}_{m+1}^{(r)}$, the diagonal matrix of the squares of the natural circular frequencies $\boldsymbol{\Omega}_{m+1}^{(r)}$ and so on.

The normal coordinates $\delta\boldsymbol{\psi}^{(r)}$ are obtained from solving the following equation.

$$\left\{\frac{1}{\beta\Delta t}\left(\frac{1}{\Delta t}+\frac{\nu}{2}\right)\mathbf{I}+\left(\frac{\mu}{2\beta\Delta t}+1\right)\boldsymbol{\Omega}_{m+1}^{(r)}\right\}\delta\boldsymbol{\psi}^{(r)} = \mathbf{X}_{m+1}^{(r)\mathrm{T}}\delta\mathbf{U}_{m+1}^{(r)}, \tag{5}$$

where $\delta\mathbf{U}_{m+1}^{(r)}$ is the unbalanced force as insufficiency in the equation of the motion at the time t_{m+1} as follows.

$$\delta\mathbf{U}_{m+1}^{(r)} = -\mathbf{M}\left(\mathbf{J}\ddot{\boldsymbol{\phi}}_{m+1}+\ddot{\mathbf{u}}_{m+1}^{(r)}\right)-\mathbf{C}_{m+1}^{(r)}\dot{\mathbf{u}}_{m+1}^{(r)}-\sum_e \alpha_e\mathbf{S}_{e,m+1}^{(r)}. \tag{6}$$

Then, the normal coordinates are corrected by the following equation,

$$\Delta\boldsymbol{\psi}^{(r)} = \mathbf{X}_{m+1}^{(r)\mathrm{T}}\mathbf{M}\mathbf{X}_{m+1}^{(r-1)}\Delta\boldsymbol{\psi}^{(r-1)}+\delta\boldsymbol{\psi}^{(r)}. \tag{7}$$

2.2 The direct method with the Newmark's β=1/4 method

There are the two of the typical direct methods that derive the responses fulfilling the equation of the motion at the time t_{m+1}. One uses the predictor-corrector method, and the other uses the secant stiffness given by joining the two quantities of structural states at the times of t_m and t_{m+1}. Though the methods are popular and famous, the paper briefly describes the theories used in the computation to compare the proposed method.

(1) The predictor-corrector method, hereinafter shortened to PCM

Applying Newmark's β=1/4 method to the equation (1) gives the predictor of the increment displacement as follows.

$$\left(\frac{4}{\Delta t^2}\mathbf{M}+\frac{2}{\Delta t}\mathbf{C}_m+\mathbf{K}_{\mathrm{T}m}\right)\Delta\mathbf{u}^{(0)}=\mathbf{M}\left(\frac{4}{\Delta t}\dot{\mathbf{u}}_m+2\ddot{\mathbf{u}}_m-\mathbf{J}\Delta\ddot{\boldsymbol{\phi}}\right)+2\mathbf{C}_m\dot{\mathbf{u}}_m. \tag{8}$$

The corrector is derived from the equation of the motion and the unbalanced force at the time t_{m+1} given by the predictor or the corrector at the iterative time number r-1.

$$\left(\frac{4}{\Delta t^2}\mathbf{M}+\frac{2}{\Delta t}\mathbf{C}_{m+1}^{(r)}+\mathbf{K}_{\mathrm{T}\,m+1}^{(r)}\right)\boldsymbol{\delta}\mathbf{u}^{(r)}=\boldsymbol{\delta}\mathbf{U}_{m+1}^{(r)}, \tag{9}$$

where $\boldsymbol{\delta}\mathbf{U}_{m+1}^{(r)}$ is the unbalanced force at the time t_{m+1}.

The increment of the nodal displacement is renewed by $\Delta\mathbf{u}^{(r)}=\Delta\mathbf{u}^{(r-1)}+\boldsymbol{\delta}\mathbf{u}^{(r)}$.

(2) The secant stiffness method, hereinafter shortened to SSM

First computing the response at the time t_{m+1} uses the tangent stiffness at the time t_m, and then the second afterward in the iterative computation uses the secant stiffness. The increment of the nodal displacement in the time interval Δt is obtained from the following equation.

$$\left(\frac{4}{\Delta t^2}\mathbf{M}+\frac{2}{\Delta t}\mathbf{C}_m^{(r)}+\mathbf{K}_{\mathrm{S}m}^{(r)}\right)\Delta\mathbf{u}^{(r)}=\mathbf{M}\left(\frac{4}{\Delta t}\dot{\mathbf{u}}_m+2\ddot{\mathbf{u}}_m-\mathbf{J}\Delta\ddot{\boldsymbol{\phi}}\right)+2\mathbf{C}_m^{(r)}\dot{\mathbf{u}}_m\ , \tag{10}$$

where $\mathbf{K}_{\mathrm{S}m}^{(r)}$ is the secant stiffness.

3 COMPUTATIONAL MODEL AND THE INELASTIC VIBRATION CHARACTERISTIC

The paper uses the same frame structure as that used to be computed in the paper [5]. The structure is the steel pier for supporting the girder as shown in **Figure 1**. The pier fixed at the base plate is 11m in height and the upper part on the ground is 126kN in weight. The pier supports the girder of 957kN in weight and the mass is added to the top node, so that the structure is comparatively a top-heavy model. The computational model is axially divided to 19 elements of the bar models that the bending behavior is only inelastic and the others are elastic. All the deformation of bending, shear and elongation is assumed to be uniform in the elements. Consequently, the element model is the simplest in various inelastic models ever proposed, whereas it could be allowable to use the model because the purpose of the paper is to discuss the accuracy of numerical integrations, and because the fine division of a structure by using the inelastic model keeps enough accuracy.

The steel pier has the cross section of the box shape and the bending stiffness before/after the yielding are 49021.56 MNm^2 and 5071.8 MNm^2 respectively, and the bending moment at the yielding is 5MNm. The law of kinematic work-hardening applies to the bending behavior of the inelasticity of the computational model, as shown in **Figure 2** describing the hysteresis of the bending moment vs. the curvature at the base in an example of seismic responses.

When the bending stiffness in the pier changes like **Figure 2**, the natural frequencies of the structure used in the modal analysis fluctuate as shown in **Figure 3**. **Figure 3(a)** shows the first natural frequency, and **Figure 3(b)** shows the third natural frequency. The figures indicate that the deterioration of the bending stiffness remarkably decreases the natural frequencies. Therefore, inelastic seismic computations should take care to adopt the modes that the natural frequencies are higher than a main frequency area of a seismic wave but the decreasing natural frequencies come within the frequency area. By the way, since the second mode relates to the vertical vibration of the structure, the natural frequency is almost flat even after the bending yielding and then omitted.

The damping constant defined by the Rayleigh damping in the paper is determined by the tow constants of ν=0.114 sec^{-1} and μ=0.001157sec. The values of the two constants keep the damping constant of 0.02 at the two frequencies of 0.5Hz and 10Hz, and then the pier has around 0.02 of the damping constant in a main frequency area of the seismic vibration used in the paper.

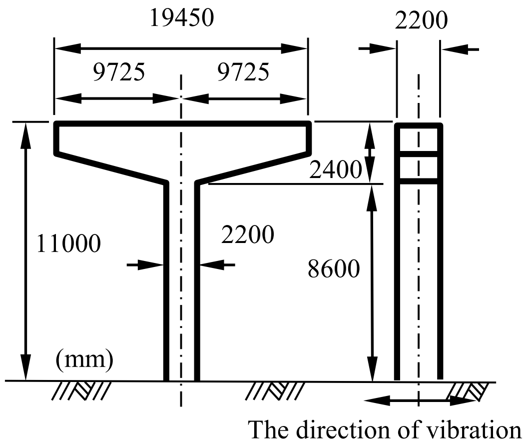

Figure 1: The steel pier used in the computation

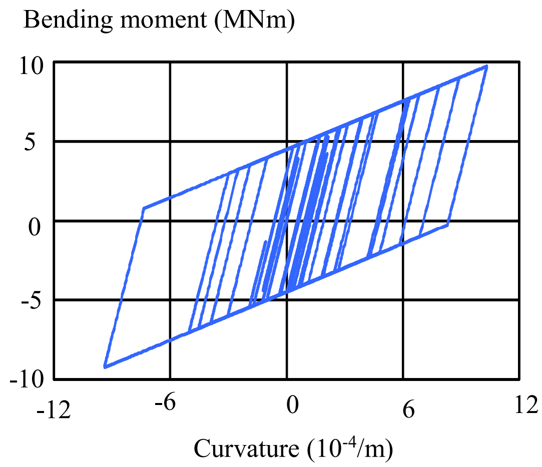

Figure 2: The hysteresis loop in the bending behavior of the element on the base

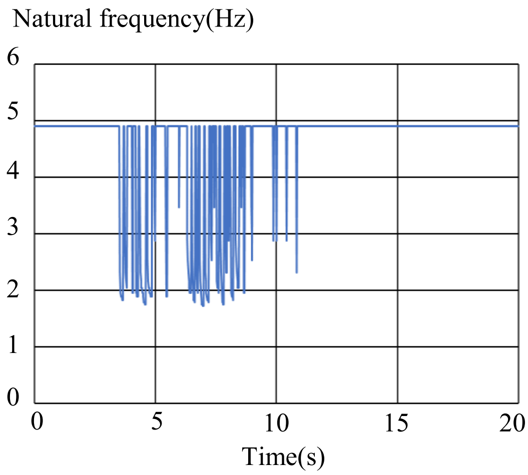

Figure 3(a): Fluctuation of the first natural frequency

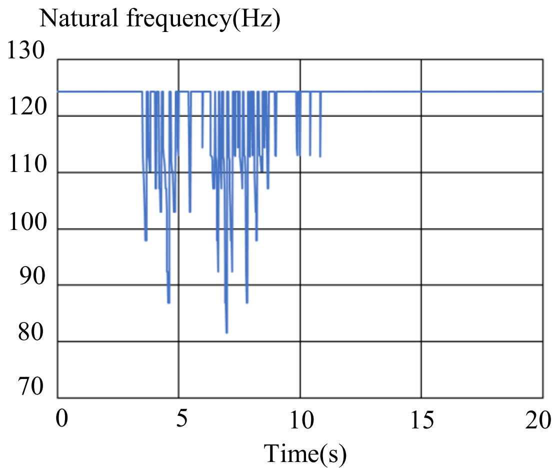

Figure 3(b): Fluctuation of the third natural frequency

4 COMPUTATIONAL RESULTS AND DISCUSSIONS

The input acceleration of the ground motion used in the computations is the acceleration record of Hyogo-ken Nanbu earthquake (1997). The acceleration of the digital data was recorded at the time interval of 0.02sec., and the computation applies the Fourier transformation to the record and the Fourier inverse transformation is used for the interpolation of the input acceleration. Most of the computations in the paper use the time interval of 0.002sec.. The reason for adopting so small time interval compared to that of the record is why the response displacement by the direct method with PCM converges at the time interval. The design criterion in Japan prescribes the convergence by the time interval. Additionally, the main frequency area of the input acceleration is around 0.5Hz to 5Hz.

The time histories of the horizontal component of the displacement at the top of the pier are shown in **Figure 4**. The figure compares the three waves of the time histories that the green line is the result by the direct method with PCM, the blue line is the results by the direct method with SSM and the dotted orange line is the result by the modal analysis with PCM.

The figure shows that the time history by the direct method with PCM is almost identical with that by the modal analysis and the direct method with SSM is only different from the others. Since the pier used for computing is a top-heavy structure, the first mode only is dominant and the third mode or higher mode in bending vibration does not almost affect the seismic response of the displacement. Therefore, although the direct method cannot filter high frequency components, the components may be so small that the direct method with PCM results in nearly equaling to the modal analysis. The reason why SSM gradually differ from PCM with the progress of time should be that a little difference of the bending stiffness affects the inelastic seismic response.

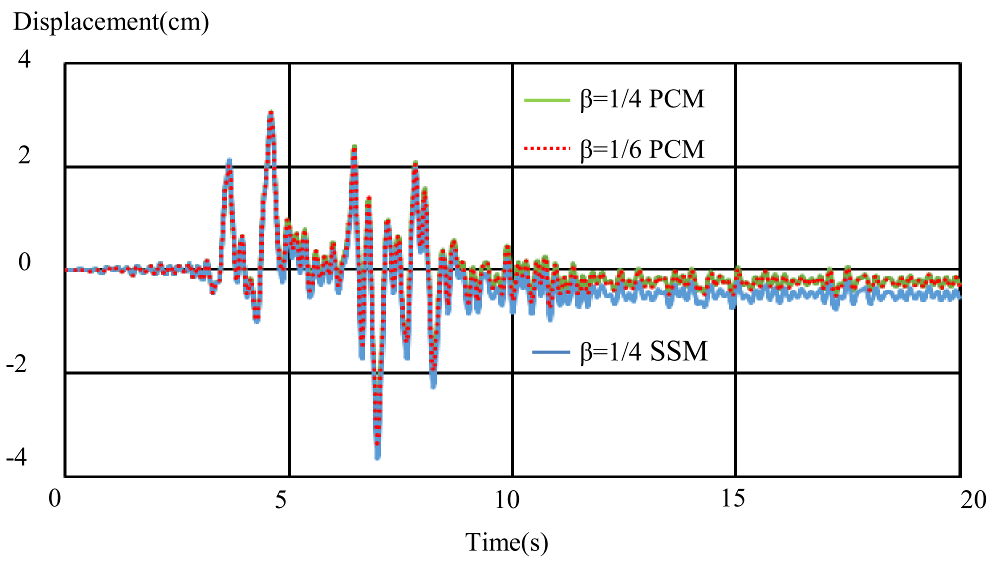

Figure 4: The time history responses of the displacement at the top in the pier

The lines in **Figure 5(a)** join the maximums of the horizontal component of displacement at all the nodes and the lines in **Figure 5(b)** join the maximums of the horizontal acceleration of all the nodes. The vertical axes in the figures indicate height in the pier.

The maximums of the horizontal displacement indicate that the result by the direct method with PCM almost agrees with that by the modal analysis but differs from the direct method with SSM as well as the time histories of the displacement.

The maximums of the horizontal acceleration indicate that the three results largely differ together except the maximums at the top of the pier. The mass at the top of the pier is especially so large that the top slowly oscillates, but the intermediate part of the pier vibrates with high frequencies. The modal analysis uses only the two modes and then the response does not include components of higher frequency more than around 125Hz. The response by direct method, however, includes the components of very high frequency and the components will contain errors.

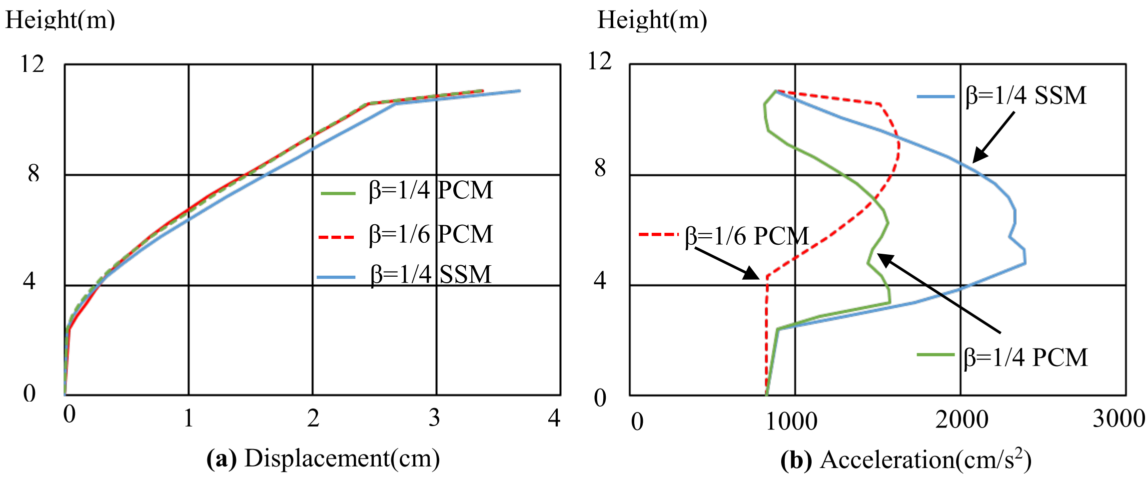

Figure 5: The maximums of the responce of the pier in height

12 CONCLUSIONS

- The paper proposed a method of the time integral with the modal analysis for computing seismic responses of inelastic structures. The advantage of the method is that the method can filter out components of high frequency from the seismic responses. The components of high frequency in the direct method tend to become errors included in the seismic responses.
- A typical steel pier with the inelasticity of bilinear characteristic was used for comparing seismic responses by the proposed method with that by existing methods.
- Since the pier was a top-heavy structure, the first mode only is dominant in the response, and then the displacement by the proposed method well agreed with the direct method with the predictor-corrector method but did not with the secant stiffness method.
- The acceleration responses of the pier computed by the three methods all differed each other.

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