LIMIT ANALYSIS AND INF-SUP CONDITIONS ON CONVEX CONES

STANISLAV SYSALA*, JAROSLAV HASLINGER*, SERGEY REPIN[†,‡]

* Institute of Geonics of the Czech Academy of Sciences, (IGN)
Department of Applied Mathematics and Computer Sciences & Department IT4Innovations,
Studentska 1768, 708 00 Ostrava, Czech Republic
e-mails: stanislav.sysala@ugn.cas.cz, hasling@karlin.mff.cuni.cz, web page:
http://www.ugn.cas.cz

[†] St. Petersburg Department of V.A. Steklov Institute of Mathematics of the Russian
Academy of Sciences,
27 Fontanka, 191023 St.Petersburg, Russia
e-mail: repin@pdmi.ras.ru - web page: http://www.pdmi.ras.ru

[‡] University of Jyväskylä,
Mattilanniemi 2, 40100 Jyväskylä, Finland
e-mail: sergey.s.repin@jyu.fi - web page: https://www.jyu.fi

Key words: perfect plasticity, limit analysis, conic optimization, inf-sup condition, finite element method, mesh adaptivity

Abstract. This paper is focused on analysis and reliable computations of limit loads in perfect plasticity. We recapitulate our recent results arising from a continuous setting of the so-called limit analysis problem. This problem is interpreted as a convex optimization subject to conic constraints. A related inf-sup condition on a convex cone is introduced and its importance for theoretical and numerical purposes is explained. Further, we introduce a penalization method for solving the kinematic limit analysis problem. The penalized problem may be solved by standard finite elements due to available convergence analysis using a simple local mesh adaptivity. This solution concept improves the simplest incremental method of limit analysis based on a load parametrization of an elastic-perfectly plastic problem.

1 INTRODUCTION

Stability of a structure is analyzed in many engineering areas. From this analysis, one can get various safety parameters that depend on the applied loads. One can also estimate failure zones describing collapse of the structure. Limit analysis is one of the main methods in stability assessment and is based on a parametrization of the load by a scalar factor. The related safety parameter is defined as a limit (ultimate) value of this factor. Beyond this value, the structure collapses. The limit load factor can be

determined by solving a specific optimization problem, which is called the limit analysis problem. This problem can be formulated either in terms of stresses (the static approach) or in terms of kinematic fields. The static and kinematic approaches are in mutual duality and lead to lower and upper bounds of the limit load factor, respectively. For literature survey, we refer to [6, 15].

Although limit analysis was originally developped as an analytical method [6], now it is widely used in computations. The simplest numerical scheme is based on solution of an elastoplastic problem by standard finite elements and incremental methods. However, such an approach may lead to an overestimation of the limit factor caused by locking effects. More sophisticated numerical schemes take into account two important features of limit analysis problems: possible discontinuities of kinematical fields along surfaces and the presence of conic contraints. The discontinuities influence a choice of finite elements or mixed finite elements. One can also choose one type of finite elements to find upper bounds of the limit factor and another type for lower bounds. Further, it is possible to transform the limit analysis problem to the second order cone programming, semidefinite programming or other conic optimization and to use convenient algorithms for such problems like interior point methods. Finally, mesh adaptivity is used to get the lower and upper bounds close each other or to visualize expected failure zones more accurately. For the numerical treatment, we refer to, e.g., [2, 12, 5, 11, 15].

This contribution arises from the classical theory of limit analysis which is based on perfect plasticity and the associative plastic flow rule. Although mathematical theory for continuous setting of the classical limit analysis problem is known for a long time, see, e.g., [21, 7, 13], some new results have appeared quite recently [8, 9, 14, 10]. We describe two of them and show their importance for numerical analysis and computations.

The first result is inspired by current computational experiences based on conic optimization, see the text above. We introduce a specific inf-sup condition related to the conic optimization, discuss its validity for basic yield criteria and show its consequences. In particular, by using this condition, one can prove the equivalence between the static and kinematic approaches and derive the analytical upper bounds of the limit load using functions which need not belong to the restrictive conic set. These bounds can be computable and used in a posteriori error analysis. The reader will find a systematic exposition in [14, 10].

The second result is inspired by incremental methods and construction of global working diagrams describing the material response during the loading process. To construct the global working diagram for continuous setting of the problem, we penalize the kinematic limit analysis problem (in order to release the conic constraints) and study the relation between the load factor λ and the penalty parameter α. The resulting $\lambda - \alpha$ curve is continuous, nondecreasing and tending to the limit load factor λ^* as $\alpha \to +\infty$. The parameter α may be also interpreted as the work of external forces. In addition, convergence results for standard finite elements and fixed α are available. The penalized problem may be solved by standard elastoplastic solvers and combined with a mesh adaptivity in order to estimate the limit factor λ^* from below within the kinematic approach. For a more detailed discussion and examples we refer to [8, 9].

This paper is organized as follows. Continuous setting the limit analysis problems and their duality relationship are summarized in Section 2. The respective inf-sup conditions on convex cones and their consequences are introduced in Section 3. The penalization of kinematic limit analysis problem is discussed in Section 4. A finite element discretization of the problem and its convergence analysis are presented in Section 5. This section also contains brief notes concerning used mesh adaptivity, solvers and implementation. Illustrative numerical examples to the presented solution concept can be found in recent papers [8, 9, 14, 10, 20, 16].

2   LIMIT ANALYSIS PROBLEM: CONTINUOUS SETTINGS

Let Ω be a bounded domain in $\mathbb{R}^3$ representing the investigated body. Its boundary $\partial\Omega$ is split into two nonempty parts Γ_D, Γ_N where the body is fixed and surface tractions $\boldsymbol{f} : \Gamma_N \to \mathbb{R}^3$ are prescribed, respectively. The volume forces are denoted as $\boldsymbol{F} : \Omega \to \mathbb{R}^3$. Both external forces $\boldsymbol{f}, \boldsymbol{F}$ are multiplied by a positive scalar factor $\lambda \mapsto \lambda\boldsymbol{f}, \lambda\boldsymbol{F}$.

By $\mathbb{M}^{3\times3}_{sym}$, we denote the space of symmetric 3×3 matrices (second order tensors) equipped with the scalar product $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sum_{i,j} \sigma_{ij}\varepsilon_{ij}$ and the norm $|\boldsymbol{\varepsilon}|^2 = \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}$. This space will be used for Cauchy stress tensors and infinitesimal small strain tensors, respectively. We assume that the Cauchy stress tensor $\boldsymbol{\sigma}$ belongs to a closed, convex set $B \subset \mathbb{M}^{3\times3}_{sym}$. This set is usually defined by a yield criterion and we let it independent of the space variable $\boldsymbol{x} \in \Omega$, for the sake of simplicity. Other assumptions on B will be specified in Section 3.

We use notation $\boldsymbol{\sigma}$ also for stress tensor fields, i.e. $\boldsymbol{\sigma} := \boldsymbol{\sigma}(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$. These fields belong to the space $\mathbb{S} := L^2(\Omega, \mathbb{M}^{3\times3}_{sym})$ of Lebesgue integrable functions [21]. The set of statically admissible stress fields reads as

$$Q_\lambda = \{\boldsymbol{\sigma} \in \mathbb{S} \,| \; -\operatorname{Div}\boldsymbol{\sigma} = \lambda\boldsymbol{F} \;\text{ in } \Omega, \;\; \boldsymbol{\sigma}\boldsymbol{n} = \lambda\boldsymbol{f} \;\text{ on } \Gamma_N\}, \quad \lambda \geq 0.$$

where Div denotes the divergence operator, i.e. $(\operatorname{Div}\boldsymbol{\sigma})_i = \sum_j \partial\sigma_{i,j}/\partial x_j$, and $\boldsymbol{n}$ is the unit outward normal vector to Ω. The set of plastically admissible stress fields is defined as follows:

$$P_B = \{\boldsymbol{\sigma} \in \mathbb{S} \,| \;\boldsymbol{\sigma}(\boldsymbol{x}) \in B, \;\; \boldsymbol{x} \in \Omega\} \,.$$

The limit load factor λ^* is defined as the supremum over all λ, for which there exists simultaneously a statically and plastically admissible stress field:

$$\lambda^* := \sup\{\lambda \geq 0 \,| \; \text{subject to } \exists\boldsymbol{\sigma} \in Q_\lambda \cap P_B\}. \tag{2.1}$$

Problem (2.1) represents the static principle of limit analysis. To derive its dual kinematic form, we introduce the space of kinematically admissible vector fields:

$$\mathbb{V} = \{\boldsymbol{v} \in W^{1,2}(\Omega, \mathbb{R}^3) \,| \; \boldsymbol{v} = \boldsymbol{0} \;\text{ on } \Gamma_D\},$$

where $W^{1,2}$ is standard notation for Sobolev spaces [21]. Within limit analysis, functions $\boldsymbol{v}$ rather represent velocity fields or displacement rates because displacement fields are unbounded as $\lambda \to \lambda^*$. The load functional reads as:

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v} \, dV + \int_{\Gamma_N} \boldsymbol{f} \cdot \boldsymbol{v} \, dS, \quad \boldsymbol{v} \in \mathbb{V}.$$

Using Green's theorem, one can rewrite the set Q_λ to a more convenient form [21, 7]:

$$Q_\lambda = \left\{ \boldsymbol{\sigma} \in \mathbb{S} \mid \int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v}) \, dV = \lambda L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V} \right\}, \quad \lambda \geq 0,$$

where $\varepsilon(\boldsymbol{v}) = (\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^\top)/2$ stands for the linearized strain tensor related to $\boldsymbol{v} \in \mathbb{V}$ and ∇ denotes the gradient operator. Further, it is well-known that

$$\boldsymbol{\sigma} \in Q_\lambda \quad \text{if and only if} \quad \inf_{\substack{\boldsymbol{v} \in \mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v}) \, dV = \lambda.$$

Using this equivalence, one can write the static principle as follows:

$$\lambda^* = \sup_{\boldsymbol{\sigma} \in P_B} \inf_{\substack{\boldsymbol{v} \in \mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v}) \, dV. \tag{2.2}$$

The dual (kinematic) principle arises from (2.2) by interchanging sup and inf:

$$\zeta^* = \inf_{\substack{\boldsymbol{v} \in \mathbb{V} \\ L(\boldsymbol{v})=1}} J_\infty(\boldsymbol{v}), \quad J_\infty(\boldsymbol{v}) = \sup_{\boldsymbol{\sigma} \in P_B} \int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v}) \, dV. \tag{2.3}$$

Here, J_∞ denotes the dissipation potential which need not be finite on $\mathbb{V}$ and thus some additional constraints on $\boldsymbol{v}$ must be taken into account, see Sections 3.1 and 3.2. The value ζ^* is called the kinematic limit factor. Since sup inf $\leq$ inf sup, we have $\lambda^* \leq \zeta^*$, thus ζ^* is only an upper bound of the true limit factor λ^*. Therefore, to use the kinematic principle correctly, it is important to study when $\lambda^* = \zeta^*$. Such a result was proven for specific sets B, but not in general. In particular, $\lambda^* = \zeta^*$ holds for B representing the von Mises and Tresca yield criteria [21, 7] or for bounded B [9]. In [13, 10], this equality was established for the Drucker-Prager yield criterion provided that the slope of the Drucker-Prager cone is sufficiently small. In the next section, we summarize recent results from [10] enabling us to study this equality in a more general way.

It is also worth noticing that the space $\mathbb{V}$ is sufficient for the definition of the kinematic limit value ζ^* but $\mathbb{V}$ may not contain minimizers of (2.3), which in general belong to an extended space $BD(\Omega)$ (BD – bounded deformations) associated with the so–called relaxed variational problem, see [21, 7, 13] and the references therein. The space $BD(\Omega)$ contains vector valued functions, which may have jumps along some surfaces. In limit analysis, these discontinuities are associated with failure zones.

3 INF-SUP CONDITION FOR LIMIT ANALYSIS PROBLEM

From now on, we assume that the set B can be split as follows:

$$B = \mathcal{C} + A = \{\boldsymbol{\sigma} \in \mathbb{M}^{3 \times 3}_{sym} \mid \boldsymbol{\sigma} = \boldsymbol{\sigma}_\mathcal{C} + \boldsymbol{\sigma}_A, \ \boldsymbol{\sigma}_\mathcal{C} \in \mathcal{C}, \ \boldsymbol{\sigma}_A \in A\}, \quad A = B \cap \mathcal{C}^-, \tag{3.1}$$

where $\mathcal{C} \subset \mathbb{M}^{3 \times 3}_{sym}$ is the largest cone with vertex at zero which is contained in B and A is a bounded subset of the polar cone

$$\mathcal{C}^- = \{\boldsymbol{\eta} \in \mathbb{M}^{3 \times 3}_{sym} \mid \boldsymbol{\sigma}_\mathcal{C} : \boldsymbol{\eta} \leq 0 \ \forall \boldsymbol{\sigma}_\mathcal{C} \in \mathcal{C}\}.$$

This split of B is illustrated in Sections 3.1 and 3.2 below for the von Mises and Drucker-Prager yield criteria, respectively. Similarly, one can also split sets B represented by Tresca and Mohr-Coulomb yield criteria.

For B satisfying (3.1), the kinematic limit analysis problem can be written in a more convenient form. First, we have

$$\zeta^* = \inf_{\substack{\boldsymbol{v} \in \mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega j_\infty(\varepsilon(\boldsymbol{v}))\, dV, \quad j_\infty(\boldsymbol{\varepsilon}) = \sup_{\boldsymbol{\sigma} \in B} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}. \tag{3.2}$$

Further, since

$$j_\infty(\boldsymbol{\varepsilon}) = \sup_{\boldsymbol{\tau} \in B} \boldsymbol{\tau} : \boldsymbol{\varepsilon} = \begin{cases} j_\infty^A(\boldsymbol{\varepsilon}), & \boldsymbol{\varepsilon} \in \mathcal{C}^-, \\ \\ +\infty, & \boldsymbol{\varepsilon} \notin \mathcal{C}^-, \end{cases} \quad j_\infty^A(\boldsymbol{\varepsilon}) := \max_{\boldsymbol{\tau} \in A} \boldsymbol{\tau} : \boldsymbol{\varepsilon}, \quad \forall \boldsymbol{\varepsilon} \in \mathbb{M}_{sym}^{3\times3}, \tag{3.3}$$

one can specify admissible kinematic fields in (3.2) by

$$\mathcal{K} = \{\boldsymbol{v} \in \mathbb{V} \mid \varepsilon(\boldsymbol{v}) \in \mathcal{C}^- \text{ in } \Omega\}, \tag{3.4}$$

and write

$$\zeta^* = \inf_{\substack{\boldsymbol{v} \in \mathcal{K} \\ L(\boldsymbol{v})=1}} J_\infty^A(\boldsymbol{v}), \quad J_\infty^A(\boldsymbol{v}) := \int_\Omega j_\infty^A(\varepsilon(\boldsymbol{v}))\, dV. \tag{3.5}$$

Let us note that the functional J_∞^A is finite-valued for any $\boldsymbol{v} \in \mathbb{V}$ unlike J_∞. Therefore, the setting (3.5) of the kinematic limit analysis problem is more transparent than the one in (2.3) or (3.2). The contraint set $\mathcal{K}$ has a cone property. It holds that $\boldsymbol{v} \in \mathcal{K}$ if and only if $\int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v})\, dV \leq 0$ for any $\boldsymbol{\sigma} \in P_\mathcal{C}$, where $P_\mathcal{C} := \{\boldsymbol{\sigma} \in \mathbb{S} \mid \boldsymbol{\sigma}(\boldsymbol{x}) \in \mathcal{C}, \ \boldsymbol{x} \in \Omega\}$.

The other and crucial assumption for the limit analysis problem is the satisfaction of the following inf-sup condition:

$$\inf_{\substack{\boldsymbol{\sigma} \in P_\mathcal{C} \\ \boldsymbol{\sigma} \neq 0}} \sup_{\substack{\boldsymbol{v} \in \mathbb{V} \\ \boldsymbol{v} \neq 0}} \frac{\int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v})\, dV}{\|\boldsymbol{\sigma}\|_\Omega \|\nabla \boldsymbol{v}\|_\Omega} = c_* > 0, \tag{3.6}$$

where $\|\cdot\|_\Omega$ denotes the standard L^2-norm of scalar, vector or tensor variables. The inf-sup condition (3.6) is non-standard in literature because the infimum is taken over the cone $P_\mathcal{C}$ instead of a current linear space. We have the following fundamental result [10].

Theorem 3.1. *Let (3.1) and (3.6) be satisfied. Then $\lambda^* = \zeta^*$.*

It is well-known that inf-sup conditions are also important in numerical analysis, see, e.g., [1]. Here, we use the inf-sup value c_* to find computable majorants of $\lambda^* = \zeta^*$.

From (3.5), it follows that any function $\boldsymbol{v} \in \mathcal{K}$, $L(\boldsymbol{v}) = 1$, defines an upper bound $J_\infty^A(\boldsymbol{v}) \geq \zeta^*$. In general, it is practically impossible to find $\boldsymbol{v} \in \mathcal{K}$ to make this bound sufficiently sharp. Moreover, the constraint set $\mathcal{K}$ has to be approximated within the numerical process. Therefore, we have derived the following upper bound of ζ^*, which uses a larger class of functions than the ones from $\mathcal{K}$, see [14, 10].

Theorem 3.2. *Let (3.1) and (3.6) be satisfied. Then for any $\boldsymbol{v} \in \mathbb{V}$ such that $L(\boldsymbol{v}) > c_*^{-1}\|L\|_*\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{v})\|_{\Omega}$, it holds:*

$$\zeta^* \leq \frac{J_\infty^A(\boldsymbol{v}) + c_*^{-1}\rho_A|\Omega|^{1/2}\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{v})\|_{\Omega}}{L(\boldsymbol{v}) - c_*^{-1}\|L\|_*\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{v})\|_{\Omega}}, \tag{3.7}$$

where $\|L\|_*$ is the norm of the load functional, ρ_A is the diameter of the bounded set A and Π_C denotes the projection of $\mathbb{M}_{sym}^{3\times3}$ onto $\mathcal{C}$.

Let us note that Π_C and ρ_A are known explicitly for classical yield criteria. Computable or analytical bounds of $\|L\|_*$ are standard in literature focused on a posteriori error analysis. Computable majorants of c_*^{-1} have been derived in [14, 10]. Further, for $\boldsymbol{v} \in \mathcal{K}$, $L(\boldsymbol{v}) = 1$, we have $|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{v})| = 0$ and thus the bound on the right of (3.7) coincides with $J_\infty^A(\boldsymbol{v})$ mentioned above. In Section 5, we combine (3.7) with our solution concept in limit analysis.

3.1 Example 1 – von Mises yield criterion

The set B corresponding to the von Mises criterion is defined by

$$B = \left\{\boldsymbol{\sigma} \in \mathbb{M}_{sym}^{3\times3} \mid |\boldsymbol{\sigma}^D| \leq \gamma\right\},$$

where $\gamma > 0$ represents the uniaxial yield stress, $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{1}{3}(\text{tr}\,\boldsymbol{\sigma})\boldsymbol{I}$ stands for the deviatoric part of $\boldsymbol{\sigma}$, $\text{tr}\,\boldsymbol{\sigma}$ denotes the trace of $\boldsymbol{\sigma}$ and $\boldsymbol{I}$ is the unit 3×3 matrix. This set is an unbounded cylinder aligned with the hydrostatic axis, see Figure 1.
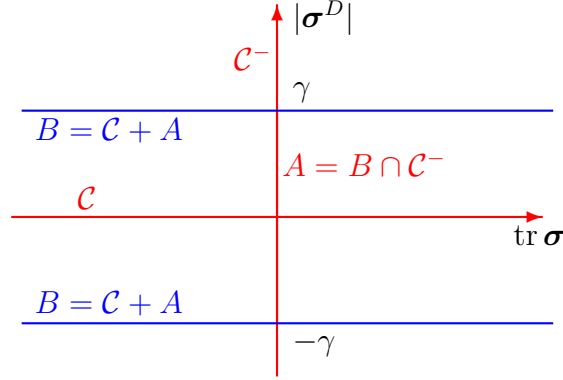

Figure 1: Decomposition of the set B for the von Mises yield criterion.

It is easy to see that B satisfies (3.1) with

$$\mathcal{C} = \{\boldsymbol{\sigma} \in \mathbb{M}_{sym}^{3\times3} \mid \boldsymbol{\sigma} = q\boldsymbol{I},\ q \in \mathbb{R}\}, \quad A = \{\boldsymbol{\eta} \in \mathcal{C}^- \mid |\boldsymbol{\eta}| \leq \gamma\}, \quad \mathcal{C}^- = \{\boldsymbol{\eta} \in \mathbb{M}_{sym}^{3\times3} \mid \text{tr}\,\boldsymbol{\eta} = 0\}.$$

Hence, we arrive at the following kinematic limit analysis problem:

$$\zeta^* = \inf_{\substack{\boldsymbol{v}\in\mathcal{K}\\L(\boldsymbol{v})=1}} \int_{\Omega} \gamma|\varepsilon^D(\boldsymbol{v})|\,dV, \quad \mathcal{K} = \{\boldsymbol{v} \in \mathbb{V} \mid \text{div}\,\boldsymbol{v} = 0 \ \text{ in } \ \Omega\}.$$

Further, (3.6) becomes the well-known inf-sup condition used in incompressible media problems:

$$\inf_{\substack{q \in L^2(\Omega) \\ q \neq 0}} \sup_{\substack{\boldsymbol{v} \in \mathbb{V} \\ \boldsymbol{v} \neq 0}} \frac{\int_\Omega q \operatorname{div} \boldsymbol{v} \, dV}{\|q\|_\Omega \|\nabla \boldsymbol{v}\|_\Omega} = c_\Omega > 0. \tag{3.8}$$

It is easy to see that $c_\Omega = \sqrt{3} c_*$ for the von Mises yield criterion, where c_* is from (3.6). This inf-sup condition holds if the body is fixed only on a part of the boundary (as assumed in Section 2). Nevertheless, the equality $\lambda^* = \zeta^*$ holds even if $\partial \Omega = \Gamma_D$, see [21, 7, 14]. In [14] and the references therein, there are discussed ways how to get computable majorants of c_Ω^{-1}. Finally, the upper bound (3.7) holds for the von Mises yield criterion with $\rho_A = \gamma$ and $\|\Pi_\mathcal{C} \varepsilon(\boldsymbol{v})\|_\Omega = \frac{1}{\sqrt{3}} \|\operatorname{div} \boldsymbol{v}\|_\Omega$.

3.2 Example 2 – Drucker-Prager yield criterion

The set B corresponding to the Drucker-Prager yield criterion is defined by

$$B := \left\{ \boldsymbol{\sigma} \in \mathbb{M}^{3 \times 3}_{sym} \mid |\boldsymbol{\sigma}^D| + \frac{a}{3} \operatorname{tr} \boldsymbol{\sigma} \leq \gamma \right\}, \quad a, \gamma > 0.$$
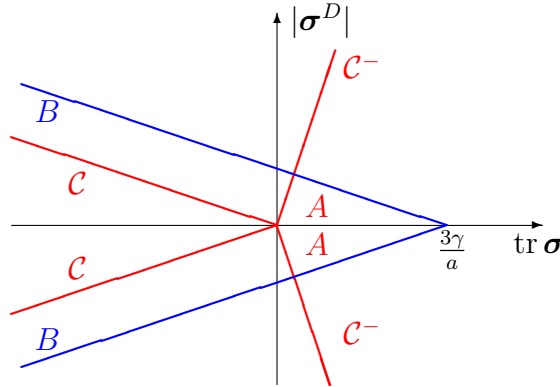

Figure 2: Decomposition of the set B for the Drucker-Prager yield criterion.

It is the cone with vertex at $\frac{\gamma}{a} \boldsymbol{I}$, see Figure 2. Again, B satisfies (3.1) with

$$\mathcal{C} = \{ \boldsymbol{\sigma} \in \mathbb{M}^{3 \times 3}_{sym} \mid |\boldsymbol{\sigma}^D| + \frac{a}{3} \operatorname{tr} \boldsymbol{\sigma} \leq 0 \}, \quad \mathcal{C}^- = \{ \boldsymbol{\eta} \in \mathbb{M}^{3 \times 3}_{sym} \mid \operatorname{tr} \boldsymbol{\eta} \geq a |\boldsymbol{\eta}^D| \}, \quad A = B \cap \mathcal{C}^-.$$

Hence, the kinematic limit analysis problem reads as follows:

$$\zeta^* = \inf_{\substack{\boldsymbol{v} \in \mathcal{K} \\ L(\boldsymbol{v}) = 1}} \int_\Omega \frac{\gamma}{a} \operatorname{div} \boldsymbol{v} \, dV, \quad \mathcal{K} = \{ \boldsymbol{v} \in \mathbb{V} \mid \operatorname{div} \boldsymbol{v} \geq a |\varepsilon^D(\boldsymbol{v})| \text{ in } \Omega \}.$$

It can be shown that the inf–sup condition (3.6) is valid under the additional assumption on the material parameter a defining the slope of the Drucker-Prager cone [10]:

$$a < (c_\Omega^{-2} - 1/3)^{-1/2},$$

where c_Ω is the inf-sup value from (3.8). Thus (3.6) and $\zeta^* = \lambda^*$ hold for a sufficiently small slope of the Drucker-Prager cone. If the body is fixed on the whole boundary then (3.6) does not hold. In this case, $\mathcal{K} = \{0\}$ and $\lambda^* = \zeta^* = +\infty$ as was shown in [13].

Computable majorants of c_*^{-1} use the bounds of the inf-sup value c_Ω, see [10]. Finally, the upper bound (3.7) for the Drucker-Prager yield criterion holds with $\rho_A = \frac{\gamma}{\alpha}\sqrt{3}$ and

$$
\Pi_\mathcal{C}\boldsymbol{\varepsilon} = \begin{cases} \boldsymbol{\varepsilon}, & \boldsymbol{\varepsilon} \in \mathcal{C}, \\ 0, & \boldsymbol{\varepsilon} \in \mathcal{C}^-, \\ \frac{a|\boldsymbol{\varepsilon}^D|-\mathrm{tr}\,\boldsymbol{\varepsilon}}{3+a^2}\left(a\frac{\boldsymbol{\varepsilon}^D}{|\boldsymbol{\varepsilon}^D|} - I\right), & \boldsymbol{\varepsilon} \notin \mathcal{C} \cup \mathcal{C}^-. \end{cases} \tag{3.9}
$$

4 PENALIZATION OF KINEMATIC LIMIT ANALYSIS PROBLEM

This section summarizes the results published in [8, 9, 18]. Recall that

$$
\zeta^* = \inf_{\substack{\boldsymbol{v}\in\mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega j_\infty(\varepsilon(\boldsymbol{v}))\,dV, \quad j_\infty(\boldsymbol{\varepsilon}) = \sup_{\boldsymbol{\sigma}\in B}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}, \tag{4.1}
$$

where j_∞ may be infinite for some $\boldsymbol{v} \in \mathbb{V}$. One can penalize the problem (4.1) as follows:

$$
\tilde{\psi}(\alpha) := \inf_{\substack{\boldsymbol{v}\in\mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega j_\alpha(\varepsilon(\boldsymbol{v}))\,dx, \quad j_\alpha(\boldsymbol{\varepsilon}) = \max_{\boldsymbol{\sigma}\in B}\{\boldsymbol{\sigma}:\boldsymbol{\varepsilon} - \frac{1}{2\alpha}\mathbb{C}^{-1}\boldsymbol{\sigma}:\boldsymbol{\sigma}\}, \tag{4.2}
$$

where $\alpha > 0$ is the penalization parameter and $\mathbb{C}$ is a fourth order tensor representing the Hooke law (known from elasticity). Let us note that limit analysis and also the results summarized below are independent of elastic material parameters defining $\mathbb{C}$.

It holds that j_α is finite-valued, convex, smooth and $j_\alpha \to j_\infty$ pointwisely in $\mathbb{M}_{sym}^{3\times3}$ as $\alpha \to +\infty$. It is readily seen that $j_\alpha \le j_\infty$ so that $\tilde{\psi}(\alpha) \le \zeta^*$ for any $\alpha > 0$. In addition, the function $\tilde{\psi}$ is continuous, nondecreasing and $\lim_{\alpha\to+\infty}\tilde{\psi}(\alpha) = \lambda^*$. Therefore, the values $\tilde{\psi}(\alpha)$ define lower bounds of the static limit factor λ^* although we used the kinematic principle. For smaller values of α, the function $\tilde{\psi}$ is usually linear and $\lim_{\alpha\to0}\tilde{\psi}(\alpha) = 0$. The properties of $\tilde{\psi}$ are depicted in Figure 3.
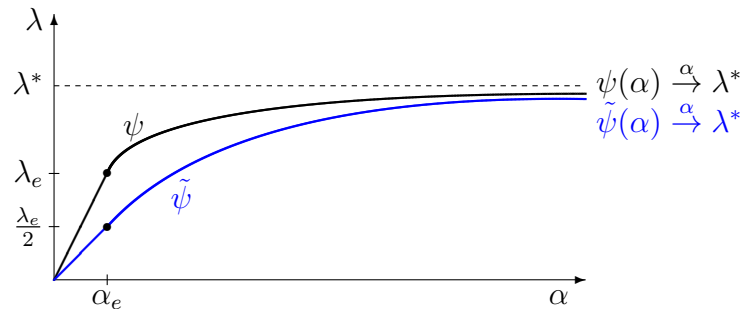

Figure 3: Properties of the functions $\tilde{\psi}$ and ψ. The value α_e is the threshold of purely elastic response.

Let $\boldsymbol{u}_\alpha$ denote a minimizer to the problem (4.2). To simplify the next presentation, we assume that $\boldsymbol{u}_\alpha$ belongs to $\mathbb{V}$ although, in general, $\boldsymbol{u}_\alpha$ belongs to the space $BD(\Omega)$ mentioned above. For a more general treatment, we refer to [8].

Next, we show that the penalized problem (4.2) is closely related to a simplified static version of the elastic-perfectly plastic problem (Hencky's problem) and to incremental limit analysis. To this end, we denote j as the function j_α for $\alpha = 1$. Clearly, $j_\alpha(\boldsymbol{\varepsilon}) = \frac{1}{\alpha} j(\alpha\boldsymbol{\varepsilon})$ for any $\alpha > 0$ and any $\boldsymbol{\varepsilon} \in \mathbb{M}^{3\times3}_{sym}$. Let $\Pi_B \colon \mathbb{M}^{3\times3}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ stand for the (Fréchet) derivative of j. The function Π_B represents the elastic-perfectly plastic constitutive operator, i.e., the projection of the elastic stress tensor $\mathbb{C}\boldsymbol{\varepsilon}$ onto the set B of admissible stress tensors. Notice that the value $\Pi_B(\alpha\boldsymbol{\varepsilon})$ is the derivative of j_α at $\boldsymbol{\varepsilon}$. Then (4.2) can be written as the following saddle-point problem: *given $\alpha > 0$, find $\boldsymbol{u}_\alpha \in \mathbb{V}$ and $\lambda_\alpha \in \mathbb{R}$ such that $L(\boldsymbol{u}_\alpha) = 1$ and*

$$\int_\Omega \Pi_B(\alpha\varepsilon(\boldsymbol{u}_\alpha)) : \varepsilon(\boldsymbol{v}) \, dV = \lambda_\alpha L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V}. \tag{4.3}$$

The above mentioned elastic-perfectly plastic problem reads as: *given $\lambda > 0$, find $\boldsymbol{u}_\lambda \in \mathbb{V}$:*

$$\int_\Omega \Pi_B(\varepsilon(\boldsymbol{u}_\lambda)) : \varepsilon(\boldsymbol{v}) \, dV = \lambda L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V}. \tag{4.4}$$

We see that the equations (4.3) and (4.4) are almost identical. In particular, if $\lambda = \lambda_\alpha$ then $\boldsymbol{u}_\lambda = \alpha\boldsymbol{u}_\alpha$. From this, we see that the displacement field $\boldsymbol{u}_\lambda$ is unbounded as $\lambda \to \lambda^*$, i.e. $\alpha \to +\infty$. In [8], it was shown that there exists a unique solution component λ_α in (4.3) and $\lambda_\alpha = \int_\Omega \Pi_B(\alpha\varepsilon(\boldsymbol{u}_\alpha)) : \varepsilon(\boldsymbol{u}_\alpha) \, dV \geq 0$. Therefore, one can introduce the function $\psi : \alpha \mapsto \lambda_\alpha$. It holds that $\psi \geq \tilde{\psi}$. In addition, ψ has analogous properties as $\tilde{\psi}$, i.e. it is a continuous, nondecreasing function satisfying $\psi(0) = 0$ and $\lim_{\alpha\to+\infty} \psi(\alpha) = \lambda^*$, see Figure 3.

Since the function ψ transforms the penalization parameter α into the load factor λ it describes the material response during loading. The inverse mapping $\lambda \mapsto \alpha$ is defined by the relation $\alpha = L(\boldsymbol{u}_\lambda)$. Therefore, the penalty parameter α may be interpreted as the work of external forces. The knowledge of ψ enables us to control the loading process indirectly through α. This idea was originally introduced in [19, 3] for a discrete setting of the problem.

5   DISCRETIZATION AND SOLVERS

The penalized problem (4.2) is solved by standard finite elements which are conforming with respect to $\mathbb{V}$. Let $\mathbb{V}_h$ be the corresponding finite-dimensional subspace of $\mathbb{V}$ and $\boldsymbol{u}_{h,\alpha}$, ψ_h, λ_h^* and ζ_h^* be the discrete counterparts of u_α, ψ, λ^*, and ζ^*, respectively.

We have $\lambda_h^* = \zeta_h^*$. Further, ψ_h is a continuous and nondecreasing function satisfying $\psi_h(0) = 0$ and $\lim_{\alpha\to+\infty} \psi_h(\alpha) = \lambda_h^*$. If we consider a regular system of finite element partitions of Ω then $\psi_h(\alpha) \to \psi(\alpha)$ as $h \to 0_+$ for any $\alpha > 0$. Hence, $\lambda_h^* \geq \lambda^* \geq \psi(\alpha)$ for any $\alpha > 0$ and $h > 0$. These results are depicted in Figure 4 and were proven in [8, 9, 18]. We can see that the standard discretization may lead to reliable results if we fix the parameter α and use a mesh adaptivity. On the other hand, the discrete limit value λ_h^* can significantly overestimate λ^* since convergence $\lambda_h^* \to \lambda^*$ is not guaranteed.
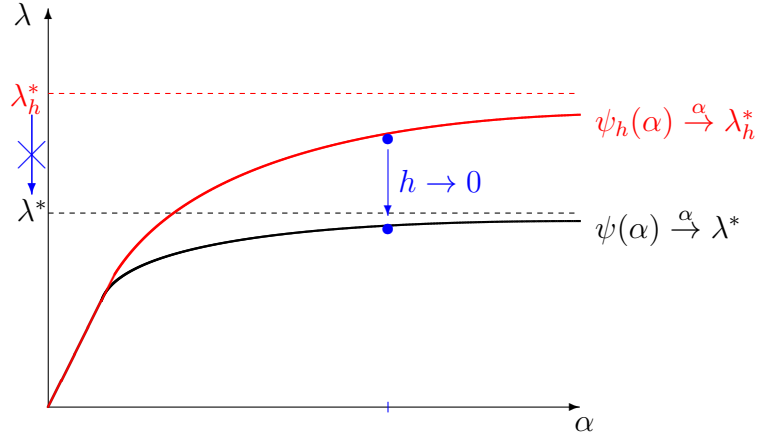
Figure 4: Properties of the functions ψ_h and ψ.

If a positive lower estimate of the inf-sup value c_* is available, one can use the upper bound (3.7) of ζ^* with $\boldsymbol{v} := \boldsymbol{u}_{h,\alpha}$:

$$\lambda^* = \zeta^* \leq \frac{J_\infty^A(\boldsymbol{u}_{h,\alpha}) + c_*^{-1}\rho_A|\Omega|^{1/2}\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{u}_{h,\alpha})\|_\Omega}{L(\boldsymbol{u}_{h,\alpha}) - c_*^{-1}\|L\|_*\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{u}_{h,\alpha})\|_\Omega}. \tag{5.1}$$

This bound is well-defined for larger values of α since $\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{u}_{h,\alpha})\|_\Omega \to 0$ as $\alpha \to +\infty$. In addition, it can be sharp and detect a possible overestimation of λ_h^* as was shown in [14, 10, 20].

In limit analysis, the process of failure is usually localized in a vicinity of a surface discontinuity while rigid deformations appear far from the failure. Therefore, it is quite natural to use mesh adaptive techniques, which can significantly reduce the number of unknowns and improve accuracy of the results. Using a mesh adaptivity, one can also achieve convergence $\psi_h(\alpha) \to \psi(\alpha)$ as $h \to 0_+$ in order to receive a lower bound of λ^*.

We use the following simple strategy of mesh adaptation. First, a sufficiently large value α is found by continuation on the coarsest mesh and then this value is fixed. For the mesh level $k = 0, 1, 2, \ldots$, we find the solution $\boldsymbol{u}_k$. Then $\int_e\, j_\infty^A(\varepsilon(\boldsymbol{u}_k))\, dV$ is evaluated for any element e and 10% of elements with the highest values is selected. Alternatively, one can also use the criterion based on the values $\int_e\, j_\alpha(\varepsilon(\boldsymbol{u}_k))\, dV$.

The discrete version of the penalized problem (4.2) is solved by a variant of the semismooth Newton method (SSNM) proposed in [3]. SSNM may be interpreted as a sequential quadratic programming in context of this optimization problem. In order to receive convergence for large α, SSNM is supplied with damping and/or regularized stiffness matrices. Continuation using α also improves convergence. For more details, we refer to [19, 3, 9, 17].

The presented numerical strategy is implemented in Matlab within our codes. Tangential stiffness matrices and load vectors are assembled by vectorized codes described in [4]. These codes are available for P1, P2, Q1 and Q2 elements in 2D and 3D. Illustrative numerical examples for various yield criteria and benchmarks are presented in recent papers [8, 9, 14, 17, 10, 20, 16]. In particular, [16] illustrates that the solution concept

works well even for composite materials with the von Mises yield criterion for one material component and the Drucker-Prager yield criterion for another one.

6 CONCLUSION

Two recent theoretical results in classical limit analysis have been presented: the related inf-sup condition and its importance, and the existence of a continuous loading path in perfect plasticity. Using these results, we have proposed a solution concept based on a penalization of the kinematic limit analysis problem. We have shown that the penalization method may be interpreted as an improved incremental approach and can be solved similarly as standard elastoplastic problems. Therefore, our solution strategy is convenient for scientists familiar with current computational plasticity. This strategy can be realized within commercial softwares which are not specialized in limit analysis.

REFERENCES

[1] Boffi, D., Brezzi, F., and Fortin, M. *Mixed finite element methods and applications.* Springer, Heidelberg (2013).

[2] Borges, L.A., Zouain, N., Costa, C., et al. An adaptive approach to limit analysis. *Int. J. Solids Struct.* (2001) **38**: 1707–1720.

[3] Cermak, M., Haslinger, J., Kozubek, T. and Sysala, S. Discretization and numerical realization of contact problems for elastic-perfectly plastic bodies. PART II – numerical realization. *ZAMM - Z. Angew. Math. Mech.* (2015) **95**: 1348–1371.

[4] Čermák, M., Sysala, S. and Valdman, J. Efficient and flexible MATLAB implementation of 2D and 3D elastoplastic problems. *Applied Mathematics and Computation* (2019) **355**: 595–614.

[5] Ciria, H., Peraire, J. and Bonet, J. Mesh adaptive computation of upper and lower bounds in limit analysis. *Int. J. Numer. Methods Engng* (2008) **75**: 899–944.

[6] Chen, W.F. and Liu, X. *Limit analysis in soil mechanics.* Elsevier, 2012.

[7] Christiansen, E. Limit analysis of colapse states. In: Ciarlet, P.G. and Lions, J.L. (eds), *Handbook of Numer. Anal.*, Vol IV, Part 2, North-Holland (1996), 195–312.

[8] Haslinger, J., Repin, S. and Sysala, S. A reliable incremental method of computing the limit load in deformation plasticity based on compliance: Continuous and discrete setting. *Journal of Computational and Applied Mathematics* (2016) **303**: 156–170.

[9] Haslinger, J., Repin, S. and Sysala, S. Guaranteed and computable bounds of the limit load for variational problems with linear growth energy functionals. *Applications of Mathematics* (2016) **61**: 527–564.

[10] Haslinger, J., Repin, S. and Sysala, S. Inf-sup conditions on convex cones and applications to limit load analysis. *Mathematics and Mechanics of Solids* (2019), on-line available at https://doi.org/10.1177/1081286519843969.

[11] Krabbenhoft, K., Lyamin, A. V. and Sloan, S.W. Three-dimensional Mohr–Coulomb limit analysis using semidefinite programming. *Communications in Numerical Methods in Engineering*, (2008) **24**: 1107–1119.

[12] Makrodimopoulos, A. and Martin, C.M. Upper bound limit analysis using simplex strain elements and second-order cone programming. *Int. J. Numer. Analyt. Methods Geomech.* (2007) **31**: 835–865.

[13] Repin, S. and Seregin, G. Existence of a weak solution of the minimax problem arising in Coulomb-Mohr plasticity. *Nonlinear evolution equations*, 189–220, Amer. Math. Soc. Transl. (2), **164**, Amer. Math. Soc., Providence, RI, (1995).

[14] Repin, S., Sysala, S. and Haslinger, J. Computable majorants of the limit load in Hencky's plasticity problems. *Comp. & Math. with Appl.* (2018) **75**: 199–217.

[15] Sloan, S. Geotechnical stability analysis, *Géotechnique* (2013) **63**, 531–572.

[16] Sysala, S., Blaheta, R., Kolcun, A., Sčučka, J., Souček, K. and Pan, P. Computation of Composite Strengths by Limit Analysis. *Key Engineer. Mat.* (2019). Accepted.

[17] Sysala, S., Čermák, M., and Ligurský, T. Subdifferential-based implicit return-mapping operators in Mohr-Coulomb plasticity. *ZAMM - Z. Angew. Math. Mech.* (2017) **97**: 1502–1523.

[18] Sysala, S. and Haslinger, J. Truncation and Indirect Incremental Methods in Hencky's Perfect Plasticity. In: *Mathematical Modelling in Solid Mechanics*. Springer, Singapore, (2017), 265–284.

[19] Sysala, S., Haslinger, J., Hlaváček, I. and Čermák, M. Discretization and numerical realization of contact problems for elastic-perfectly plastic bodies. PART I – discretization, limit analysis. *ZAMM - Z. Angew. Math. Mech.* (2015) **95**: 333–353.

[20] Sysala, S., Haslinger, J. and Repin, S. Reliable computation and local mesh adaptivity in limit analysis. In *Programs and Algorithms of Numerical Mathematics 19*, Institute of Mathematics of the Czech Academy of Sciences, Prague (2019), 149–158.

[21] Temam, R. *Mathematical Problems in Plasticity*. Gauthier-Villars, Paris, (1985).