On the $\mathcal{O}_\beta$-hull of a planar point set*

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Abstract

We study the $\mathcal{O}_\beta$-hull of a planar point set, a generalization of the Orthogonal Convex Hull where the coordinate axes form an angle β. Given a set P of n points in the plane, we show how to maintain the $\mathcal{O}_\beta$-hull of P while β runs from 0 to π in $\Theta(n \log n)$ time and $O(n)$ space. With the same complexity, we also find the values of β that maximize the area and the perimeter of the $\mathcal{O}_\beta$-hull and, furthermore, we find the value of β achieving the best fitting of the point set P with a two-joint chain of alternate interior angle β.

1 Introduction

Let $\mathcal{O}_\beta$ be a set of two lines with slopes 0 and $\tan(\beta)$, where $0 < \beta < \pi$. A region in the plane is said to be *$\mathcal{O}_\beta$-convex*, if its intersections with all translations of any line in $\mathcal{O}_\beta$ are either empty or connected. An *$\mathcal{O}_\beta$-quadrant* is a translation of one of the ($\mathcal{O}_\beta$-convex) open regions that result from subtracting the lines in $\mathcal{O}_\beta$ from the plane. We call the quadrants of $\mathcal{O}_\beta$ as *top-right*, *top left*, *bottom-right*, and *bottom-left* according to their position with respect to the elements of $\mathcal{O}_\beta$, see Figure 1(a). Let P be a set of n points, and $\mathcal{Q}$ the set of all $\mathcal{O}_\beta$-quadrants that are *P-free*; i.e., that contain no elements of P. The *$\mathcal{O}_\beta$-hull* of P is the set

$$\mathcal{O}_\beta\mathcal{H}(P) = \mathbb{R}^2 - \bigcup_{q \in \mathcal{Q}} q$$

of points in the plane belonging to all connected supersets of P which are $\mathcal{O}_\beta$-convex [3, 11]. See Figure 1(b).

*In memorial of professor Ferran Hurtado, inspirational friend and colleague, acknowledging his key contribution to the development of Computational Geometry.

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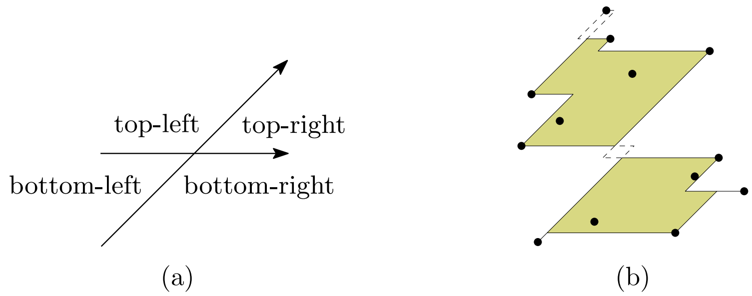


Figure 1: (a) A set $\mathcal{O}_\beta$-hull, the top-right, top-left, bottom right, and bottom left quadrants. (b) The corresponding $\mathcal{O}_\beta$-hull of a point set.

The concept of $\mathcal{O}_\beta$-convexity stemmed from the notion of *restricted orientations* [9], where geometric objects comply with a property (or a set of properties) related to some fixed set of lines. Researchers have extensively studied this notion by considering restricted-oriented polygons [9], proximity [18], visibility [17], and both restrictions and generalizations of $\mathcal{O}_\beta$-convexity. The particular case of *orthogonal convexity* [16] considers β to be fixed at $\frac{\pi}{2}$. In the more general $\mathcal{O}$*-convexity* [15, 16], $\mathcal{O}_\beta$ is replaced by a (possibly infinite) set of lines with arbitrary orientations. Other restricted-oriented notions of convexity include *D-convexity* [8] and $\mathcal{O}$*-convexity* [14]. The former is based in a functional (rather than set-theoretical) definition, while the latter (unlike $\mathcal{O}_\beta$-convexity) always leads to connected sets. For a comprehensive compilation of studies on the area please refer to Fink and Wood [7]. Some recent computational results can be found in [1, 2, 3, 12].

In this paper, we solve the problem of maintaining the combinatorial structure of $\mathcal{O}_\beta\mathcal{H}(P)$ while β goes from 0 to π, and apply this result to some optimization problems. Following the lines of Bae et al. [5], we find the values of β that maximize the area and the perimeter of $\mathcal{O}_\beta\mathcal{H}(P)$. In addition, we include an appendix extending the results from Díaz-Báñez et al. [6] to fit a two-joint not-necessarily orthogonal polygonal chain to a point set. See Figure 2.

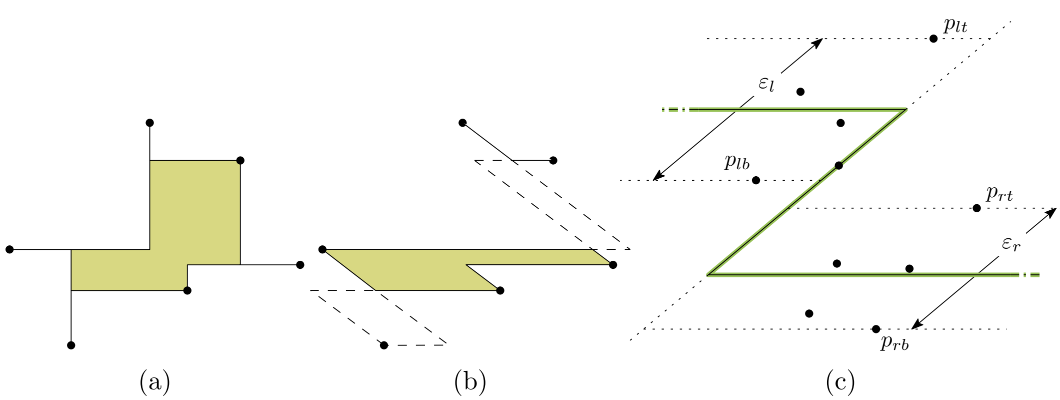


Figure 2: (a) $\mathcal{O}_{\frac{\pi}{2}}\mathcal{H}(P)$. (b) $\mathcal{O}_{\beta_0}\mathcal{H}(P)$, where $\beta_0 > \frac{\pi}{2}$. (c) A two-joint non-orthogonal polygonal chain fitting a point set.

In all cases, our general approach is to perform an angular sweep. We first discretize the set $\{\beta : \beta \in (0, \pi)\}$ into a linear sequence of increasing angles $\{\beta_1, \beta_2, \dots, \beta_{O(n)}\}$.

While β runs from 0 to π, each β_i corresponds to an angle where there is a change in the combinatorial structure of $\mathcal{O}_\beta\mathcal{H}(P)$. We then solve the particular problem for any $\beta \in [\beta_1, \beta_2)$ in $O(n \log n)$ time, and show how to update our solution in logarithmic time in the subsequent intervals $[\beta_i, \beta_{i+1})$. All our algorithms run in $O(n \log n)$ time and $O(n)$ space.

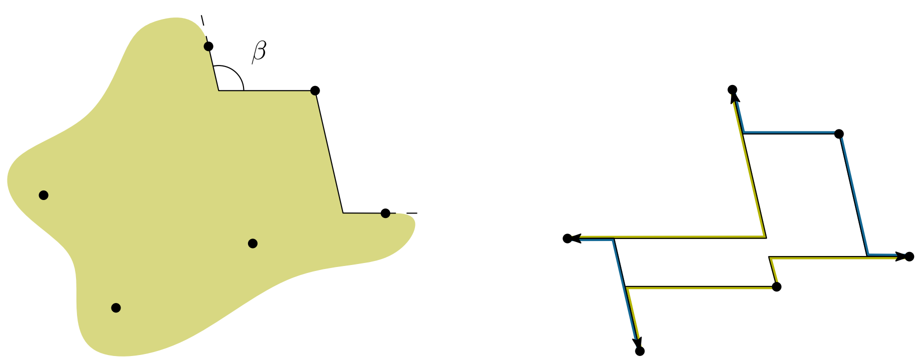
Outline of the paper. In Section 2 we show how to maintain the $\mathcal{O}_\beta$-hull of P while β goes from 0 to π. In Section 3 we extend this result to solve the optimization problems we mentioned above. We end in Section 4 with our concluding remarks.

2 The $\mathcal{O}_\beta$-hull of P

In this section we introduce definitions that are central to our results. We also show how to compute $\mathcal{O}_\beta\mathcal{H}(P)$ for a fixed value of β, and how to maintain its combinatorial structure while β runs from 0 to π.

2.1 Preliminaries

For the sake of simplicity, we will assume P to have no three colinear points, and no pair of points on a horizontal line. Consider the region $\mathcal{R}$ obtained by removing from the plane all top-right $\mathcal{O}_\beta$-quadrants free of elements of P. The *top-right $\mathcal{O}_\beta$-staircase* of P is the directed polygonal chain formed by the segment of the boundary of $\mathcal{R}$ that starts at the rightmost and ends at the topmost vertex (element of P that lies over the boundary) of $\mathcal{O}_\beta\mathcal{H}(P)$, with respect to the coordinate system defined by the lines in $\mathcal{O}_\beta$. We further define the *top-left*, *bottom-left*, and *bottom-right* $\mathcal{O}_\beta$-staircases in a similar way. See Figure 3.



(a) (b)

Figure 3: (a) Construction of the top-right $\mathcal{O}_\beta$-staircase. (b) The four $\mathcal{O}_\beta$-staircases of P.

Observation 1. *A point in P is a vertex of $\mathcal{O}_\beta\mathcal{H}(P)$ if, and only if, it is the apex of at least one P-free $\mathcal{O}_\beta$-quadrant free of elements of P. Conversely, a point in the plane lies in the interior of $\mathcal{O}_\beta\mathcal{H}(P)$ if, and only if, every $\mathcal{O}_\beta$-quadrant with apex on it contains at least one point in P.*

We say that an $\mathcal{O}_\beta$-quadrant is *maximal* if its boundary joins two consecutive elements in the sequence of vertices found while traversing an $\mathcal{O}_\beta$-staircase in its corresponding direction. Two $\mathcal{O}_\beta$-quadrants are *opposite* to each other if, after placing their apices over a common point, their rays bound opposite angles. Similarly, we say that two $\mathcal{O}_\beta$-staircases are opposite to each other, if they were constructed using opposite $\mathcal{O}_\beta$-quadrants. It is easy to see that $\mathcal{O}_\beta\mathcal{H}(P)$ is disconnected when the intersection of two opposite maximal $\mathcal{O}_\beta$-quadrants is not empty. In such case we say that both $\mathcal{O}_\beta$-quadrants *overlap*, and refer to their intersection as an *overlapping region*. See the regions bounded by dashed lines in Figures 1(b) and 2(b).

Observation 2. *Non-opposite $\mathcal{O}_\beta$-staircases cannot generate overlapping regions. Moreover, only one pair of $\mathcal{O}_\beta$-staircases can intersect at the same time.*

We will specify $\mathcal{O}_\beta\mathcal{H}(P)$ in terms of its vertices and its overlapping regions. From Observation 1, the set of vertices of $\mathcal{O}_\beta\mathcal{H}(P)$ is the set of maximal elements of P under vector dominance [4]. Thus they can be computed for a fixed value of β in $\Theta(n \log n)$ time and $O(n)$ space [10, 13]. Note that $\mathcal{O}_\beta$-staircases are monotone with respect to both lines in $\mathcal{O}_\beta$ (they could not bound $\mathcal{O}_\beta$-convex regions otherwise), so any pair of them intersect with each other at most a linear number of times. From Observation 2, in a fixed value of β there is at most a linear number of overlapping regions. Thus, if the vertices of $\mathcal{O}_\beta\mathcal{H}(P)$ are sorted according to either the x- or the y-axis, we can compute from them the set of overlapping regions in linear time. We get then the following theorem where the $\Omega(n \log n)$ time lower bound comes from the fact that from $\mathcal{O}_\beta\mathcal{H}(P)$ we can compute the convex hull of P in linear time.

Theorem 3. *For a fixed value of β, the sets of vertices and overlapping regions of $\mathcal{O}_\beta\mathcal{H}(P)$ can be computed in $\Theta(n \log n)$ time and $O(n)$ space.*

2.2 The angular sweep

The $\mathcal{O}_\beta$-hull of P is shown in Figure 4 at the *initial increasing configuration*, that is, where β is equal to an angle $\beta_I = 0 + \varepsilon$ for a small enough ε. Note that every point in P is the apex of a P-free $\mathcal{O}_\beta$-quadrant, and is thus contained in at least one $\mathcal{O}_\beta$-staircase: both top-right and bottom-left $\mathcal{O}_\beta$-staircases contain the whole set P, and the top-left and bottom-right $\mathcal{O}_\beta$-staircases are formed respectively, by the topmost and bottom-most points in P. Also, the intersection between the top-right and bottom-left $\mathcal{O}_\beta$-staircases generate a linear number of overlapping regions.

By performing an *increasing sweep* (where β goes from 0 to π), the initial increasing configuration is gradually transformed to the *initial decreasing configuration*, where β is equal to a value $\beta_D = \pi - \varepsilon$ for a small enough ε (see Figure 5). At this configuration, the top-left and bottom-right $\mathcal{O}_\beta$-staircases contain P and generate a linear number of overlapping regions, and the top-right and bottom-left $\mathcal{O}_\beta$-staircases contain respectively, the topmost and bottom-most points in P. Clearly, the converse of the above discussion holds: from the initial decreasing configuration, a *decreasing sweep* (where β goes from π to 0) will gradually transform $\mathcal{O}_{\beta_D}\mathcal{H}(P)$ into $\mathcal{O}_{\beta_I}\mathcal{H}(P)$.

During the transition between initial configurations, we recognize four types of events that modify the set of vertices and overlapping regions of $\mathcal{O}_\beta\mathcal{H}(P)$. An *insertion* (resp.

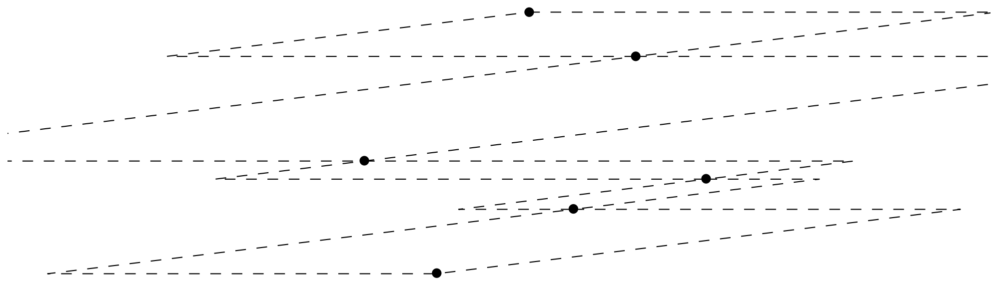


Figure 4: The initial increasing configuration.

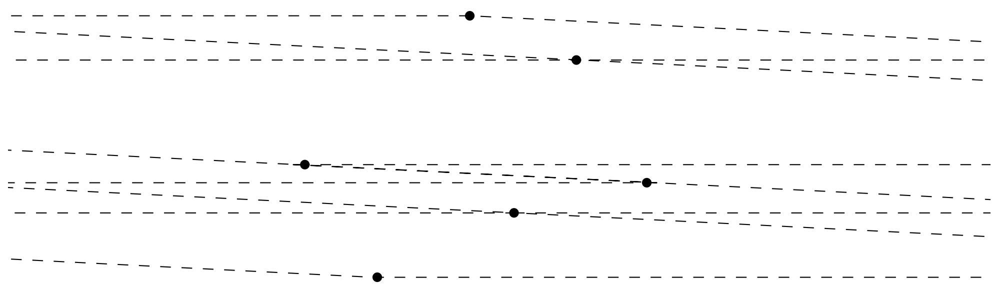


Figure 5: The initial decreasing configuration.

deletion) event occurs when a vertex joins (resp. leaves) a $\mathcal{O}_\beta$-staircase. At *overlap* (resp. *release*) events, an overlapping region is created (resp. destroyed).

Note that a vertex leaves (resp. joins) the same $\mathcal{O}_\beta$-staircase at most once, and thus, there is in total a linear number of insertion (resp. deletion) events. From Observation 2, between β_I and β_D there is always an interval $\phi = [\beta_1, \beta_2]$ such that, for any $\beta \in \phi$, the $\mathcal{O}_\beta$-hull of P contains no overlapping regions. Let us consider the angular intervals $\phi_I = [\beta_I, \beta_{N_1}]$ and $\phi_D = [\beta_{N_2}, \beta_D]$. An angular sweep in ϕ_I results in a linear number of releasing events caused by the deletion of all overlapping regions present at the initial increasing configuration. As any vertex supports at most two maximal $\mathcal{O}_\beta$-quadrants, an additional linear number of region events are generated by vertex events and, therefore, region events in ϕ_I add up to $O(n)$. Using the same argument on ϕ_D, we can count a linear number of these events during an angular sweep.

Lemma 1. *There are $O(n)$ events during an angular sweep.*

We now show how to compute the sequence of increasing angles that mark vertex and overlapping events during an angular sweep.

Insertion and deletion events. The set of vertices of $\mathcal{O}_\beta\mathcal{H}(P)$ on the top-right $\mathcal{O}_\beta$-staircase has a total ordering that, at any value of β is given by traversing the staircase along its direction. At the initial configuration, the order is also given by the sequence $p_1, \ldots, p_n$ of points in P labeled in ascending vertical order.

Let us consider the set $\alpha(P) = \{\alpha_1, \ldots, \alpha_{n-1}\}$ where for each α_i, the slope of the line through p_i and p_{i+1} equals $\tan(\alpha_i)$. In an increasing sweep, the first point leaving the top-right $\mathcal{O}_\beta$-staircase is p_i. Indeed, for any $\beta > \alpha_i$, a top-right $\mathcal{O}_\beta$-quadrant with apex over p_i is not P-free. This is not the case for points corresponding to any α_j such that $\alpha_j > \alpha_i$ and $\alpha_j > \beta$. See Figure 6.

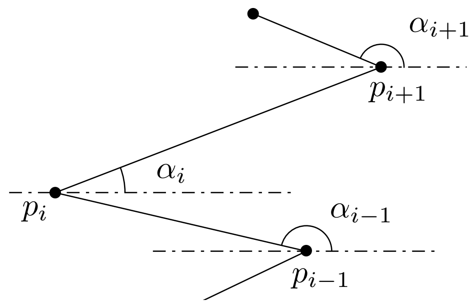


Figure 6: Insertion and deletion events for the top-right $\mathcal{O}_\beta$-staircase.

To compute the next value of β where a point will leave the top-right $\mathcal{O}_\beta$-staircase, we must remove α_i from $\alpha(P)$, update α_{i-1} to the angle where the slope of the line through p_{i-1} and p_{i+1} equals $\tan(\alpha_{i-1})$, and compute the new smallest element of $\alpha(P)$. A recursive repetition of this computation allows us to obtain all deletion events corresponding to the top-right $\mathcal{O}_\beta$-staircase.

Lemma 2. *All insertion and deletion events can be computed in $O(n \log n)$ time and $O(n)$ space.*

Proof. Store the points in P in a balanced search tree ordered according to the y-axis, and the set $\alpha(P)$ in a priority queue. From Lemma 1, the algorithm described above requires $O(n \log n)$ time and $O(n)$ space to compute the sets of insertion and deletion events, associated with the top-right $\mathcal{O}_\beta$-staircase. Considering the angles shown in Figure 7, a similar algorithm can be used to obtain the corresponding events for the remaining $\mathcal{O}_\beta$-staircases in the same time and space complexity. □

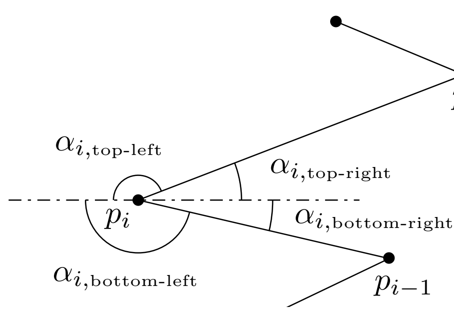


Figure 7: Lemma 2.

Overlap and release events. Let Q_r and Q_l be respectively, a pair of overlapping top-right and bottom-left maximal $\mathcal{O}_\beta$-quadrants. Consider that Q_r is supported by the vertices p_j, p_{j+1}, and Q_l by the vertices p_k, p_{k+1}. Also, assume the supporting points are labeled according to the total ordering of their corresponding staircases (see Figure 8).

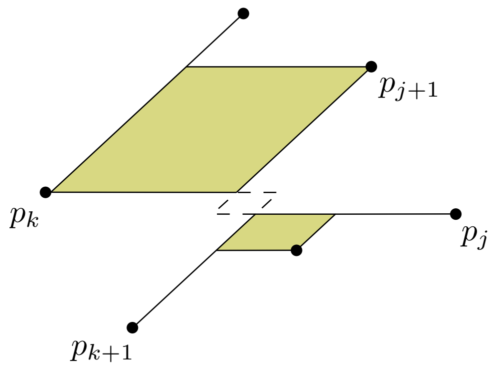


Figure 8: An overlapping region (bounded by dashed lines) generated by the intersection between a top-right and a bottom-left maximal $\mathcal{O}_\beta$-quadrants.

The *full overlap event* for the overlapping region defined by Q_r and Q_l is the angle ω for which the slope of the line through p_{j+1} and p_{k+1} equals $\tan(\omega)$. If the supporting points do not leave their corresponding staircases, this event marks the value of β where the overlapping region disappears.

Let $\omega(P)$ be the set of full overlap events for all the overlapping regions at the initial increasing configuration, and $\alpha_d(P)$ the set of all deletion events corresponding to the vertices over the top-right and bottom-left $\mathcal{O}_\beta$-staircases. Let ω_m and α_m be the smallest values in $\omega(P)$ and $\alpha_d(P)$, respectively. Performing an increasing sweep, to obtain the first release event, we need to deal with the following cases:

1. α_m corresponds to a supporting point, and $\alpha_m \leq \omega_m$. In this case, α_m needs to be processed and $\omega(P)$ needs to be updated. By removing a supporting point, at most two overlapping regions are terminated (two release events are added to $\omega(P)$), and at most one new overlapping region is generated (one overlapping event and one full overlap event are added to $\omega(P)$). After updating $\omega(P)$, ω_m and α_m are recomputed and the test is repeated.

2. α_m does not correspond to a supporting point. In this case, ω_m is the first release event.

To compute the next release event, we must remove the current release event from $\omega(P)$, and recompute ω_m as described above. A recursive repetition of these steps allow us to obtain all release events caused by intersections between the top-right and bottom-left $\mathcal{O}_\beta$-staircases.

Lemma 3. *All overlap and release events can be computed in $O(n \log n)$ time and $O(n)$ space.*

Proof. Store the points in P in a balanced search tree ordered according to the y-axis, and the sets $\alpha_d(P), \omega(P)$ in priority queues. From Lemma 1, the algorithm described above requires $O(n \log n)$ time and $O(n)$ space to compute the sets of overlap and release events associated with the top-right and bottom-left $\mathcal{O}_\beta$-staircases. A similar algorithm can be used to obtain the events associated to the top-right and bottom-left $\mathcal{O}_\beta$-staircases, with the same time and space upper bounds. $\square$

Maintaining $\mathcal{O}_\beta\mathcal{H}(P)$. Considering the previous results, the maintenance of $\mathcal{O}_\beta\mathcal{H}(P)$ is straightforward:

1. Compute all vertex and overlap events, and store them in a list sorted by appearance during an increasing sweep.

2. Compute $\mathcal{O}_{\beta_I}\mathcal{H}(P)$. Store in height balanced trees the total orders of the sets of vertices lying over the four $\mathcal{O}_\beta$-staircases. Store the set of overlapping regions in any constant-time access data structure (such as a hash table).

3. Simulate the angular sweep by traversing the list of events. At each insertion and deletion event, update the corresponding set of vertices. At each overlap and release event, update the set of overlapping regions.

From Lemmas 2 and 3, to compute the sets of vertex and overlap events, we require $O(n \log n)$ time and $O(n)$ space. As we have a linear number of elements on each set, we can merge them into a single ordered set using $O(n \log n)$ time. Thus, item 1 requires $O(n \log n)$ time and $O(n)$ space.

From Theorem 3, computing $\mathcal{O}_\beta\mathcal{H}(P)$ for any fixed value of β takes $O(n \log n)$ time and $O(n)$ space. Every $\mathcal{O}_\beta$-staircase contains at most n elements and therefore, to store their total order in a height balanced tree we require $O(n \log n)$ time. Using a hash table, we can initialize the set of overlapping regions in $O(n)$ time. Therefore, item 2 requires $O(n \log n)$ time and $O(n)$ space.

At each insertion and deletion event, updating the corresponding set of $\mathcal{O}_\beta$-maximal elements requires $O(\log n)$ time per operation. Updates on the set of overlapping regions takes constant time, so item 3 takes $O(n \log n)$ time. From this analysis we get that, in total, we can compute and maintain $\mathcal{O}_\beta\mathcal{H}(P)$ through an angular sweep in $O(n \log n)$ time and $O(n)$ space. From Theorem 3, this time complexity is optimal.

Theorem 4. *Computing and maintaining $\mathcal{O}_\beta\mathcal{H}(P)$ through an angular sweep requires $\Theta(n \log n)$ time and $O(n)$ space.*

3 Application problems

In this section we extend the results from Section 2 to the solution of related optimization problems. We deal with the problem of maximizing the area and the perimeter of $\mathcal{O}_\beta\mathcal{H}(P)$ (Sections 3.1 and 3.2, respectively). As an extra application, in Appendix A we deal with the problem of fitting a two-joint polygonal chain to a point set.

3.1 Area optimization.

In this section we solve the following problem:

Problem 1 (Maximum area)**.** Given a set P of n points in the plane, compute the value of β for which $\mathcal{O}_\beta\mathcal{H}(P)$ has maximum area.

Let $\{\beta_1, \ldots, \beta_{O(n)}\}$ be the sequence of (vertex and overlapping) events, ordered by appearance during an increasing sweep. Following the lines of Bae et al. [5] (see also Figure 9), we express the area of $\mathcal{O}_\beta\mathcal{H}(P)$ for any $\beta \in [\beta_i, \beta_{i+1})$ as

$$\text{area}(\mathcal{O}_\beta\mathcal{H}(P)) = \text{area}(\mathcal{P}(\beta)) - \sum_i \text{area}(\triangle_i(\beta)) + \sum_j \text{area}(\varparallelogram_j(\beta)), \tag{1}$$

where $\mathcal{P}(\beta)$ denotes the (simple) polygon having the same vertices as $\mathcal{O}_\beta\mathcal{H}(P)$ and an edge connecting two vertices if they are consecutive in a $\mathcal{O}_\beta$-staircase. The term $\triangle_i(\beta)$ is the i-th triangle defined by two consecutive vertices in a $\mathcal{O}_\beta$-staircase, and $\varparallelogram_j(\beta)$ is the j-th overlapping region defined by the intersection of two opposite $\mathcal{O}_\beta$-staircases.

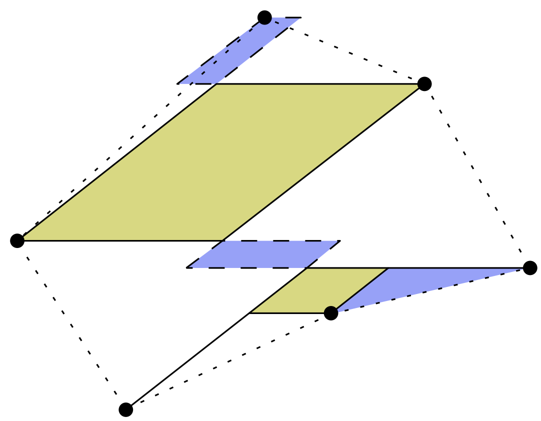
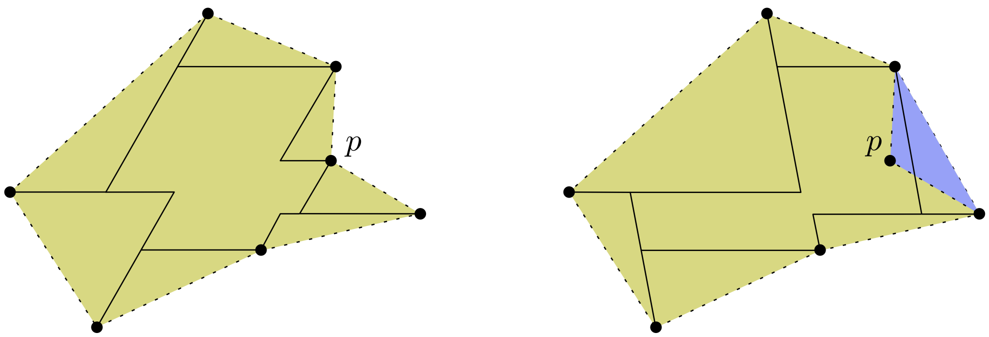


Figure 9: The area of $\mathcal{O}_\beta\mathcal{H}(P)$. The polygon $\mathcal{P}(\beta)$ is bounded by dotted lines. A triangle $\triangle_i(\beta)$ and two parallelograms $\varparallelogram_j(\beta)$ are filled in blue.

Our general approach is to maintain the terms of Equation (1) during a complete angular sweep. We first compute the optimal value of β for $[\beta_1, \beta_2)$. We then traverse the event sequence, updating the affected terms in Equation (1) at each event. At the same time, we compute the local angle of maximum area for each $[\beta_i, \beta_{i+1})$. With any new computation, we keep the local optimal angle only if the previous maximum area is improved.

The polygon $\mathcal{P}(\beta)$. At any fixed value of β, the polygon can be constructed from the vertices of $\mathcal{O}_\beta\mathcal{H}(P)$ in linear time. Once constructed, it takes a second linear run to compute its area. During an interval between events the area does not change. As $\mathcal{P}(\beta)$ only depends on the vertices of $\mathcal{O}_\beta\mathcal{H}(P)$, it is only modified by insertion and deletion events. Each event can be handled in constant time: the area of a triangle needs to be added (deletion event) or subtracted (insertion event) from the previous value of the area of $\mathcal{P}(\beta)$. See Figure 10.

The triangles $\triangle_i(\beta)$. A triangle is defined by a pair of consecutive vertices of $\mathcal{P}(\beta)$. If we consider a top-right $\mathcal{O}_\beta$-staircase, the area of $\triangle_i(\beta)$ is bounded by a line through p_i and p_{i+1}, an horizontal line through p_i, and a line with slope $\tan(\beta)$ through p_{i+1}. In this context, the area of $\triangle_i(\beta)$ is given by

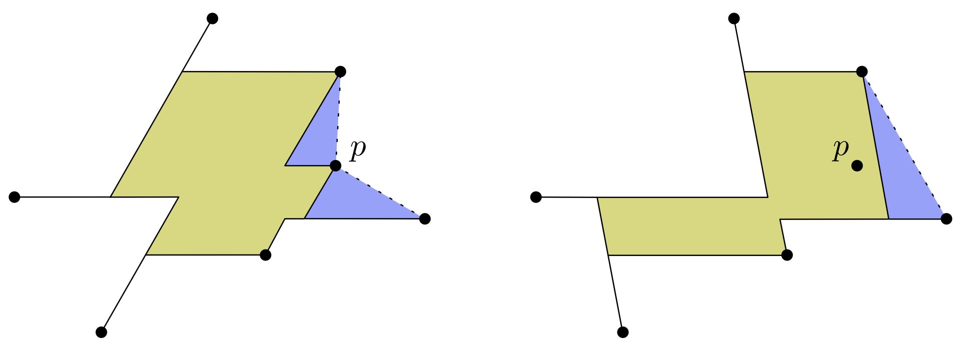


(a) (b)

Figure 10: Updating area($\mathcal{P}(\beta)$). (a) The vertex p will leave the top-right $\mathcal{O}_\beta$-staircase in an increasing sweep. (b) The area of a triangle needs to be added after the deletion event from area($\mathcal{P}(\beta)$), once p is no longer a vertex.

$$\begin{aligned} \text{area}(\triangle_i(\beta)) &= \left|(x_i - x_{i+1})(y_{i+1} - y_i) + (y_{i+1} - y_i)^2 \cot(\beta)\right| \\ &= \left|a_i \pm b_i \cot(\beta)\right|, \end{aligned} \tag{2}$$

with a_i, b_i constants, where (x_i, y_i) and (x_{i+1}, y_{i+1}) are respectively, the coordinates of the points p_i and p_{i+1}.



(a) (b)

Figure 11: Updating the term $\sum_i \text{area}(\triangle_i(\beta))$. (a) The point p will leave the top-right $\mathcal{O}_\beta$-staircase during an increasing sweep. (b) When p is no longer a vertex, two triangles are deleted, and a new triangle is created.

The term $\sum_i \text{area}(\triangle_i(\beta))$ is impacted by insertion and deletion events and, at each event, it needs to be modified a constant number of times. As any vertex of $\mathcal{O}_\beta\mathcal{H}(P)$ supports at most two maximal $\mathcal{O}_\beta$-quadrants, at a deletion event two triangles are removed and one triangle is added. The converse occurs for insertion events. See Figure 11.

The overlapping regions $\varangle_j(\beta)$. An overlapping region is defined by two pairs of consecutive vertices of $\mathcal{O}_\beta\mathcal{H}(P)$ belonging to opposite $\mathcal{O}_\beta$-staircases. Overlapping regions are bounded by parallelograms with sides parallel to the lines in $\mathcal{O}_\beta$. If we consider top-right and bottom-left $\mathcal{O}_\beta$-staircases intersecting as shown in Figure 12, the area of a parallelogram is given by

$$\begin{aligned}\operatorname{area}(\varnothing_j(\beta)) &= |(x_{k+1}-x_{i+1})(y_k-y_i)+(y_{k+1}-y_{i+1})(y_k-y_i)\cot(\beta)| \\ &= |a_j \pm b_j \cot(\beta)|\,, \end{aligned} \tag{3}$$

with a_i, b_i constants, where $p_i = (x_i, y_i), p_{i+1} = (x_{i+1}, y_{i+1})$ and $p_k = (x_k, y_k), p_{k+1} = (x_{k+1}, y_{k+1})$ are respectively, the supporting vertices of the overlapping maximal opposite $\mathcal{O}_\beta$-quadrants.

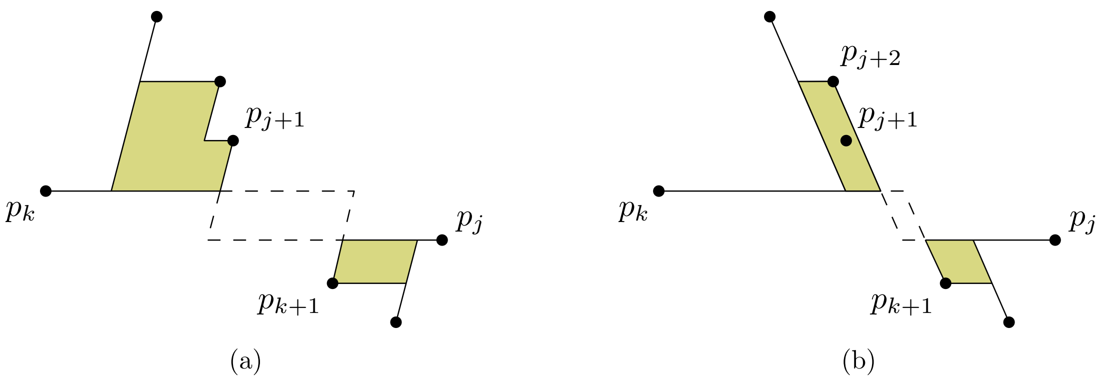


Figure 12: An overlapping region destroyed because of the vertex p_{j+1} leaving the top-right $\mathcal{O}_\beta$-staircase, during an increasing sweep.

The term $\sum_j \operatorname{area}(\varnothing_j(\beta))$ is impacted by all types of events. Overlap and release events require a single overlapping region to be added or deleted. For insertion and deletion events, at most two new overlaps are created, or destroyed.

Characterization. Before describing our algorithm, in the following lemmas we answer some basic questions about the behavior of area($\mathcal{O}_\beta\mathcal{H}(P)$). Lemmas 4 and 5 imply that it seems not possible to restrict the number of candidate angles of maximum area. On the other hand, Lemma 6 shows that the angle of maximum area is actually located at an event.

Lemma 4. *For any $\beta_0 \in (0, \pi)$ there exists a point set P such that*

$$\max_\beta \operatorname{area}(\mathcal{O}_\beta\mathcal{H}(P)) \neq \operatorname{area}(\mathcal{O}_{\beta_0}\mathcal{H}(P)).$$

Proof. Consider the coordinate system formed by $\mathcal{O}_{\beta_0}$. Place one point over the y^+-, y^--, and x^+-semiaxes, and a point over the second quadrant (see Figure 13(a)). From this position, note that area($\mathcal{O}_\beta\mathcal{H}(P)) = 0$ for any $\beta \leq \beta_0$ (Figure 13(b)), and there exists at least one $\beta_1 > \beta_0$ such that area($\mathcal{O}_{\beta_1}\mathcal{H}(P)) \neq 0$ (Figure 13(c)). Hence β_0 cannot be the angle of maximum area. □

Lemma 5. *For any $\beta_0, \beta_1 \in (0, \pi)$, there exists a point set P for which* area($\mathcal{O}_\beta(P)$) *has local maxima in β_0 and β_1.*

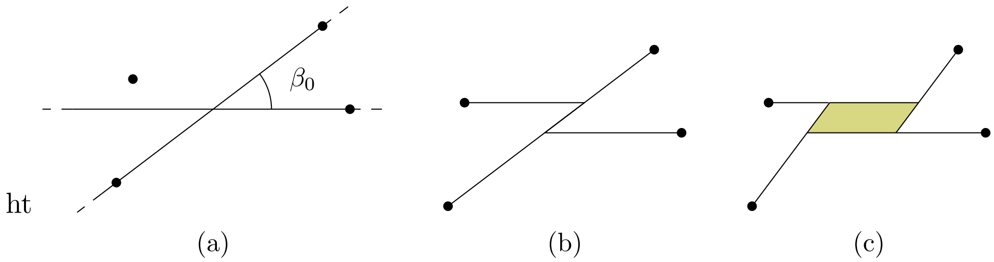


Figure 13: Lemma 4. (a) The set of points. (b) area($\mathcal{O}_\beta\mathcal{H}(P)$) = 0 for $\beta \leq \beta_0$. (c) area($\mathcal{O}_\beta\mathcal{H}(P)$) $\neq 0$ for some $\beta > \beta_0$.

Proof. Let ℓ_0 be a line with slope $\tan(\beta_0)$, ℓ_1 a line with slope $\tan(\beta_1)$, and without loss of generality, let us assume that $\beta_0 < \beta_1$. We define p_l, p_r, p_t, and p_c to be the points located respectively, at the left corner, right corner, top corner, and the interior of the triangle bounded by the x-axis, ℓ_0, and ℓ_1. See Figure 14.

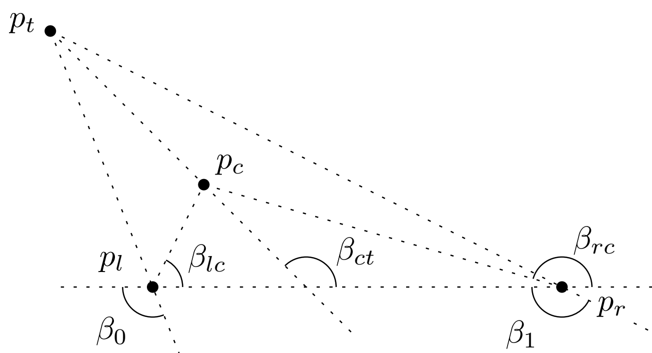


Figure 14: The points configuration.

Consider the angles β_{lc}, β_{ct}, and β_{rc} as in Figure 14. Note that $\beta_{lc} < \beta_0 < \beta_{ct} < \beta_1 < \beta_{rc}$. Using an increasing sweep from the initial increasing configuration the first release event is β_{lc}. From there, the area of $\mathcal{O}_\beta\mathcal{H}(P)$ is given by a parallelogram $\varparallelogram_{lc}$ of constant height, so both the base of $\varparallelogram_{lc}$ and the area of $\mathcal{O}_\beta\mathcal{H}(P)$ increase or decrease together as β changes. As β goes from β_{lc} to β_0, the base of $\varparallelogram_{lc}$ increases up to β_0, there exist a local maximum. The base of $\varparallelogram_{lc}$ then decreases from β_0 to β_{ct}, to increase again from β_{ct} to β_1. At β_1 there is a second local maximum, as the base of $\varparallelogram_{lc}$ starts decreasing again after β_1 up to the last construction event at β_{rc}, where the area of $\mathcal{O}_\beta\mathcal{H}(P)$ is zero. See Figure 15. $\square$

Lemma 6. *The area of $\mathcal{O}_\beta\mathcal{H}(P)$ reaches its maximum at values of β belonging to the sequence of events.*

Proof. Let us consider the area of $\mathcal{O}_\beta\mathcal{H}(P)$ given by Equation (1). From Equations (2) and (3), the area of $\mathcal{O}_\beta\mathcal{H}(P)$ can be rewritten as

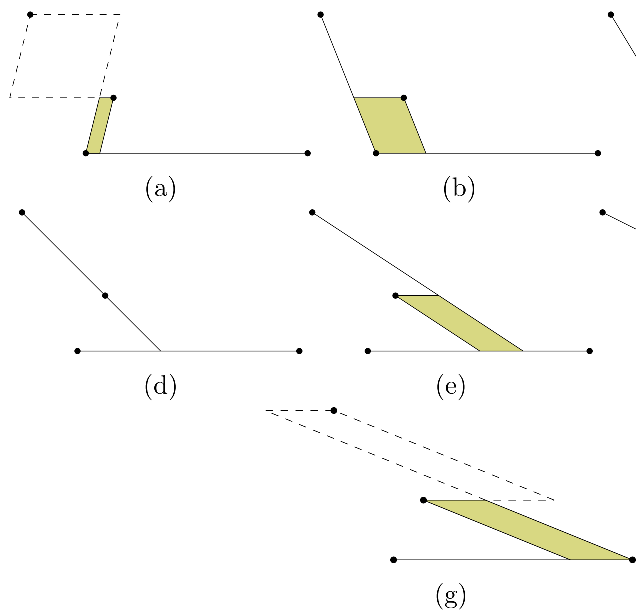


Figure 15: Increasing sweep over the point set of Figure 14. (a) $\beta = \beta_0 - \epsilon$. (b) A local maximum on $\beta = \beta_0$. (c) $\beta \in (\beta_0, \beta_{ct})$. (d) A local minimum on $\beta = \beta_{ct}$. (e) $\beta \in (\beta_{ct}, \beta_1)$. (f) A second local maximum on $\beta = \beta_1$. (g) $\beta = \beta_1 + \epsilon$.

$$\begin{aligned}\text{area}(\mathcal{O}_\beta\mathcal{H}(P)) &= \text{area}(\mathcal{P}(\beta)) - \sum_i \text{area}(\triangle_i(\beta)) + \sum_j \text{area}(\varparallelogram_j(\beta)) \\ &= \text{area}(\mathcal{P}(\beta)) - \sum_i |a_i \pm b_i \cot(\beta)| + \sum_j |a_j \pm b_j \cot(\beta)|\,. \end{aligned} \tag{4}$$

If we consider the different point configurations that define a triangle (see Figure 16), we can express $|a_i \pm b_i \cot(\beta)|$ as $a_i + b_i \cot(\beta)$ or $a_i - b_i \cot(\beta)$, according to the specific configuration. Thus, we have

$$\begin{aligned}\sum_i \text{area}(\triangle_i(\beta)) &= \sum_i |a_i \pm b_i \cot(\beta)| \\ &= \sum_{i_0} (a_{i_0} + b_{i_0} \cot(\beta)) + \sum_{i_1} (a_{i_1} - b_{i_1} \cot(\beta)) = a + b\cot(\beta).\end{aligned}$$

It is possible to make a similar case-by-case analysis for the overlapping regions, to obtain from Equation (3) an expression with the form $c + d\cot(\beta)$. Within an interval between events P does not change, and its area remains constant. Therefore, in an interval $[\beta_i, \beta_{i+1})$ we can rewrite:

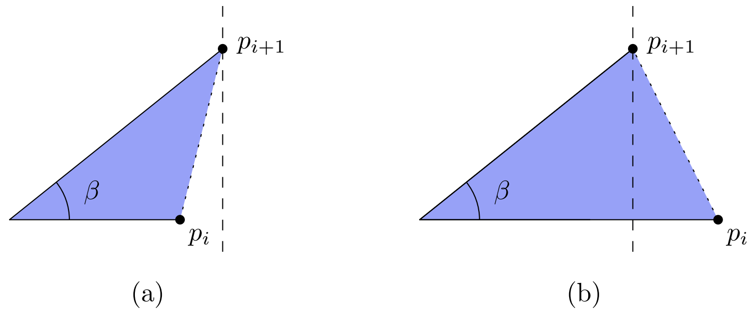


Figure 16: Relative positions between the vertices of the triangle $\triangle_i(\beta)$.

$$
\begin{aligned}
\operatorname{area}(\mathcal{O}_\beta\mathcal{H}(P)) &= \operatorname{area}(\mathcal{P}(\beta)) - \sum_i |a_j \pm b_j \cot(\beta)| + \sum_j |a_j \pm b_j \cot(\beta)| && (5)\\
&= \operatorname{area}(\mathcal{P}(\beta)) - (a + b\cot(\beta)) + (c + d\cot(\beta)) \\
&= \operatorname{area}(\mathcal{P}(\beta)) + (c-a) + (d-b)\cot(\beta) \\
&= A + B\cot(\beta), && (6)
\end{aligned}
$$

where A and B contain the sum of all constants from the terms in Equation (5). Note that Equation (6) is monotone at any interval $[\beta_i, \beta_{i+1})$, as it is monotone in $(0, \pi)$. Depending on the particular values of A and B, area($\mathcal{O}_\beta\mathcal{H}(P)$) might be non-decreasing or non-increasing. Thus, the local maximum is given either by β_i or β_{i+1}. □

The search algorithm. The algorithm to compute the angle of optimum area is outlined as follows.

1. Traverse the sequence of events to identify the first release event β_d, and the last overlap event β_c. Restrict the sequence to start with β_d and finish with β_c, so that $\mathcal{O}_\beta(P)$ has at least one connected component in every interval. Ignored events have no effect in the result, as they belong to an initial (increasing or decreasing) configuration, where area($\mathcal{O}_\beta\mathcal{H}(P)$) $= 0$.

2. At the first interval, compute $\mathcal{O}_\beta\mathcal{H}(P)$ and using Equation (1) compute area($\mathcal{O}_\beta\mathcal{H}(P)$), keeping the angle β_m of maximum area.

3. Traverse the sequence of events. At each event:

   (a) Update the set of vertices and overlapping regions of $\mathcal{O}_\beta\mathcal{H}(P)$.

   (b) Handle each event updating Equation (1) as explained above.

   (c) Compute the local angle of maximum area. Replace β_m only if the area of $\mathcal{O}_\beta\mathcal{H}(P)$ is improved.

There is a linear number of events in total, so step 1 requires $O(n)$ time. Equation (1) contains at most a linear number of terms, as there is at most a linear number of vertices and overlapping regions. Thus, from Theorem 3 and previous discussions, step 2 requires $\Theta(n \log n)$ time and $O(n)$ space.

From Section 2.2, the updates on step 3a require logarithmic time. Every event results in a constant number of modifications to Equation (1), as we described previously in this section. From Lemma 6 we can obtain the angle of maximum area in constant time. As there is a linear number of events, step 3 requires a total of $O(n \log n)$ time. From this analysis we obtain the following Theorem, where the lower bound comes from the maintenance of $\mathcal{O}_\beta\mathcal{H}(P)$.

Theorem 5. *Computing the value(s) of $\beta \in (0, \pi)$ for which $\mathcal{O}_\beta\mathcal{H}(P)$ has maximum area, requires $\Theta(n \log n)$ time and $O(n)$ space.*

3.2 Perimeter optimization.

In this section we solve the following problem:

Problem 2 (Maximum perimeter)**.** Given a set P of n points in the plane, compute the value of β for which $\mathcal{O}_\beta\mathcal{H}(P)$ has maximum perimeter.

The perimeter of $\mathcal{O}_\beta\mathcal{H}(P)$ is given by

$$\mathrm{perim}(\mathcal{O}_\beta\mathcal{H}(P)) = \sum_i \mathrm{perim}(\angle_i(\beta)) - \sum_j \mathrm{perim}(\varangle_j(\beta)) - \sum_k \mathrm{perim}(\diagdown_k(\beta)), \tag{7}$$

where the $\angle_i(\beta)$ and the $\varangle_j(\beta)$ denote the *steps* and parallelograms, respectively, defined by the staircases, and $\diagdown_k$ denotes one of the (at most four) *antennas* of $\mathcal{O}_\beta\mathcal{H}(P)$, that is, a segment of an $\mathcal{O}_\beta$-staircase bounding a zero-area region of $\mathcal{O}_\beta\mathcal{H}(P)$. See again Figure 9.

The same approach, and most of the arguments we used to maximize the area can be applied here. Following the same ideas, we will first analyze the computation and maintenance of Equation (7), we then present adaptations of lemmas 4 to 6, and finalize outlining the search algorithm.

The steps $\angle_i(\beta)$. Considering a top-right $\mathcal{O}_\beta$-staircase (see again Figure 11), the perimeter of $\angle_i(\beta)$ is given by Equation (8), where $p_i = (x_i, y_i)$ and $p_{i+1} = (x_{i+1}, y_{i+1})$ are the points supporting the i-th step. Vertices over the staircase have non-decreasing y coordinates, so a_i is always positive. Event handling is done in the same way as we did with triangles in the previous section.

$$\begin{aligned}\mathrm{perim}(\angle_i(\beta)) &= |(y_{i+1} - y_i)\cot(\beta) + (y_{i+1} - y_i)\csc(\beta) + (x_i - x_{i+1})| \\ &= |a_i\,(\cot(\beta) + \csc(\beta)) \pm b_i|\,. \end{aligned} \tag{8}$$

The overlapping regions $\varangle_j(\beta)$. If we consider top-right and bottom-left $\mathcal{O}_\beta$-staircases intersecting as shown in Figure 12, the perimeter of an overlapping region is given by Equation (9). The constants c_j and d_j are always positive. Event handling is done in the same way as we handled overlapping regions to optimize the area of $\mathcal{O}_\beta\mathcal{H}(P)$.

$$
\begin{aligned}
\operatorname{perim}(\varangle_j(\beta)) &= |2(y_{i+1} - y_{k+1})\cot(\beta) + 2(y_k - y_i)\csc(\beta) - (x_{i+1} - x_{k+1})| \\
&= |c_j \cot(\beta) + d_j \csc(\beta) \pm e_j| \,. \qquad (9)
\end{aligned}
$$

The antennas $\diagdown_k(\beta)$. An antenna is a semistep at one of the extremes of an $\mathcal{O}_\beta$-staircase. Just as steps and triangles, an antenna is defined by two consecutive $\mathcal{O}_\beta$-maximal points. If we consider a top-right $\mathcal{O}_\beta$-staircase, the perimeter of an antenna is given by Equation (10) if it is the first semistep of the staircase, and by Equation (11) if it is the last one (see Figure 17). In both equations we consider $p_i = (x_i, y_i)$ to be the point supporting the corresponding semistep. The constant f_k is always positive.

$$
\begin{aligned}
\operatorname{perim}_f(\diagdown_k) &= |(y_{i+1} - y_i)\cot(\beta) + (x_i - x_{i+1})| \\
&= |f_k \cot(\beta) \pm g_k| \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{perim}_l(\diagdown_k) &= (y_{i+1} - y_i)\csc(\beta) \\
&= f_k \cot(\beta) \qquad (11)
\end{aligned}
$$

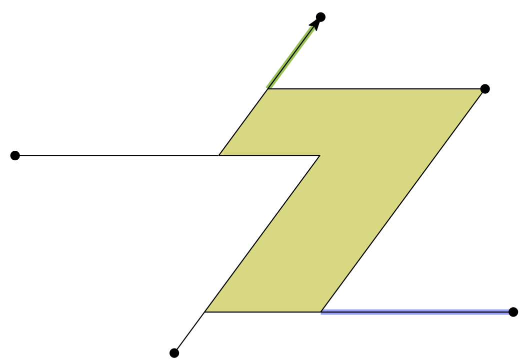


Figure 17: Two antennas respectively, at the first (horizontal segment) and last (non-horizontal segment) semisteps of the top-right $\mathcal{O}_\beta$-staircase.

Considering the case-by-case analysis we did in the previous section, we can rewrite Equations 8 to 11 as

$$
\begin{aligned}
\sum_i \operatorname{perim}(\angle_i(\beta)) &= \sum_i |a_i \cot(\beta) + a_i \csc(\beta) \pm b_i| \\
&= \sum_{i_0} a_{i_0} \cot(\beta) + a_{i_0} \csc(\beta) + b_{i_0} + \sum_{i_1} a_{i_1} \cot(\beta) + a_{i_1} \csc(\beta) - b_{i_1} \\
&= a \cot(\beta) + a \csc(\beta) + b \qquad (12)
\end{aligned}
$$

$$\begin{aligned}\sum_j \text{perim}(\varPhi_j(\beta)) &= \sum_j |c_j \cot(\beta) + d_j \csc(\beta) \pm e_j| \\ &= \sum_{j_0} c_{j_0} \cot(\beta) + d_{j_0} \csc(\beta) + e_{j_0} + \sum_{j_1} c_{j_1} \cot(\beta) + d_{j_1} \csc(\beta) - e_{j_1} \\ &= c \cot(\beta) + d \csc(\beta) + e \qquad (13)\end{aligned}$$

$$\begin{aligned}\sum_k \text{perim}_l(\diagdown_k) &= |f_k \cot(\beta) \pm g_k| \\ &= \sum_{k_0} f_{k_0} \cot(\beta) + g_{k_0} + \sum_{k_1} f_{k_1} \cot(\beta) - g_{k_1} \\ &= f \cot(\beta) + g, \qquad (14)\end{aligned}$$

and use Equations (12) to (14) to rewrite Equation (7) as

$$\begin{aligned}\text{perim}(\mathcal{O}_\beta\mathcal{H}(P)) &= \sum_i \text{perim}(\angle_i(\beta)) - \sum_j \text{perim}(\varPhi_j(\beta)) - \sum_k \text{perim}(\diagdown_k(\beta)) \\ &= (a+c+f)\cot(\beta) + (a+d)\csc(\beta) + (b+e+g) \\ &= A\cot(\beta) + B\csc(\beta) + C. \qquad (15)\end{aligned}$$

Note that all the constants in Equation (15) adding up to A and B are always positive, so $A, B > 0$. Moreover, within an interval there are at most four antennas, as they contain one of the left-most, right-most, top-most, and bottom-most points in P (see again Figure 17). Therefore, the number of terms contributed by antennas to Equation (7) is constant and, except for C, they do not modify the original signs of any other term.

For simplicity, we will avoid antennas in the optimization of the perimeter of $\mathcal{O}_\beta\mathcal{H}(P)$, by using a version of Equation (7) not containing the term $\sum_k \text{perim}(\diagdown_k(\beta))$. From the discussion above, both expressions have maxima at the same values of β.

Characterization. We next answer questions about the behavior of $\text{perim}(\mathcal{O}_\beta\mathcal{H}(P))$, similar to the ones answered with lemmas 4 to 6 in Section 3. Specifically, we show that the angle of maximum perimeter corresponds to an event (Lemma 9) and, other than that, no restriction on the candidate angles seems to be possible (Lemmas 7 and 8).

Lemma 7. *For any $\beta_0 \in (0, \pi)$, there exists a point set such that*

$$\max_\beta \text{perim}(\mathcal{O}_\beta\mathcal{H}(P)) \neq \text{perim}(\mathcal{O}_{\beta_0}\mathcal{H}(P)).$$

Proof. Consider the coordinate system formed by $\mathcal{O}_{\beta_0}$. Place one point on the origin, and a point on the second and fourth quadrants (Figure 18(a)). As the set of points is monotone with respect of the x- and y-axes, $\mathcal{O}_{\beta_0}\mathcal{H}(P) = P$. Therefore, $\text{perim}(\mathcal{O}_{\beta_0}\mathcal{H}(P))$ is equal to zero.

From this position, note that $\text{perim}(\mathcal{O}_\beta\mathcal{H}(P)) = 0$ for any $\beta \leq \beta_0$ (Figure 18(b)), and there exists at least one $\beta_1 > \beta_0$ such that $\text{perim}(\mathcal{O}_{\beta_1}\mathcal{H}(P)) \neq 0$ (Figure 18(c)). Clearly, β_0 is not the angle of maximum perimeter. $\square$

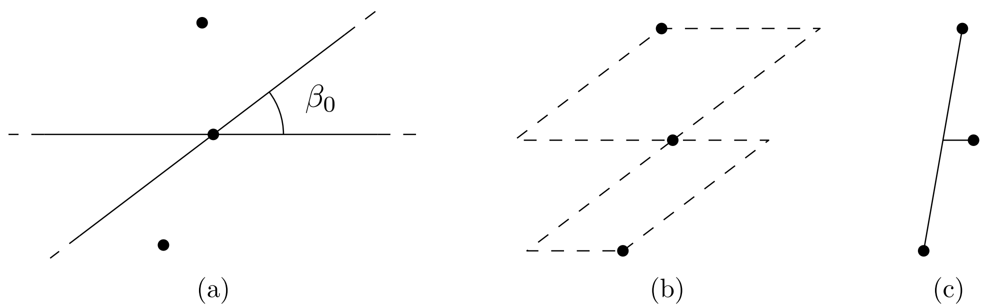


Figure 18: Lemma 7. (a) A set of points. (b) $\mathrm{perim}(\mathcal{O}_\beta\mathcal{H}(P)) = 0$ for $\beta \leq \beta_0$. (c) $\mathrm{perim}(\mathcal{O}_{\beta_0}\mathcal{H}(P)) \neq 0$ for some $\beta > \beta_0$.

Lemma 8. *For any $\beta_0, \beta_1 \in (0, \pi)$, there exists a point set P for which* $\mathrm{area}(\mathcal{O}_\beta(P))$ *has local maxima in β_0 and β_1.*

Proof. Let $\triangle_t$ be an acute triangle bounded by the x-axis, and two lines ℓ_{tl}, ℓ_{tr} with slopes $\tan(\beta_{tl})$ and $\tan(\beta_{tr})$, respectively. Without loss of generality, we assume that $\beta_{tl} < \beta_{tr}$, and the intersection point between ℓ_{tl} and ℓ_{tr} lies on the y^+-semiplane.

Let us consider the set $P' = \{p_l, p_r, p_t\}$ of points located respectively, over the left, right, and top vertices of $\triangle_t$. Note that, at any starting position, the perimeter of $\mathcal{O}_\beta\mathcal{H}(P')$ is constant and equal to the base of $\triangle_t$. Using an increasing sweep, from β_{tl} to β_{tr} the perimeter is formed additionally by a line segment $\ell_{t,b}$ joining p_t, and a point p_b traversing the base of $\triangle_t$ from p_l to p_r. During this interval, both $\ell_{t,b}$ and the perimeter of $\mathcal{O}_\beta\mathcal{H}(P')$ increase or decrease together as β changes. On this conditions, the perimeter of $\mathcal{O}_\beta\mathcal{H}(P')$ has a local minimum on $\beta = \frac{\pi}{2}$ and thus, a local maximum on β_{tl} ($\mathrm{perim}(\mathcal{O}_{\beta_{tl}}\mathcal{H}(P')) > \mathrm{perim}(\mathcal{O}_{\beta_{tl}-\varepsilon}\mathcal{H}(P'))$), and a second local maximum on β_{tr} ($\mathrm{perim}(\mathcal{O}_{\beta_{tr}}\mathcal{H}(P')) > \mathrm{perim}(\mathcal{O}_{\beta_{tr}+\varepsilon}\mathcal{H}(P'))$). See Figure 19.

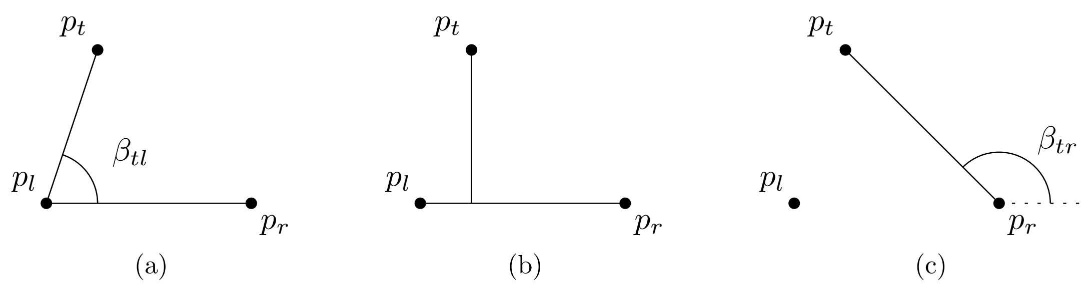


Figure 19: (a) and (c) Maxima on β_{tl} and β_{tr}. (b) A minima on $\frac{\pi}{2}$.

Let $\triangle_b$ be a second acute triangle bounded by the x-axis, and a second pair of lines ℓ_{bl}, ℓ_{br} with slopes $\tan(\beta_{bl})$ and $\tan(\beta_{br})$ that pass through p_l and p_r, respectively. The angles are such that $\beta_{bl} > \beta_{br}$, and the intersection point p_b between ℓ_{bl} and ℓ_{br} lie on the Y^- semiplane. Note that, if we add p_b to the set P', the arguments from the above discussion hold for both $\triangle_t$ and $\triangle_b$, so the perimeter of $\mathcal{O}_\beta\mathcal{H}(P')$ has now local maxima on $\beta_{tl}, \beta_{tr}, \beta_{bl}$, and β_{br}. See Figure 20.

Given the angles β_0 and β_1, construct the previous point set as explained. In this construction, $\beta_{tl} \leq \frac{\pi}{2} < \beta_{tr}$ and $\beta_{bl} > \frac{\pi}{2} \geq \beta_{br}$. Set two of $\beta_{tl}, \beta_{tr}, \beta_{bl}, \beta_{br}$ to the values

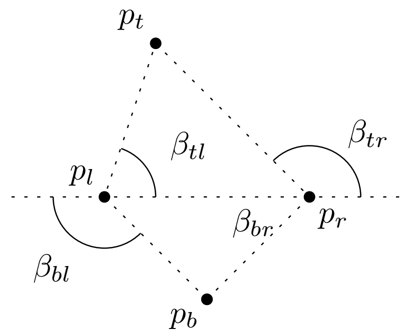


Figure 20: The angles with local maxima.

of β_0 and β_1 appropriately, according to the cases *i)* $\beta_0, \beta_1 < \frac{\pi}{2}$, *ii)* $\beta_0, \beta_1 > \frac{\pi}{2}$, and *iii)* $\beta_0 \leq \frac{\pi}{2} < \beta_1$ or viceversa. The perimeter of $\mathcal{O}_\beta\mathcal{H}(P')$ will have local maxima at β_0 and β_1. □

Lemma 9. *The perimeter of $\mathcal{O}_\beta\mathcal{H}(P)$ reaches its maximum at values of β corresponding to sequence events.*

Proof. From Equation (15) we know that the perimeter of $\mathcal{O}_\beta\mathcal{H}(P)$ is given by

$$\mathrm{perim}(\mathcal{O}_\beta\mathcal{H}(P)) = A\cot(\beta) + B\csc(\beta) + C,$$

where $A, B \geq 0$. Looking for critical points in this expression, we arrive to

$$\cos(\beta) = -\frac{A}{B}, \tag{16}$$

where $\beta \neq 0, \pi$. By analyzing the possible roots in Equation (16), we deal with the following cases:

1. $A > B$. There are no roots in this case, as $\frac{A}{B} > 1$. The length of the perimeter is monotonic in an interval between events.

2. $A = B$. There are again no roots in this case, as β cannot be 0 nor π. The length of the perimeter is again monotonic in an interval between events.

3. $A < B$. There is one root at $\beta = \cos^{-1}(-\frac{A}{B})$, as A and B are always positive and different from zero. In an interval between events we have one inflection point, so there are either two local maxima or two local minima, located at the endpoints of the interval.

□

The search algorithm. We look for the maximum perimeter angles in the same way as we obtained the values for maximum area. We first compute the list of events, obtain the maximum perimeter angle for the first interval between events, and repeat the procedure for the remaining events. While traversing the event list, we update the optimum value angle only if the previous value is improved. A similar complexity analysis is also valid.

Theorem 6. *Computing the value(s) of $\beta \in (0, \pi)$ for which $\mathcal{O}_\beta\mathcal{H}(P)$ has maximum perimeter takes $O(n \log n)$ time and $O(n)$ space.*

4 Concluding remarks

We presented an algorithm to maintain the $\mathcal{O}_\beta$-hull of a planar point set while β runs from 0 to π and extended this result to solve related optimization problems. We considered the maximization of the area and the perimeter of $\mathcal{O}_\beta\mathcal{H}(P)$, and presented a variation of the 2-fitting problem studied in [6]. In our version, the fitting curve is an alternating polygonal chain with segments forming an angle β.

A natural extension of this work is to replace $\mathcal{O}_\beta$ with a set $\mathcal{O}$ containing more than two lines. Different variations can be obtained by restricting the orientations and (or) the number of lines in $\mathcal{O}$. In particular, the characterization of the area and perimeter functions on each variation, seems an interesting and non-trivial problem.

As the Orthogonal Convex Hull, the $\mathcal{O}_\beta$-hull is suitable to be used as a separator or an enclosing shape. As it is always contained in the standard convex hull (and therefore, in several other traditional enclosing shapes), it is relevant in applications where the separator or enclosing shape is required to have minimum area. Finally, note that we can easily extend the results from Section 2 to optimize the number of vertices of $\mathcal{O}_\beta\mathcal{H}(P)$, by keeping track of the vertex count at each interval between events. Without much effort, the approach and arguments from Alegría-Galicia et al. [2] can be extended to $\mathcal{O}_\beta$-convexity, and applied to problems related to containment relations between $\mathcal{O}_\beta$-hulls of colored point sets.

References

[1] C. Alegría-Galicia, T. Garduño, A. Rosas-Navarrete, C. Seara, and J. Urrutia. Rectilinear convex hull with minimum area. In *Special LNCS Festschrift volume in honor of F. Hurtado's 60th Birthday. Lecture Notes in Computer Science*, volume 7579, pages 226–235. Springer-Verlag, 2012.

[2] C. Alegría-Galicia, C. Seara, and J. Urrutia. Computing containment relations between rectilinear convex hulls. In *Mexican Conference on Discrete Mathematics and Computational Geometry, 60th birthday of Jorge Urrutia*, November 2013.

[3] C. Alegría-Galicia, D. Orden, C. Seara, and J. Urrutia. On the $\mathcal{O}$-hull of planar point sets. In *30th European Workshop on Computational Geometry (EuroCG)*, March 2014.

[4] D. Avis, B. Beresford-Smith, L. Devroye, H. Elgindy, E. Guévremont, F. Hurtado, and B. Zhu. Unoriented Θ-maxima in the plane: complexity and algorithms. *SIAM J. Comput.*, 28(1):278–296, February 1999. ISSN 0097-5397.

[5] S.W. Bae, C. Lee, H.K. Ahn, S. Choi, and K.Y. Chwa. Computing minimum-area rectilinear convex hull and l-shape. *Computational Geometry: Theory and Applications*, 42:903–912, November 2009. ISSN 0925-7721.

[6] J. M. Díaz-Báñez, M. A. López, M. Mora, C. Seara, and I. Ventura. Fitting a two-joint orthogonal chain to a point set. *Computational Geometry: Theory and Applications*, 44(3):135–147, April 2011. ISSN 0925-7721.

[7] E. Fink and D. Wood. *Restricted-orientation convexity.* Monographs in Theoretical Computer Science (An EATCS Series). Springer-Verlag, 2004.

[8] V. Franěk and J. Matoušek. Computing d-convex hulls in the plane. *Computational Geometry: Theory and Applications*, 42(1):81–89, January 2009. ISSN 0925-7721.

[9] R.H. Güting. *Conquering contours: efficient algorithms for computational geometry.* PhD thesis, Fachbereich Informatik, Universität Dortmund, 1983.

[10] H. T. Kung, F. Luccio, and F.P. Preparata. On finding the maxima of a set of vectors. *Journal of the Association for Computer Machinery*, 22(4):469–476, October 1975.

[11] T. Ottmann, E. Soisalon-Soininen, and D. Wood. On the definition and computation of rectilinear convex hulls. *Information Sciences*, 33(3):157–171, 1984.

[12] C. Peláez, A. Ramírez-Vigueras, C. Seara, and J. Urrutia. On the rectilinear convex layers of a planar set. In *Mexican Conference on Discrete Mathematics and Computational Geometry, 60th birthday of Jorge Urrutia*, pages 195–202, November 11–15 2013.

[13] F.P. Preparata and M.I. Shamos. *Computational Geometry: An Introduction.* Springer–Verlag, 1985.

[14] G. J. E. Rawlins. *Explorations in restricted-orientation geometry.* PhD thesis, School of Computer Science, University of Waterloo, 1987.

[15] G. J. E. Rawlins and D. Wood. Optimal computation of finitely oriented convex hulls. *Information and Computation*, 72:150–166, February 1987. ISSN 0890-5401.

[16] G. J. E. Rawlins and D. Wood. Ortho-convexity and its generalizations. *Computational Morphology: A Computational Geometric Approach to the Analysis of Form*, pages 137–152, 1988.

[17] S. Schuierer. *On generalized visibility.* PhD thesis, Institut für Informatik, Universität Freiburg, 1991.

[18] P. Widmayer, Y.F. Wu, and C.K. Wong. On some distance problems in fixed orientations. *SIAM Journal on Computing*, 16(4):728–746, 1987.

Appendix A The oriented $(2, \beta)$-fitting problem

For $k \geq 1$, $\theta \in [0, \pi)$, and $\beta \in (0, \pi)$, a (k, β)*-polygonal chain with orientation* θ, $\mathcal{C}_{k,\beta}(\theta)$, is a chain with $2k - 1$ consecutive alternating links with slopes $\tan(\theta)$ and $\tan(\theta + \beta)$ such that the extreme links are half-lines with orientation $\tan(\theta)$. Let us define $\ell_{i,\beta}(\theta)$ as the line passing through $p_i \in P$ with slope $\tan(\theta + \beta)$. The *fitting distance* between p_i and $\mathcal{C}_{k,\beta}(\theta)$ is given by

$$d_f(p_i, \mathcal{C}_{k,\beta}(\theta)) = \min_{p \in \ell_{i,\beta}(\theta) \cap \mathcal{C}_{k,\beta}(\theta)} d(p_i, p),$$

where $d(p_i, p)$ represents the Euclidean distance between p_i and p. The *error tolerance* of $\mathcal{C}_{k,\beta}(\theta)$ with respect to P is the maximum fitting distance between $\mathcal{C}_{k,\beta}(\theta)$ and the elements in P, that is

$$\mu(\mathcal{C}_{k,\beta}(\theta), P) = \max_{p_i \in P} \;\; d_f(p_i, \mathcal{C}_{k,\beta}(\theta)).$$

The (k, β)-fitting problem for P with the Min-Max criterion, consists on finding a polygonal chain $\mathcal{C}_{k,\beta}(\theta)$ with minimum error tolerance $\mu(\mathcal{C}_{k,\beta}(\theta), P)$. See Figure 21.

Theorem 7 ([6]). *The $\left(2, \frac{\pi}{2}\right)$-fitting problem can be solved in $\Theta(n \log n)$ time and $O(n)$ space.*

We consider here the case where θ has a constant value, namely 0, and we want to find the chain $\mathcal{C}_{2,\beta}(0) = \mathcal{C}_{2,\beta}$ of optimal error tolerance. More formally, we solve the following problem.

Problem 3 (Oriented $(2, \beta)$-fitting). Given a set P of n points in the plane, compute a polygonal chain $\mathcal{C}_{2,\beta}$ such that $\mu(\mathcal{C}_{2,\beta}, P)$ has minimum value.

Consider the algorithm used in [6] to obtain the $O(n \log n)$ time bound for the $\left(2, \frac{\pi}{2}\right)$-fitting problem used to prove Theorem 7. The $\mathcal{O}_{\frac{\pi}{2}}$-hull of P is used as a tool to solve the problem in $O(\log n)$ time for a fixed value of θ in a closed orientation interval $[\theta_i, \theta_{i+1}]$. An event sequence of a linear number of orientation intervals is created to maintain $\mathcal{O}_{\frac{\pi}{2}}\mathcal{H}(P)$ as θ grows from 0 to 2π.

To solve Problem 3 we can follow exactly the same techniques. We refer the reader to reference [6] just to see the evident changes coming from the use of a different structure. More concretely, the structure $\mathcal{O}_{\frac{\pi}{2}}\mathcal{H}(P)$ is replaced by $\mathcal{O}_{\beta}\mathcal{H}(P)$ which needs also a linear number of *interval events* $[\beta_i, \beta_{i+1}]$ to be maintained, and where the angular sweep is performed over β. Thus, Lemmas 3 and 4 in [6] can be now stated as follows:

(i) *Given a value $\beta \in [\beta_i, \beta_{i+1}]$, an optimal solution of the $(2, \beta)$-fitting problem for β is defined by a line $\ell_{i,\beta}$ with slope $\tan(\beta)$ passing through a point p_i of P which gives the bipartition of P.*

(ii) *The optimal solution of the $(2, \beta)$-fitting problem for an interval event $[\beta_i, \beta_{i+1}]$ occurs either at an endpoint of the interval, i.e., at β_i or β_{i+1}, or at a value $\beta_0 \in [\beta_i, \beta_{i+1}]$ when the left and right error tolerance are equal.*

Using the properties (i),(ii) and following the maintenance of $\mathcal{O}_{\beta}\mathcal{H}(P)$, the problem is solved as follows:

1. Compute $\mathcal{O}_{\beta}\mathcal{H}(P)$ and the optimal error tolerance for the first interval between events.

2. Traverse the event sequence, obtaining the optimal error tolerance at each interval between events.

3. Update the previous solution only when it is improved.

Thus, the approach and arguments used in Theorem 7 hold in the case of the $(2, \beta)$-fitting problem See Figure 21. As a consequence, we get the following theorem.

Theorem 8. *The $(2, \beta)$-fitting problem can be solved in $O(n \log n)$ time and $O(n)$ space.*

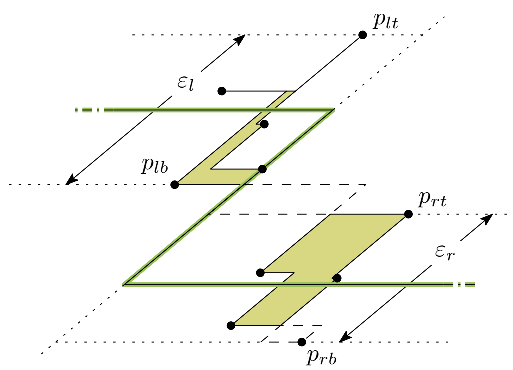


Figure 21: The polygonal chain $\mathcal{C}_{2,\beta}$ and the $\mathcal{O}_\beta$-hull of P.