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MODELS OF NONLINEAR KINEMATIC HARDENING BASED ON DIFFERENT VERSIONS OF RATE-INDEPENDENT MAXWELL FLUID

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Abstract. Different models of finite strain plasticity with a nonlinear kinematic hardening are analyzed in a systematic way. All the models are based on a certain formulation of a rate-independent Maxwell fluid, which is used to render the evolution of backstresses. The properties of each material model are determined by the underlying formulation of the Maxwell fluid. The analyzed approaches include the multiplicative hyperelasto-plasticity, additive hypoelasto-plasticity and the use of generalized strain measures. The models are compared with respect to different classification criteria, such as the objectivity, thermodynamic consistency, pure volumetric-isochoric split, shear stress oscillation, exact integrability, and w-invariance.

1 INTRODUCTION

As is well known, a correct numerical analysis of residual stresses and springback is possible only if the material model accounts for the nonlinear kinematic hardening. Nowadays, there is a big variety of phenomenological approaches to the nonlinear kinematic hardening and different formulations may be available for the same approach. Even more, some new approaches are occasionally developed, which are effectively equivalent to the already existing ones. Thus, there is a need for a unifying classification study. Here, some of the basic approaches are compared in a qualitative way, using a set of classification criteria. These criteria include the objectivity (frame invariance), thermodynamic consistency, pure split of the stress response into volumetric and isochoric parts, stress oscillation under simple shear, integrability of the elastic formulation, and w-invariance. We discuss the hyperelasto-plasticity based on the multiplicative split of the deformation gradient, hypoelasto-plasticity based on the additive split of the strain rate tensor with different corotational and non-corotational stress rates, and a model employing the structure of the small-strain plasticity in combination with a generalized strain measure.

Second- and fourth-rank tensors in $\mathbb{R}^3$ are denoted by bold symbols. The trace, transposition, inverse of transposed, determinant, and Frobenius norm are denoted respectively by $\mathrm{tr}(\cdot)$, $(\cdot)^{\mathrm{T}}$, $(\cdot)^{\text{-T}}$, $\det(\cdot)$. The deviatoric part, symmetric part, and scalar product of two second-rank tensors are defined through

$$\mathbf{Y}^{\mathrm{D}} := \mathbf{Y} - \frac{1}{3}\mathrm{tr}(\mathbf{Y})\ \mathbf{1}, \quad \mathrm{sym}(\mathbf{Y}) := \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^{\mathrm{T}}), \quad \mathbf{A} : \mathbf{B}\ := \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\ \mathbf{B}). \tag{1}$$

Here, $\mathbf{1}$ stands for the identity tensor. The Frobenius norm and the unimodular part are defined as follows

$$\|\mathbf{Y}\| := \sqrt{\mathbf{Y} : \mathbf{Y}} = \sqrt{\mathrm{tr}(\mathbf{Y}^{\mathrm{T}}\ \mathbf{Y})}, \quad \overline{\mathbf{Y}} := (\det(\mathbf{Y}))^{-1/3}\ \mathbf{Y}. \tag{2}$$

2 SMALL STRAIN CASE

Let us consider a small-strain model of an elasto-plastic material with a nonlinear kinematic hardening. The overall infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is decomposed additively into the inelastic (plastic) strain $\boldsymbol{\varepsilon}_{\mathrm{i}}$ and the elastic strain $\boldsymbol{\varepsilon}_{\mathrm{e}}$. The inelastic strain, in turn, is decomposed into the dissipative part $\boldsymbol{\varepsilon}_{\mathrm{ii}}$ and the conservative part $\boldsymbol{\varepsilon}_{\mathrm{ie}}$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{i}} + \boldsymbol{\varepsilon}_{\mathrm{e}}, \quad \boldsymbol{\varepsilon}_{\mathrm{i}} = \boldsymbol{\varepsilon}_{\mathrm{ii}} + \boldsymbol{\varepsilon}_{\mathrm{ie}}. \tag{3}$$

The Helmholz free energy per unit mass is a sum of the elastic part ψ_{el} and the part ψ_{kin}, related to the kinematic hardening:

$$\psi = \psi(\boldsymbol{\varepsilon}_{\mathrm{e}}, \boldsymbol{\varepsilon}_{\mathrm{ie}}) = \psi_{\mathrm{el}}(\boldsymbol{\varepsilon}_{\mathrm{e}}) + \psi_{\mathrm{kin}}(\boldsymbol{\varepsilon}_{\mathrm{ie}}), \tag{4}$$

$$\rho\psi_{\mathrm{el}}(\boldsymbol{\varepsilon}_{\mathrm{e}}) = \frac{k}{2}(\mathrm{tr}\boldsymbol{\varepsilon}_{\mathrm{e}})^2 + \mu\boldsymbol{\varepsilon}_{\mathrm{e}}^{\mathrm{D}} : \boldsymbol{\varepsilon}_{\mathrm{e}}^{\mathrm{D}}, \quad \rho\psi_{\mathrm{kin}}(\boldsymbol{\varepsilon}_{\mathrm{ie}}) = \frac{c}{2}\boldsymbol{\varepsilon}_{\mathrm{ie}}^{\mathrm{D}} : \boldsymbol{\varepsilon}_{\mathrm{ie}}^{\mathrm{D}}, \tag{5}$$

where ρ is the mass density, k and μ are the elastic constants, and c is the bulk modulus of the substructure. The stress tensor $\boldsymbol{\sigma}$ and the backstress tensor $\mathbf{x}$ are evaluated through

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi_{\mathrm{el}}(\boldsymbol{\varepsilon}_{\mathrm{e}})}{\partial\boldsymbol{\varepsilon}_{\mathrm{e}}}, \quad \mathbf{x} = \rho\frac{\partial\psi_{\mathrm{kin}}(\boldsymbol{\varepsilon}_{\mathrm{ie}})}{\partial\boldsymbol{\varepsilon}_{\mathrm{ie}}}, \tag{6}$$

$$\boldsymbol{\sigma} = k\ \mathrm{tr}(\boldsymbol{\varepsilon}_{\mathrm{e}})\mathbf{1} + 2\mu\ \boldsymbol{\varepsilon}_{\mathrm{e}}^{\mathrm{D}}, \quad \mathbf{x} = c\ \boldsymbol{\varepsilon}_{\mathrm{ie}}^{\mathrm{D}}. \tag{7}$$

Let $K \geq 0$ be the initial uniaxial yield stress of the material. Neglecting the isotropic hardening, the yield function f is then defined by

$$f := \|(\boldsymbol{\sigma} - \mathbf{x})^{\mathrm{D}}\| - \sqrt{\frac{2}{3}}K. \tag{8}$$

The flow rule governing the evolution of $\boldsymbol{\varepsilon}_{\mathrm{i}}$ is given by

$$\dot{\boldsymbol{\varepsilon}}_{\mathrm{i}} = \lambda_{\mathrm{i}}\frac{(\boldsymbol{\sigma} - \mathbf{x})^{\mathrm{D}}}{\|(\boldsymbol{\sigma} - \mathbf{x})^{\mathrm{D}}\|}, \quad \lambda_{\mathrm{i}} \geq 0,\ f \leq 0,\ \lambda_{\mathrm{i}}f = 0. \tag{9}$$

The nonlinear kinematic hardening of Armstrong-Frederick type is described by the rule

$$\dot{\boldsymbol{\varepsilon}}_{\rm ii} = \lambda_{\rm i} \ \varkappa \ \mathbf{x}, \tag{10}$$

where $\varkappa \geq 0$ is a material parameter. Taking $(3)_2$ and $(7)_2$ into account, this equation is equivalent to

$$\dot{\mathbf{x}} = c \ \dot{\boldsymbol{\varepsilon}}_{\rm i} - c \ \lambda_{\rm i} \ \varkappa \ \mathbf{x}. \tag{11}$$

This structure corresponds to a small-strain version of a rate-independent Maxwell fluid. It is commonly employed to model the evolution of backstresses $\mathbf{x}$ in a hardening/dynamic recovery format. Depending on the formulation, the initial conditions can be set in terms of $\boldsymbol{\sigma}$ and $\mathbf{x}$, or, alternatively, in terms of $\boldsymbol{\varepsilon}_{\rm i}$ and $\boldsymbol{\varepsilon}_{\rm ii}$.

This material model is thermodynamically consistent and objective. In the following, its finite strain extensions are analyzed. The crucial part of any extension is how the rate-independent Maxwell equation (11) is modified to the geometrically nonlinear case.

3 PRINCIPLES OF CONSTITUTIVE MECHANICS

Along with the general constitutive restrictions, like objectivity and thermodynamic consistency, some more specific principles will be considered in the presented study.

3.1 W-invariance

In the case of metal plasticity, it is reasonable to consider the following property (cf. [16]). Let $\mathbf{F}$ be the deformation gradient which maps the local reference configuration $\widetilde{\mathcal{K}}$ to the current configuration $\mathcal{K}$. For a simple material with initial conditions, the current Kirchhoff stress tensor $\mathbf{S}$ is a function of the local history of $\mathbf{F}$ and a set of initial conditions $\mathcal{Z}_0$:

$$\mathbf{S}(t) = \underset{t_0 \leq t' \leq t}{\mathbf{S}} \big(\mathbf{F}(t'), \mathcal{Z}_0\big). \tag{12}$$

Next, let $\mathbf{F}_0$ be a second-rank tensor, such that $\det(\mathbf{F}_0) = 1$. Let $\widetilde{\mathcal{K}}^{\rm new} := \mathbf{F}_0 \widetilde{\mathcal{K}}$ be a new reference configuration. The corresponding new deformation gradient (also known as the relative deformation gradient) is given by

$$\mathbf{F}^{\rm new}(t) := \mathbf{F}(t) \ \mathbf{F}_0^{-1}. \tag{13}$$

The model (12) is weakly invariant under the transformation (13) if there is

$$\mathcal{Z}_0^{\rm new} = \mathcal{Z}_0^{\rm new}(\mathcal{Z}_0, \mathbf{F}_0), \tag{14}$$

such that the material model predicts the same Kirchhoff stresses:

$$\underset{t_0 \leq t' \leq t}{\mathbf{S}} \big(\mathbf{F}(t'), \mathcal{Z}_0\big) = \underset{t_0 \leq t' \leq t}{\mathbf{S}} \big(\mathbf{F}^{\rm new}(t'), \mathcal{Z}_0^{\rm new}\big). \tag{15}$$

If the model (12) is invariant under arbitrary isochoric changes of the reference configuration, we say that it is *weakly invariant* or, shortly, *w-invariant* (cf. [16]). Similar to the classical (strong) invariance, the w-invariance represents a certain symmetry of the constitutive equations. Just as any other symmetry, w-invariance provides insights into the structure of the underlying constitutive equations.

3.2 Pure volumetric-isochoric split

Again, consider the model (12). Let $\overline{\mathbf{F}}(t) := \det(\mathbf{F}(t))^{-1/3}\ \mathbf{F}(t)$ be the isochoric (unimodular) part of the deformation gradient $\mathbf{F}(t)$. We say that (12) exhibits a pure volumetric-isochoric split (v-i split) with elastic volume changes, if

i:
$$\mathrm{tr}\Big(\underset{t_0\leq t'\leq t}{\mathbf{S}}\Big(\overline{\mathbf{F}}(t'),\mathcal{Z}_0\Big)\Big) \equiv 0, \text{ whenever } \mathrm{tr}\ \mathbf{S}|_{t=t_0} = 0;$$

ii: there is $\mathcal{Z}_0^{dev} = \mathcal{Z}_0^{dev}(\mathcal{Z}_0)$ such that
$$\Big(\underset{t_0\leq t'\leq t}{\mathbf{S}}\big(\mathbf{F}(t'),\mathcal{Z}_0\big)\Big)^{\mathrm{D}} \equiv \underset{t_0\leq t'\leq t}{\mathbf{S}}\big(\overline{\mathbf{F}}(t'),\mathcal{Z}_0^{dev}\big);$$

iii: $\mathrm{tr}\big(\mathbf{S}(t)\big)$ is a function of the instant value $\det\big(\mathbf{F}(t)\big)$.

The most crucial part here is the condition i. Indeed, consider, for example, a model where the initial conditions are formulated with respect to the Kirchhoff stresses: $\mathcal{Z}_0 = \{\mathbf{S}|_{t=t_0}\}$. If the property i is satisfied for a certain model, then the properties ii and iii can be enforced by putting

$$\underset{t_0\leq t'\leq t}{\mathbf{S}}\big(\mathbf{F}(t'),\mathbf{S}|_{t=t_0}\big) := \underset{t_0\leq t'\leq t}{\mathbf{S}}\big(\overline{\mathbf{F}}(t'),(\mathbf{S}|_{t=t_0})^{\mathrm{D}}\big) + p\Big(\det(\mathbf{F}(t))\Big)\mathbf{1}, \tag{16}$$

where $p = \frac{1}{3}\mathrm{tr}\ \mathbf{S}$ is a suitable function of the current $\det \mathbf{F}$.

Note that a certain volumetric-isochoric split is satisfied by the small strain model presented in the previous section. Therefore, it is natural to expect the v-i split in the finite strain context as well.

4 GENERALIZATIONS TO FINITE STRAINS

4.1 Hyperelasto-plasticity with a nested multiplicative split

We discuss here a special case of a multiplicative viscoplasticity, which was proposed in [17]. First, consider a multiplicative split of the deformation gradient $\mathbf{F}$ into the inelastic (plastic) part $\mathbf{F}_{\mathrm{i}}$ and the elastic part $\mathbf{F}_{\mathrm{e}}$. Next, basing on the seminal idea of Lion [8], the inelastic part $\mathbf{F}_{\mathrm{i}}$ is decomposed into the dissipative part $\mathbf{F}_{\mathrm{ii}}$ and the conservative (energetic) part $\mathbf{F}_{\mathrm{ie}}$:

$$\mathbf{F} = \mathbf{F}_{\mathrm{e}}\mathbf{F}_{\mathrm{i}}, \quad \mathbf{F}_{\mathrm{i}} = \mathbf{F}_{\mathrm{ie}}\mathbf{F}_{\mathrm{ii}}. \tag{17}$$

Note that the kinematic relations (3) are restored from (17) in the small strain case.

The right Cauchy-Green tensor (RCGT) $\mathbf{C}$, the inelastic RCGT $\mathbf{C}_{\mathrm{i}}$, and the inelastic RCGT of substructure $\mathbf{C}_{\mathrm{ii}}$ are defined through

$$\mathbf{C} := \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{C}_{\mathrm{i}} := \mathbf{F}_{\mathrm{i}}^{\mathrm{T}}\mathbf{F}_{\mathrm{i}}, \quad \mathbf{C}_{\mathrm{ii}} := \mathbf{F}_{\mathrm{ii}}^{\mathrm{T}}\mathbf{F}_{\mathrm{ii}}. \tag{18}$$

Analogously to (4), the free energy per unit mass is represented in the form

$$\psi = \psi_{\mathrm{el}}(\mathbf{C}\mathbf{C}_{\mathrm{i}}{}^{-1}) + \psi_{\mathrm{kin}}(\mathbf{C}_{\mathrm{i}}\mathbf{C}_{\mathrm{ii}}{}^{-1}), \tag{19}$$

where $\psi_{\text{el}}(\cdot)$ and $\psi_{\text{kin}}(\cdot)$ are isotropic functions. To be definite, neo-Hookean assumptions are used for the deviatoric part of the free energy

$$\rho_{\text{R}}\psi_{\text{el}} = \rho_{\text{R}}\psi_{\text{vol}}(\det(\mathbf{C}\mathbf{C}_{\text{i}}^{-1})) + \frac{\mu}{2}\big(\text{tr}\overline{\mathbf{C}\mathbf{C}_{\text{i}}^{-1}} - 3\big), \quad \rho_{\text{R}}\psi_{\text{kin}} = \frac{c}{4}\big(\text{tr}\overline{\mathbf{C}_{\text{i}}\mathbf{C}_{\text{ii}}^{-1}} - 3\big), \tag{20}$$

where $\rho_{\text{R}} > 0$ is the mass density in the reference configuration, μ and c have the same meaning as in the small strain case. We do not specify the volumetric part ψ_{vol}, since it is irrelevant for the current study. The second Piola-Kirchhoff stress $\tilde{\mathbf{T}}$ and the backstress $\tilde{\mathbf{X}}$, both operating on $\tilde{\mathcal{K}}$, are computed through

$$\tilde{\mathbf{T}} = 2\rho_{\text{R}}\frac{\partial\psi_{\text{el}}(\mathbf{C}{\mathbf{C}_{\text{i}}}^{-1})}{\partial\mathbf{C}}\Big|_{\mathbf{C}_{\text{i}}=\text{const}}, \quad \tilde{\mathbf{X}} = 2\rho_{\text{R}}\frac{\partial\psi_{\text{kin}}(\mathbf{C}_{\text{i}}{\mathbf{C}_{\text{ii}}}^{-1})}{\partial\mathbf{C}_{\text{i}}}\Big|_{\mathbf{C}_{\text{ii}}=\text{const}}. \tag{21}$$

Using (20) we arrive at

$$\tilde{\mathbf{T}} = p(\det(\mathbf{C}))\ \mathbf{C}^{-1} + \mu\ \mathbf{C}^{-1}(\overline{\mathbf{C}}\mathbf{C}_{\text{i}}^{-1})^{\text{D}},\ p \in \mathbb{R}, \quad \tilde{\mathbf{X}} = \frac{c}{2}\ \mathbf{C}_{\text{i}}^{-1}(\mathbf{C}_{\text{i}}\mathbf{C}_{\text{ii}}^{-1})^{\text{D}}. \tag{22}$$

The norm of the driving force $\mathfrak{F}$ and the yield function f are defined through

$$\mathfrak{F} := \sqrt{\text{tr}\big[\big(\mathbf{C}\tilde{\mathbf{T}} - \mathbf{C}_{\text{i}}\tilde{\mathbf{X}}\big)^{\text{D}}\big]^2}, \quad f := \mathfrak{F} - \sqrt{\frac{2}{3}}K. \tag{23}$$

The inelastic flow is described by the following system of constitutive equations

$$\dot{\mathbf{C}}_{\text{i}} = 2\frac{\lambda_{\text{i}}}{\mathfrak{F}}\big(\mathbf{C}\tilde{\mathbf{T}} - \mathbf{C}_{\text{i}}\tilde{\mathbf{X}}\big)^{\text{D}}\mathbf{C}_{\text{i}}, \quad \dot{\mathbf{C}}_{\text{ii}} = 2\lambda_{\text{i}}\varkappa\big(\mathbf{C}_{\text{i}}\tilde{\mathbf{X}}\big)^{\text{D}}\mathbf{C}_{\text{ii}}, \tag{24}$$

$$\lambda_{\text{i}} \geq 0,\ f \leq 0,\ \lambda_{\text{i}}f = 0. \tag{25}$$

Finally, the initial conditions are formulated in terms of $\mathbf{C}_{\text{i}}$ and $\mathbf{C}_{\text{ii}}$.

This material model is thermodynamically consistent (cf. [17]) and objective. As shown in [20], the material model is w-invariant. Since the evolution of internal variables $\mathbf{C}_{\text{i}}$ and $\mathbf{C}_{\text{ii}}$ depends on $\bar{\mathbf{C}}$, this model exhibits the pure v-i split. Within the elastic range, the stress response is hyperelastic. The model is free from any spurious oscillations of shear stresses under monotonic simple shear. As shown in [22], the w-invariance allows one to build an efficient numerical procedure (one-equation integrator) for this model. Various extensions of this model are presented, among others, in [19, 21, 23, 18]. An alternative derivation of the model was presented in [5]. The practical application of the w-invariance of this model is discussed in [20, 14].

4.2 Logarithmic strain with the small strain structure

Another popular approach to the finite strain elasto-plasticity adopts the structure of the geometrically linear theory (3) – (11) (cf. [12, 9, 15]). Let $\mathbf{H}$ be the Lagrangian logarithmic strain (Hencky strain)

$$\mathbf{H}(t) := \frac{1}{2}\ln(\mathbf{C}(t)). \tag{26}$$

The infinitesimal strain tensor which appears in the geometrically linear theory, is replaced now by the logarithmic strain: $\boldsymbol{\varepsilon}(t) := \mathbf{H}(t)$. Let $\boldsymbol{\sigma}(t)$ be the stress tensor, computed by the small strain theory as a response to $\boldsymbol{\varepsilon}(t)$. In the finite strain case, $\boldsymbol{\sigma}$ is understood as a Lagrangian stress measure which is power conjugate to the logarithmic strain $\mathbf{H}$:

$$\boldsymbol{\sigma} : \dot{\mathbf{H}} = \widetilde{\mathbf{T}} : \Big(\frac{1}{2}\dot{\mathbf{C}}\Big) \quad \text{for all} \quad \dot{\mathbf{C}} \in Sym. \tag{27}$$

Using this identity, we obtain the following formula for the second Piola-Kirchhoff $\widetilde{\mathbf{T}}$

$$\widetilde{\mathbf{T}} = \frac{\partial \ln(\mathbf{C})}{\partial \mathbf{C}} : \boldsymbol{\sigma}. \tag{28}$$

The resulting finite strain model is objective. Since the small-strain model (3) – (11) is thermodynamically consistent, so is its finite-strain counterpart. The model is free from the shear stress oscillations (cf. Section 5). The stress response in the elastic domain is hyperelastic. The model exhibits a pure v-i split. Efficient and robust numerical procedures are available for this approach. Unfortunately, this model is not w-invariant (this can be shown using a procedure, presented in [16]). Since the constitutive equations depend on the choice of the reference configuration, one needs to specify exactly, which configuration is used as a reference.

4.3 Hypoelasto-plasticity with an additive split

Another major modelling framework is based on a nested additive split of the strain rate tensor, used in combination with hypoelastic relations (cf. [10, 11]). Let $\mathbf{L} := \dot{\mathbf{F}}\mathbf{F}^{-1}$ be the velocity gradient. Its symmetric part, called the strain rate $\mathbf{D} := \mathrm{sym}(\mathbf{L})$, is decomposed into the inelastic (plastic) part $\mathbf{D}_{\mathrm{i}}$ and the elastic part $\mathbf{D}_{\mathrm{e}}$. The inelastic part itself is decomposed into the dissipative part $\mathbf{D}_{\mathrm{ii}}$ and the conservative part $\mathbf{D}_{\mathrm{e}}$

$$\mathbf{D} = \mathbf{D}_{\mathrm{e}} + \mathbf{D}_{\mathrm{i}}, \quad \mathbf{D}_{\mathrm{i}} = \mathbf{D}_{\mathrm{ii}} + \mathbf{D}_{\mathrm{ie}}. \tag{29}$$

These relations can be seen as a generalization of (3). Let $\mathbf{S}$ and $\mathbf{X}$ be the Kirchhoff stress and the backstress, respectively, both operating on the current configuration $\mathcal{K}$. Denote by $\overset{o}{\mathbf{Y}}$ an objective time derivative of a second-rank tnesor $\mathbf{Y}$. As a generalization of (7), we consider the following hypoelastic relations

$$\overset{o}{\mathbf{S}} = k \ \mathrm{tr}(\mathbf{D}_{\mathrm{e}})\mathbf{1} + 2\mu\mathbf{D}_{\mathrm{e}}^{\mathrm{D}}, \quad \overset{o}{\mathbf{X}} = c\mathbf{D}_{\mathrm{ie}}, \tag{30}$$

where k, μ, and c were already introduced in (7). These equations corresponds to the grade-zero hypoelasticity. Next, the yield function is postulated in the form (cf. (8))

$$f := \|(\mathbf{S} - \mathbf{X})^{\mathrm{D}}\| - \sqrt{2/3}K, \tag{31}$$

where $K > 0$ is the initial uniaxial yield stress. The inelastic flow is governed by (cf. (9) and (10))

$$\mathbf{D}_{\mathrm{i}} = \lambda_{\mathrm{i}} \frac{(\mathbf{S} - \mathbf{X})^{\mathrm{D}}}{\|(\mathbf{S} - \mathbf{X})^{\mathrm{D}}\|}, \quad \mathbf{D}_{\mathrm{ii}} = \lambda_{\mathrm{i}} \ \varkappa \ \mathbf{X}, \tag{32}$$

$$f \leq 0, \ \lambda_{\mathrm{i}} \geq 0, \ f\lambda_{\mathrm{i}} = 0. \tag{33}$$

The initial conditions are imposed on the Kirchhoff stresses $\mathbf{S}$ and the backstress $\mathbf{X}$.

Different models can be build by using different objective stress rates which appears in (30). Some authors apply the so-called *yield stationarity criterion*, which was proposed by Prager in [13]. For the models, where the yield function f is a general isotropic function of $\mathbf{S}$ and $\mathbf{X}$, the yield stationarity requires that $f = const$ whenever $\overset{\circ}{\mathbf{S}} = \overset{\circ}{\mathbf{X}} = \mathbf{0}$. As shown in [26], the yield stationarity implies that the objective rates $\overset{\circ}{\mathbf{S}}$ and $\overset{\circ}{\mathbf{X}}$ must be corotational rates of the same type. In other words, the yield stationarity implies

$$\overset{\circ}{\mathbf{S}} = \dot{\mathbf{S}} + \mathbf{S}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{S}, \quad \overset{\circ}{\mathbf{X}} = \dot{\mathbf{X}} + \mathbf{X}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{X}, \quad \mathbf{\Omega} \in \mathrm{Skew}. \tag{34}$$

Here, Skew stands for the set of skew-symmetric tensors, the skew-symmetric operator $\mathbf{\Omega}$ is referred to as a spin tensor, superimposed dot stands for the material time derivative. There are infinitely many ways of defining the spin tensor $\mathbf{\Omega}$ [25, 7, 3]. Clearly, the properties of the resulting material model depend on the specific choice of the spin $\mathbf{\Omega}$. In particular, we have the following theorem (cf. [16]):

Theorem. *Constitutive relations* (29)—(34) *are w-invariant if and only if the spin tensor* $\mathbf{\Omega}$ *does not depend on the choice of the reference configuration.*

Let us consider some of the commonly used spins.

Zaremba-Jaumann rate. Fot the Zaremba-Jaumann rate (also known as the Zaremba-Jaumann-Noll rate), we put

$$\mathbf{\Omega}^{ZJ} := \mathbf{W} = \mathrm{skew}(\mathbf{L}), \quad \overset{\circ}{\mathbf{Y}}^{ZJ} := \dot{\mathbf{Y}} + \mathbf{Y}\mathbf{\Omega}^{ZJ} - \mathbf{\Omega}^{ZJ}\mathbf{Y}. \tag{35}$$

Note that the continuum spin $\mathbf{W} = \mathrm{skew}(\mathbf{L})$ does not depend on the choice of the reference configuration. Therefore, the corresponding system of equations is w-invariant. One major drawback of this approach is that the stress response exhibits non-physical oscillations under the simple shear: The shear stress oscillates like $\sin(\gamma)$, where γ is the shear strain. These oscillations may lead to absurd results in case of kinematic hardening, although the elastic strains may remain small (cf. Section 5). Another drawback is that the material response fails to become hyperelastic in case of a frozen inelastic flow (when $\lambda_{\mathrm{i}} = 0$).

Green-Naghdi rate. In order to define the Green-Naghdi rate (also known as the Green-Naghdi-Dienes rate, Green-McInnis rate or polar rate) we consider the polar decomposition of the deformation gradient: $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$. Then we put

$$\mathbf{\Omega}^{GN} := \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \in \mathrm{Skew}, \quad \overset{\circ}{\mathbf{Y}}^{GN} := \dot{\mathbf{Y}} + \mathbf{Y}\mathbf{\Omega}^{GN} - \mathbf{\Omega}^{GN}\mathbf{Y}. \tag{36}$$

Unfortunately, the spin $\mathbf{\Omega}^{GN}$ depends on the choice of the reference configuration [6, 16]. Thus, the corresponding system of equations is not w-invariant. On the other hand, such a model is free from spurious shear oscillations (see Section 5). Just as in the previous case, the corresponding material model fails to provide a hyperelastic response even for a frozen inelastic flow.

Logarithmic rate. Let $\mathbf{V}$ be the left stretch tensor ($\mathbf{V} := \sqrt{\mathbf{F}\mathbf{F}^{\mathrm{T}}}$). The logarithmic stress rate is given by

$$\overset{\circ}{\mathbf{S}}^{\log} := \dot{\mathbf{S}} + \mathbf{S}\mathbf{\Omega}^{\log} - \mathbf{\Omega}^{\log}\mathbf{S}, \tag{37}$$

where the logarithmic spin $\mathbf{\Omega}^{\log} = \mathbf{\Omega}^{\log}(\mathbf{V}, \mathbf{L})$ is uniquely defined by the relation (cf. [24, 27])

$$\mathbf{D} = (\ln \overset{o}{\mathbf{V}})^{\log}. \tag{38}$$

The following statement was proved in [2]: Dealing with grade-zero hypoelasticity with corotational rates and constant elastic stiffness, the logarithmic stress rate *is the only choice* which allows one to build integrable stress-strain relations in the elastic range. For that reason, the logarithmic rate enjoys a privileged position among all the corotational rates. As was shown in [16], the spin $\mathbf{\Omega}^{\log}$ depends on the choice of the reference configuration. Therefore, the corresponding material model is not w-invariant. In the purely elastic case, the stress response reduces to a special type of hyperelasticity, where the strain energy function is given by a quadratic function of the Hencky strain. This elastic potential is known to produce absurd results for large elastic strains.

In a summary, it is *impossible* to build a material model of type (29)–(33), which would combine the yield stationarity, w-invariance and exact integrability of the elastic part. On the other hand, a positive feature of the corotational spin (34) is that the corresponding models always exhibit the pure v-i split. Now, in an attempt to build a model, which would be objective, w-invariant, and exactly integrable in the elastic domain, we proceed to non-corotational rates.

Covariant Oldroyd rate. The covariant Oldroyd rate (also known as the lower Oldroyd rate or the Cotter-Rivlin rate) of a Eulerian tensor $\mathbf{Y}$ is defined by

$$\mathfrak{O}_{covar}(\mathbf{Y}) := \dot{\mathbf{Y}} + \mathbf{L}^{\mathrm{T}}\, \mathbf{Y} + \mathbf{Y}\, \mathbf{L}. \tag{39}$$

A material model of type (29)–(33), based on this stress rate, is objective and w-invariant. One remarkable property of this rate is that for the Almansi strain $\mathbf{A}$ we have

$$\mathfrak{O}_{covar}(\mathbf{A}) = \mathbf{D}, \quad \text{where } \mathbf{A} := \frac{1}{2}(\mathbf{1} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}). \tag{40}$$

Thus, the stress response is exactly integrable whenever $\lambda_{\mathrm{i}} = 0$. Unfortunately, the corresponding model does not exhibit the pure v-i split: Even if the prescribed strain rate $\mathbf{D}$ is trace-free and the initial stresses are deviatoric, the natural condition $\mathrm{tr}\mathbf{S} = \mathrm{tr}\mathbf{X} = 0$ is violated. Nevertheless, although Prager's yield stationarity condition is violated by this model, the model allows one to obtain plausible results (see Section 5).

The corresponding rate-independent Maxwell fluid is a scleronomous version of the covaraint Maxwell model (cf. [4]).

Deviatorized covariant Oldroyd rate. In an attempt to enforce the pure v-i split we consider a deviatorized covariant Oldroyd rate as follows

$$\mathfrak{O}^{dev}_{covar}(\mathbf{Y}) := \mathfrak{O}_{covar}(\mathbf{Y}) - \frac{2}{3}(\mathbf{Y} : \mathbf{D})\mathbf{1} = \dot{\mathbf{Y}} + \mathbf{L}^{\mathrm{T}}\, \mathbf{Y} + \mathbf{Y}\, \mathbf{L} - \frac{2}{3}(\mathbf{Y} : \mathbf{D})\mathbf{1}. \tag{41}$$

The corresponding material model is objective and w-invaraint; the pure v-i split holds true. According to the available analytical solutions (cf. [1]), the stress rate (41) produces an oscillatory response to the monotonic simple shear even in the purely elastic case. The shear stress oscillates like $\sin(\sqrt{\frac{2}{3}}\gamma)$, where γ is the shear strain. Thus, the oscillation frequency is slightly lower than in the case of the Zaremba-Jaumann stress rate. In general, this model should not be used if the elastic strains in the corresponding rate-independent Maxwell body exceed a certain limit (see Section 5).

Contravariant Oldroyd rate. The contravariant Oldroyd rate (upper Oldroyd rate) is defined as,

$$\mathfrak{O}_{contravar}(\mathbf{Y}) := \dot{\mathbf{Y}} - \mathbf{L}\mathbf{Y} - \mathbf{Y}\mathbf{L}^{\mathrm{T}}. \tag{42}$$

The corresponding material model is objective and w-invariant. The contravariant rate of the Finger tensor $\mathbf{a}$ is related to the strain rate in the following way:

$$\mathfrak{O}_{contravar}(\mathbf{a}) = -\mathbf{D}, \quad \text{where } \mathbf{a} := \frac{1}{2}(\mathbf{1} - \mathbf{F}\mathbf{F}^{\mathrm{T}}). \tag{43}$$

Thus, this stress rate allows one to obtain exactly integrable response in the elastic range. Unfortunately, just as for the covariant rate, the corresponding material model does not exhibit the pure v-i split. Although the model violates Prager's yield stationarity, it allows one to obtain a reasonable stress response, even dealing with linear and nonlinear kinematic hardening (see Section 5). The underlying Maxwell fluid is a scleronomous version of the contravaraint Maxwell model (cf. [4]).

Deviatorized contravariant Oldroyd rate. In order to enforce the pure v-i split, we consider now a deviatorized variant of the contravariant Oldroyd rate

$$\mathfrak{O}^{dev}_{contravar}(\mathbf{Y}) := \mathfrak{O}_{contravar}(\mathbf{Y}) + \frac{2}{3}(\mathbf{Y} : \mathbf{D})\mathbf{1} = \dot{\mathbf{Y}} - \mathbf{L}\mathbf{Y} - \mathbf{Y}\mathbf{L}^{\mathrm{T}} + \frac{2}{3}(\mathbf{Y} : \mathbf{D})\mathbf{1}. \tag{44}$$

The resulting system of constitutive equations is objective and w-invaraint; the pure v-i split is satisfied. An analytical solution is available for the simple shear (cf. [1]); the solution says that the stresses oscillate like $\sin(\sqrt{\frac{2}{3}}\gamma)$, where γ is the shear strain. Just as its covariant counterpart, this model should not be implemented if the elastic strains in the rate-independent Maxwell body exceed a certain limit (see Section 5).

5 NUMERICAL RESULTS

Let us simulate a stress response under the non-monotonic simple shear

$$\mathbf{F}(t) = \mathbf{1} + \gamma(t)\mathbf{e}_x \otimes \mathbf{e}_y, \quad \gamma(t) = \min(5t, 10 - 5t), \quad t \in [0, 2]. \tag{45}$$

The following material parameters are used (all quantities are non-dimensional): $K = 1$, $\mu = 10$, $c = 0.5$, $\varkappa = 0.5$. Since the simple shear is isochoric, the bulk modulus k is irrelevant. Since the elastic strains accumulated in the rate-independent Maxwell fluid are large, second-order effects like the stress oscillation and Poynting/Swift effect are clearly visible (see Figure 1). Plausible results are predicted by the multiplicative model (Section 4.1) and the hypoelasto-plasticity with logarithmic rate (Section 4.3). Unrealistic shear stresses are observed for oscillating models (Zaremba-Jaumann, deviatorized contravariant and covariant Oldroyd); the small-strain-structure-model with the logarithmic strain (Section 4.2) exhibits very strong isotropic softening, caused by the model kinematics.

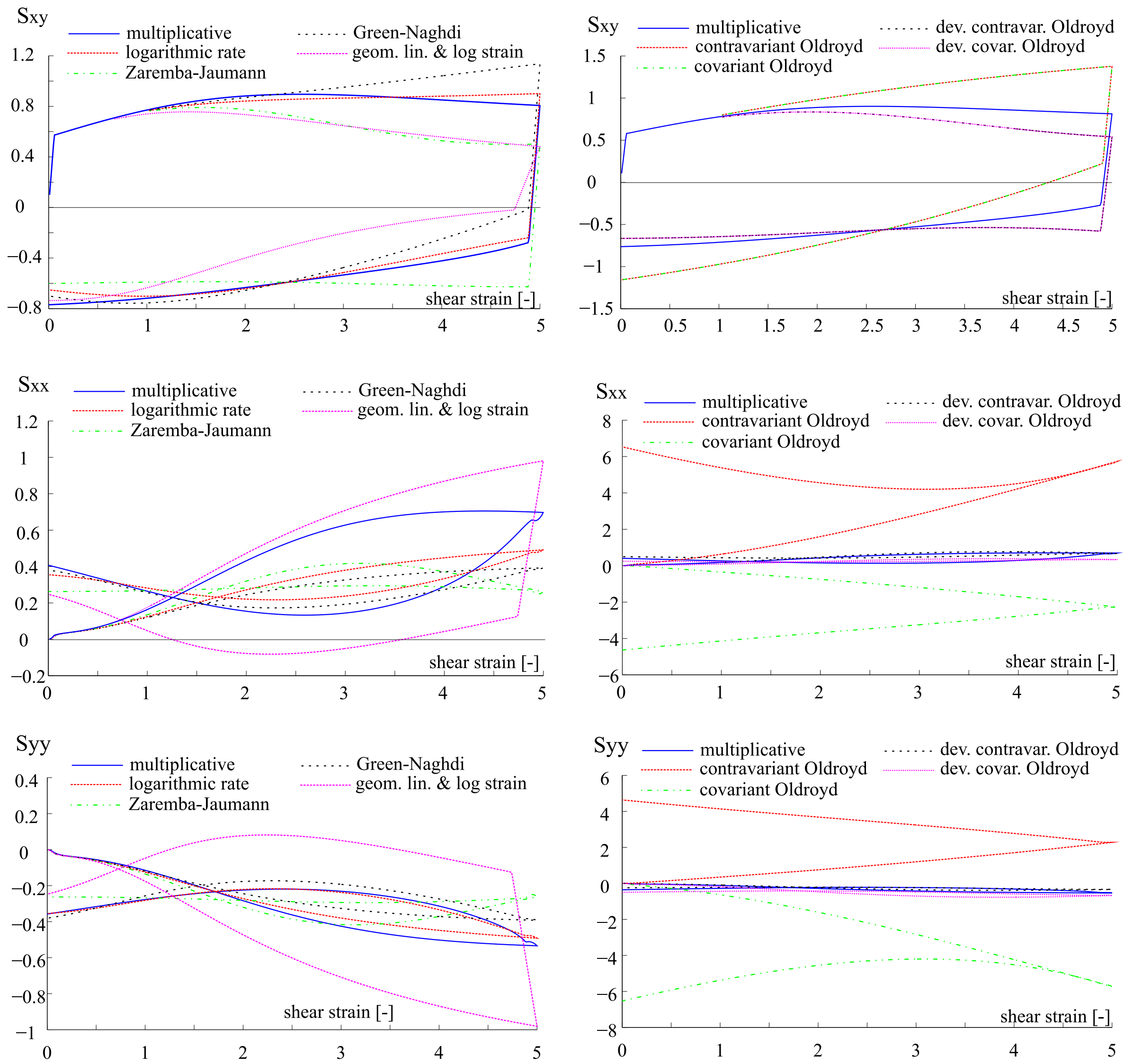

Figure 1: Simulations results for non-monotonic simple shear using different material models

6 CONCLUSIONS

Nine different models of finite strain plasticity with the nonlinear kinematic hardening are analyzed in a qualitative way, using a number of criteria. The model based on the multiplicative split (Section 4.1) is the only model which combines objectivity, thermodynamic consistency, w-invariance, and the pure v-i split.

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