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DISTINGUISHING TOURNAMENTS WITH SMALL LABEL CLASSES

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Abstract. A *d-distinguishing vertex (arc) labeling* of a digraph is a vertex (arc) labeling using d labels that is not preserved by any nontrivial automorphism. Let $\rho(T)$ ($\rho'(T)$) be the minimum size of a label class in a 2-distinguishing vertex (arc) labeling of a tournament T. Gluck's Theorem implies that $\rho(T) \leq \lfloor n/2 \rfloor$ for any tournament T of order n. We construct a family of tournaments $\mathcal{H}$ such that $\rho(T) \geq \lfloor n/2 \rfloor$ for any tournament of order n in $\mathcal{H}$. Additionally, we prove that $\rho'(T) \leq \lfloor 7n/36 \rfloor + 3$ for any tournament T of order n and $\rho'(T) \geq \lceil n/6 \rceil$ when $T \in \mathcal{H}$ and has order n. These results answer some open questions stated by Boutin.

1. Introduction

Given a *digraph* G, $V(G)$ ($A(G)$) stands for its set of vertices (arcs) and $\mathrm{Aut}(G)$ denotes the automorphism group of G. We refer to the identity automorphism in $Aut(G)$ as to the *trivial* automorphism. A tournament is a complete oriented graph, that is, a digraph T for which for every $u, v \in V(T)$, either $uv \in A(T)$ or $vu \in A(T)$ but not both.

A *vertex (arc) labeling* of a digraph G is a total function $\phi\colon V(G) \to L$ ($\phi\colon A(G) \to L$) which labels each vertex (arc) of G with a label from the set L. Given a vertex labeling ϕ for a digraph G, we say that an automorphism $\sigma \in \mathrm{Aut}(G)$ *preserves* ϕ if $\phi(\sigma(v)) = \phi(v)$ for every vertex $v \in V(G)$. Similarly, we say that $\sigma \in \mathrm{Aut}(G)$ *preserves* an arc labeling ϕ if $\phi(uv) = \phi(\sigma(u)\sigma(v))$ for every arc $uv \in A(G)$. On the contrary, a vertex or arc labeling ϕ *breaks* an automorphism $\sigma \in \mathrm{Aut}(G)$ if ϕ is not preserved by σ. A (vertex or arc) labeling ϕ of G that breaks all nontrivial automorphisms in $\mathrm{Aut}(G)$ is called *distinguishing* for G. Additionally, if ϕ uses d labels, it is called d-distinguishing for G. When discussing about 2-distinguishing labelings, our color labels will be *white* and *black*.

Albertson and Collins introduced the concept of *distinguishing number* in the seminal paper [**1**]. The *distinguishing number* $D(G)$ of a digraph G is the least cardinal d such that G has a d-distinguishing vertex labeling. In recent years, this concept has been extended to the *distinguishing index* $D'(G)$, which is defined as

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the least cardinal d such that G has an d-distinghishing arc labeling. A *distinguishing vertex class (distinguishing arc class)* of ϕ in G is any of the d subsets of $V(G)$ $(A(G))$ having the same label under ϕ. These notions have been studied in [**3, 4, 5, 6, 7, 10, 13**].

With respect to tournaments, Albertson and Collins [**2**] conjectured that every tournament T satisfies $D(T) \leq 2$. As Godsil observed in 2002 [**9**], since tournaments have odd order automorphism groups, the conjecture follows from Gluck's Theorem ([**8**], see also the shorter and self-contained proof by Matsuyama [**12**]). In the following statement of Gluck's Theorem, given a permutation group G on Ω, $S \subseteq \Omega$ is a regular subset of G if the setwise stabilizer $\{g \in G \mid Sg = S\}$ only contains the identity.

Theorem 1 ([**8, 12**, Gluck's Thm.])**.** *Let G be a permutation group of odd order on a finite set Ω. Then G has a regular subset in Ω.*

Given a tournament T, Gluck's Theorem shows the existence of a regular subset $S \subseteq \Omega = V(T)$ for $\mathrm{Aut}(T)$. Define a labeling ϕ that assigns a white label to the vertices in S and a black label to the vertices in $V(T) \setminus S$. Now, the definition of regular subset implies that the only automorphism in $\mathrm{Aut}(T)$ preserving labeling ϕ is the identity. Therefore, ϕ constitutes a 2-distinguishing vertex labeling of the vertices of T.

Corollary 1 ([**9**])**.** *If T is a tournament, then $D(T) \leq 2$.*

As an added consequence of Gluck's Theorem, we can observe that the distinguishing index of tournaments is also bounded by 2. Suppose that S is a regular subset of the vertices of a tournament T given by Gluck's Theorem. Clearly, vertices in S can be singularized if the arcs lying inside S are labeled white and the rest are labeled black. This way, the orbit of a vertex in S by any automorphism will lie inside S, and the previous arc labeling will be 2-distinguishing.

Corollary 2. *If T is a tournament, then $D'(T) \leq 2$.*

Some literature on the subject has focused on the minimum possible size of a distinguishing vertex class, which has been called *the cost of 2-distinguishing*. We define it here both for vertices and arcs. For a digraph G such that $D(G) \leq 2$, define $\rho(G)$ $(\rho'(G))$ as the minimum size of a distinguishing vertex (arc) class. Upper bounds for $\rho(T)$ and $\rho'(T)$, for any tournament T, are provided in Section 2. In Section 3 we introduce a class of tournaments $\mathcal{H}$ that will allow us to give lower bounds for $\rho(T)$ and $\rho'(T)$, for any $T \in \mathcal{H}$.

2. Upper bounds

Just by observing that distinguishing vertex classes are closed under complementation, we obtain an upper bound for their size with the help of Gluck's theorem.

Theorem 2. *For any tournament T of order n, $\rho(T) \leq \lfloor n/2 \rfloor$.*

To get an upper bound of the cost of 2-distinghishing a tournament by means of the arcs, we will use the concept of determining set. Given a digraph G, a subset $S \subseteq V(G)$ is a *determining set* of G if for any $\varphi, \psi \in \mathrm{Aut}(G)$ such that $\varphi(x) = \psi(x)$ for all $x \in S$, then $\varphi = \psi$. Thus, the action of an automorphism on S determines its action on $V(G)$. In particular, every distinguishing set is a determining set. The *determining number* of a digraph G, denoted by $\mathrm{Det}(G)$, is defined as the minimum size of a determining set for G. We will use the following result from [**11**].

Theorem 3 ([**11**, Thm. 8])**.** *For every tournament T of order n,* $\mathrm{Det}(T) \leq \lfloor n/3 \rfloor$.

To get an upper bound for $\rho'(T)$, where T is a tournament of order n, we start considering a determining set $S \subseteq V(T)$ that, according to Theorem 3, can be selected with size at most $\lfloor n/3 \rfloor$. We can now singularize the vertices in S by coloring some of the arcs in the subtournament of T induced by S, $T[S]$. An easy way to do it is by coloring the arcs of a Hamiltonian path in $T[S]$ in black while coloring the rest of the arcs in T in white. This way, all the vertices in S will be at a different distance through the black arcs from the beginning of the path, and therefore, S will be fixed pointwise and $\rho'(T) \leq \lfloor n/3 \rfloor - 1$. However, we can push the upper bound further down by combining determining and distinguishing sets. By Theorem 2, $\rho(T[S]) \leq \lfloor |S|/2 \rfloor \leq \lfloor n/6 \rfloor$ and, then, there exists a distinguishing set $R \subseteq S$ that proves it. Now, by conveniently grouping the vertices in R by disjoint black paths of length two and the vertices in $S \setminus R$ by disjoint black arcs, we can show the following improved bound.

Theorem 4. *For any tournament T of order n, $\rho'(T) \leq \lfloor 7n/36 \rfloor + 3$.*

3. Lower bounds

We introduce a class of tournaments that is needed to provide our lower bounds for the cost of 2-distinguishing tournaments. By $\vec{C}_3$ we denote the directed triangle, that is, the tournament containing the vertices x_1, x_2, and x_3 and the arcs x_1x_2, x_2x_3, and x_3x_1.

Definition 1. The family $\mathcal{H} = \{H_k\}_{k \geq 0}$ of tournaments is inductively defined as follows. Tournament H_0 consists of a single vertex. For $k > 0$, H_k is the tournament consisting of a copy of $\vec{C}_3$ in which every vertex x_i in $\vec{C}_3$ is substituted by a copy of H_{k-1}, called *tertian* T_i, and an arc $x_ix_j \in A(C_3)$ is substituted by all possible arcs from T_i to T_j.

Observation 1. For any $k \geq 0$, $|V(H_k)| = 3^k$.

A *module* in a tournament T is a set X of vertices such that each vertex in $V(T) \setminus X$ has a uniform relationship to all vertices in X, that is, for every vertex $v \in V(T) \setminus X$, either $uv \in A(T)$ for all $u \in X$ or $vu \in A(T)$ for all $u \in X$. Note that both T and the sets $\{u\}$, where $u \in V(T)$, are modules. Furthermore, modularity is transitive: if Y is a module in the subtournament $T[X]$ induced by module X, then Y is a module in T.

According to the definition of H_k, each of its three tertians are modules. By transitivity of modularity we can make the following observation.

Observation 2. For every $k \geq 1$, H_k can be decomposed into 3^{k-1} pairwise disjoint modules isomorphic to $\vec{C}_3$.

We also need the following fact on how vertices in H_k can move in an automorphism.

Proposition 1. *Let $\sigma \in \mathrm{Aut}(H_k)$ be an automorphism and let T_1, T_2, T_3 be the tertians of H_k. Then, any tertian is mapped by σ into another tertian as a whole, that is, for any $u, v \in T_i$, $\sigma(u), \sigma(v) \in T_j$, for $1 \leq i, j \leq 3$.*

The following labelings play an important role in the proof of the lower bounds (see Figure 1).

Definition 2. A *black* (*white*) *labeling* of H_0 consists of labeling its unique vertex black (white). If $k > 0$, then a *black* (*white*) labeling of H_k contains two copies of H_{k-1} with a black (white) labeling and one copy of H_{k-1} with a white (black) labeling.

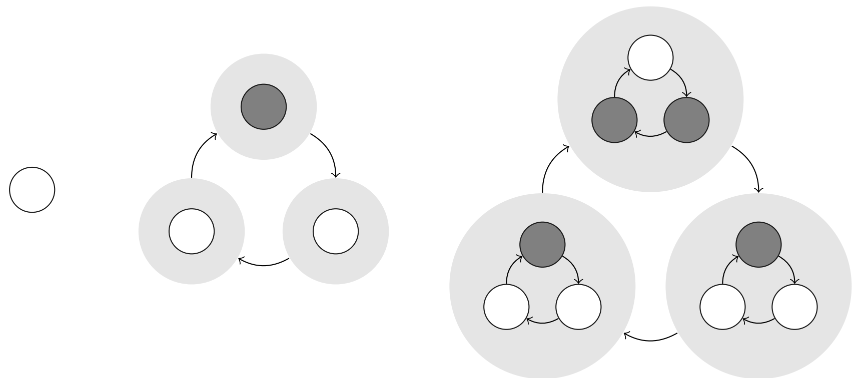

Figure 1. From left to right, white labelings for H_0, H_1, and H_2. Tertians are shadowed in grey. Arcs between tertians imply all arcs between their vertices in the same direction.

The bound given in Theorem 2 is optimal for the family $\mathcal{H} = \{H_k\}_{k \geq 0}$. The proof is by induction on k based on counting the number of black (white) vertices in a black (white) labeling of H_k.

Proposition 2. *For every $k \geq 0$, $\rho(H_k) \geq \lfloor 3^k/2 \rfloor$.*

As a consequence, the upper bound from Theorem 2 is tight.

Theorem 5. *For every $k \geq 0$, there is a tournament T of order $n = 3^k$ such that $\rho(T) = \lfloor n/2 \rfloor$.*

We now show a lower bound for the distinguishing index of tournaments. We refer to any of the pairwise disjoint modules mentioned in Observation 2 as to a *basic module*. Note that a nontrivial automorphism in any basic module trivially extends to H_k by definition. This fact leads to the following lower bound for $\rho'(H_k)$.

Proposition 3. *For every $k \geq 1$, $\rho'(H_k) \geq \lceil 3^{k-1}/2 \rceil$.*

We show that $\lceil 3^{k-1}/2 \rceil$ is also un upper bound for the family of tournaments $\{H_k\}_{k\geq 1}$. We prove it by induction based on counting the number of black arcs whose endpoints belong to the same basic module in any 2-distinguishing arc labelling of H_k (see an illustration of the labeling in Figure 2). Proposition 1 is also needed here to argue about the orbits of vertices in H_k.

Proposition 4. *For every $k \geq 1$, $\rho'(H_k) \leq \lceil 3^{k-1}/2 \rceil$.*

As a consequence, we obtain the following result.

Theorem 6. *For every $k \geq 0$, there is a tournament T of order $n = 3^k$ such that $\rho'(T) = \lceil n/6 \rceil$.*

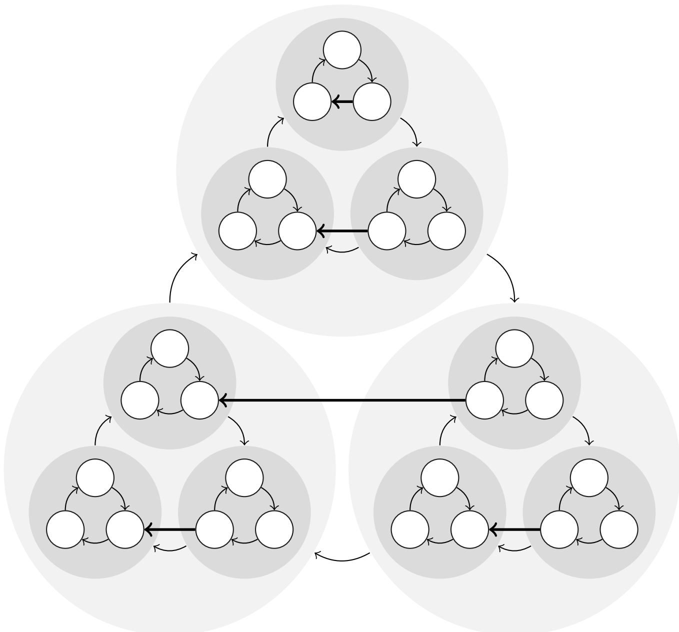

Figure 2. Arc labeling for tournament H_3. The five straight thick arcs represent the only black arcs. Arcs between tertians imply all arcs between their vertices in the same direction.

4. Conclusions and open questions

In [**3**], Boutin proves that $\rho(Q_n) = \mathcal{O}(\mathrm{Det}(Q_n))$, where Q_n is the hypercube of dimension n, and asks in Question 9 whether this is also the case of other graph families. In relation with this question, Problem 4 in [**4**] asks whether there are graphs G such that $\rho(G)$ is arbitrarily larger than $\mathrm{Det}(G)$. But for tournaments T of order n belonging to $\mathcal{H}$, we have, on the one hand, that $\rho(T) = \lfloor n/2 \rfloor$ (as seen in Section 3) and, on the other hand, it is easy to see that $\mathrm{Det}(T) = \lfloor n/3 \rfloor$.

Therefore, $\rho(T)$ and $\mathrm{Det}(T)$, for any $T \in \mathcal{H}$, are related by a factor of $3/2$ and we can answer affirmatively to both questions. We conclude with an obvious question left open in this work.

Question 1. Can the bound in Theorem 4 be improved? In particular, is $\rho'(T) \leq \lceil \frac{n}{6} \rceil$ for any tournament T of order n?

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