Analysis and design of quadratically bounded QPV control systems ⋆

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Abstract: A nonlinear system is said to be quadratically bounded (QB) if all its solutions are bounded and this is guaranteed using a quadratic Lyapunov function. This paper considers the QB analysis and state-feedback controller design problems for quadratic parameter varying (QPV) systems. The developed approach, which relies on a linear matrix inequality (LMIs) feasibility problem, ensures that the QB property holds for an invariant ellipsoid which contains a predefined polytopic region of the state space. An example is used to illustrate the main characteristics of the proposed approach and to confirm the validity of the theoretical results.

Keywords: Quadratic boundedness, quadratic systems, gain-scheduling, linear matrix inequalities

1. INTRODUCTION

The notion of quadratic boundedness (QB) was first introduced by Brockman and Corless (1995, 1998). Roughly speaking, a system is said to be quadratically bounded if all its solutions are bounded and this behavior can be guaranteed with a quadratic Lyapunov function. Several results have exploited this concept for different purposes. For example, a few works have used QB for the design of state estimators, see e.g., Alessandri et al. (2004, 2006), Cayero et al. (2019). Output feedback stabilization by means of QB was investigated by a few papers, see e.g. Ding (2009, 2013), Ping (2017), Ping et al. (2017). Recently, QB was applied to problems related to fault estimation, as in Buciakowski et al. (2017a,b), Witczak et al. (2018).

In the last decades, linear parameter varying (LPV) systems have shown to be a valid framework for extending LTI techniques in order to address the identification, control and estimation of nonlinear systems (Rotondo, 2017). The main strength of this framework is that it has proved to be suitable for controlling nonlinear systems by embedding the nonlinearities within some varying parameters that will depend on endogenous signals, e.g., states, inputs or outputs (Hoffmann and Werner, 2015). In this case, the obtained system is referred to as *quasi-LPV*, to make a distinction from *pure LPV* systems, for which the varying parameters depend only on exogenous signals (Marcos and Balas, 2004).

However, in many cases, maintaining a nonlinear structure instead of reducing to a linear one, can be more appropriate and lead to less conservativeness and over-approximation. For this reason, a few recent works have investigated nonlinear parameter varying (NLPV) systems (Cai et al., 2015, Blesa et al., 2015). In particular, in the last years, some results about quadratic parameter varying (QPV) systems have appeared (Chen et al., 2017, Rotondo and Johansen, 2018, Kanarachos et al., 2018). The QPV framework can be used to characterize nonlinear systems, such as robotic manipulators (Siciliano et al., 2010) and inverted pendula (Siciliano et al., 2010). Additionally, QPV systems could be obtained by calculating the first and second order terms of the Taylor expansion of a nonlinear plant about a family of operating points (this approach is akin to the *linearization scheduling* for LPV systems, see e.g. Rugh and Shamma (2000)).

The main contribution of this paper lies in considering the QB analysis and state-feedback controller design problems for QPV systems. The developed approach, which relies on a linear matrix inequality (LMIs) feasibility problem, ensures that the QB property holds for an invariant ellipsoid which contains a predefined polytopic region of the state space.

The paper is structured as follows. Section 2 provides the formal definition of quadratic boundedness, and shows how QB analysis conditions for QPV systems can be derived. The controller design problem is described and solved in Section 3. Reduction of the LMI-based conditions from an infinite number to a finite number, computationally tractable, is performed in Section 4 using a polytopic approach. An example is used to illustrate the main characteristics of the proposed approach and to confirm the validity of the theoretical results in Section 5. Finally, Section 6 outlines the main conclusions.

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2. QUADRATIC BOUNDEDNESS ANALYSIS

Let us consider the following continuous-time quadratic parameter-varying (QPV) system:

$$\dot{x}(t) = A(\theta(t))x(t) + N(\theta(t),x(t)) + G(\theta(t))w(t) \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ is the system's state, $w \in \Omega \subset \mathbb{R}^{n_w}$ represents an exogenous disturbance, and $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ is the varying parameter vector that schedules both the matrix functions of appropriate dimensions $A(\theta(t)), G(\theta(t))$ and the nonlinear function $N(\theta(t),x(t))$ defined as:

$$N(\theta(t),x(t)) = \begin{bmatrix} x(t)^T N_1(\theta(t))x(t) \\ x(t)^T N_2(\theta(t))x(t) \\ \cdots \\ x(t)^T N_{n_x}(\theta(t))x(t) \end{bmatrix} \quad (2)$$

where $N_1(\theta(t)), N_2(\theta(t)), \ldots, N_{n_x}(\theta(t))$ are matrix functions of appropriate dimensions. In the following, it will be assumed that:

$$\Omega = \{w \in \mathbb{R}^{n_w} : \|w\| \le 1\} \quad (3)$$

where w denotes the Euclidean norm of the vector w. Note that the case in which $\|w\| \le d \in \mathbb{R}_+$ can be brought to (3) by means of an appropriate rescaling of the matrix $G(\theta(t))$ in (1). The concept of quadratic boundedness was introduced by Brockman and Corless (1995), and can be extended for QPV systems in the form (1)-(2) as follows:

Definition 1. The QPV system (1)-(2) is quadratically bounded (QB) with Lyapunov matrix P if P is a positive definite symmetric matrix and $\forall w \in \Omega$ and $\forall \theta \in \Theta$:

$$V(x(t)) = x(t)^T P x(t) > 1 \rightarrow \dot{V}(x(t),w(t),\theta(t)) < 0 \quad (4)$$

Using arguments of Lyapunov analysis, Brockman and Corless (1995) showed that if a system is QB with Lyapunov matrix P, then the set $\mathscr{E}_P = \{x \in \mathbb{R}^{n_x} : x^T P x \le 1\}$ is positively invariant and attractive.

Given two scalars $\gamma, \rho \in \mathbb{R}$, let us define:

$$Q_{\gamma,\rho}(x,\theta) \triangleq -He\left\{\gamma P A(\theta) + P\rho \begin{bmatrix} x^T N_1(\theta) \\ x^T N_2(\theta) \\ \cdots \\ x^T N_{n_x}(\theta) \end{bmatrix}\right\} \quad (5)$$

where $He\{\cdot\}$ stands as a shorthand notation for $(\cdot) + (\cdot)^T$. Also, let us recall the following lemma (Brockman and Corless, 1998).

Lemma 1. Let $P, B \succeq 0$ and $Q \succ 0$. Then:

$$x^T Q x - 2\left(x^T B x\right)^{1/2} > 0 \quad (6)$$

for $x^T P x > 1$ if and only if there exists $\alpha > 0$ such that:

$$Q - \alpha P - \alpha^{-1} B \succeq 0 \quad (7)$$

Then, the following theorem yields a sufficient condition for (1)-(2) to be *globally* QB for some $P \succ 0$.

Theorem 1. Let $P \succ 0$. Then, system (1)-(2) is globally QB with Lyapunov matrix P if there exists a scalar $\alpha > 0$ such that:

$$\begin{bmatrix} -Q_{1,1}(x,\theta) + \alpha P & PG(\theta) \\ G(\theta)^T P & -\alpha I \end{bmatrix} \preceq 0 \quad \forall \theta \in \Theta \quad (8)$$

Proof: Let us calculate the time derivative of the Lyapunov function $V(x(t)) = x(t)^T P x(t)$ which, taking into account the system's dynamics described by (1)-(2), is given by [1]:

$$\dot{V}(x,w,\theta) = 2x^T P[A(\theta)x + N(\theta,x) + G(\theta)w] \quad (9)$$

[1] In the following, dependence of signals and matrices on time is omitted.

It follows from Definition 1 that QB is equivalent to the following condition holding $\forall \theta \in \Theta$ when $x^T P x > 1$:

$$\max\{2x^T P(A(\theta)x + N(\theta,x)) + 2x^T PG(\theta)w : \|w\| \le 1\} < 0 \quad (10)$$

Following Brockman and Corless (1998), the above condition is true if and only if for $x^T P x > 1$:

$$x^T Q_{1,1}(x,\theta)x - 2\|x^T PG(\theta)\| > 0 \quad (11)$$

or, equivalently:

$$x^T Q_{1,1}(x,\theta)x - 2\left(x^T PG(\theta)G(\theta)^T P x\right)^{1/2} > 0 \quad (12)$$

Using Lemma 1 and Schur complements, from (12) follows that (1)-(2) is globally QB with Lyapunov matrix P if matrix inequality (8) holds. □

Due to the presence of the product between the unknown variables α and P, (8) represents a parameterized bilinear matrix inequality (BMI), which can be converted into a parameterized linear matrix inequality (LMI) if one fixes beforehand the value of the scalar α. However, (8) must hold $\forall x \in \mathbb{R}^{n_x}$ and $\forall \theta \in \Theta$, which cannot be assessed $\forall x \in \mathbb{R}^{n_x}$ using available computational tools. For this reason, following some ideas developed in Rotondo and Johansen (2018), the following theorem provides conditions for analyzing whether the property of QB holds for an invariant ellipsoid which contains the polytope $\mathscr{P} \subset \mathbb{R}^{n_x}$, described by:

$$\mathscr{P} = Co\left\{x_{(1)}, x_{(2)}, \ldots, x_{(p)}\right\} = \left\{x \in \mathbb{R}^{n_x} : a_k^T x \le 1, k = 1, \ldots, q\right\} \quad (13)$$

where p and q are suitable integer numbers, $x_{(i)}$ denotes the i-th vertex of $\mathscr{P}$, $Co\{\cdot\}$ denotes the convex hull, and a_k denote the coefficients of the equivalent half-space representation.

Theorem 2. Let $P \succ 0$. Then, system (1)-(2) is QB in the polytope $\mathscr{P}$ defined by (13) if there exist scalars $\alpha > 0$, $0 < \gamma < 1$ and $\mu > 1$ such that $\forall i \in \{1, \ldots, p\}$, $\forall k \in \{1, \ldots, q\}$ and $\forall \theta \in \Theta$:

$$\begin{bmatrix} -Q_{\gamma,1}\left(x_{(i)},\theta\right) + \gamma\alpha P & PG(\theta) \\ G(\theta)^T P & -\gamma\alpha I \end{bmatrix} \preceq 0 \quad (14)$$

$$x_{(i)}^T P x_{(i)} \le \mu \quad (15)$$

$$\begin{bmatrix} 1/\mu & \gamma a_k^T \\ \gamma a_k & P \end{bmatrix} \succeq 0 \quad (16)$$

Proof: Let us define $\tilde{\mathscr{P}}$ as an enlarged version of $\mathscr{P}$ obtained by multiplying all the coordinates of its vertices by $\rho = \gamma^{-1} > 1$:

$$\tilde{\mathscr{P}} = Co\left\{\rho x_{(1)}, \rho x_{(2)}, \ldots, \rho x_{(p)}\right\} \quad (17)$$

$$= \left\{x \in \mathbb{R}^{n_x} : \gamma a_k^T x = \frac{a_k^T}{\rho} x \le 1, k = 1, \ldots, q\right\}$$

and let us note that (14) implies:

$$\begin{bmatrix} -Q_{1,\rho}(x_{(i)},\theta) + \alpha P & PG(\theta) \\ G(\theta)^T P & -\alpha I \end{bmatrix} \preceq 0 \quad (18)$$

which, in virtue of (13), is equivalent to:

$$\begin{bmatrix} -Q_{1,\rho}(x,\theta) + \alpha P & PG(\theta) \\ G(\theta)^T P & -\alpha I \end{bmatrix} \preceq 0 \quad \begin{matrix} \forall x \in \mathscr{P} \\ \forall \theta \in \Theta \end{matrix} \quad (19)$$

that, according to Theorem 1, proves that $\dot{V}(x(t)) < 0$ $\forall x \in \tilde{\mathscr{P}}$ such that $x^T P x > 1$. The remaining of the proof aims at demonstrating that $\tilde{\mathscr{P}}$ contains the level curve $V(x) = \mu$ which contains $\mathscr{P}$.

It is straightforward that a necessary and sufficient condition for the polytope $\mathscr{P}$ to be contained within the level curve $V(x) = \mu$ is (15). On the other hand, by means of Schur complements, (16) is equivalent to:

$$\gamma a_k^T P^{-1} \gamma a_k \le 1/\mu \quad (20)$$

which guarantees that the level curve $V(x) = \mu$ is contained within $\tilde{\mathscr{P}}$ (Boyd et al., 1994). Hence, the state trajectory starting from any state inside $\mathscr{P}$ will be contained within the level curve $V(x) = \mu > 1$, for which the QB condition $\dot{V}(x) < 0$ if $V(x) > 1$ is ensured, which completes the proof. □

3. CONTROLLER DESIGN

Let us now consider a QPV system with the following structure:

$$\begin{aligned}\dot{x}(t) &= A(\theta(t))x(t) + N(\theta(t),x(t)) + B(\theta(t))u(t) \\ &+ M(\theta(t),x(t),u(t)) + G(\theta(t))w(t)\end{aligned} \tag{21}$$

where $x \in \mathbb{R}^{n_x}$ is the system's state, $u \in \mathbb{R}^{n_u}$ is the control input and $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ is the varying parameter vector that schedules both the matrix functions $A(\theta(t))$, $B(\theta(t))$, $G(\theta(t))$ and the nonlinear functions $N(\theta(t),x(t))$ and $M(\theta(t),x(t),u(t))$, which are defined as in (2) and:

$$M(\theta(t),x(t),u(t)) = \begin{bmatrix} x(t)^T M_1(\theta(t))u(t) \\ x(t)^T M_2(\theta(t))u(t) \\ \vdots \\ x(t)^T M_{n_x}(\theta(t))u(t) \end{bmatrix} \tag{22}$$

respectively, where $N_1(\theta(t)),\ldots,M_1(\theta(t)),\ldots,M_{n_x}(\theta(t))$ are matrix functions of appropriate dimensions. As in previous section, it is assumed that the exogenous disturbance w belongs to the set Ω defined in (3).

Hereafter, we will provide design conditions to ensure that a state-feedback controller in the form:

$$u(t) = K(\theta(t))x(t) \tag{23}$$

where $K(\theta(t))$ is a matrix of appropriate dimensions to be designed, such that the closed-loop system obtained as the interconnection of (21)-(23) is QB in a polytope $\mathscr{P}$ defined by (13).

Theorem 3. Let $R \succ 0$, $\alpha > 0$, $0 < \gamma < 1$, $\mu > 1$ and a matrix function $\Gamma(\theta)$ be such that $\forall i \in \{1,\ldots,p\}$, $\forall k \in \{1,\ldots,q\}$ and $\forall \theta \in \Theta$:

$$\begin{bmatrix} S_{\gamma,1}(x_{(i)},\theta) + \gamma\alpha R & G(\theta) \\ G(\theta)^T & -\alpha\gamma I \end{bmatrix} \preceq 0 \tag{24}$$

$$\begin{bmatrix} \mu & x_{(i)}^T \\ x_{(i)} & R \end{bmatrix} \succeq 0 \tag{25}$$

$$\begin{bmatrix} 1/\mu & \gamma a_k^T R \\ \gamma R a_k & R \end{bmatrix} \succeq 0 \tag{26}$$

where:

$$\begin{aligned} S_{\gamma,1}(x_{(i)},\theta) &= -He\{\gamma(A(\theta)R + B(\theta)\Gamma(\theta))\} \\ &- He\left\{\begin{bmatrix} x_{(i)}^T(N_1(\theta)R + M_1(\theta)\Gamma(\theta)) \\ x_{(i)}^T(N_2(\theta)R + M_2(\theta)\Gamma(\theta)) \\ \vdots \\ x_{(i)}^T(N_{n_x}(\theta)R + M_{n_x}(\theta)\Gamma(\theta)) \end{bmatrix}\right\} \end{aligned} \tag{27}$$

Then, the system obtained as the interconnection of (21)-(23), with $K(\theta(t)) = \Gamma(\theta(t))R^{-1}$, is QB in the polytope $\mathscr{P}$ defined by (13).

Proof: The closed-loop system obtained as the interconnection of (21)-(23) is the following:

$$\dot{x}(t) = A_{cl}(\theta(t))x(t) + N_{cl}(\theta(t),x(t)) + G(\theta(t))w(t) \tag{28}$$

with:

$$A_{cl}(\theta(t)) = A(\theta(t)) + B(\theta(t))K(\theta(t)) \tag{29}$$

$$N_{cl}(\theta(t),x(t)) = \begin{bmatrix} x(t)^T N_{cl,1}(\theta(t))x(t) \\ x(t)^T N_{cl,2}(\theta(t))x(t) \\ \vdots \\ x(t)^T N_{cl,n_x}(\theta(t))x(t) \end{bmatrix} \tag{30}$$

$$N_{cl,i}(\theta(t)) = N_i(\theta(t)) + M_i(\theta(t))K(\theta(t)) \tag{31}$$

Hence, according to Theorem 2, (29) is QB in the polytope $\mathscr{P}$ defined by (13) if there exist scalars $\alpha > 0$, $0 < \gamma < 1$ and $\mu > 1$ such that $\forall i \in \{1,\ldots,p\}$, $\forall k \in \{1,\ldots,q\}$ and $\forall \theta \in \Theta$:

$$\begin{bmatrix} -\tilde{Q}_{\gamma,1}(x_{(i)},\theta) + \gamma\alpha P & PG(\theta) \\ G(\theta)^T P & -\gamma\alpha I \end{bmatrix} \preceq 0 \tag{32}$$

and Eqs. (15)-(16) hold, where:

$$\tilde{Q}_{\gamma,1}(x_{(i)},\theta) \triangleq -He\left\{\gamma P A_{cl}(\theta) + P\begin{bmatrix} x_{(i)}^T N_{cl,1}(\theta) \\ x_{(i)}^T N_{cl,2}(\theta) \\ \cdots \\ x_{(i)}^T N_{cl,n_x}(\theta) \end{bmatrix}\right\} \tag{33}$$

By pre- and post-multiplying (33) by diag(P^{-1}, I), and defining $R \triangleq P^{-1}$, we obtain:

$$\begin{bmatrix} \tilde{S}_{\gamma,1}(x_{(i)},\theta) + \gamma\alpha R & G(\theta) \\ G(\theta)^T & -\alpha\gamma I \end{bmatrix} \preceq 0 \tag{34}$$

where:

$$\tilde{S}_{\gamma,1}(x_{(i)},\theta) = -He\left\{\gamma A_{cl}(\theta)R + \begin{bmatrix} x_{(i)}^T N_{cl,1}(\theta)R \\ x_{(i)}^T N_{cl,2}(\theta)R \\ \cdots \\ x_{(i)}^T N_{cl,n_x}(\theta)R \end{bmatrix}\right\} \tag{35}$$

from which, by means of the change of variables $\Gamma(\theta) \triangleq K(\theta)R$ ($K(\theta) = \Gamma(\theta)R^{-1}$), (24) is obtained. On the other hand, (25) is easily obtained from (15) by means of Schur complements. Finally, by pre- and post-multiplying (16) by diag(I, R), (26) is obtained, which completes the proof. □

4. POLYTOPIC CONDITIONS

The conditions provided by Theorem 3 cannot be applied directly to the design of the controller gain $K(\theta(t))$, since they represent an infinite number of constraints due to the continuous variability of θ within Θ. However, under the assumption that the QPV system is *polytopic*, which means that the matrix functions $A(\theta), B(\theta), M_1(\theta), \ldots, M_{n_x}(\theta), N_1(\theta), \ldots, N_{n_x}(\theta), G(\theta)$ can be written as an affine combination of vertex matrices, such that:

$$\begin{aligned}\dot{x}(t) = \sum_{j=1}^{N} \lambda_j(\theta(t))\,[&A_j x(t) + N_j(x(t)) + B_j u(t) \\ &+ M_j(x(t),u(t)) + G_j w(t)]\end{aligned} \tag{36}$$

with:

$$N_i(\cdot) = \begin{bmatrix} x(t)^T N_{1,j} x(t) \\ x(t)^T N_{2,j} x(t) \\ \cdots \\ x(t)^T N_{n_x,j} x(t) \end{bmatrix} \quad M_j(\cdot) = \begin{bmatrix} x(t)^T M_{1,j} u(t) \\ x(t)^T M_{2,j} u(t) \\ \cdots \\ x(t)^T M_{n_x,j} u(t) \end{bmatrix} \tag{37}$$

and:

$$\sum_{j=1}^{N} \lambda_j(\theta(t)) = 1, \lambda_j(\theta(t)) \geq 0, \forall j = 1,\ldots,N, \forall \theta \in \Theta \tag{38}$$

then it is possible to assess the conditions provided by Theorem 3 using a finite number of conditions. if the controller gain $K(\theta(t))$ is chosen to be polytopic as well, i.e.:

$$K(\theta(t)) = \sum_{j=1}^{N} \lambda_i(\theta(t)) K_j \tag{39}$$

In the general case, the polytopic assumption would require assessing the negativity of the double polytopic sums arising from condition (24). This assessment can be done by applying Polya's theorem on positive forms in the standard simplex, as suggested by Sala and Ariño (2007), thus obtaining a set of sufficient conditions to assess the definiteness of double sums, which are progressively less conservative when a complexity parameter increases. However, in order to keep the mathematical complexity simple, hereafter we will describe the special case in which the matrix functions $B(\theta), M_1(\theta), \dots, M_{n_x}(\theta)$ are constant with respect to θ, in which case (36) becomes:

$$\dot{x}(t) = \sum_{j=1}^{N} \lambda_j(\theta(t)) \left[A_j x(t) + N_j(x(t)) + G_j w(t)\right] + Bu(t) + M(x(t), u(t)) \tag{40}$$

Hence, the following corollary can be obtained.

Corollary 1. Let $R \succ 0$, $\alpha > 0$, $0 < \gamma < 1$, $\mu > 1$ and matrix functions $\Gamma_1, \dots, \Gamma_N$ be such that $\forall i \in \{1, \dots, p\}$, $\forall j \in \{1, \dots, N\}$ and $\forall k \in \{1, \dots, q\}$:

$$\begin{bmatrix} S_j\left(x_{(i)}\right) + \gamma\alpha R & G_j \\ G_j^T & -\alpha\gamma I \end{bmatrix} \preceq 0 \tag{41}$$

and (25)-(26) hold, where:

$$S_j\left(x_{(i)}\right) = -He\left\{\gamma(A_j R + B\Gamma_j)\right\} - He\left\{\begin{bmatrix} x_{(i)}^T\left(N_{1,j}R + M_1\Gamma_j\right) \\ x_{(i)}^T\left(N_{2,j}R + M_2\Gamma_j\right) \\ \vdots \\ x_{(i)}^T\left(N_{n_x,j}R + M_{n_x}\Gamma_j\right) \end{bmatrix}\right\} \tag{42}$$

Then the system obtained as the interconnection of (23) and (40), with controller gain chosen as in (39) with $K_j = \Gamma_j R^{-1}$, is QB in the polytope $\mathscr{P}$ defined as (13).

Proof: It follows from the basic property of matrices that any linear combination of negative semidefinite matrices with nonnegative coefficients is negative semidefinite. Hence, using the linear combination brought by (40), (41) leads to (24). □

5. EXAMPLE

Let us consider a QPV system as in (21), with:

$$A(\theta(t)) = \begin{bmatrix} -4-\theta_1(t) & 10 & 2+2\theta_2(t) \\ -1 & -1-\theta_2(t) & 1.5+2\theta_1(t) \\ 1 & 1 & -4-3\theta_1(t) \end{bmatrix}$$

$$N_1(\theta(t)) = \begin{bmatrix} 0.5 & 1+\theta_1(t) & 0 \\ 0 & 0 & -\theta_2(t) \\ 0 & 1+\theta_2(t) & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1.2 & 0 & 0.7 \\ 1 & 0.8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$N_2(\theta(t)) = \begin{bmatrix} -0.4 & 0 & 1-\theta_1(t) \\ 1.5 & 0 & 1+\theta_2(t) \\ 2+\theta_2(t) & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$N_3(\theta(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $M_2 = M_3 = 0_{3\times3}$, with $\theta_1, \theta_2 \in [0,1]$. Let us consider the following polytope $\mathscr{P}$:

$$\mathscr{P} = [-0.5, 0.5] \times [-0.5, 0.5] \times [-0.5, 0.5]$$

The open-loop system (with $u(t) = 0$) is not QB in $\mathscr{P}$. Indeed, Fig. 1 shows the open-loop state trajectory obtained starting from the initial condition $x(0) = [0.5, 0.5, 0.5]^T$ with $\theta_1(t) = 0.5 + 0.5\sin(2\pi t/5)$, $\theta_2(t) = 0.5 + 0.5\cos(2\pi t/3)$ and $w(t) = \sin(3t)$, which is divergent.

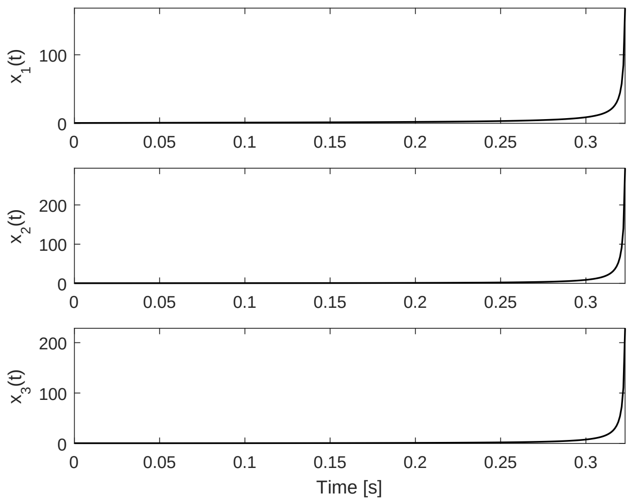

Fig. 1. Open-loop state trajectories.

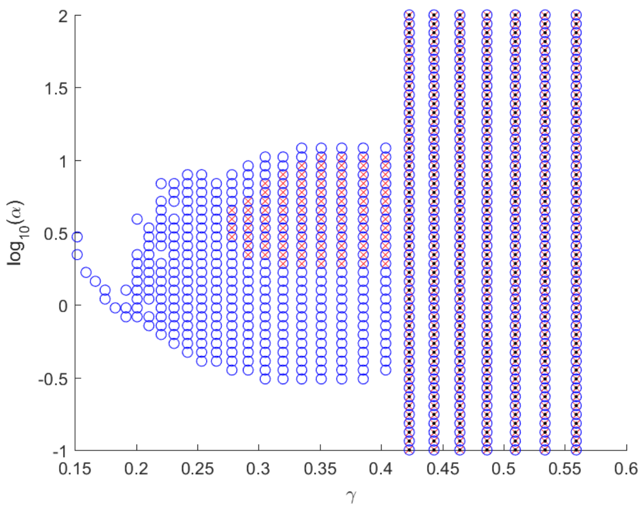

Fig. 2. LMI feasibility for $\mu = 1$ (blue circles), $\mu = 5$ (red crosses) and $\mu = 10$ (black dots).

Afterwards, using LMI conditions brought by Corollary 1, feasibility of a design that guarantees the closed-loop system to be QB in $\mathscr{P}$ has been checked for different values of γ, α and μ. It can be seen that the least conservative design is the one performed using $\mu = 1$, for which a wider set of pairs of values (γ, α) leads to feasibility (in order to choose candidate values for γ and α, a gridding with linear and logarithmic scale, respectively, has been used).

Fig. 3 shows the closed-loop state trajectory obtained from the previously considered initial condition $x_0 = [0.5, 0.5, 0.5]^T$ when a controller designed to guarantee QB in $\mathscr{P}$ is used. Since $x_0 \in \mathscr{P}$, boundedness of the state remains guaranteed (actually, it converges to a quite close neighborhood of the origin of the state space). The evolution of the Lyapunov function with the matrix P returned by the LMI solver:

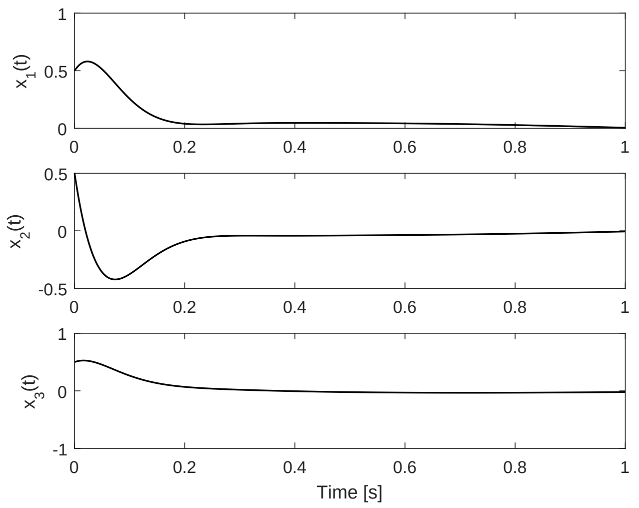

Fig. 3. Closed-loop state trajectories ($x_0 = [0.5, 0.5, 0.5]^T$).

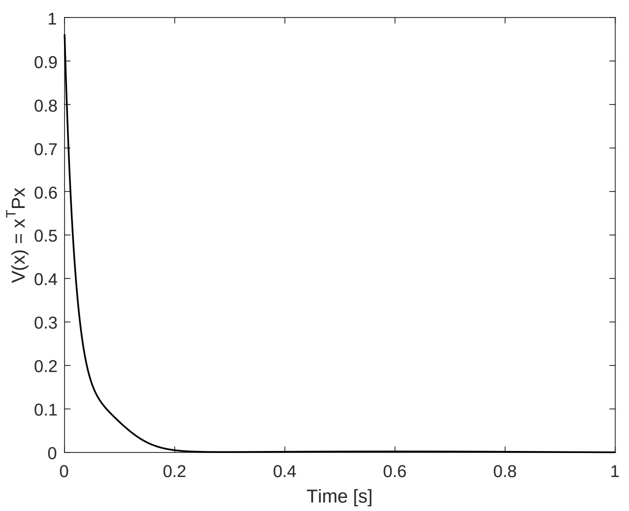

Fig. 4. Lyap. function $V(x) = x^T Px$ ($x_0 = [0.5, 0.5, 0.5]^T$).

$$P = \begin{bmatrix} 0.9094 & 0.6085 & -0.0804 \\ -0.6085 & 0.9576 & 0.1826 \\ -0.0804 & 0.1826 & 0.5576 \end{bmatrix}$$

is shown in Fig. 4.

When the initial condition is outside of $\mathscr{P}$, nothing can be forecasted. Actually, one can find some initial conditions starting from which the state trajectory remains bounded, such as $x_0 = [6.71, 6.71, 6.71]^T$ (see Fig. 5), as well as other initial conditions starting from which the state trajectory is divergent, e.g. $x_0 = [6.72, 6.72, 6.72]^T$ (see Fig. 6).

Indeed, when checking the evolution of the function $V(x) = x^T Px$ in the first case, one can find that the condition $\dot{V} < 0$ does not hold everywhere, although it holds everywhere after a certain transient, and actually $V(x)$ eventually converges into the region of the state-space where $V(x) \leq 1$ such that boundedness becomes guaranteed by design (see Fig. 7). On the other hand, in the divergent case, the function $V(x) = x^T Px$ diverges as well, as shown in Fig. 8.

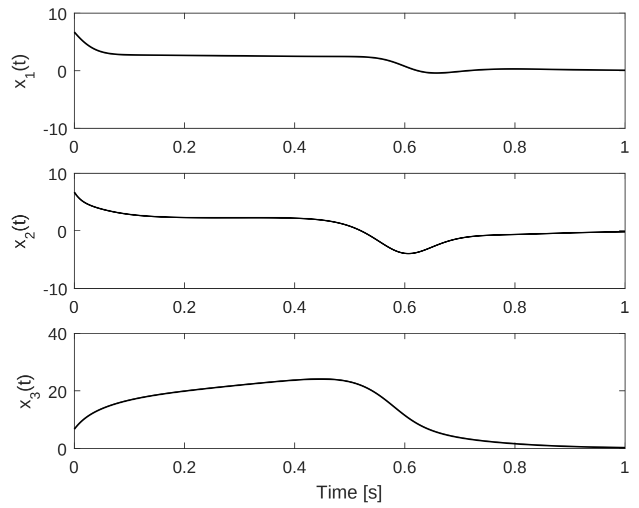

Fig. 5. Closed-loop state trajectories ($x_0 = [6.71, 6.71, 6.71]^T$).

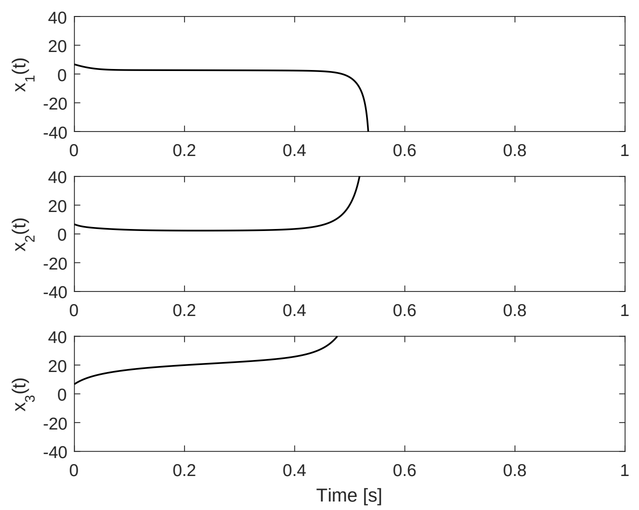

Fig. 6. Closed-loop state trajectories ($x_0 = [6.72, 6.72, 6.72]^T$).

6. CONCLUSIONS

This paper has considered the quadratic boundedness analysis and state-feedback controller design problems for quadratic parameter varying systems. An approach based on linear matrix inequalities has been proposed in order to ensure that the quadratic boundedness property holds for an invariant ellipsoid which contains a predefined polytopic region of the state space. The simulation results obtained with a numerical example have confirmed the validity of the theoretical results.

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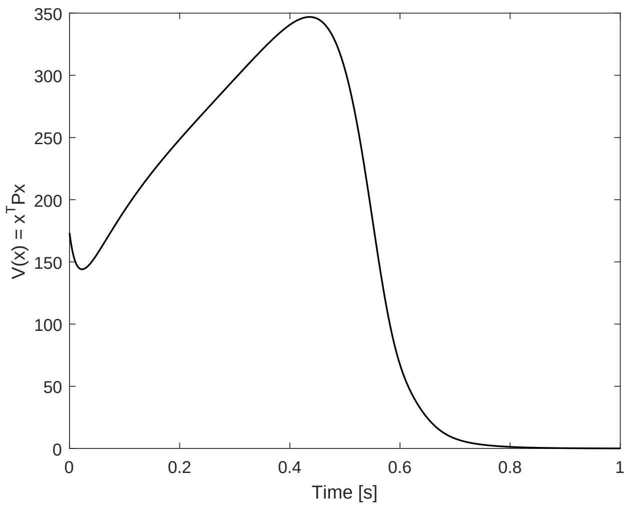

Fig. 7. Lyap. function $V(x) = x^T Px$ ($x_0 = [6.71, 6.71, 6.71]^T$).

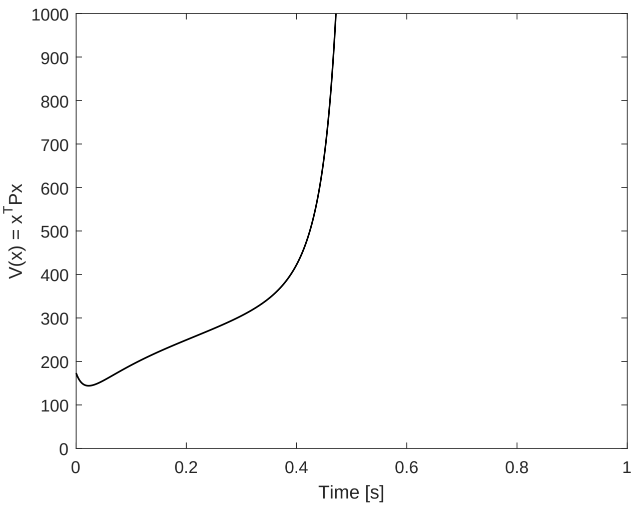

Fig. 8. Lyap. function $V(x) = x^T Px$ ($x_0 = [6.72, 6.72, 6.72]^T$).

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