Matching random colored points with rectangles

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Abstract

Let $S \subset [0,1]^2$ be a set of n points, randomly and uniformly selected. Let $R \cup B$ be a random partition, or coloring, of S in which each point of S is included in R uniformly at random with probability $1/2$. We study the random number $M(n)$ of points of S that are covered by the rectangles of a maximum strong matching of S with axis-aligned rectangles. The matching consists of closed rectangles that cover exactly two points of S of the same color. A matching is strong if all its rectangles are pairwise disjoint. We prove that almost surely $M(n) \geq 0.83\,n$ for n large enough. Our approach is based on modeling a deterministic greedy matching algorithm, that runs over the random point set, as a Markov chain.

1 Introduction

Given a point set $S \subset \mathbb{R}^2$ of n points, and a class $\mathcal{C}$ of geometric objects, a *geometric matching* of S is a set $M \subseteq \mathcal{C}$ such that each element of M contains exactly two points of S and every point of S lies in at most one element of M. A geometric matching is *strong* if the geometric objects are pairwise disjoint, and *perfect* if every point of S belongs to (or is covered by) some element of M. This type of geometric matching problems was considered by Ábrego et al. [1], who studied the existence and properties of matchings for point sets in the plane when $\mathcal{C}$ is the class of axis-aligned squares, or the class of disks.

Let $S = R \cup B \subset \mathbb{R}^2$ be a set of n colored points in the plane, each point colored red or blue, where R and B are the sets of red and blue points, respectively. A geometric matching of S is called *monochromatic* if all matching objects cover points of the same color, and *bichromatic* if all matching objects cover points of different colors. For example, monochromatic matchings of two-colored point sets in the plane with straight segments have been studied [4, 5]. In the case of bichromatic matchings with straight segments, a classical result in discrete geometry asserts that for any planar point set S consisting of n red points and n blue points in general position (i.e., no three points of S are collinear) there exists a perfect, strong bichromatic matching of S with straight segments.

In this paper, we consider strong monochromatic matchings with axis-aligned rectangles. Every rectangle will be axis-aligned and a closed set.

Caraballo et al. [2] studied monochromatic strong matchings of S with rectangles from the algorithmic point of view. That is, the problem of finding a monochromatic strong matching of S with the maximum number of rectangles; proving that the problem is NP-hard and giving a polynomial-time 4-approximation algorithm. As noted by Caraballo et al., this problem is a special case of the Maximum Independent Set of Rectangles problem (MISR): Given a finite set $\mathcal{R}$ of rectangles in the plane, find a subset $\mathcal{R}' \subseteq \mathcal{R}$ of maximum cardinality, denoted $\alpha(\mathcal{R})$, such that every pair of rectangles in $\mathcal{R}'$ are disjoint.

Indeed, suppose that we want to find a monochromatic matching of S with the maximum number of rectangles. For every distinct $p, q \in \mathbb{R}^2$, let $D(p,q)$ be the minimum axis-aligned rectangle that encloses p and q. Let $\mathcal{R}(S)$ be the set of all rectangles $D(p,q)$ such that $p, q \in S$, p and q have the same color, and $D(p,q)$ contains no points of S different from p and q. Finding a monochromatic strong matching of S with the maximum number of rectangles is equivalent to finding in $\mathcal{R}(S)$ a maximum subset of pairwise disjoint rectangles, whose size is $\alpha(\mathcal{R}(S))$. That is, to solving the MISR problem in $\mathcal{R}(S)$.

We study monochromatic strong matchings of S with rectangles from the combinatorial point of view, and from this point forward, every rectangle will cover precisely two points of S. Point sets $S = R \cup B$ exist

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in which no matching rectangle is possible (e.g., S is a color-alternating sequence of points on the line $y = x$), and point sets in which a perfect strong matching with rectangles exists (e.g., an even number of red points in the negative part of the line $y = x$, and an even number of blue points in the positive part). These two extreme cases show that it is not worth studying the number $\alpha(\mathcal{R}(S))$ for fixed, or given, colored point sets S. Instead, we want to study $\alpha(\mathcal{R}(S))$ when S is a random point set in the square $[0,1]^2$, in which the positions of the n points of S are random and the color of each point of S is also random. Formally:

Let $n > 0$, and let $S \subset [0,1]^2$ be a set of n points, randomly and uniformly selected. Let $R \cup B$ be a random partition (i.e., coloring) of S in which each point of S is included in R uniformly at random with probability 1/2. We study the random variable $M(n) = 2 \cdot \alpha(\mathcal{R}(S))$ equal to the number of points of S that are covered by the rectangles of a maximum monochromatic strong matching of S with rectangles.

Given a set S of n points, randomly and uniformly selected in the square $[0,1]^2$, Chen et al. [3] studied a similar variable: the random variable $\alpha(D(S))$, where $D(S)$ is the random graph with vertex set S and two points $p, q \in S$ define an edge if and only if $D(p,q) \cap S = \{p, q\}$. Here, $\alpha(D(S))$ denotes the size of a maximum independent set of $D(S)$.

One result of Chen et al. [3, Theorem 1] states that if n tends to infinity, then we have $\alpha(D(S)) = O(n(\log^2 \log n)/\log n)$ with probability tending to 1. This result implies that if $C(n)$ denotes the number of points of S that are covered by a maximum monochromatic matching of S with rectangles, where the rectangles may overlap (i.e., the matching is not necessarily strong), then $C(n) = n - o(n)$ with probability tending to 1. In fact, let M' be a maximum monochromatic matching of S with rectangles, where M' is not necessarily strong, and let $S' \subset S$ be the points not covered by M'. Note that at least $|S'|/2$ points of S' have the same color, and they form an independent set in the graph $D(S)$. Then, with probability tending to 1, we have that M' covers at least $n - |S'| = n - O(n(\log^2 \log n)/\log n) = n - o(n)$ points.

2 Preliminaries

Since for matching S with rectangles, only the left-to-right and bottom-to-top orders of S are relevant, and since the probability that two points of S are in the same vertical or horizontal line is zero, we consider S equal to the point set $S_\pi = \{(i, \pi(i)) \mid i = 1, 2, \ldots, n\}$, where $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a randomly and uniformly selected permutation. This assumption was also done by Chen et al. [3].

We have implemented a program that, given n, generates a uniform random permutation π, and selects the color of each $p \in S_\pi$ (red or blue) randomly and uniformly. The program then runs a deterministic algorithm on $S_\pi = R \cup B$ that greedily finds a maximum independent subset of rectangles in $\mathcal{R}(S_\pi)$. The algorithm iterates the points of S_π from left to right, and for each point p in the iteration, it performs the following action: If p is not matched with any point prior to p in the iteration, it finds the first point q to the right of p such that $D(p,q) \in \mathcal{R}(S_\pi)$ and $D(p,q)$ has empty intersection with all matching rectangles already reported. If q exists, the algorithm reports $D(p,q)$ as a matching rectangle. In any case, regardless of whether q exists, the algorithm continues the iteration to the next unmatched point p.

	$n = 1000$		$n = 10000$	
k	mean	sdev	mean	sdev
1	0.6653	0.0175	0.6673	0.0052
2	0.7948	0.0104	0.7934	0.0036
3	0.8301	0.0097	0.8304	0.0034
4	0.8555	0.0094	0.8562	0.0028
5	0.8727	0.0090	0.8736	0.0026
6	0.8860	0.0087	0.8864	0.0026
∞	0.9724	0.0062	0.9780	0.0022

Table 1: The experimental results obtained when running the greedy matching algorithm for $n \in \{1000, 10000\}$, parameterized with $k \in [1..6]$, or not parameterized ($k = \infty$). For each combination n, k, we run the algorithm 100 times, and measured the mean and standard deviation of the ratio between the total number of matched points and n.

For large n, say $n = 10000$, the implemented algorithm reports a matching covering approximately $\frac{97}{100}n$ of the points. Then, it seems that $M(n) \geq \frac{97}{100}n$ for n large enough and probability close to 1. Analyzing the algorithm, when run over the random S_π, seems to be a good approach for obtaining a high lower bound for $M(n)$. One way to analyze the algorithm is to consider a parameterized version of it, with a parameter k, such that each unmatched point p finds its match point q among only the next k points of S_π to the right of p. Let $\mathcal{A}_k$ denote this parameterized algorithm. For experimental results, see Table 1.

We show how to model (an adaptation of) $\mathcal{A}_k$ as a Markov chain, for any $k \in \{1, 2, \ldots\}$. Then, we show that $\mathcal{A}_3$ almost surely guarantees $M(n) \geq \frac{83}{100}n$, for n large enough, by computing the stationary distribution of the Markov chain and applying the Ergodic theorem. See [6] for the theory on Markov chains.

3 The Markov chains

We consider $S = S_\pi$, and whenever we say point i, for $i \in \{1, 2, \ldots, n\}$, or just i when it is clear from the context, we are referring to the point $p_i := (i, \pi(i)) \in S$. Let color$(i) \in \{R, B\}$ denote the color of point i.

Let $k \in \{1, 2, 3, \ldots\}$ be a constant, and let $\tilde{\mathcal{A}}_k$ be the following adaptation of algorithm $\mathcal{A}_k$, consisting in the next idea: Suppose that $\mathcal{A}_k$ matches points i and j, with $i < j \leq i + k$, when the iteration of S_π is on point i. $\tilde{\mathcal{A}}_k$ iterates S_π from left to right, and will also match i and j but, in contrast with $\mathcal{A}_k$, when the iteration is on j, or on a point to the right of j. Using $\tilde{\mathcal{A}}_k$ instead of $\mathcal{A}_k$, allows us to describe in a more compact way the states of the memory of the algorithm during the iteration of the elements of S_π.

Let $E(j)$ be the data structure associated with point $j \in \{1, 2, \ldots, n\}$, that is maintained by $\tilde{\mathcal{A}}_k$ during the iteration of S_π. For any j, let $i = i(j)$ be the smallest element in the set $\{\max(1, j-(k-1)), \ldots, j\}$ such that the point i is not matched, and each point in $\{i+1, \ldots, j\}$ is matched with a point to the left of i or is not yet matched. If i exists, then $E(j)$ consists of the following elements:

- The set $U(j) \subseteq \{i, i+1, \ldots, j\}$ of the points that are not matched, with $i \in U(j)$.
- The set $\text{Rect}(j)$ of the (pairwise disjoint) rectangles that match the points in $\{i+1, \ldots, j\} \setminus U(j)$ with points to the left of i.
- The number $f(j)$ of points of S_π that are matched while the iteration is at point j.

If i does not exist, then $E(j)$ consists of the same three above elements with $U(j) = \emptyset$ and $\text{Rect}(j) = \emptyset$.

For $j = 1$, we have $U(1) = \{1\}$, $\text{Rect}(1) = \emptyset$, and $f(1) = 0$. We show now how to obtain $E(j+1)$ from $E(j)$, for any $j \in \{1, \ldots, n-1\}$. First, we match points i and $j+1$ if and only if $j+1 \leq i+k$, $\text{color}(i) = \text{color}(j+1)$, and the rectangle $D(p_i, p_{j+1})$ does not overlap any rectangle in $\text{Rect}(j)$. After that, we match other points in $(U(j) \setminus \{i\}) \cup \{j+1\}$ if and only if i was matched in the previuos step, or we have finished with point i. We say that we have *finished* with point i if there do not exist more chances for point i to be matched, which is equivalent to $i + k \leq j + 1$. This final matching procedure consists in running the original algorithm $\mathcal{A}_k$ with input the points $\{i+1, \ldots, j, j+1\}$, but with the extra condition that the algorithm terminates if the current point t on the iteration of $\{i+1, \ldots, j, j+1\}$ from left to right, cannot be matched with any other one to its right. This is because t must find its match among the points in $\{j+2, \ldots, t+k\}$, before any matching between points in $\{t+1, \ldots, j+1\}$ occurs. We set $f(j+1)$ equal to the total number of points matched in the above steps. Obtaining $U(j+1)$ and $\text{Rect}(j+1)$ is straightforward.

Let $j \in \{1, 2, \ldots, n\}$. The *state* of $E(j)$ is the 2-tuple formed by: As first component, (a certificate of) the relative positions between the points of $U(j)$ and the rectangles of $\text{Rect}(j)$, together with the color of each point of $U(j)$. If the leftmost point is blue, then we switch the color of every point such that the leftmost one is always red. As second component, $f(j)$. We say that two states e and e' are *equal* (i.e., $e = e'$) if: (i) the first components are equal, or one first component is symmetric to the other in the vertical direction, and (ii) the second components are equal.

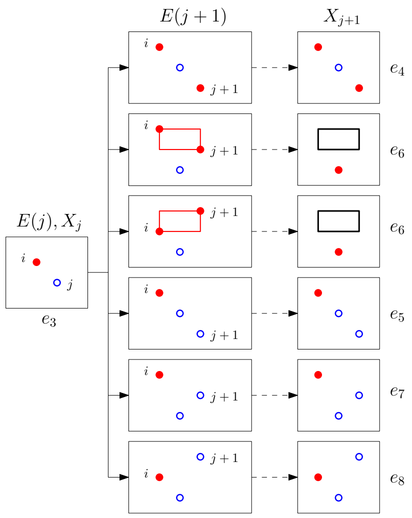


Figure 1: Example of the data structure $E(j)$, its state $X_j = e_3$, and the states in the neighborhood $N(e_3) = \{e_4, e_5, e_6, e_6, e_7, e_8\}$ corresponding to $E(j+1)$, for each position and color of point $j+1$. Note that $f(e_6) = 2$, and $f(e) = 0$ for all $e \in \{e_4, e_5, e_7, e_8\}$.

Let $\mathcal{E} = \{e_1, e_2, \ldots, e_N\}$ be the set of all possible states of $E(j)$, which is a finite set, and let $X_j \in \mathcal{E}$ be the random variable equal to the state of $E(j)$. Let $e \in \mathcal{E}$ be a state, and assume that e is the state of $E(j)$ for some j. Let $f(e) = f(j)$ (with abuse of notation), and let $N(e)$ be the *neighborhood* of e, which is the multiset consisting of the state of $E(j+1)$ for every color and every different relative position, with respect to the elements of both $U(j)$ and $\text{Rect}(j)$, of point $j+1$. See for example Figure 1.

Lemma 1 *Let $e, e' \in \mathcal{E}$ be two states. For every $j \geq 2$, we have:*

$$\text{Prob}(X_{j+1} = e' \mid X_j = e) = \frac{m}{2\left(|U(j)| + 2|\text{Rect}(j)| + 1\right)},$$

where m is the multiplicity of e' in $N(e)$.

Proof. Through each point of $U(j)$ draw a horizontal line, and for each rectangle of $\text{Rect}(j)$ draw a horizontal line through the top side and a horizontal line through the bottom side. Each of these $K = |U(j)| + 2|\text{Rect}(j)|$ lines goes through a different element of S_π, subdividing the plane into $K+1$ strips. Since the point $j+1$ is to the right of every point of $U(j)$ and every rectangle of $\text{Rect}(j)$, its relative position w.r.t. the elements of $U(j)$ and $\text{Rect}(j)$ is to be in one of these strips, and this happens with probability $1/(K+1)$. Furthermore, the color of point $j+1$ is given with probability $1/2$. The lemma follows. □

Note that $\text{Prob}(X_{j+1} = x_{j+1} \mid X_j = x_j, \ldots, X_1 = x_1) = \text{Prob}(X_{j+1} = x_{j+1} \mid X_j = x_j)$ for all

$x_1, \ldots, x_{j+1} \in \mathcal{E}$ such that $\text{Prob}(X_j = x_j, \ldots, X_1 = x_1) > 0$. Thus, $(X_n)_{n \geq 1}$ is a Markov chain, denoted $\mathcal{C}_k$, over the set $\mathcal{E} = \{e_1, e_2, \ldots, e_N\}$ of states. Let P be the transition matrix, of dimensions $N \times N$, such that $P_{i,j} = \text{Prob}(X_{\ell+1} = e_j \mid X_\ell = e_i)$. The key observation is that the total number of points matched by $\tilde{\mathcal{A}}_k$ is precisely $M_k(n) := \sum_{j=1}^{n} f(X_j)$.

A Markov chain is *irreducible* if with positive probability any state can be reached from any other state [6]. It can be proved that $\mathcal{C}_k$ is irreducible. Since $\mathcal{C}_k$ has a finite state set, it has a unique stationary distribution $s = (s_1, s_2, \ldots, s_N)$, which is the solution of the system $s = s \cdot P$, $s_1 + s_2 + \cdots + s_N = 1$ of linear equations [6]. Furthermore, since $f(e) \in \{0, 2, 4, \ldots, 2\lceil \frac{k+1}{2} \rceil\}$ for all $e \in \mathcal{E}$, the function f is bounded and then the Ergodic theorem ensures

$$\lim_{n\to\infty} \frac{M_k(n)}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(X_j) = \sum_{i=1}^{N} s_i f(e_i),$$

almost surely [6]. Let $\alpha_k = \sum_{i=1}^{N} s_i f(e_i)$. We then arrive to the main result of this paper:

Theorem 2 *Let $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a uniform random permutation. Let $S_\pi = \{(i, \pi(i)) \mid i = 1, 2, \ldots, n\}$ be a random point set, where the color (red or blue) of each point of S_π is selected randomly and uniformly with probability $1/2$. Let $k \in \{1, 2, 3, \ldots\}$ be a constant. For all constant $\varepsilon > 0$ and n large enough, almost surely the number $M_k(n)$ of points of S_π that are matched by the algorithm $\tilde{\mathcal{A}}_k$ satisfies $M_k(n) \geq (\alpha_k - \varepsilon)n$.*

4 The Markov chain for $k = 3$

Using algorithm $\tilde{\mathcal{A}}_3$, we give a precise value for α_3. In Table 2, we describe the states, and the transitions between the states, of the Markov chain $\mathcal{C}_3$. Since $f(e) = 2$ for all $e \in \{e_2, e_6, e_9, e_{10}, e_{16}, e_{17}, e_{18}\}$, $f(e_{11}) = 4$, $f(e) = 0$ for all other state e, and the stationary distribution $s = (s_1, \ldots, s_{18})$ satisfies

$$s_2 = \tfrac{167959}{816233},\ s_6 = \tfrac{69640}{816233},\ s_9 = \tfrac{6800}{816233},\ s_{10} = \tfrac{58650}{816233},$$
$$s_{11} = \tfrac{13600}{816233},\ s_{16} = \tfrac{5950}{816233},\ s_{17} = \tfrac{1360}{816233},\ s_{18} = \tfrac{1190}{816233},$$

we obtain

$$\begin{aligned} \alpha_3 &= 2(s_2 + s_6 + s_9 + s_{10} + s_{16} + s_{17} + s_{18}) + 4s_{11} \\ &= \tfrac{677498}{816233} \approx 0.830030151. \end{aligned}$$

By Theorem 2, taking $\varepsilon = \alpha_3 - 0.83 > 0$, for n large enough we have almost surely that $M(n) \geq M_3(n) \geq 0.83\,n$. It can be noted in Table 1 that in practice this lower bound is satisfied.

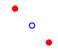
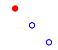
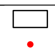
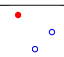
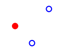
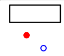
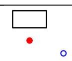
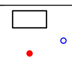
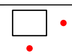
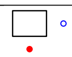
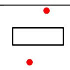
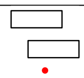
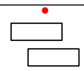
e_i	elem. of e_i	$f(e_i)$	neighbors of e_i
e_1		0	$(e_2, 1/2), (e_3, 1/2)$
e_2	$\emptyset$	2	$(e_1, 1)$
e_3		0	$(e_4, 1/6), (e_5, 1/6), (e_6, 1/3), (e_7, 1/6), (e_8, 1/6)$
e_4		0	$(e_4, 1/8), (e_5, 1/8), (e_6, 3/8), (e_7, 1/8), (e_9, 1/4)$
e_5		0	$(e_{10}, 3/4), (e_{11}, 1/4)$
e_6		2	$(e_2, 1/4), (e_{12}, 1/8), (e_{13}, 1/8), (e_{14}, 1/4), (e_{15}, 1/4)$
e_7		0	$(e_{10}, 3/4), (e_{11}, 1/4)$
e_8		0	$(e_{10}, 3/4), (e_{16}, 1/4)$
e_9		2	$(e_2, 3/10), (e_{12}, 1/10), (e_{14}, 1/5), (e_{15}, 1/5), (e_{17}, 1/5)$
e_{10}		2	$(e_2, 1/2), (e_3, 1/2)$
e_{11}	$\emptyset$	4	$(e_1, 1)$
e_{12}		0	$(e_2, 1/2), (e_3, 3/10), (e_6, 1/5)$
e_{13}		0	$(e_2, 1/2), (e_3, 3/10), (e_6, 1/5)$
e_{14}		0	$(e_2, 3/10), (e_3, 1/2), (e_6, 1/5)$
e_{15}		0	$(e_2, 1/2), (e_3, 3/10), (e_6, 1/5)$
e_{16}		2	$(e_2, 1/5), (e_{12}, 1/10), (e_{13}, 1/10), (e_{14}, 1/10), (e_{15}, 3/10), (e_{18}, 1/5)$
e_{17}		2	$(e_1, 5/6), (e_2, 1/6)$
e_{18}		2	$(e_1, 5/6), (e_2, 1/6)$

Table 2: The 18 states of the Markov chain for $k = 3$. In the 2nd column we show the first component of e_i, and in the 3rd column the second component $f(e_i)$. In the last column we show the neighbor states of e_i as a list of tuples of the form $(e_j, P_{i,j})$, where $P_{i,j} = \text{Prob}(X_{\ell+1} = e_j \mid X_\ell = e_i) > 0$ is the transition probability from e_i to e_j.

References

[1] B. M. Ábrego, E. M. Arkin, S. Fernández-Merchant, F. Hurtado, M. Kano, J. S. Mitchell, and J. Urrutia. Matching points with squares. *Discrete & Computational Geometry*, 41(1):77–95, 2009.

[2] L. E. Caraballo, C. Ochoa, P. Pérez-Lantero, and J. Rojas-Ledesma. Matching colored points with rectangles. *Journal of Combinatorial Optimization*, 33(2):403–421, 2017.

[3] X. Chen, J. Pach, M. Szegedy, and G. Tardos. Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles. *Random Structures and Algorithms*, 34(1):11–23, 2009.

[4] A. Dumitrescu and R. Kaye. Matching colored points in the plane: Some new results. *Computational Geometry*, 19(1):69–85, 2001.

[5] A. Dumitrescu and W. L. Steiger. On a matching problem in the plane. *Discrete Mathematics*, 211:183–195, 2000.

[6] J. R. Norris. *Markov chains*. Number 2. Cambridge University Press, 1998.