A Disutility-Based Drift Control for Exchange Rates

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Abstract

In this paper we propose an exchange rate model as solution of a disutility based drift control problem. Given the exchange rate is a function of the fundamental, we assume Government Authorities control the fundamental dynamics aimed at minimizing the discounted expected disutility caused by the distance between the fundamental and some specific target. The theoretical model is solved using the dynamic programming approach and introducing the concept of viscosity solution. We contribute to research on exchange rate control policies by deriving the optimal interventions aimed at stabilizing the exchange rate and preserving macroeconomic stability. We also show that, under particular conditions, it is possible to derive the optimal width of the currency band.

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Keywords: drift control; dynamic programming; viscosity solutions;

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stabilization policies; exchange rates.

1 Introduction

Since the breakdown of the Bretton Woods system, the relevance of the exchange rate stabilization policies has been causing frequent and forceful interventions of the Government Authorities. The reactions to the Asian financial crisis or the European Monetary System (ERM) accession represent only some recent examples. The economies of East Asia have adopted a variety of foreign exchange rate policies, ranging from currency board system to "independently floating" exchange rates. Most of the Asian economies have implemented "managed floats" that allow their local currency to fluctuate over time within a limited range (Rajan and Zhang, 2002). The recent enlargement of the European Union to 27 countries requires that the new Member States fulfill a period of managed floating regime (ERM II) before the adoption of the Euro. In this context, together with monetary and fiscal challenges, exchange rate policies have become a key tool for the new EU members. Optimal exchange rate policies have to be set to manage the hardening against the Euro. Interventions by Government Authorities are required to stabilize the exchange rate even before the participation in ERM II (Dean, 2004).

Another example is provided by the Chinese exchange rate system. On July 2005, the China's Authorities announced that the Renmibi (RMB) would be managed "with reference to a basket of currencies" rather than being pegged to the dollar. According to the Public Announcement of the People's Bank of China (PBOC) on reforming the RMB Exchange Rate Regime, the Chinese Authorities "make adjustment of the RMB exchange rate band when necessary according to market developments as well as the economic and financial situation" and maintain "the RMB exchange rate basically stable at an adaptive and equilibrium level, so as to promote the basic equilibrium of the balance of payments and safeguard macroeconomic and financial stability"[1]. Although the RMB exchange rate adjustments initially were too cautious, the announcement made possible transitional arrangements like those applied in other emerging countries showing the PBOC's awareness of the unsustainability of the US Dollar pegging. The managed floating exchange rate system, together with a more independent monetary policy, might help the Chinese economy to cope better with both the internal and external macroeconomic shocks to which a developing country may be exposed (Goldstein and Lardy, 2009).

Exchange rate stabilization policies represent a crucial issue, and they have been largely analyzed in the literature. Krugman (1991) emphasized the role of official interventions at the margin of a currency band, when the fundamentals driving the exchange rate follow a random walk with constant variance. Most empirical results are controversial, leaving many questions unanswered, such as the issues of the optimal monetary policy and the optimal width of the currency band (if adopted). Improvements of Krugman's framework are obtained thank to the extensions of the basic model (amongst others: Jeanblanc-Picqué, 1993; Miller and Zhang, 1996; Mundaca and Oksendal, 1998; Im, 2001; Zampolli, 2006; Castellano and D'Ecclesia, 2007). Jeanblanc-Picqué (1993) applies impulse control methods to show that using a diffusion process with constant coefficients it is possible to keep the exchange rate in a given target zone with discrete interventions. Miller et

[1] http://www.pbc.gov.cn

al. (1996) find a subgame-perfect solution for a Central Bank aiming at stabilizing the exchange rate in a target zone, given proportional costs of intervention. Mundaca and Oksendal (1998), using a jump diffusion process for the exchange rate dynamics, combine continuous and impulse controls to stabilize the exchange rate. Im (2001) presents the central bank optimal intervention strategies to find the policy which minimizes the value of the loss function. Assuming the economy randomly switches between different regimes, with time-invariant transition probabilities, Zampolli (2006) examines the trade-offs deriving from deviations of the exchange rate from fundamentals and from extreme changes. Castellano and D'Ecclesia (2007) solve a stochastic optimal control model to describe exchange rate dynamics in a managed floating regime assuming Government Authorities aim to keep the aggregate fundamental not too far from a predetermined target and within an optimal currency band.

In this paper, optimal exchange rate stabilization policies are taken into account. We assume that the exchange rate is a function of the *aggregate fundamental* whose dynamics is described by a stochastic differential equation (SDE) with a general functional shape for the state-dependent drift and variance. The drift of the fundamental is the control variable to maintain the fundamental level as close as possible to a time-varying target. We introduce a disutility function that depends: 1) on the difference between the aggregate fundamental and its target dynamics; 2) on the control variable. The implicit costs associated with the interventions are measured in terms of disutility. The stochastic control problem is solved using the dynamic programming approach. The optimal strategies are obtained in two steps: first, deriving the unique solution of the Hamilton Jacobi Bellman (HJB) in the viscosity sense (Barles and Rouy, 1998); second, formalizing the existence of the optimal strategies and their related paths according to the regularity properties of the value function. The optimal trajectory of the exchange rate is fully characterized. We also show that, under particular conditions, the optimal width of the currency band can be determined.

The main innovations of this paper are represented by: 1) the choice of a general shaped function for the stochastic dynamics of the aggregate fundamental; 2) the introduction of a disutility function which measures the implicit costs of the intervention; 3) the definition of the endogenous currency band.

The work is organized as follows: the next section describes the model and the related optimal control problem; section 3 presents the properties of the value function and the optimal strategies; in section 4 a particular case is discussed; some concluding remarks are presented in section 5, and the mathematical derivations are reported in the Appendix.

2 The Model

This section describes the model developed to study the interventions of Government Authorities in a managed floating regime. The building blocks of the model are given by the exchange rate dynamics depending on some random fundamental, the presence of a time dependent target and the optimization problem.

2.1 The exchange rate dynamics

We assume that the exchange rate depends on both some current fundamentals and expectations of future values of the exchange rate. The (log) of

the spot exchange rate at any time t, s_t, is assumed to depend on an aggregate "fundamental", f_t, and a speculative term proportional to the expected change in the exchange rate. As stated in Svensson (1992), the fundamental absorbs the driving forces of the exchange rate (i.e. monetary and fiscal policy variables, domestic output, price level, foreign interest rate, etc.).
Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a simple representation of the spot exchange rate dynamics is given by:

$$s_t dt = f_t dt + \lambda \mathbb{E}_t[ds_t] \qquad \lambda > 0, \tag{1}$$

where:

- s_t is the logarithm of the exchange rate defined as unit of domestic currency per unit of the reference currency;
- f_t denotes the logarithm of the aggregate fundamental;
- λ is a constant positive parameter which can be interpreted as the semielasticity of the exchange rate with respect to the instantaneous rate of currency depreciation;
- $\mathbb{E}_t[ds_t]$ measures the expected change of the exchange rate with respect to time t.

The process for the fundamental, f_t, is given by:

$$df_t = \mu_f(f_t, \theta_t)dt + \sigma_f(f_t)dB_t, \tag{2}$$

where:

- $\theta_t \in \Theta$, represents the control variable, used by Monethary Authorities to manage the current fundamental dynamics, Θ is the admissible region defined as

$$\Theta := \Big\{\theta : [0,+\infty)\times\Omega \to [\theta_m, \theta_M]\ \ \mathcal{F}_t-\text{adapted processes},\ \theta_m < \theta_M\Big\}; \tag{3}$$

- $\mathbb{E}[f_t^2] < +\infty$;
- $\mu_f : \mathbb{R} \times [\theta_m, \theta_M] \to \mathbb{R}$;
- $\sigma_f : \mathbb{R} \to \mathbb{R}$;
- B_t is a standard Brownian Motion.

We assume that the initial value of the fundamental, f_0, is deterministic. The effective aggregate fundamental, f_t, consists of exogenous and endogenous components. We further assume that Government Authorities, using monetary, economic and fiscal policies, monitor the exchange rate and may intervene in order to maintain the fundamental, f_t, broadly in line with its target. In particular, equation (2) states that Government Authorities intervene on the control variable, θ_t, to manage the drift of the fundamental, $\mu_f(f_t, \theta_t)$.

We set a target for the fundamental, $\tilde{f}_t$, which includes a set of variables affecting the exchange rate. For instance, some of the parameters set by the European Commission during the process of EU accession have to show some specific behavior, or some macroeconomic variables have to perform according to given targets. In the case of China, the PBOC officially sets targets for money supply (Burderkin and Siklos, 2008) and credit growth "to maintain stability of the value of the currency and thereby promote economic growth"[2].

[2] http://www.pbc.gov.cn

The evolution of the potential target is described by an ordinary differential equation:

$$d\tilde{f}_t = \beta_t dt, \tag{4}$$

where:

- β is defined on $[0, +\infty)$;
- $\tilde{f}_0$ is the deterministic initial value of $\tilde{f}_t$.

We define the state-variable $x_t := f_t - \tilde{f}_t$ whose dynamics, given (2) and (4), on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, is given by:

$$dx_t = d(f_t - \tilde{f}_t) = \mu(x_t, \theta_t)dt + \sigma(x_t)dB_t, \qquad t > 0 \tag{5}$$

where:

- $\mu\,(x_t, \theta_t) = \mu_f(f_t, \theta_t) - \beta_t$;
- $\sigma(x_t) = \sigma_f(f_t)$ and $\sigma(x) \neq 0, \;\; \forall\, x \in \mathbb{R}$;
- μ is a continuous real value bounded function with respect to the process θ;
- $x_0 = x$, is the deterministic starting point of the dynamics x_t.

μ and σ satisfy the usual regularity conditions for the existence and uniqueness of the solution for (5).

2.2 The optimization problem

The decision maker aims to reduce its disutility intervening on its preferences as well as on the fundamental through the control variable θ. The expected disutility allows an assessment of the Government Authorities' policies and the total "*social*" costs of the stabilization process.

The disutility function depends on the distance of the fundamental value from its target, and it is controlled by θ_t. The larger the distance between the fundamental and the target, the lower the satisfaction and the higher the disutility. The control problem is solved finding the optimal control rule θ_t, as a function of the state variable x_t, that minimizes the expected discounted disutility and the implicit costs of the control policies. We formalize the dynamic optimization problem in terms of the value function, $V : \mathbb{R} \to \mathbb{R}$, presented as:

$$V(x) := \inf_{\theta \in \Theta} J(\theta, x), \tag{6}$$

with

$$J(\theta, x) := \mathbb{E}^x \Big\{ \int_0^{+\infty} e^{-\gamma t} u(|x_t|, \theta_t) \mathrm{d}t \Big\}, \qquad \text{for } x_0 = x, \tag{7}$$

where:

- $e^{-\gamma t}$, with $\gamma > 0$, is the discount factor;
- $u : [0, +\infty) \times [\theta_m, \theta_M] \to \mathbb{R}$ is the Government Authority's disutility function;
- $|x_t| = |f_t - \widetilde{f}_t|$ is the distance of the fundamental from its target;
- $\mathbb{E}^x$ is the expected value of the disutility, u, depending on the absolute value of x_t, whose dynamics are given by (5), with initial position x.

Since V is symmetric with respect to the origin we do not lose any generality assuming $x \geq 0$.

As stated above, u is increasing with respect to $|x_t|$ and continuous with respect to θ_t. With no loss of generality, we assume that the disutility function is essentially bounded with respect to x and this implies, together with the continuity of u with respect to θ in $[\theta_m, \theta_M]$, that also V is essentially bounded with respect to x.

To clarify the concept of disutility function it may be useful to list some specific functions:

- $u(|x_t|, \theta_t) = \theta_t^2 \cdot \mathrm{e}^{|x_t|}$; the disutility grows rapidly as the distance of the fundamental from its target, $|x_t|$, increases; Government Authorities should intervene as soon as possible to avoid an explosion of the disutility; the control variable θ_t should be pushed downward to compensate the growth of $\mathrm{e}^{|x_t|}$.
- $u(|x_t|, \theta_t) = |x_t| \exp\left[\frac{1}{\theta_t}\right]$; the disutility grows linearly as the gap $|x_t|$ increases at a rate depending on θ_t; Government Authorities should intervene to reduce the growth of the disutility, by pushing upward the control variable θ_t.

The introduction of the disutility in the objective functional and its dependence on the control variable guarantees that the optimal solution is also the one minimizing the costs of the controls. The smaller is the distance between the observed fundamental and the target, the smaller is the cost of the control measured in terms of disutility.

3 The optimal policies

In this section, following the dynamic programming approach, we study the properties of the value function and derive the implied Government Authority's optimal strategies.

Theorem 1 *The value function V is the unique classical solution of the HJB equation:*

$$\gamma V(x) = \frac{\sigma^2(x)}{2} V''(x) + \min_{\theta \in [\theta_m, \theta_M]} \Big\{ u(x,\theta) + \mu(x,\theta)V'(x) \Big\}; \tag{8}$$

in $(0, +\infty)$, with boundary condition $V(0) = 0$ and $V(+\infty) = M > 0$.

The proof is reported in the Appendix.

By Theorem 1, the optimal strategies in feedback form can be obtained.

Theorem 2 *Consider $x \in [0, +\infty)$ and define*

$$\theta^* \in \mathrm{argmin}_\theta \Big\{ \frac{\sigma^2(x_t)}{2} V''(x_t) + u(|x_t|, \theta) + \mu(x_t, \theta)V'(x_t) \Big\}.$$

1. *The closed loop equation:*

$$\begin{cases} dx_t = \mu(x_t, \theta^*)dt + \sigma(x_t)dB_t, & t > 0 \\ \\ x_0 = x, \end{cases} \tag{9}$$

   admits a unique solution.

2. *Assuming that $\bar{x}_t$ is the solution of the closed loop equation, we obtain θ_t depending on $\bar{x}_t$; so we set $\bar{\theta}_t := \theta^*(\bar{x}_t)$ and obtain:*

$$\begin{cases} d\bar{x}_t = \mu(\bar{x}_t, \bar{\theta}_t)dt + \sigma(\bar{x}_t)dB_t, & t > 0 \\ \\ \bar{x}_0 = x. \end{cases}$$

   Since $J(x, \bar{\theta}) = V(x)$ holds, $\bar{\theta}_t$ is the optimal value for the control variable, with the related optimal trajectory, $\bar{x}_t$.

The proof is reported in the Appendix.

The existence of the optimal strategy $\bar{x}_t$ implies the existence of an optimal trajectory for the fundamental, $f_t^* := \bar{x}_t + \tilde{f}_t$, given the relationship between x_t and f_t. Starting from the optimal fundamental, a characterization of the optimal exchange rate dynamics, s_t^*, can be provided.

Next result allows to define the optimal exchange rate dynamics.

Proposition 3 *Given the optimal fundamental f_t^*, then the optimal exchange rate dynamics can be written as $s_t^* = h(f_t^*)$, where the function h is the solution of the following second order differential equation:*

$$\frac{\sigma_f^2(f_t^*)}{2} h''(f_t^*) + \mathbb{E}_t[\mu_f(f_t^*, \bar{\theta}_t)] h'(f_t^*) - \frac{1}{\lambda} h(f_t^*) = -\frac{f_t^*}{\lambda}. \tag{10}$$

Proof. We look for solution of (1) introducing a function h:

$$s_t^* = h(f_t^*). \tag{11}$$

Applying Ito's Lemma to (11), we have:

$$ds_t^* = h'(f_t^*) df_t^* + \frac{1}{2} h''(f_t^*)(df_t^*)^2 =$$

$$= h'(f_t^*)[\mu_f(f_t^*, \bar{\theta}_t)dt + \sigma_f(f_t^*)dB_t] + \frac{1}{2} h''(f_t^*)\sigma_f^2(f_t^*)dt. \tag{12}$$

The conditional expectation of ds_t^* is given by:

$$\mathbb{E}_t[ds_t^*] = h'(f_t^*)\mathbb{E}_t[\mu_f(f_t^*, \bar{\theta}_t)]dt + \frac{1}{2} h''(f_t^*)\sigma_f^2(f_t^*)dt.$$

Therefore, equation (1) can be rewritten as

$$h'(f_t^*)\mathbb{E}_t[\mu_f(f_t^*, \bar{\theta}_t)]dt + \frac{1}{2} h''(f_t^*)\sigma_f^2(f_t^*)dt = \frac{1}{\lambda}(s_t^* - f_t^*)dt. \tag{13}$$

Given (11) and (13), h can be found as solution of (10). ■

4 Some applications: derivation of the optimal currency band

To provide an explicit formalization of the optimal values for the control variable, θ, as given in Theorem 1, in this section some particular cases are discussed. For our purpose, we remove the assumption that $x > 0$ and consider $x \in \mathbb{R}$.

First of all, we argue that μ and u are assumed to exhibit the same behavior w.r.t. θ in $[\theta_m, \theta_M]$, and this means that if $|x|$ increases, then u and μ increase. Therefore, the intervention of the Government Authorities through the control θ should push downward simultaneously μ and u. In this particular example we assume the existence of $A, B, C \subseteq \mathbb{R}$ such that $A \cup B \cup C = \mathbb{R}$ and

$$A = \{x \in \mathbb{R} \,|\, \mu(x,\theta),\ u(|x|,\theta) \text{ increase w.r.t. } \theta \text{ in } [\theta_m, \theta_M]\}\,;$$

$$B = \{x \in \mathbb{R} \,|\, \mu(x,\theta),\ u(|x|,\theta) \text{ are constant w.r.t. } \theta \text{ in } [\theta_m, \theta_M]\}\,;$$

$$C = \{x \in \mathbb{R} \,|\, \mu(x,\theta),\ u(|x|,\theta) \text{ decrease w.r.t. } \theta \text{ in } [\theta_m, \theta_M]\}\,.$$

The optimization problem can be represented introducing the map $g_x : [\theta_m, \theta_M] \to \mathbb{R}$ such that:

$$g_x(\theta) = u(|x|,\theta) + \mu(x,\theta)V'(x), \qquad \forall\, x \in \mathbb{R}. \tag{14}$$

According to Theorem 1, the optimization problem is solved by minimizing the function g_x w.r.t. θ. By assuming the right regularity for the functions μ and u and applying the first order condition we get:

$$g_x'(\theta) = \frac{\partial u(|x|,\theta)}{\partial \theta} + V'(x)\frac{\partial \mu(x,\theta)}{\partial \theta} = 0. \tag{15}$$

Under particular conditions, we are able to derive some *intervention bands* for x, i.e. the regions where the optimal control rule is invariant. In particular, it is easy to choose μ and u such that two thresholds $\underline{x}_1, \underline{x}_2 \in \mathbb{R}$, with $\underline{x}_1 < 0 < \underline{x}_2$exist, with:

(i) $A = (-\infty, \underline{x}_1)$, $B = [\underline{x}_1, \underline{x}_2]$, $C = (\underline{x}_2, +\infty)$;

(ii) $A = (\underline{x}_2, +\infty)$, $B = [\underline{x}_1, \underline{x}_2]$, $C = (-\infty, \underline{x}_1)$.

A and C represent two intervention bands for x. The optimal strategies can be written as follows:

(i)

$$\theta^*(x) = \begin{cases} \theta_m & \text{when } x < \underline{x}_1 \\ \\ \theta_M & \text{when } x > \underline{x}_2. \end{cases} \tag{16}$$

(ii)

$$\theta^*(x) = \begin{cases} \theta_M & \text{when } x < \underline{x}_1 \\ \\ \theta_m & \text{when } x > \underline{x}_2. \end{cases} \tag{17}$$

When $x \in [\underline{x}_1, \underline{x}_2]$, then $\theta^*(x)$ can freely fluctuate. If $x \in [\underline{x}_1, \underline{x}_2]$, then $f_0^* \in [\underline{x}_1 + \tilde{f}_0, \underline{x}_2 + \tilde{f}_0]$. Furthermore, the behavior of the optimal exchange rate dynamics is fully described by the function h in (11), which depends on μ_f, σ_f, λ. Thus, when h is strictly monotonic, for instance increasing, then $f_0^* \in [\underline{x}_1 + \tilde{f}_0, \underline{x}_2 + \tilde{f}_0]$ implies that $s_0^* \in [h(\underline{x}_1 + \tilde{f}_0), h(\underline{x}_2 + \tilde{f}_0)]$. The interval $[h(\underline{x}_1 + \tilde{f}_0), h(\underline{x}_2 + \tilde{f}_0)]$ represents the optimal currency band for the exchange rate dynamics, where no interventions occur.

To provide an intuitive understanding of the optimal strategies, we introduce the functions: $u_1, \mu_1, \alpha : \mathbb{R} \to \mathbb{R}$ and $u_2, \mu_2 : [\theta_m, \theta_M] \to \mathbb{R}$, such that the drift and the disutility functions can be defined, respectively, as:

$$\mu(x, \theta) = \mu_1(x)\mu_2(\theta), \tag{18}$$

$$u(|x|, \theta) = u_1(|x|)u_2(\theta) + \alpha(|x|), \tag{19}$$

where (18) and (19) satisfy the regularity conditions given in Section 2 and μ_1 is increasing in $[0, +\infty)$ and decreasing in $(-\infty, 0)$.
In (18), the Government Authorities may apply a control θ, through μ_2, in order to let $\mu(x, \theta)$ be close to $\mu(0, \theta)$ and drive the process of the fundamental, f_t, closer to its target, $\tilde{f}_t$. Equation (19) provides a general example of the specific disutility functions introduced in subsection 2.2.
Given (18) and (19), the map g_x becomes:

$$g_x(\theta) = u_1(|x|)u_2(\theta) + \mu_1(x)\mu_2(\theta)V'(x) + \alpha(|x|), \tag{20}$$

and applying the first order condition we get:

$$g'_x(\theta) = u_1(|x|)u'_2(\theta) + \mu_1(x)\mu'_2(\theta)V'(x) = 0, \tag{21}$$

from which, by assuming $u_1(|x|) \neq 0$,

$$\frac{u'_2(\theta)}{\mu'_2(\theta)} = -\frac{\mu_1(x)V'(x)}{u_1(|x|)}. \tag{22}$$

Assume that $\mu'_2, u'_2 \neq 0$ and the convexity of u_2 and μ_2 in $[\theta_m, \theta_M]$. If

$$\frac{u''_2(\theta)}{u'_2(\theta)} > \frac{\mu''_2(\theta)}{\mu'_2(\theta)} \qquad \text{or} \qquad \frac{u''_2(\theta)}{u'_2(\theta)} < \frac{\mu''_2(\theta)}{\mu'_2(\theta)}, \qquad \forall\, \theta \in [\theta_m, \theta_M], \tag{23}$$

then the function $\rho(\theta) = \frac{u'_2(\theta)}{\mu'_2(\theta)}$ is invertible and the optimal control $\theta^*(x)$ is given by:

$$\theta^*(x) = \rho^{-1}\left(-\frac{\mu_1(x)V'(x)}{u_1(|x|)}\right). \tag{24}$$

It is possible to consider the limit case of two dominant optimal policies: one expansionary and the other restrictive (i.e.: optimal policies of *bang-bang* type). In this case, the optimal currency band collapses to a single value that is necessarily 0.

Now, assume $\mu_2(\theta) = u_2(\theta) = n(\theta)$ twice differentiable and convex in (θ_m, θ_M).

The map g_x becomes:

$$g_x(\theta) = n(\theta)[u_1(|x|) + \mu_1(x)V'(x)] + \alpha(x), \tag{25}$$

and the first order condition gives:

$$g'_x(\theta) = n'(\theta)[u_1(|x|) + \mu_1(x)V'(x)] = 0. \tag{26}$$

For $\mu_1(x) \neq 0$ in $\mathbb{R}$, we have two cases:

- if
$$V'(\underline{x}) + \frac{u_1(|\underline{x}|)}{\mu_1(\underline{x})} = 0, \qquad \text{for } \underline{x} \in \mathbb{R}, \tag{27}$$
then (26) is satisfied for each $\theta \in [\theta_m, \theta_M]$ and the value $\underline{x}$ represents a specific distance between the fundamental and its target for which Government Authorities may apply arbitrary decision rules. It is natural that it must be $\underline{x} = 0$: here, the target of the Government Authority is reached and no intervention is needed;

- if
$$V'(x) + \frac{u_1(|x|)}{\mu_1(x)} \neq 0, \qquad \text{for } x \in \mathbb{R}, \tag{28}$$
then (26) cannot be satisfied assuming an increasing (decreasing) n, i.e. when $n'(\theta) > 0 (< 0)$ for $\theta \in [\theta_m, \theta_M]$. However, the continuity of g_x and Weierstrass' Theorem guarantee the existence of the optimal

strategies, belonging to $\{\theta_m, \theta_M\}$. More precisely, a *critical region* $\Gamma \subseteq \mathbb{R}$ can be defined as follows:

$$\Gamma := \Big\{x \in [0,+\infty) \,|\, V'(x) + \frac{u_1(|x|)}{\mu_1(x)} > 0\Big\} \cup \Big\{x \in (-\infty,0) \,|\, V'(x) + \frac{u_1(|x|)}{\mu_1(x)} < 0\Big\}. \tag{29}$$

We have:

– if $n'(\theta) > 0$ in $[\theta_m, \theta_M]$, then

$$\theta^*(x) = \begin{cases} \theta_m & \text{when } x \in \Gamma \\ \\ \theta_M & \text{when } x \in \mathbb{R} \setminus (\Gamma \cup \{\underline{x}\}) \end{cases} \tag{30}$$

– if $n'(\theta) < 0$ in $[\theta_m, \theta_M]$, then

$$\theta^*(x) = \begin{cases} \theta_M & \text{when } x \in \Gamma \\ \\ \theta_m & \text{when } x \in \mathbb{R} \setminus (\Gamma \cup \{\underline{x}\}) \,. \end{cases} \tag{31}$$

Since u is an increasing function of $|x|$, then by (6) and (7) V increases in $[0,+\infty)$ and decreases in $(-\infty, 0)$. As a consequence, further assumptions on u_1 and μ_1 allow to derive some intervention bands for x. As a particular example, we have that if $\mu_1(x) \cdot u_1(|x|) > 0$, for each $x \in [0,+\infty)$ and $\mu_1(x) \cdot u_1(|x|) < 0$, for each $x \in (-\infty, 0)$, then $V' + \frac{u_1}{\mu_1} > 0$ in $[0,+\infty)$ and $V' + \frac{u_1}{\mu_1} < 0$ in $(-\infty, 0)$. Hence, $\Gamma = \mathbb{R}$.
By (30) and (31), when $\Gamma = \emptyset$ or $\Gamma = \mathbb{R}$, then there exists a unique optimal strategy $\theta^* \in \{\theta_m, \theta_M\}$ that the Government Authority can apply. In particular:

- for $\Gamma = \emptyset$ and n increasing (decreasing), then $\theta^*(x) = \theta_M$ ($\theta^*(x) = \theta_m$) for each $x \in \mathbb{R}$;

- for $\Gamma = \mathbb{R}$ and n increasing (decreasing), then $\theta^*(x) = \theta_m$ ($\theta^*(x) = \theta_M$), for each $x \in \mathbb{R}$.

When $\Gamma = (-\infty, \underline{x})$, then Γ and $(\underline{x}, +\infty)$ represent two optimal intervention bands for x.
In this particular case, the optimal currency band for the exchange rates collapses to a singleton. Given the optimal fundamental path $f_t^* := \bar{x}_t + \tilde{f}_t$, then $x = \underline{x}$ implies $f_0^* = \underline{x} + \tilde{f}_0$. By definition of the function h in (11), we have that $s_0^* = h(\underline{x} + \tilde{f}_0)$ is the threshold for the exchange rate where no intervention is applied by the Government Authority. The set $\{h(\underline{x} + \tilde{f}_0)\}$ is the degenerate currency band.

4.1 Interpretation of the results

The optimal control is defined as a function of the gap registered between the theoretical and the observed fundamental, (24).
To fix ideas, assume $x > 0$ (the case $x < 0$ is analogous). Given (23), ρ^{-1} is an increasing function of its argument and the relationship between the optimal control, θ^*, and the variable x can be derived. $\theta^*(x)$ is directly related to the disutility function, $u_1(|x|)$, and inversely related to the drift $\mu_1(x)$ of the state variable, x_t, and to the growth rate of the expected disutility function $V'(x)$.
In other words, if the Government Authority's disutility is large, need for strong interventions occur and a large value of θ^* is chosen; on the other hand, if the deterministic trend of x_t or the change in the disutility is large, then there may be a need for a small intervention of the Government Authority.
For instance, considering the current international context, one can observe

that some countries are experiencing a much lower growth rate than expected, and this causes a large value of the state variable, i.e. the distance between theoretical and observed fundamental, and high level of disutility. According to our model, this would require incisive interventions represented by large value of θ, i.e strong measures of fiscal and monetary policy.

The optimal control, θ^*, described by (24) is applied whenever $\theta \in (\theta_m, \theta_M)$. This means that theoretical and observed fundamentals may differ one from the other, but the difference is still within the optimal tolerance band: $x \in [\underline{x}_1, \underline{x}_2]$.

When x becomes too large ($x > \underline{x}_2$), or too small ($x < \underline{x}_1$), the optimal strategies are given by (16) and (17) -or (30) and (31)-, which may represent the extreme interventions that Government Authority have to choose in order to bring the fundamental closer to its target value.

In the case of the current economic situation, the observed fundamental may result very far from the theoretical one, therefore strong interventions have to be adopted, meaning large values of θ have to be chosen. For instance, in terms of monetary policies this may be translated in monetary control exercised via manipulation of the monetary base or control of domestic credit or bank credit or net foreign assets. The monetary policies to be adopted are defined according to the each country's internal targets such as inflation and unemployment, and external targets such as competitiveness, the current accounts, and reserves.

5 Conclusions

This paper presents a disutility based drift control model for exchange rate dynamics, in the framework of managed floating regimes. The dynamics of the exchange rate is described as a function of the aggregate fundamental at time t, f_t, which follows a Brownian Motion with state dependent drift and volatility. The process for the fundamental dynamics are obtained as the solution of a stochastic control problem describing the Government Authorities' aim to keep the value of the fundamental as close as possible to its target. An expected disutility function minimization problem is developed, and the related Hamilton Jacobi Bellman equation is solved in viscosity sense.

We show that under particular conditions, it is possible to obtain the optimal width of the currency band. The model is realistic since it suggests a more adequate process to describe the exchange rate dynamics and provides an accurate analysis of the observed phenomenon with respect to simple diffusion processes which may lack in economic content. The model takes into account the time-varying features of the dynamics of the exchange rates and the optimal strategies that can be applied by Government Authorities to stabilize the exchange rate within a band.

Appendix

Proof of Theorem 1

By Dynamic Programming Principle (see Yong and Zhou, 1999), we have the following result:

Proposition 4 *If* $V \in C^0[0, +\infty) \cap C^2(0, +\infty)$*, then (8) holds in* $(0, +\infty)$*, with the boundary condition* $V(0) = 0$ *and* $V(+\infty) = M$*.*

Equation (8) with the boundary condition holds formally, in the sense that the regularity conditions required for the function V are assumed. Since

V is generally not twice differentiable, then we proceed by proving the existence and uniqueness of the solution of (8) with boundary conditions in a weak sense. To this end we use the concept of the viscosity solutions (for a complete survey on viscosity solutions we refer to Crandall et al., 1992; Barles, 1994; Fleming and Soner, 2006). The following result states the existence and uniqueness of the solution of the HJB (8) in the viscosity sense. Such solution coincides with V.

Theorem 5 *The value function V is continuous in $(0, +\infty)$ and can be extended continuously on $[0, +\infty)$. Moreover, V is the unique viscosity solution of the HJB equation (8) with the boundary condition $V(0) = 0$ and $V(+\infty) = M$.*

Proof. The proof is a direct consequence of a result in Barles and Rouy (1998). ■

We now need to discuss the regularity properties of the value function to prove Theorem 1. In fact, if V is at least twice differentiable, then Theorem 5 guarantees that it is the unique classical solution of (8) with boundary condition $V(0) = 0$ and $V(+\infty) = M$.

We firstly need to prove that V is concave. To this end, we fix $x \in [0, +\infty)$ and real-valued function $v \in C^0[0, +\infty) \cap C^2(0, +\infty)$ and define the Hamiltonian:

$$H(x, v(x), v'(x), v''(x)) :=$$
$$\gamma v(x) - \frac{\sigma^2(x)}{2} v''(x) - \min_{\theta \in [\theta_m, \theta_M]} \Big[u(x, \theta) + \mu(x, \theta) v'(x) \Big]. \tag{32}$$

Writing

$$-H(x, -v(x), -v'(x), -v''(x)) = 0, \qquad \forall\, x \in (0, +\infty),$$

we obtain:

$$\gamma v(x) - \frac{1}{2}\sigma^2(x)v''(x) + \min_{\theta\in[\theta_m,\theta_M]}\Big[u(x,\theta) - \mu(x,\theta)v'(x)\Big] = 0, \qquad (33)$$

for each $x \in (0,+\infty)$. The following lemma holds:

Lemma 6 (Barles, 1994) *$\varphi \in C^0(0,+\infty)$ is a viscosity supersolution (subsolution) of (8) if and only if $\psi := -\varphi$ is a subsolution (supersolution) of (33).*

The previous result implies the following corollary.

Corollary 7 *If φ is the unique viscosity solution of (8), then $\psi := -\varphi$ is the unique viscosity solution of (33).*

In the following lemma we recall an important general result due to Alvarez et al. (1997). This result is useful to prove concavity.

Lemma 8 (Alvarez et al., 1997) *Let us consider an interval $I \subseteq \mathbb{R}$ and define an hamiltonian operator*

$$\tilde{H} : \bar{I} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Assume that $\tilde{H}$ satisfies the following properties:

- *there holds*

$$\tilde{H}(x,v,p,q) = 0 \qquad \forall\, x \in I; \qquad (34)$$

- *$\tilde{H} \in C^0(\bar{I} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$;*
- *$\tilde{H}$ is elliptic;*

- *It results*

$$(x,v) \mapsto \tilde{H}(x,v,p,0)$$

concave, for every p.

Let v *lower semi-continuous in* $\bar{I}$ *be a viscosity supersolution of (34) and define the convex envelope* v_{**} *of* v *as*

$$v_{**}(x) := \inf \Big\{ \lambda_1 v(x_1) + \lambda_2 v(x_2) \,|\, x = \lambda_1 x_1 + \lambda_2 x_2,$$

$$\text{with } x_i \in I,\ \lambda_i \geq 0,\ i = 1,2,\ \lambda_1 + \lambda_2 = 1 \Big\}.$$

Then v_{**} *is lower semi-continuous in* $\bar{I}$ *and it is a viscosity supersolution of (34).*

Theorem 9 V *is a concave function in* $[0,+\infty)$.

Proof. In order to prove the theorem, it is sufficient to prove that $u := -V$ is a convex function. We use Corollary 7 and apply it to equation (33).

Let us now define:

$$0 = \gamma v(x) - \frac{1}{2}\sigma^2(x) q + \min_{\theta \in [\theta_m, \theta_M]} \Big[u(x,\theta) - \mu(x,\theta) p \Big] =:$$

$$=: \tilde{H}(x,v,p,q) \qquad \forall\, x \in [0,+\infty). \tag{35}$$

It results:

$$\tilde{H}(x,v,p,0) = \gamma v(x) + \min_{\theta \in [\theta_m, \theta_M]} \Big[u(x,\theta) - \mu(x,\theta) p \Big].$$

A direct computation gives us that the map

$$(x,v) \mapsto \tilde{H}(x,v,p,0)$$

is concave for every p.

Furthermore, since $\sigma \neq 0$, for each $x \in [0, +\infty)$, then $\tilde{H}$ is an elliptic operator.

Since the hypotheses of Lemma 8 hold, the convex envelope v_{**} of v is a viscosity supersolution of (35).

Using the definition of convex envelope, for each $x \in [0, +\infty)$, we have:

$$v_{**}(x) = \inf \Big\{ \lambda_1 v(x_1) + \lambda_2 v(x_2) \,|\, x = \lambda_1 x_1 + \lambda_2 x_2 \Big\} \leq v(x), \qquad (36)$$

with the choice $\lambda_1 = 1$, $\lambda_2 = 0$, $x_1 = x$, x_2 arbitrary in $[0, +\infty)$.

If w_1 is a viscosity subsolution and w_2 is a viscosity supersolution of (35) then, from the Existence and Uniqueness Theorem 5, we get $w_1 \leq w_2$.

Given this result and (36), the convex envelope v_{**} of v is a viscosity subsolution of (35).

By Theorem 5 and Corollary 7, v_{**} is the unique viscosity solution of (35) and, hence, the unique viscosity solution of (33). Therefore $v = -V$ is convex in $[0, +\infty)$ and the theorem is completely proved. ■

Next result guarantees that the viscosity solution of the HJB equation is a classical solution.

Theorem 10 V *is twice differentiable in* $(0, +\infty)$.

Proof. Since $\sigma(x) \neq 0$, for each $x \in (0, +\infty)$, equation (8) is uniformly elliptic in $(0, +\infty)$. Furthermore, given the concavity/continuity and adapting Alexandrov's Theorem to this case (see Fleming and Soner, 2006), we know that V is twice differentiable a.e. in $(0, +\infty)$. Therefore, it follows that $V' \in L^\infty(0, +\infty)$, given $\mu, \sigma, u \in L^\infty(0, +\infty)$, by definition.

Moreover, we can write, a.e. in $(0, +\infty)$,

$$V''(x) = \frac{2}{\sigma^2(x)} \Big\{ \gamma V(x) - \min_{\theta \in [\theta_m, \theta_M]} \Big[u(x, \theta) + \mu(x, \theta) V'(x) \Big] \Big\}. \qquad (37)$$

The right-hand side of the (37) is the sum of functions that are in $L^\infty(0, +\infty)$ and, hence, we can state that $V'' \in L^\infty(0, +\infty)$.

Using previous arguments, we obtain that V is a function in the Sobolev space $W^{2,\infty}(0, +\infty)$.

Since $(0, +\infty)$ is an interval, the hypotheses of the Sobolev's Embedding Theorem (see Gilbarg and Trudinger, 1977) are trivially true and we get $V \in C^m(0, +\infty)$, $\forall m \in [0, 2)$. Therefore, V' is a continuous function, and the second term of (37) is a combination of continuous functions: $V'' \in C^0(0, +\infty)$.

The result is proved. ■

Proof of Theorem 1. By Theorems 5 and 10, we have the thesis. ■

Proof of Theorem 2

We first present a Verification Theorem to identify the optimal strategies and the related optimal trajectories.

Lemma 11 *Assume that* $v \in C^0[0, +\infty) \cap C^2(0, +\infty)$ *is the (classical) solution of (8) with the boundary condition* $V(0) = 0$ *and* $V(+\infty) = M$*. Then:*

- $v(x) \leq V(x), \;\; \forall x \in [0, +\infty)$.

Let us, now, consider a pair of stochastic processes, (θ^*, x^*) *with* $x_0^* = x$*, such that*

$$\theta^* \in \text{argmin}_\theta \Big\{ \frac{\sigma^2(x_t^*)}{2} v''(x_t^*) + u(|x_t^*|, \theta) + \mu(x_t^*, \theta) v'(x_t^*) \Big\},$$

then, θ^* *is optimal in* x*, and* x^* *is the related optimal trajectory, if and only if* $v(x) = V(x), \;\; \forall x \in [0, +\infty)$.

A detailed proof can be found in Fleming and Soner, 2006.

Given Theorems 1, 5 and 10, we can rewrite the HJB as:

$$0 = H(x, V(x), V'(x), V^{''}(x)) = \inf_{\theta\in[\theta_m,\theta_M]} H_\theta(x, V(x), V'(x), V^{''}(x)), \quad (38)$$

where

$$H_\theta(x, V(x), V'(x), V^{''}(x)) :=$$
$$\gamma V(x) - \frac{\sigma^2(x)}{2} V''(x) - u(x,\theta) - \mu(x,\theta)V'(x). \quad (39)$$

Since $\mu, u \in C^0[\theta_m, \theta_M]$, then the function $H_\theta \in C^0[\theta_m, \theta_M]$ and Weierstrass's Theorem guarantees the existence of the absolute minimum point $\theta^* \in [\theta_m, \theta_M]$ of the function H_θ defined in (39).

Proof of Theorem 2.

1. The proof follows from the existence of θ^*, shown in Lemma 11, and by the existence and uniqueness of the solution for the state equation (5).

2. The proof is due to Lemma 11.

■

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