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Asymptotic solutions of a generalized eigenvalue problem

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Abstract

This paper provides a solution of a generalized eigenvalue problem for a fractional integrated processes. To this end two random matrices are constructed in order to take into account the stationarity properties of the differences of a fractional p-variate integrated process. The matrices are defined by some weight functions and the difference orders are assumed to vary in a continuous and discrete range. The asymptotic behavior of these matrices is obtained imposing some conditions on the weight functions. Using Bierens (1987) and Andersen *et al.* (1983) results, a generalized eigenvalues problem is solved.

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1 Introduction and preliminaries

This paper proposes a solution of a generalized eigenvalues for fractional integrated process. The eigenvalue problem is solved by considering a combination of two random matrices constructed by taking into account the stationarity properties of the differences of a fractional p-variate integrated process. The random matrices are defined using weight functions and the difference orders are assumed to vary in a continuous and discrete range. The continuous case is

general since it takes into account the whole set of information. A discretization of the continuous case based on the set of the rational number $\mathbf{Q}$, dense in $\mathbf{R}$, is also provided and not irrelevant number of difference orders is considered. The asymptotic behavior of these random matrices are then obtained and a solution of a generalized eigenvalues is given.
The paper is organized as follows. Section 2 presents the data generating process. In Section 3 the convergence of random matrices is studied. Last Section concludes.

2 Data generating process

In this section the data generating process is described. We consider a p-variate fractional, non explosive, non stationary integrated process Y_t satisfying the following definition.

Definition 2.1 *Given $p \in \mathbf{N}$, a p-variate time series $\{Y_t\}$ is a fractional integrated process with fractional degree of integration $1/2 < d \leq 1$ if*

$$Y_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j} \qquad \text{with } c_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \tag{1}$$

where $\{\epsilon_t\}_{t>0}$ is an i.i.d. p-variate vector sequence with zero mean. We denote $Y_t \sim I(d)$.

The following Assumptions are required.

Assumption 2.2 *There exists a p-squared matrix of lag polynomials in the lag operator L such that*

$$\epsilon_t = \sum_{j=0}^{\infty} C_j v_{t-j} =: C(L)v_t, \quad t = 1, \dots, n, \tag{2}$$

where v_t is a p-variate stationary white noise process.

Assumption 2.3 *The process ϵ_t can be written as in (2), where v_t are i.i.d. zero-mean p-variate gaussian variables with variance equals to the identity matrix of order p, I_p, and there exist $C_1(L)$ and $C_2(L)$ p-squared matrices of lag polynomials in the lag operator L such that all the roots of $\det C_1(L)$ are outside the complex unit circle and $C(L) = C_1(L)^{-1}C_2(L)$.*

The lag polynomial $C(L) - C(1)$ attains value zero at $L = 1$ with fractional algebraic multiplicity equals to d. Thus there exists a lag fractional polynomial

$$D(L) = \sum_{k=0}^{\infty} D_k L^{\zeta_k}, \qquad D_k,\ \zeta_k \in \mathbf{R},\ \forall\, k = 1, \dots, +\infty,$$

such that $C(L) - C(1) = (1 - L)^d D(L)$ and ζ_k is increasing.
Therefore, we have

$$\epsilon_t = C(L)v_t = C(1)v_t + [C(L) - C(1)]v_t = C(1)v_t + D(L)(1 - L)^d v_t. \quad (3)$$

Let us define $w_t := D(L)v_t$. Then substituting w_t into (3), we obtain

$$\epsilon_t = C(1)v_t + (1 - L)^d w_t. \quad (4)$$

(4) implies that, given $Y_t \sim I(d)$, we can write recursively

$$Y_t = Y_0 + (1 - L)^{2-d} w_t - (1 - L)^{1-d} w_0 + C(1) \cdot (1 - L)^{1-d} \sum_{j=1}^{t} v_j. \quad (5)$$

Assumption 2.4 *Let us consider R_r the matrix of the eigenvectors of $C(1)C(1)^T$ corresponding to the r zero eigenvalues. Then the matrix $R_r^T D(1)D(1)^T R_r$ is nonsingular.*

Assumption 2.4 implies that Y_t cannot be integrated of order $\bar{d}$, with $\bar{d} > d$. In fact, if there exists $\bar{d} > d$ such that $Y_t \sim I(\bar{d})$, then the lag polynomial $D(L)$ admits a unit root with algebraic multiplicity $\bar{d} - d$, and so $D(1)$ is singular. Therefore $R_r^T D(1)D(1)^T R_r$ is singular, and Assumption 2.4 does not hold.

3 Convergence of a pair of random matrices and their generalized eigenvalue

In this section two random matrices which takes into account the stationary and nonstationary part of the data generating process are constructed using the α-th differences of Y_t. Depending on the choice of α, ΔY_t can be stationary or non stationary:

- if $\alpha < d - 1/2$, then $\Delta^\alpha Y_t$ is nonstationary;
- if $d - 1/2 \leq \alpha \leq d + 1/2$, then $\Delta^\alpha Y_t$ is stationary.

The difference orders are assumed to vary in a continuous and discrete range.

3.1 The continuous case

In this section the entire set of the admissible differences of Y_t is considered. Fixed $\alpha \in (-\infty, d + 1/2]$, the α-th difference of the process Y_t is opportunely weighted by some functions depending on α. Then, all these terms are aggregated by integrating on α. The random matrices are assumed to be dependent

on an integer number $m \geq p$.
Let us fix $k = 1, \ldots, m$, and define the functions

$$F_k : [0,1] \to \mathbf{R}; \tag{6}$$

$$G_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad \alpha \in (-\infty, d-1/2);$$
$$H_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad \alpha \in [d-1/2, d+1/2].$$

Moreover, we consider a couple of sequences:

$$\{\phi_1(n,\alpha)\} \subseteq \mathbf{R}, \qquad \alpha \in (-\infty, d-1/2),\ n \in \mathbf{N};$$

$$\{\phi_2(n,\alpha)\} \subseteq \mathbf{R}, \qquad \alpha \in [d-1/2, d+1/2],\ n \in \mathbf{N}.$$

By using the previous definitions of functions and sequences, two random matrices are constructed:

$$A_m := \sum_{k=1}^{m} a_{n,k} a_{n,k}^T; \tag{7}$$

$$B_m := \sum_{k=1}^{m} b_{n,k} b_{n,k}^T, \tag{8}$$

where

$$a_{n,k} := \frac{M_n^{nonst}/\sqrt{n}}{\sqrt{\int\int F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}}; \tag{9}$$

$$b_{n,k} := \frac{\sqrt{n} M_n^{st}}{\sqrt{\int F_k(x)^2 \mathrm{d}x}}, \tag{10}$$

and

$$M_n^{nonst} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^{d-1}Y_t + \int_{-\infty}^{d-1/2} \Big[\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha; \tag{11}$$

$$M_n^{st} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^{d}Y_t + \int_{d-1/2}^{d+1/2} \Big[\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha, \tag{12}$$

The main result of this work is obtained by an asymptotic analysis of a particular combination of the random matrices. These random matrices are defined on the basis of the weight functions F's, G's and H's.[1] Two definitions are proposed in order to describe three functional classes in which the weight functions lies.

Definition 3.1 *Let us fix $m \in \mathbf{N}$, $k = 1, \ldots m$.*

[1]Differently from our approach, that follows Bierens' one, one could adopt Breitung (2002), that is based on functionals of the partial sums of the process.

(i) There exists a function $\theta_1 : (-\infty, d-1/2) \to \mathbf{R}$ *and* $\phi_1 : \mathbf{N} \times (-\infty, d-1/2) \to \mathbf{R}$ *such that*

$$\alpha \mapsto \theta_1(\alpha), \qquad \theta_1 \in L^1(-\infty, d-1/2)$$

and

$$\Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| \le \theta_1(\alpha), \qquad \forall\, \alpha \in (-\infty, d-1/2),\, \forall\, n \in \mathbf{N}.$$

(ii) For each $\alpha \in (-\infty, d-1/2)$, *it results*

$$\lim_{n\to+\infty} \sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n) = 0; \tag{13}$$

(iii) There exists a function $\theta_2 : [d-1/2, d+1/2] \to \mathbf{R}$ *and* $\phi_2 : \mathbf{N} \times [d-1/2, d+1/2] \to \mathbf{R}$ *such that*

$$\alpha \mapsto \theta_2(\alpha), \qquad \theta_2 \in L^1[d-1/2, d+1/2]$$

and

$$\Big|n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Big| \le \theta_2(\alpha), \qquad \forall\, \alpha \in [d-1/2, d+1/2],\, \forall\, n \in \mathbf{N}.$$

(iv) For each $\alpha \in [d-1/2, d+1/2]$, *it results*

$$\lim_{n\to+\infty} n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n) = 0; \tag{14}$$

The functional classes $\mathcal{G}_{m,\alpha}$ *and* $\mathcal{H}_{m,\alpha}$ *are*

$$\mathcal{G}_{m,\alpha} := \Big\{G_{k,\alpha} : [0,1] \to \mathbf{R} \,|\, (i),\ (ii) \text{ hold}\Big\}. \tag{15}$$

$$\mathcal{H}_{m,\alpha} := \Big\{H_{k,\alpha} : [0,1] \to \mathbf{R},\, |\, (iii),\ (iv) \text{ hold}\Big\}. \tag{16}$$

Definition 3.2 *Consider the following conditions:*

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} F_k(t/n) = o(1) \qquad \text{as}\, n \to +\infty; \tag{17}$$

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} tF_k(t/n) = o(1) \qquad \text{as}\, n \to +\infty; \tag{18}$$

$$\int\!\!\int F_i(x)F_j(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y = 0, \qquad i,j \in \mathbf{N},\ i \ne j; \tag{19}$$

$$\int F_i(x)\int_0^x F_j(y)\mathrm{d}x\mathrm{d}y = 0, \qquad i,j\in\mathbf{N},\ i\neq j; \tag{20}$$

$$\int F_i(x)F_j(x)\mathrm{d}x = 0, \qquad i,j\in\mathbf{N},\ i\neq j. \tag{21}$$

The functional class $\mathcal{F}_m$ is

$$\mathcal{F}_m := \Big\{F_k : [0,1]\to\mathbf{R},\ F_k\in C^1(0,1)\,|\,(17)\ -\ (21)\text{ hold},\ k=1\ldots,m\Big\}. \tag{22}$$

Bierens (1997) shows that the functional class $\mathcal{F}_m$ is not empty. He points out that if one define

$$\bar{F}_k : \mathbf{R}\to\mathbf{R}$$

such that

$$\bar{F}_k(x) = cos(2k\pi x), \tag{23}$$

and taking the restriction

$$F_k := \bar{F}_k|_{[0,1]},$$

then $F_k\in\mathcal{F}_m$.
Moreover $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are not empty and contain a huge number of elements. Therefore it is not restrictive to assume that the weights G's and H's belong to these spaces. Some properties of $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are showed in order to evidence the big cardinality of these spaces.

Proposition 3.3 *$\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are closed with respect to the linear combination.*

Proof.
We provide only the proof for the functional space $\mathcal{G}_{m,\alpha}$, being the one for $\mathcal{H}_{m,\alpha}$ analogous.
Given $k=1,\ldots,m$ and $\alpha\in(-\infty,d-1/2)$, let us consider

$$G^j_{k,\alpha} : [0,1]\to\mathbf{R}, \qquad j=1,\ldots,N,\ N\in\mathbf{N}$$

such that $G^j_{k,\alpha}\in\mathcal{G}_{m,\alpha}$.
Define

$$G_{k,\alpha} := \sum_{j=1}^{N} q_j G^j_{k,\alpha}, \qquad q_j\in\mathbf{R},\ \forall\, j=1,\ldots,N.$$

Conditions (i) and (ii) of Definition 3.1 can be rewritten by indexing with j the sequence ϕ_1 and the function θ_1, for $j=1,\ldots,N$, where $N\in\mathbf{N}$,

(i) There exists a function $\theta^j_1 : (-\infty,d-1/2)\to\mathbf{R}$ and $\phi^j_1 : \mathbf{N}\times(-\infty,d-1/2)\to\mathbf{R}$ such that

$$\alpha\mapsto\theta^j_1(\alpha), \qquad \theta_1\in L^1(-\infty,d-1/2)$$

and

$$\Big|\sqrt{n}\phi^j_1(n,\alpha)\sum_{t=1}^{n} G^j_{k,\alpha}(t/n)\Big| \le \theta^j_1(\alpha), \qquad \forall\,\alpha\in(-\infty,d-1/2).$$

(ii) For each $\alpha \in (-\infty, d-1/2)$, it results

$$\lim_{n\to+\infty} \sqrt{n}\phi_1^j(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}^j(t/n) = 0; \tag{24}$$

Condition (ii) is fulfilled for $G_{k,\alpha}$. In fact, by choosing ϕ_1 such that

$$\phi_1(n,\alpha) = o(\phi_1^j(n,\alpha)), \qquad \forall\, j = 1,\dots,N, \text{ as } n \to +\infty,$$

then

$$\lim_{n\to+\infty} \sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n) = \lim_{n\to+\infty} \sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n}\Big[\sum_{j=1}^{N} q_j G_{k,\alpha}^j(t/n)\Big] =$$

$$= \sum_{j=1}^{N} q_j\Big[\lim_{n\to+\infty} \sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}^j(t/n)\Big] = 0.$$

Furthermore, by using ϕ_1 as above it results

$$\Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| = \Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n}\Big[\sum_{j=1}^{N} q_j G_{k,\alpha}^j(t/n)\Big]\Big| \leq$$

$$\leq \sum_{j=1}^{N} |q_j|\Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}^j(t/n)\Big| \leq \sum_{j=1}^{N} |q_j|\theta_1^j(\alpha).$$

Since L^1 is closed with respect to the linear combinations, then

$$\sum_{j=1}^{N} |q_j|\theta_1^j(\alpha) \in L^1(-\infty, d-1/2),$$

and condition (i) holds.
As a consequence of the previous result, the following topological property of $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ can be obtained.

Corollary 3.4 *$\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$ are convex sets.*

Proof.
Only the proof for the functional space $\mathcal{G}_{m,\alpha}$ is provided.
For $k = 1,\dots,m$ and $\alpha \in (-\infty, d-1/2)$, we define a couple of functions

$$G_{k,\alpha}^j : [0,1] \to \mathbf{R}, \qquad j = 1, 2.$$

such that $G_{k,\alpha}^j \in \mathcal{G}_{m,\alpha}$.
Define $q_1, q_2 \in [0,1]$ such that $q_1 + q_2 = 1$, and the convex linear combination function

$$G_{k,\alpha} := q_1 G_{k,\alpha}^1 + q_2 G_{k,\alpha}^2.$$

Since Proposition 3.3 implies $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$, we have the thesis.
Now we wish to show a sufficient condition to characterize $\mathcal{G}_{m,\alpha}$ and $\mathcal{H}_{m,\alpha}$.

Theorem 3.5 *Fix* $\alpha \in (-\infty, d+1/2]$ *and* $k = 1, \dots, m$. *Define* $\varrho_{k,\alpha} : [0,1] \to \mathbf{R}$, *and assume that there exists* $M > 0$ *such that*

$$|\varrho_{k,\alpha}(x)| \leq M, \qquad \forall\, x \in [0,1].$$

Then:

- $\varrho_{k,\alpha}$ *belongs to* $\mathcal{G}_{m,\alpha}$ *if* $\alpha \in (-\infty, d-1/2)$*;*
- $\varrho_{k,\alpha}$ *belongs to* $\mathcal{H}_{m,\alpha}$ *if* $\alpha \in [d-1/2, d+1/2]$.

Proof.
We denote ϱ as H and G when $\alpha \in [d-1/2, d+1/2]$ and $\alpha \in (-\infty, d-1/2)$ respectively.
Standard analysis provides that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{t=1}^{n} G_{k,\alpha}(t/n) = \int_0^1 G_{k,\alpha}(x)\mathrm{d}x. \tag{25}$$

Fixed $\alpha \in (-\infty, d-1/2]$, the sequence $\{\psi_1(n,\alpha)\}_{n\in\mathbf{N}}$ is defined such that

$$\phi_1(n,\alpha) = \frac{1}{n^{3/2}} \cdot \psi_1(n,\alpha), \tag{26}$$

$$\lim_{n\to+\infty} \psi_1(n,\alpha) = 0. \tag{27}$$

Moreover we assume that $\psi_1(n,\alpha) \in L^1(-\infty, d-1/2]$, $\forall\, n \in \mathbf{N}$.
By (26), we have

$$\Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| = \Big|\psi_1(n,\alpha)\frac{1}{n}\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| =$$

$$= |\psi_1(n,\alpha)|\Big|\frac{1}{n}\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| \leq |\psi_1(n,\alpha)|\Big|\frac{1}{n}\cdot n \cdot M\Big| = M|\psi_1(n,\alpha)|.$$

By assuming $\theta_1(\alpha) = |\psi_1(n,\alpha)|$, condition (i) of Definition 3.1 holds.
Furthermore, it results

$$0 \leq \Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| \leq M|\psi_1(n,\alpha)|.$$

Using (27) and a comparison principle, we obtain

$$\lim_{n\to+\infty} \sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n) = 0.$$

(ii) of Definition 3.1 holds, and $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$.
Now, fixed $\alpha \in [d-1/2, d+1/2]$, a sequence $\{\psi_2(n,\alpha)\}_{n\in\mathbf{N}}$ is defined such that

$$\phi_2(n,\alpha) = \frac{1}{n^2} \cdot \psi_2(n,\alpha), \tag{28}$$

$$\lim_{n\to+\infty} \psi_2(n,\alpha) = 0. \tag{29}$$

Furthermore, we assume that $\psi_2(n,\alpha) \in L^1[d-1/2, d+1/2]$, $\forall\, n \in \mathbf{N}$.
By (28), some algebra gives

$$\Big|n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Big| = \Big|\psi_2(n,\alpha)\frac{1}{n}\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Big| \leq$$

$$\leq |\psi_2(n,\alpha)|\Big|\frac{1}{n}\sum_{t=1}^{n} H_{k,\alpha}(t/n)\Big| \leq |\psi_2(n,\alpha)|\Big|\frac{1}{n}\cdot n \cdot M\Big| = M|\psi_2(n,\alpha)|.$$

By assuming $\theta_2 = |\psi_2|$, condition (iii) of Definition 3.1 holds.
Furthermore (29) and a comparison principle give

$$\lim_{n\to+\infty} n\phi_2(n,\alpha)\sum_{t=1}^{n} H_{k,\alpha}(t/n) = 0.$$

Then, (iv) of Definition 3.1 is satisfied, and so $H_{k,\alpha} \in \mathcal{H}_{m,\alpha}$.

3.1.1 Asymptotic results

This section presents the main asymptotic results. Two random vectors dependent on the weight functions F's are defined as follows:

$$\Psi_k := \frac{\int_0^1 F_k(x)W(x)\mathrm{d}x}{\sqrt{\int_0^1\int_0^1 F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}},$$

$$\Phi_k := \frac{F_k(1)W(1) - \int_0^1 f_k(x)W(x)\mathrm{d}x}{\int_0^1 F_k(x)^2\mathrm{d}x},$$

where f_k is the derivative of F_k.
Moreover, the following p-variate standard normally distributed random vectors is considered:

$$\Psi_k^* := \left(R_{p-r}^T C(1)C(1)^T R_{p-r}\right)^{\frac{1}{2}} R_{p-r}^T C(1)\Psi_k,$$

$$\Phi_k^* := \left(R_{p-r}^T C(1)C(1)^T R_{p-r}\right)^{\frac{1}{2}} R_{p-r}^T C(1)\Phi_k,$$

$$\Phi_k^{**} := (R_r^T D(1)D(1)^T R_r)^{-\frac{1}{2}} R_r^T D(1)\Phi_k,$$

and we construct the matrix $V_{r,m}$ as

$$V_{r,m} := (R_r^T D(1)D(1)^T R_r)^{\frac{1}{2}} V_{r,m}^* (R_r^T D(1)D(1)^T R_r)^{\frac{1}{2}},$$

with

$$V_{r,m}^* = \Big(\sum_{k=1}^m \gamma_k^2 \Phi_k^{**}\Phi_k^{**T}\Big) - \Big(\sum_{k=1}^m \gamma_k \Phi_k^{**}\Psi_k^{*T}\Big)\Big(\sum_{k=1}^m \Psi_k^{*}\Psi_k^{*T}\Big)^{-1}\Big(\sum_{k=1}^m \gamma_k \Psi_k^{*}\Phi_k^{**T}\Big),$$

where

$$\gamma_k = \frac{\sqrt{\int_0^1 F_k^2(x)\mathrm{d}x}}{\sqrt{\int_0^1\int_0^1 F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}}.$$

The following result summarizes the eigenvalue problem and provide a non-parametric solution for it.

Theorem 3.6 *Assume that $F_k \in \mathcal{F}_m$, $G_{k,\alpha} \in \mathcal{G}_{m,\alpha}$ and $H_{k,\alpha} \in \mathcal{H}_{m,\alpha}$. If Assumptions 2.2, 2.3 and 2.4 are true, then:*

(I) suppose that $\hat{\lambda}_{1,m} \geq \ldots \geq \hat{\lambda}_{p,m}$ are the ordered solutions of the generalized eigenvalue problem

$$\det\Big[A_m - \lambda(B_m + n^{-2}A_m^{-1})\Big] = 0, \tag{30}$$

and $\lambda_{1,m} \geq \ldots \geq \lambda_{p-r,m}$ the ordered solutions of

$$\det\Big[\sum_{k=1}^m \Psi_k^*\Psi_k^{*T} - \lambda \sum_{k=1}^m \Phi_k^*\Phi_k^{*T}\Big] = 0. \tag{31}$$

Then we have the following convergence in distribution

$$(\hat{\lambda}_{1,m}, \ldots, \hat{\lambda}_{p,m}) \to (\lambda_{1,m}, \ldots, \lambda_{p-r,m}, 0, \ldots, 0);$$

(II) let us consider $\lambda_{1,m}^* \geq \ldots \geq \lambda_{r,m}^*$ the ordered solutions of the generalized eigenvalue problem

$$\det\Big[V_{r,m}^* - \lambda(R_r^T D(1)D(1)^T R_r)^{-1}\Big] = 0. \tag{32}$$

Then the following convergence in distribution holds

$$n^2(\hat{\lambda}_{p-r+1,m}, \ldots, \hat{\lambda}_{p,m}) \to (\lambda_{1,m}^{*2}, \ldots, \lambda_{r,m}^{*2}).$$

Proof.
Due to Lemmas 1, 2 and 4 (Bierens, 1997), it is sufficient to study the asymptotic behavior of $\sqrt{n}M_n^{nonst}$ and nM_n^{st}.
We have

$$\lim_{n\to+\infty}\sqrt{n}M_n^{nonst} = \lim_{n\to+\infty}\frac{1}{\sqrt{n}}\sum_{t=1}^n F_k(t/n)\Delta^{d-1}Y_t +$$

$$+\lim_{n\to+\infty}\int_{-\infty}^{d-1/2}\Big[\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n}G_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha =: L_1+L_2$$

By Bierens (1997), we have to show that $L_2=0$.
Since $G_{k,\alpha}\in\mathcal{G}_{m,\alpha}$, then the existence of the function θ_1 (Definition 3.1-(i)) guarantees, that the Lebesgue Theorem on the dominate convergence holds. Therefore we can write

$$L_2=\int_{-\infty}^{d-1/2}\lim_{n\to+\infty}\Big[\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n}G_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha.$$

Hence, the fractional lag-difference process $\Delta^{\alpha}Y_t$ is well defined. Definition 3.1-(ii) assures that $L_2=0$, and the first part of the proof is complete.
Now,

$$\lim_{n\to+\infty}nM_n^{st}=\lim_{n\to+\infty}\sum_{t=1}^{n}F_k(t/n)\Delta^{d}Y_t+$$

$$+\lim_{n\to+\infty}\int_{d-1/2}^{d+1/2}\Big[n\phi_2(n,\alpha)\sum_{t=1}^{n}H_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha =: L_3+L_4.$$

By Lemmas 1, 2 and 4 (Bierens, 1997), we need $L_4=0$.
Since $H_{k,\alpha}\in\mathcal{H}_{m,\alpha}$, the existence of the function θ_2 (Definition 3.1-(ii)) implies that the hypotheses of the Lebesgue's Theorem on the dominate convergence are fulfilled. Thus we have

$$L_4=\int_{d-1/2}^{d+1/2}\lim_{n\to+\infty}\Big[n\phi_2(n,\alpha)\sum_{t=1}^{n}H_{k,\alpha}(t/n)\Delta^{\alpha}Y_t\Big]\mathrm{d}\alpha.$$

The condition (ii) of the Definition 3.1 assures that $L_4=0$.
The result is completely proved.

3.2 The discrete case

The analysis carried out in the previous subsection deals with all differences of the fractional integrated process Y_t. This section provides a discretization of the continuous case using the M_n's described by (11) and (12) and it is made with respect to the difference order, named α, of the process Y_t. The discrete set of rational numbers $\mathbf{Q}$, that is infinite, countable and dense in $\mathbf{R}$ is used. The density property of $\mathbf{Q}$ in $\mathbf{R}$ permits to have a set of information not too restrictive, maintaining the model in line with the general features of the continuous case.
Fix $k=1,\ldots,m$, where $m\in\mathbf{N}$. Let us consider F_k as in (6), and

$$\tilde{G}_{k,\alpha}:[0,1]\to\mathbf{R},\qquad \alpha\in(-\infty,d-1/2);$$

$$\tilde{H}_{k,\alpha} : [0,1] \to \mathbf{R}, \qquad \alpha \in [d-1/2, d+1/2].$$

Moreover, we define two sequences:

$$\{\zeta_1(n,\alpha)\} \subseteq \mathbf{R}, \qquad \alpha \in (-\infty, d-1/2);$$

$$\{\zeta_2(n,\alpha)\} \subseteq \mathbf{R}, \qquad \alpha \in [d-1/2, d+1/2].$$

The terms M_n's defined in (11) and (12) can be rewritten as

$$M_n^{nonst} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^{d-1}Y_t + \sum_{j=1}^{+\infty}\Big[\zeta_1(n,\alpha_{1,j})\sum_{t=1}^{n}\tilde{G}_{k,\alpha_{1,j}}(t/n)\Delta^{\alpha_{1,j}}Y_t\Big], \quad (33)$$

and

$$M_n^{st} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^{d}Y_t + \sum_{j=1}^{+\infty}\Big[\zeta_2(n,\alpha_{2,j})\sum_{t=1}^{n}\tilde{H}_{k,\alpha_{2,j}}(t/n)\Delta^{\alpha_{2,j}}Y_t\Big], \quad (34)$$

where

$$\{\alpha_{1,j}\}_{j\in\mathbf{N}} \equiv \mathbf{Q} \cap (-\infty, d-1/2)$$

and

$$\{\alpha_{2,j}\}_{j\in\mathbf{N}} \equiv \mathbf{Q} \cap [d-1/2, d+1/2].$$

A discrete version of the functional classes $\mathcal{G}$'s and $\mathcal{H}$'s is needed. We rewrite Definition 3.1 in the discrete case as follows:

Definition 3.7 *Let us fix $m \in \mathbf{N}$, $k = 1, \dots m$.*

(i)' There exists a function $\theta_1 : (-\infty, d-1/2) \to \mathbf{R}$ and $\phi_1 : \mathbf{N} \times (-\infty, d-1/2) \to \mathbf{R}$ such that

$$\alpha \mapsto \theta_1(\alpha), \qquad \theta_1 \in l^1(-\infty, d-1/2)$$

and

$$\Big|\sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n)\Big| \leq \theta_1(\alpha), \qquad \forall\, \alpha \in (-\infty, d-1/2).$$

(ii)' For each $\alpha \in (-\infty, d-1/2)$, it results

$$\lim_{n\to+\infty} \sqrt{n}\phi_1(n,\alpha)\sum_{t=1}^{n} G_{k,\alpha}(t/n) = 0; \quad (35)$$

(iii)' There exists a function $\theta_2 : [d-1/2, d+1/2] \to \mathbf{R}$ and $\phi_2 : \mathbf{N} \times [d-1/2, d+1/2] \to \mathbf{R}$ such that

$$\alpha \mapsto \theta_2(\alpha), \qquad \theta_2 \in l^1[d-1/2, d+1/2]$$

and

$$\left| n\phi_2(n,\alpha) \sum_{t=1}^{n} H_{k,\alpha}(t/n) \right| \leq \theta_2(\alpha), \qquad \forall\, \alpha \in [d-1/2, d+1/2].$$

(iv)' For each $\alpha \in [d-1/2, d+1/2]$, it results

$$\lim_{n\to+\infty} n\phi_2(n,\alpha) \sum_{t=1}^{n} H_{k,\alpha}(t/n) = 0; \tag{36}$$

The functional classes $\mathcal{G}^d_{m,\alpha}$ and $\mathcal{H}^d_{m,\alpha}$ are

$$\mathcal{G}^d_{m,\alpha} := \Big\{ G_{k,\alpha} : [0,1] \to \mathbf{R} \,|\, (i),\, (ii) \text{ hold} \Big\}. \tag{37}$$

$$\mathcal{H}^d_{m,\alpha} := \Big\{ H_{k,\alpha} : [0,1] \to \mathbf{R},\, |\, (iii),\, (iv) \text{ hold} \Big\}. \tag{38}$$

The main properties of the discrete functional spaces $\mathcal{G}^d_{m,\alpha}$ and $\mathcal{H}^d_{m,\alpha}$ of Definition 3.7 are the same of the continuous case. They are summarize in the following three results.

Proposition 3.8 *$\mathcal{G}^d_{m,\alpha}$ and $\mathcal{H}^d_{m,\alpha}$ are closed with respect to the linear combination.*

Corollary 3.9 *$\mathcal{G}^d_{m,\alpha}$ and $\mathcal{H}^d_{m,\alpha}$ are convex sets.*

Theorem 3.10 *Fix $\alpha \in (-\infty, d+1/2]$ and $k = 1, \ldots, m$. Define $\varrho_{k,\alpha} : [0,1] \to \mathbf{R}$, and assume that there exists $M > 0$ such that*

$$|\varrho_{k,\alpha}(x)| \leq M, \qquad \forall\, x \in [0,1].$$

Then:

- *$\varrho_{k,\alpha}$ belongs to $\mathcal{G}^d_{m,\alpha}$ if $\alpha \in (-\infty, d-1/2)$;*
- *$\varrho_{k,\alpha}$ belongs to $\mathcal{H}^d_{m,\alpha}$ if $\alpha \in [d-1/2, d+1/2]$.*

The proofs are analogous to the ones of Proposition 3.3, Corollary 3.4 and Theorem 3.5 and are omitted.
Theorem 3.6 is translated in the discrete case.

Theorem 3.11 *Assume that* $F_k \in \mathcal{F}_m$, $\tilde{G}_{k,\alpha_j} \in \mathcal{G}^d_{m,\alpha_j}$ *and* $\tilde{H}_{k,\alpha_j} \in \mathcal{H}^d_{m,\alpha_j}$ *and Assumptions 2.2, 2.3 and 2.4 are true. Then the thesis of Theorem 3.6 holds.*

Proof.
Using the proof of Theorem 3.6, we have just to prove that

$$\tilde{L}_2 := \lim_{n\to+\infty}\sum_{j=1}^{+\infty}\Big[\sqrt{n}\zeta_1(n,\alpha_{1,j})\sum_{t=1}^{n}\tilde{G}_{k,\alpha_{1,j}}(t/n)\Delta^{\alpha_{1,j}}Y_t\Big] = 0, \tag{39}$$

and

$$\tilde{L}_4 := \lim_{n\to+\infty}\sum_{j=1}^{+\infty}\Big[n\zeta_2(n,\alpha_{2,j})\sum_{t=1}^{n}\tilde{H}_{k,\alpha_{2,j}}(t/n)\Delta^{\alpha_{2,j}}Y_t\Big] = 0. \tag{40}$$

Let us assume the existence of a couple of functions

$$\gamma_1 : \mathbf{N} \to \mathbf{R}, \qquad \gamma_2 : (-\infty, d-1/2) \to \mathbf{R},$$

such that $\zeta_1(n,\alpha_{1,j}) = \gamma_1(n)\cdot\gamma_2(\alpha_{1,j})$. Consider the partial sums of the series in (39) as

$$\tilde{L}_2^\mu := \lim_{n\to+\infty}\gamma_1(n)\sqrt{n}\sum_{j=1}^{\mu}\Big[\gamma_2(\alpha_{1,j})\sum_{t=1}^{n}\tilde{G}_{k,\alpha_{1,j}}(t/n)\Delta^{\alpha_{1,j}}Y_t\Big] = 0. \tag{41}$$

There exists $K > 0$ such that

$$\Big|\gamma_2(\alpha_{1,j})\sum_{t=1}^{n}\tilde{G}_{k,\alpha_{1,j}}(t/n)\Delta^{\alpha_{1,j}}Y_t\Big| \leq$$

$$\leq K\cdot\Big|\gamma_2(\alpha_{1,j})\sum_{t=1}^{n}\tilde{G}_{k,\alpha_{1,j}}(t/n)\Big|. \tag{42}$$

By using Definition 3.7-$(i)'$, then there exist a nonnegative function θ_1 and $M > 0$ such that

$$(42) \leq \theta_1(\alpha_{1,j}), \qquad \sum_{j=1}^{+\infty}\theta_1(\alpha_{1,j}) = M < +\infty.$$

Hence, by assuming that $\gamma_1 \sim o(n^\eta)$ with $\eta > 1/2$, as $n \to +\infty$, we obtain

$$0 \leq |\tilde{L}_2| \leq M\cdot\Big|\lim_{n\to+\infty}\gamma_1(n)\sqrt{n}\Big| = 0. \tag{43}$$

Analogously, by using the conditions $(iii)'$ and $(iv)'$ in Definition 3.7, it is easy to show that (40) holds.
The proposition is completely proved.

4 Conclusions

In this paper we solve a generalized eigenvalues problem for fractional integrated process by constructing two random matrices. Such matrices are constructed by taking into account the stationarity properties of the differences of a fractional p-variate integrated process. The random matrices are defined by some weight functions and the difference orders are assumed to vary in a continuous and discrete range. The asymptotic behavior of the random matrices are thus obtained.

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