# Productivity and costs for firms in presence of technology renewal processes

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**Summary.** Wide empirical analyses investigated size and growth rate distribution of business firms, providing a relevant empirical support to economic theory. We rely on such analyses and on studies on technology renewal costs and productivity, in order to draw sufficient conditions for the optimality of firms' profit with respect to the time. The relationships that hold among productivity, costs of renewal and growth rates of the companies at the optimal profit time are shown and suggestions for firms' policies are proposed.

**Keyword and Phrases:** Optimal profit model, Technological renewal, Aggregate productivity, Firm growth rate, Firm size.

**JEL Classification Numbers:** C68, O14, O47.

# 1 Introduction

One of the possible targets of a firm holder is to maximize his/her expected profit. Several strategies can be implemented to achieve such target. Firms' financial policies are strongly correlated to investments for technology renewal processes. Such investments are usually driven by the firm productivity, which depends on the growth rate and the size of the company.

This paper aims at analyzing the profit of a firm producing a single commodity. The analysis relies on firms' renewal and productivity cost theories, combined with the work of statistical physics research groups who discovered empirical regularities about the growth of the firms. The approach presented here thus, while maintaining the theoretical approach, is driven by empirical facts on size and growth rate leading to conclusions closer to the real estate of the business firms than the ones that can be drawn by pure theoretical models.

We use the technological renewal costs model presented in Chambers (2004). This approach differs from the pioneering work of Schumpeter (1939) about the innovation theory, because the introduction of a new technology is not considered as an exogenous shock leading the economic system to jump from a walrasian equilibrium to another.

Our model also differs from the approach followed in Caballero and Hammour (1994), because we do not consider responses to external demands.

The modeling of productivity takes into account the suggestions pointed out in Diewert et al. (2001), about the separation of effects due to productivity price changes and growth, and in Ahn (2001), who proposes a complete survey on empirical studies showing how productivity depends on companies' size and growth rate.

This paper starts reviewing the main features of the firms and the related empirical literature, and it continues introducing the model and drawing the relationships among costs and productivity parameters at the optimal profit time.

# 2 Main features of the firms and related literature

This section gives an overview of both theoretical and empirical studies reported by the literature on business firms, and outlines the approach used in the present paper.

## 2.1 Analysis of firms' distribution

This section reviews literature results on size of business firms, pointing out the analysis of the validity of Gibrat's law.

Many empirical studies on size and growth rate of firms discuss the validity of Gibrat's rule of proportionate growth (also called Gibrat's law), which states that the proportional change in the size of a company in an industry is the same for all companies irrespective of their original size (Gibrat, 1931). Depending on some assumptions on the growth rates, this statement implies that firms' sizes are log-normally distributed (Gibrat's law in strong form) or at least skewed (Gibrat's law in weak form). In both cases, a small number of large firms coexists alongside a large number of small firms. Thus, the availability of databases containing information on small firms becomes relevant for this analysis. Most data have been obtained by Census and COMPUSTAT. The former has more data on small firms than the latter. Empirical analyses that examine the firms' size distribution give different results also because of the choice of the measurement unit for the size of a firm, for the hypotheses tested and the data set used. Gibrat's law in a weak form is confirmed by the detection of right skewness in the distribution of firms' size independently on the adopted definition of firms' size (Amaral et al. 1997a, 1997b, Axtell 2001, Gaffeo et al. 2003, Stanley et al. 1996). Moreover, the skewness has been shown to be robust over time (Axtell 2001).

Gaffeo et al. (2003) extend the analysis of Axtell (2001) performing empirical analyses showing that firms are more evenly distributed during recessions than during expansions of markets. Therefore, their statistics add an economic meaning to the skewness proving its relevance as an indicator of economic cycles.

The investigation of the companies distribution plays an important role in order to provide estimates of the reaction of the market to external shocks: as an example, Delli Gatti et al. (in press) show that, if data are log-normally distributed, shocks are absorbed, whilst shocks lead to strong oscillations in the case of Pareto distribution.

Although previous studies (Hart and Oulton 1996, 1997) do not completely reject the log-normal hypothesis, or discover it to be valid on limited time data (Stanley et al. 1995) or categories (Amaral et al. 1997), more recent papers (Axtell 2001, Gaffeo et al. 2003) propose the power law distribution instead of the log-normal one. This result points out the limit of the validity of the strong form of Gibrat's law, reducing it to an approximate description. The detection of the power law distribution is consistent with the weak form of Gibrat's law. As an example in Simon's model (Delli Gatti et al. in press), Gibrat's law is combined with an entry process to obtain a Levy distribution for firms' size. Under particular hypothesis on the growth rate, Gibrat's law and the power law have been shown to be both acceptable description for the distribution of firms bigger than a fixed threshold (Fujiwara et al. 2000). This property is not valid in general (Kertesz et al. 2003), but the behavior of biggest companies plays a crucial role in order to describe the macroeconomic situation. The power law behavior seems to be common also to parameters involving the economies of the G7 countries (Gaffeo et al. 2003). The results reported in Di Guilmi et al. (2003) can be interpreted as the existence of a significant range of the world GDP distribution, where countries share a common, size-independent average growth rate.

Therefore, empirical analyses show the log-normal distribution to appear only on particular data, whilst the power law behavior is much more widely spread, and thus robust to the industry sector data choice.

## 2.2 Relationship between growth rate and size

The limits of Gibrat's law on the firms' size have been pointed out allowing the modeling of firms' sizes through a power law. It can be shown that the logarithm of a power-law distributed variable obeys a Laplace distribution; thus, the Laplace law for firms' growth rate represents a consequence of power laws for firms' size.

Sutton (2003) proposes a model that explains the power law relation between firms' size and the variance of growth rates.

Stanley et al. (1996) show that the variance $\sigma(s)$ of the growth rate decreases as the size of the firm $s$ increases. Moreover, such decay can be described by a power law with an exponent $\beta$ close to 0.20 (Amaral et al. 1997a, De Fabritiis et al. 2003), that opens the way to further universality assumptions. In the framework that we are going to set up, we suppose that the growth rate is a deterministic parameter.

Despite this distinction between our case and the stochastic growth rate models, we assume that the number of companies with fixed size can be described functionally by the Laplace's law depending on the growth rate, in accordance with the approach of Stanley et al. (1996).

## 2.3 Productivity and technology

The analyses explained in the literature involve several different measures for innovation technology, and examine data about firms operating in different sectors of the economy and belonging to several different countries. Technology diffusion patterns are not the same across the surveyed technologies. The reaction to innovation technology also depends on countries (Sharpe, 1998, Milana and Zeli, 2002) and the causality is explored through linear or log regression models (McGuckin et al. 1996). Whilst the data are not homogeneous, the emerging results through all these papers are quite homogeneous: there exists a positive correlation between the adoption of a new technology and the productivity, and establishments adopting advanced technology exhibit high productivity levels (McGuckin et al. 1996).

However, the positive correlations could be a consequence either of the independent effects of innovation on performance, or of the contributions of good managers tending to adopt the best practices. Therefore, the relationship between productivity growth and renewal processes can't be thought as completely explored. The comprehension of such cause-effect phenomena could provide very important information on social policies. Several authors focus on the key role played by improving education and training policies, representing sometimes a preferred strategy rather than research support. McGuckin et al. (1996), and Milana and Zeli, (2002) have developed this aspect in the case of dominant sources of enhanced performances strongly related to good management and skilled workforce. Moreover, the age of technology is not connected to the age of the firm adopting it: new firms could adopt an old toolkit because of budget constraints.

Information technology plays a crucial role in the economic system. Studies on companies' productivity performances reveal that the industries with the largest productivity acceleration in the 1990s have been at the same time the main producers and most intensive users of information technology.

On the basis of this analysis, we assume that the productivity function decays as the technological renewal process age increases, as it happens for information science technology. Such decay can be modeled by a power law.

## 2.4 Costs of renewal of technology

The basic level of technology usage in society is continuously growing. Also companies whose workforce improve their skills and expertise with the age, like law firms, and whose character do not seem to rely on sophisticated machinery, are obliged to go on with technological renewal processes. The above remark suggests that innovation spreads across firms and is positively connected both to their size and their growth rate.

Many authors analyze the relationship between technological improvements and renewal costs.

Balcer and Lippman (1984) investigate the profitability of investments in the creation rather than in the purchasing new technology depending on time. Substantially, a technology could become profitable as time passes without new technological advances.

The problem of profit maximization considering the costs of the renewal of technology has been exploited by Chambers and Kouvelis (2003), in the case of a single firm in a fixed sized market. The production costs are significantly affected by the acquisition of new production technologies and by the accumulation of the firm experience in the production of its product.

Chambers (2004) focuses on temporal persistence of technological renewal processes, decisions on multiple adoption over a long horizon, impact of the knowledge due to production on decisions and relative strategies, in the case of a company producing a single product. In agreement with this paper, we assume that the dynamic of the costs is decreasing as the age of technology increases.

Moreover, we suppose a dependence of technological costs and firms' growth rate, consistently with most of the part of the technological improvements cases.

## 3 The Model

In this section we set up an optimal profit model for firms producing a single commodity, and we analyze the related policies on productivity and costs management.

For each time $t$, we need to provide a function approximating the number of production units, based on $t - \tau$ aged technology.

For each fixed firms' size $s$, empirical studies reviewed in a previous section provide an analytical approximation for the number of companies, supporting the choice of a Laplace density.

We base on this functional approximation, in order to describe the number of units (for each fixed size) contributing to the production of one commodity. We remark that this approach is in agreement with the actual trend of firms' management, to distribute the work across self-organized firms' networks, more than merging into new companies. Moreover, we take into account the positive correlation between growth rate $r$ and age of technology $t-\tau$ and the evidence of a Laplace distribution with linear variance with respect to $s$ for growth rates. Therefore, we use an exponential function $f(r,\tau,t)$ defined as follows:

$$f(r,\tau,t)=\lambda \exp\Big[c|r|(t-\tau)\Big], \tag{1}$$

where $c<0$ and $\lambda>0$. Fixing the growth rate $r\in(-1,1)$, we obtain for $f$ an exponential decay with respect to $t-\tau$.

Amaral et al. (1997a-b) choose $\lambda \sim \sigma(s)^{-1}$ and $c \sim -\sigma(s)^{-1}$. Thus, for each fixed $s,t,\tau$, (1) is in accordance with the empirical results. For sake of simplicity, we drop the constants and we set $\lambda=\sigma(s)^{-1}$ and $c=-\sigma(s)^{-1}$. We remark that $f(r,\tau,t)$ is not a density distribution.

Therefore, by Amaral et al. (1997a), De Fabritiis et al. (2003), we can assume $\sigma(s) = s^{-\beta}$, and we obtain the relationship between the parameters in (1) and the size $s$ of the firm under consideration. We have

$$c = -s^{\beta}, \qquad \lambda = s^{\beta}. \tag{2}$$

Formulas in (2) allow to reach a very general description of the relationships that must hold at the optimal profit date, providing the financial sense of the best strategies on productivity and costs. At first the size will be assumed fixed, then a comparison analysis with respect to the change of the size will be performed.

In this framework, the output produced depends on the age $t - \tau$ of the last technological improvement. The productivity is strongly influenced by events happening at the observation date $t$: a monetary government policy, the aggregation of different companies with different sizes and growth rates, the fluctuations of the prices of the gold or the oil, a report on the expected forecasts for the agriculture products are just a few examples.

According to Ramsden and Kiss-Haypal (2000), we distinguish a term linked to the observation time $t$, a term associated to the growth rate of the firm $r$ and one linked to the distance of the company from the technological frontier $t - \tau$. Thus, we propose a productivity function $G$ splitted into three terms related to $t$, $t-\tau$ and $r$, as follows:

$$G(r,\tau,t) = g_1(r,t)g_2(r)g_3(t-\tau).$$

The aggregated output at time $t$ is given by

$$Q(r,t) = \int_0^t \Big[g_1(r,t)g_2(r)g_3(t-\tau)\cdot f(r,\tau,t)\Big]\mathrm{d}\tau. \tag{3}$$

We don't make further hypotheses on the behavior on the function $g_2$.

The function $g_3$ decays as the distance of the firm from the technological frontier increases. Such decay can be described using a power law.

Thus, in our model, there exists $\alpha_g \in (0,1)$ such that

$$g_3(t-\tau) = \alpha_g^{(t-\tau)}. \tag{4}$$

Therefore, (3) can be rewritten as

$$Q(r,t) = \lambda g_2(r)\int_0^t \Big[g_1(r,t)\alpha_g^{(t-\tau)}\cdot \exp\Big[c|r|(t-\tau)\Big]\Big]\mathrm{d}\tau. \tag{5}$$

In order to describe the technological renewal costs, we build a function, that captures the idea of the dependence of such amount on the distance from the technological frontier $t-\tau$, on the firm growth rate $r$ and on the observation time $t$. We refer at this aim to Chambers (2004).

The technological costs are assumed to be decreasing with respect to the distance from the technological frontier $t-\tau$. Specifically, we define the marginal cost per unit in $t-\tau$ as $\alpha_1^{t-\tau}$, where $\alpha_1 \in (0,1)$ is the learning rate. Furthermore, if a company adopts a new technology in the date $t$, then it is obliged to pay an adoption cost. In this sense we define the cost function at time $t$, assuming that a new technology has been adopted at a previous time $\tau$.

Differently with respect to the previous literature, we assume that the cost function depends on the growth rate $r$ of the referred firm, in order to analyze the renewal process by this economically relevant viewpoint. This assumption is totally consistent with most of the part of the technological improvements cases.

Therefore, the renewal process aggregate cost function $K$ is

$$K(r,t) = \int_0^t k(r,t,t-\tau) \cdot f(r,\tau,t)\mathrm{d}\tau. \tag{6}$$

For each $t > 0$, $r \in (-1,1)$ and $\tau \leq t$, we define

$$k(r,t,t-\tau) = k_1(r,t) + k_2(r) \cdot \alpha_1^{t-\tau}, \tag{7}$$

Given $t > 0$, $r \in (-1,1)$, $k_1(r,t)$ and $k_2(r)$ are terms related to the cost of the renewal process for companies, depending on the observation date $t$ and on the growth rate $r$.

Thus (6) can be written as:

$$K(r,t) = \lambda \int_0^t \Big[ k_1(r,t) + k_2(r) \cdot \alpha_1^{t-\tau} \Big] \cdot \exp \Big[ c|r|(t-\tau) \Big] \mathrm{d}\tau. \tag{8}$$

Although it is not requested by a mathematical viewpoint, it is reasonable to assume that the technological renewal costs are always positive. In fact, it is economically consistent to assume, that the adoption of a new technology admits always the payment of a positive amount. Therefore

$$k_1(r,t) > 0, \qquad k_2(r) > 0, \qquad \forall\, r, t.$$

# 4 The optimal profit

This section investigates the relations on firm productivity and costs, allowing the formalization of the conditions for the aggregate profit optimization.

As usual, the aggregate profit function is the difference between the aggregate output and the aggregate costs. We denote the profit at time $t$, for companies with growth rate $r$, as $P(r,t)$. Therefore, in our model, we write

$$P(r,t) = Q(r,t) - K(r,t), \tag{9}$$

where $Q$ and $K$ are defined respectively in (5) and (8).

We search for an optimal time variable value $t^* > 0$ such that

$$P(r,t) \leq P(r,t^*), \qquad \forall\, t > 0,\, r \in (-1,1). \tag{10}$$

Moreover, we use standard analytical arguments in order to provide a characterization at $t^*$ of relationships among productivity and cost function. We start determining an uniqueness condition for the existence of an unique time $t^*$ for the optimal profit.

**Remark 1** *For each $r \in (-1,1)$, let us consider $P(r,\cdot) \in C^1(0,+\infty)$ and $t^*$ such that*

$$\frac{\partial}{\partial t} P(r,t^*) = 0 \tag{11}$$

*and*

$$\begin{cases} \frac{\partial}{\partial t} P(r,t) > 0, & \text{for } t < t^*, \\ \\ \frac{\partial}{\partial t} P(r,t) < 0, & \text{for } t > t^*. \end{cases} \tag{12}$$

*Then $t^*$ is the unique point such that (10) holds.*

**Remark 2** *A sufficient condition for (11) is that $t^*$ satisfies the following* *system:*

$$\begin{cases} \frac{\partial}{\partial t}Q(r,t^*) = 0; \\ \\ \frac{\partial}{\partial t}K(r,t^*) = 0. \end{cases} \tag{13}$$

*As long as aggregate output $Q$ and aggregate costs $K$ are mutually independent, the condition is also necessary.*

A sufficient condition for $t^*$ fulfilling (13) is formalized in the next proposition, and is related to the functional shape of $g_1$ and $k_1$ for $t = t^*$.

**Proposition 3** *Let us assume that*

$$\begin{cases} g_1(r,t^*) = \dfrac{1}{|(\mathrm{e}^{c|r|}\alpha_g)^{t^*} - 1|}, \\ \\ k_1(r,t^*) = -\dfrac{c|r|k_2(r)}{(c|r| + \log\alpha_1)} \cdot \dfrac{(\mathrm{e}^{c|r|}\alpha_1)^{t^*}}{\mathrm{e}^{c|r|t^*} - 1}. \end{cases} \tag{14}$$

*Then (13) holds.*

**Proof.** By a direct computation, we have

$$\begin{cases} \frac{\partial}{\partial t}Q(r,t) = \lambda g_2(r)\Big\{g_1(r,t) + \displaystyle\int_0^t \frac{\partial}{\partial t}\Big[g_1(r,t)\alpha_g^{(t-\tau)} \cdot \exp\big[c|r|(t-\tau)\big]\Big]\mathrm{d}\tau\Big\}, \\ \\ \frac{\partial}{\partial t}K(r,t) = \lambda\Big\{k_1(r,t) + k_2(r) \\ \quad + \displaystyle\int_0^t \frac{\partial}{\partial t}\Big\{\Big[k_1(r,t) + k_2(r)\alpha_1^{t-\tau}\Big]\exp\big[c|r|(t-\tau)\big]\Big\}\mathrm{d}\tau\Big\}. \end{cases} \tag{15}$$

(13) is equivalent to

$$\begin{cases} g_1(r,t^*) + \dfrac{1}{c|r|+\log\alpha_g}\Big[g_1(r,t^*)(c|r|+\log\alpha_g) + \dfrac{\partial}{\partial t}g_1(r,t^*)\Big][(\mathrm{e}^{c|r|}\alpha_g)^{t^*}-1] = 0, \\ \\ k_1(r,t^*) + k_2(r) + \frac{1}{c|r|}\Big[[c|r|k_1(r,t^*)+ \\ +\frac{\partial}{\partial t}k_1(r,t^*)](\mathrm{e}^{c|r|t^*}-1) + k_2(r)\Big[(\mathrm{e}^{c|r|}\alpha_1)^{t^*}-1\Big] = 0. \end{cases} \tag{16}$$

Putting in order the terms of (16), we get the following system of ordinary differential equations:

$$\begin{cases} \dfrac{(\mathrm{e}^{c|r|}\alpha_g)^{t^*}-1}{c|r|+\log\alpha_g}\dfrac{\partial}{\partial t}g_1(r,t^*) + (\mathrm{e}^{c|r|}\alpha_g)^{t^*}g_1(r,t^*) = 0, \\ \\ \frac{\partial}{\partial t}k_1(r,t^*)\cdot\Big[\frac{1}{c|r|}(\mathrm{e}^{c|r|t^*}-1)\Big] + k_1(r,t^*)\mathrm{e}^{c|r|t^*} + k_2(r)(\mathrm{e}^{c|r|}\alpha_1)^{t^*} = 0. \end{cases} \tag{17}$$

(14) is a solution of (17), and the proposition is completely proved. ■

As shown in Lemma 1, the fact that $t^*$ satisfies the system (13) is a stationarity condition, and it is not sufficient to state that $t^*$ is the optimal profit date. We have to formalize a result related to the validity of monotonicity conditions on the profit functions stated in (12).

**Proposition 4** *Let us consider*

- *for* $t = t^*$*,*

*$g_1$ and $k_1$ as in (14);*

- *for $t < t^*$,*

$$g_2(r) > 0, \qquad g_1(r,t) > 0, \qquad \frac{\partial}{\partial t} g_1(r,t) > 0 \tag{18}$$

*and*

$$\frac{\partial}{\partial t} k_1(r,t) < \frac{c|r|(k_1(r,t) + k_2(r))}{1 - \mathrm{e}^{c|r|t}}. \tag{19}$$

- *for $t > t^*$,*

$$g_2(r) > 0, \qquad g_1(r,t) > 0, \qquad \frac{\partial}{\partial t} g_1(r,t) < -\frac{c|r| + \log \alpha_g}{(\mathrm{e}^{c|r|}\alpha_g)^t - 1} \cdot g_1(r,t) \tag{20}$$

*and*

$$\frac{\partial}{\partial t} k_1(r,t) > 0. \tag{21}$$

*Then $t^*$ is the unique optimal profit date.*

**Proof.** By Proposition 3, we need only to check the validity of conditions (12).

A sufficient condition for

$$\frac{\partial}{\partial t} P(r,t) > 0 \qquad \forall\, t < t^*$$

is that

$$\begin{cases} \frac{\partial}{\partial t}Q(r,t) > 0, & \text{for } t < t^*, \\ \\ \frac{\partial}{\partial t}K(r,t) < 0, & \text{for } t < t^*. \end{cases} \tag{22}$$

We use the explicit expression of the partial derivatives of $Q$ and $K$ with respect to $t$, given in (15).

Therefore, a sufficient condition for

$$\frac{\partial}{\partial t}Q(r,t) > 0 \qquad \forall\, t < t^*$$

is the following:

$$g_2(r) > 0, \qquad \frac{\partial}{\partial t}g_1(r,t) > -\frac{c|r| + \log\alpha_g}{(\mathrm{e}^{c|r|}\alpha_g)^t - 1} \cdot (\mathrm{e}^{c|r|}\alpha_g)^t g_1(r,t). \tag{23}$$

Conditions stated in (18) imply (23).

Analogously, the relation formalized in (19) implies

$$\frac{\partial}{\partial t}k_1(r,t) < \frac{c|r|\left[k_1(r,t)\mathrm{e}^{c|r|t} + k_2(r)(\mathrm{e}^{c|r|}\alpha_1)^t\right]}{1 - \mathrm{e}^{c|r|t}}, \tag{24}$$

and from (24) we obtain $\frac{\partial}{\partial t}K(r,t) < 0$ for $t < t^*$.

Similar arguments provide the analysis of the case $\frac{\partial}{\partial t}P(r,t) < 0$, for $t > t^*$.

By (20), we obtain

$$g_2(r) > 0, \qquad \frac{\partial}{\partial t}g_1(r,t) < -\frac{c|r| + \log\alpha_g}{(\mathrm{e}^{c|r|}\alpha_g)^t - 1} \cdot (\mathrm{e}^{c|r|}\alpha_g)^t g_1(r,t). \tag{25}$$

(25) implies that $\frac{\partial}{\partial t}Q(r,t) < 0$ in $(t^*, +\infty)$.

Furthermore, the relation formalized in (21) implies

$$\frac{\partial}{\partial t}k_1(r,t) > \frac{c|r|\left[k_1(r,t)\mathrm{e}^{c|r|t} + k_2(r)(\mathrm{e}^{c|r|}\alpha_1)^t\right]}{1-\mathrm{e}^{c|r|t}}, \tag{26}$$

that is a sufficient condition for $\frac{\partial}{\partial t}K(r,t) > 0$ in $(t^*, +\infty)$. ■

# 5 Economic interpretation

This section aims at providing the economic sense of the analytic development of the model.

## 5.1 The situation before and after the optimal profit date

Proposition 4 describes the behavior of the outputs and costs before and after reaching the optimal profit date.

The productivity functions $g_1$ and $g_2$ must be in all the cases positively signed, because we don't address here any meaning to negative production. Before $t^*$, the productivity grows with respect to the time. On the other hand, the costs have to decrease with the time. The reduction of the costs is driven by the costs level, and it depends on the time $t$ and on the growth rate $r$. To optimize his/her profit, a firm holder has to reduce drastically and very fast her costs, if they are huge. Analogously, if such costs are small, a smaller reduction is requested.

After $t^*$, firm output decreases with respect to the time. The rate of decay has to be proportional to the productivity level. On the other hand, the costs have to increase, and the speedy of such growth has to be not less than a threshold, depending on the growth rate $r$ and the costs levels.

## 5.2 The situation at the optimal profit date

The productivity and cost function $k_1$ and $g_1$ allow to state relationships between growth rate, size and optimal profit date. The explicit expressions of $k_1$ and $g_1$ at the optimal profit date $t^*$, that have been provided in (14) in Proposition 3, admit an economic interpretation. First of all, a preliminary remark is needed. Since $k_1(r,t) > 0$, for each $t$ and $r$, the second formula in (14) can be written if

$$c|r| + \log \alpha_1 < 0,$$

that is trivially true. Moreover, the following result holds:

**Proposition 5** *Let us consider $t^*$ the optimal profit date. Then*

- $g_1(r,t^*)$ *is increasing for $r>0$, decreasing for $r<0$.*

- *If it results*

$$\begin{cases} k_2(r) = \log \frac{1}{r} & \text{for } r>0 \\ \\ k_2'(r) < 0 & \text{for } r<0. \end{cases} \tag{27}$$

*then $k_1(r,t^*)$ is decreasing w.r.t. $r$.*

**Proof.** Fixed $t=t^*$, the first order derivatives with respect to $r$ are

$$\begin{cases} \frac{\partial}{\partial r} g_1(r,t^*) = -\dfrac{\frac{\partial}{\partial r}(|(\mathrm{e}^{c|r|}\alpha_g)^{t^*}-1|)}{|(\mathrm{e}^{c|r|}\alpha_g)^{t^*}-1|^2}, \\ \\ \frac{\partial}{\partial r} k_1(r,t^*) = \frac{c(\alpha_1 \mathrm{e}^{c|r|})^{t^*}}{(1-\mathrm{e}^{c|r|t^*})(c|r|+\log\alpha_1)} \cdot \Big\{ k_2(r)\Big[ -\frac{c|r|\mathrm{sign}(r)}{c|r|+\log\alpha_1} + \\ +\mathrm{sign}(r) + \frac{c\mathrm{e}^{c|r|t^*}|r|\mathrm{sign}(r)}{1-\mathrm{e}^{c|r|t^*}} + c|r|t^*\mathrm{sign}(r)\Big] + |r|k_2'(r)\Big\}. \end{cases} \tag{28}$$

By the first condition of (28) we have

$$\frac{\partial}{\partial r} g_1(r,t^*) = -\frac{ct^*\mathrm{sign}(r)(\mathrm{e}^{c|r|}\alpha_g)^{t^*}}{|(\mathrm{e}^{c|r|}\alpha_g)^{t^*}-1|^2}. \tag{29}$$

Therefore

$$\frac{\partial}{\partial r} g_1(r,t^*) \begin{cases} >0 & \text{for } r>0 \\ \\ <0 & \text{for } r<0 \end{cases}$$

and the first part of the proposition is proved.

A direct computation shows that, if $k_2(r)$ satisfies the conditions listed in

(27), then

$$\frac{\partial}{\partial r} k_1(r, t^*) < 0, \qquad \text{for } r \in \mathbf{R}.$$

that is the second part of the proposition. ■

**Remark 6** *The conditions stated in (27) allow to obtain a complete information on the costs at the optimal profit date $t^*$, in both cases of companies in expansion and recession. The cost function $k_2$ follows a logarithmic law, for companies in expansion, and is decreasing, for companies in recession. Furthermore, at the optimal profit date, the productivity function $g_1$ grows for firms in expansion, decreases for firms in recession.*

A further analysis of the behavior of $g_1$ and $k_1$ with respect to the size of the firms $s$, at the optimal profit date $t^*$ can be done. Referring to the relationships between the parameters of the function $f$ and the companies' size, given by (2), we can define the functions

$$\begin{cases} \bar{g}_1(r, s, t^*) = \dfrac{1}{|(\mathrm{e}^{-s^\beta |r|} \alpha_g)^{t^*} - 1|}, \\ \\ \bar{k}_1(r, s, t^*) = -\dfrac{-s^\beta |r| k_2(r)}{(-s^\beta |r| + \log \alpha_1)} \cdot \dfrac{(\mathrm{e}^{-s^\beta |r|} \alpha_1)^{t^*}}{\mathrm{e}^{-s^\beta |r| t^*} - 1}. \end{cases} \tag{30}$$

**Proposition 7** *Fixed $r \in \mathbf{R}$, for $t = t^*$, it results*

- *$\bar{g}_1(r, s, t^*)$ is increasing with respect to $s$.*

- *For $s \in \mathcal{S} \subseteq \mathbf{R}$, then $\bar{k}_1(r,s,t^*)$ is increasing with respect to $s$ and it is decreasing otherwise, where*

$$\mathcal{S} = \Big\{ s \in \mathbf{R} \mid \frac{|r|s^\beta t^*}{\mathrm{e}^{-|r|s^\beta t^*}-1} + \frac{|r|s^\beta}{\log\alpha_1 - |r|s^\beta} + 1 > 0 \Big\}. \tag{31}$$

**Proof.** We have

$$\frac{\partial}{\partial s}\bar{g}_1(r,s,t^*) = \frac{\partial}{\partial s}\frac{1}{|(\mathrm{e}^{-s^\beta|r|}\alpha_g)^{t^*}-1|} =$$
$$= \frac{1}{|(\mathrm{e}^{-s^\beta|r|}\alpha_g)^{t^*}-1|^2}\cdot(\mathrm{e}^{-s^\beta|r|}\alpha_g)^{t^*}\cdot|r|\beta s^{\beta-1}, \tag{32}$$

that is always greater than 0. Therefore, $\bar{g}_1$ is an increasing function of the variable $s$.

Moreover, it results

$$\frac{\partial}{\partial s}\bar{k}_1(r,s,t^*) = |r|k_2(r)\alpha_1^{t^*}\frac{\beta s^{-1+\beta}\mathrm{e}^{-|r|s^\beta t^*}}{(\mathrm{e}^{-|r|s^\beta t^*}-1)(\log\alpha_1-|r|s^\beta)}\cdot$$
$$\cdot\Big[|r|s^\beta\Big(\frac{t^*}{\mathrm{e}^{-|r|s^\beta t^*}-1}+\frac{1}{\log\alpha_1-|r|s^\beta}\Big)+1\Big]. \tag{33}$$

Numerical computations considering bounds on the parameters guarantee that $\mathcal{S} \neq \emptyset$. Therefore, a direct computation shows that, if $s \in \mathcal{S}$, then it results

$$\frac{\partial}{\partial s}\bar{k}_1(r,s,t^*) > 0,$$

and for $s \notin \mathcal{S}$ the converse inequality holds.

This completes the proof. ■

**Remark 8** *The renewal costs grow with the firms' size, depending on the belonging of $s$ to a critical region $\mathcal{S}$, that depends on the learning rate, the company growth rate and the optimal profit date. Under parameters calibration, it can numerically be shown that $\mathcal{S} = (s_1, s_2)$, with $s_1 < s_2$. Therefore, there exists a critical size range in which firms' holders should exit or entry, depending on the different adopting policies.*

*The productivity increases with the size: at the optimal profit date, the bigger is the company, the higher is the productivity level.*

# 6 Conclusions

This work proposes a model for profit optimization, investigating the case of companies adopting new technologies. The main feature of such model is the introduction of the dependence of productivity and costs on growth rate and size of the firms.

The novelty of the paper is the use of empirical facts for the modeling of the number of production units.

Analytic expression for the functions that model productivity and costs as function of the growth rate and time are drawn at the optimal profit time.

Moreover, the consequences of such a modeling in terms of growth rate and size are examined.

A firm holder can use the analysis addressed in this paper and the results formalized in the previous section in order to calibrate his/her financial policies to reach productivity levels and to control renewal costs at the optimal profit time, by using appropriate policies on growth rate and size.

The effects of external events on the costs and productivity of the firms as size and growth rate change can also be examined. In this case, more appropriately, a dynamic stochastic model can be proposed and analyzed. We leave this topic to future research.

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