Dynamics of financial time series in an inhomogeneous aggregation framework

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Summary. In this paper we provide a microeconomic model to investigate the long term memory of financial time series of one share. In the framework we propose, each trader selects a volume of shares to trade and a strategy. Strategies differ for the proportion of fundamentalist/chartist evaluation of price. The share price is determined by the aggregate price. The analyses of volume distribution give an insight of imitative structure among traders. The main property of this model is the functional relation between its parameters at the micro and macro level. This allows an immediate calibration of the model to the long memory degree of the time series under examination, therefore opening the way to the understanding the emergence of stylized facts of the market through opinion aggregation.

Key words: Long memory, financial time series, fundamentalist and chartist agents.

1 Introduction

Long term memory in financial data has been studied through several papers. Wide sets of analyses are available in literature on time series of stock market indices, shares prices, price increments, volatility, returns, as well as several functions of returns (absolute returns, squared returns, powered returns). The analyses of long term memory can be refined considering different time scales as well as considering moving windows with different lenghts. The former have evidenced the multifractal structure, validating the efficient market hypothesis at long enough time scales [Be, MS, SSL]. The latter report a wide range of the self similarity parameter [BPS, BP, R]. Deviations from the gaussian case can be addressed at the microeconomic level to the lack of efficiency in markets, and to self-reinforcing imitative behaviour, as it also happens during large financial crashes due to endogenous causes [R].
We aim at providing a mathematically tractable financial market model that

can give an insight on the market microstructure that captures some characteristics of financial time series. In our model agents decide their trading price choosing among a set of price forecasts based on mixed chartist/fundamentalist strategy [FHK]. Agents will switch from one price to the other varying the volume to trade at the forecasted price. The distribution of the parameter that regulates the mixture chartist/fundamentalist forecasts reports the confidence of investors on each approach. We differ from [FHK] in that we don't introduce a performance index on a strategy.
Traders are not pure price taker: agents' opinions contribute to the market price in accord with price and volume distribution. The distribution of volumes size evidences the occurrence of imitative behavior, and the spreading of the confidence of traders on "gurus". Aggregation and spreading of opinions give an insight of social interactions. Models that allow for a opinion formation are mostly based on random interaction among agents, and they were refined considering constraints to the social contact, as an example modeled through scale free networks. It has already been shown that the relevant number of social contact in financial markets is very low, being between 3 and 4 [RB, AIV, VDA], opening the way to lattice-based models. We are going to discuss at a general level the case of random interactions of agents, and then its consequences on aggregation and disaggregation of opinions on some strategy. We are not aimed at exploring also the bid/ask spread: our price is just considered as the market price given by the mean of agents' prices. The proposed theoretical approach allows to avoid numerical calibration procedures. The paper is organized as follows. The first part of section 2 describes the model, and the subsection 2.1 and 2.2 are devoted to the study of such a model, in the case of independence and dependence, respectively. Section 3 concludes.

2 Market price dynamics

We consider N investors trading in the market, and we assume that $\omega_{i,t}$ is the size of the order placed on the market by agent i at time t. This choice allows to model individual traders as well as funds managers, that select the trading strategy on behalf of their customers. In the present analysis we consider investors getting information from two different sources: observation of the macroeconomic fundamentals and adjustment of the forecast performed at the previous time. Other markets characteristics, like as the presence of a market maker, are not considered here, and they will be studied elsewhere. Let us define with $P_{i,t}$ the forecast of the market price performed by the investor i at time t. Each of them relies on a proportion of fundamentalist $P^f_{i,t}$ and of a chartist $P^c_{i,t}$ forecast. We can write

$$P_{i,t} = (1 - \beta_i)P^f_{i,t} + \beta_i P^c_{i,t}, \qquad (1)$$

where β_i are sampled by a random variable β with compact support equals to $[0,1]$, i.e. $\beta_i \sim \beta \in D[0,1]$, for each $i = 1, \ldots, N$.
Parameter β_i in equation (1) regulates the proportion of fundamentalist/chartist in each agent forecast. The most β_i is to 0, the most is the confidence in the return to fundamentals. The most β_i is to 1, the most the next price is estimated to be the actual price. The shape of the distribution used for sampling the β_i gives relevant information on the overall behavior of agents.
In the fundamentalist analysis the value of the market fundamentals is known, and so the investor has a complete information on the risky asset (he understand over or under estimation of price). Given the market price P_t we have the following fundamentalist forecast relation:

$$P^f_{i,t} = \nu(\tilde{P}_{i,t-1} - P_{t-1}), \tag{2}$$

where $\nu \in \mathbf{R}$ and $\tilde{P}_{i,t}$ is a series of fundamentals observed with a stochastic error from the agent i at time t, i.e.

$$\tilde{P}_{i,t} = \bar{P}_{i,t} + \alpha_{i,t},$$

with $\alpha_{i,t} = \zeta_i P_t$ and ζ_i are sampled by a real random variable ζ with finite expected value $\bar{\zeta}$ and independent on β. The fundamental variables $\bar{P}_{i,t}$ can be described by the following random walk:

$$\bar{P}_{i,t} = \bar{P}_{i,t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2_\epsilon).$$

Thus

$$P^f_{i,t} = \nu \bar{P}_{i,t-1} + \nu(\zeta_i - 1)P_{t-1}. \tag{3}$$

The chartist forecast at time t is limited to an adjustment of the forecast made by the investor at the previous time. The adjustment factor related to the i-th agent is a random variable γ_i. We assume that γ_i are i.i.d, with support in the interval $(1-\delta, 1+\delta)$, with $\delta \in [0,1]$. Moreover, we suppose that

$$\mathbf{E}[\gamma_i] = \bar{\gamma}, \qquad i = 1, \ldots, N,$$

and γ_i are independent on ζ_i and β_i. Then we can write

$$P^c_{i,t} = \gamma_i P_{i,t-1}. \tag{4}$$

We assume, that the aggregate size of the order placed by the agents at a fixed time t depends uniquely on t. We denote it as $\tilde{\omega}_t$, and we have

$$\tilde{\omega}_t = \sum_{i=1}^{N} \omega_{i,t}.$$

We assume that such aggregate size is uniformly bounded. Therefore, there exists a couple of thresholds $\underline{\omega}$ and $\overline{\omega}$ such that, for each $t > 0$, $\underline{\omega} < \tilde{\omega}_t < \overline{\omega}$.

Market price is given by the weighted mean of trading prices associated to the agents. The weights are given by the size of the order. We do not consider here the bid-ask spread, and mechanisms related to the limit order book, leaving them to future studies. Summing up the components, we can write

$$P_t = \sum_{i=1}^{N} \omega_{i,t} P_{i,t}. \tag{5}$$

Then, by (1), (3) and (5)

$$P_t = \sum_{i=1}^{N} \omega_{i,t} \Big[\nu(1-\beta_i)\bar{P}_{i,t-1} + \nu(1-\beta_i)(\zeta_i - 1)P_{t-1} + \gamma_i \beta_i P_{i,t-1} \Big]. \tag{6}$$

2.1 Model property: the case of independence

The scope of this section is to describe the memory property of the financial time series P_t, in the case of absence of relations between the strategy β_i, adopted by the agent i, and the weight $\omega_{i,t}$ of the agent i at time t.
The following result holds.

Theorem 1. *Given $i = 1, \dots, N$, let β_i be a sampling drawn from a random variable β such that*

$$\mathbf{E}[\beta^k] \sim O(c) k^{-1-p} + o(k^{-1-p}) \ \text{ as } k \to +\infty. \tag{7}$$

Moreover, given $i = 1, \dots, N$, let ζ_i be a sampling drawn from a random variable ζ.
Let us assume that β and ζ are mutually independent.
Furthermore, suppose that there exists $q > 0$ such that

$$(\mathbf{E}[\gamma_i])^{k-1} = \bar{\gamma}^{k-1} \sim k^{-q}, \qquad \text{as } k \to +\infty.$$

Then, for $N \to +\infty$ and $q + p \in [-\frac{1}{2}, \frac{1}{2}]$, we have that P_t has long memory with Hurst exponent given by $H = p + q + \frac{1}{2}$.

Proof. Let L be the time-difference operator such that $LP_{i,t} = P_{i,t-1}$.
By definition of $P_{i,t}$, we have

$$(1 - \gamma_i \beta_i L) P_{i,t} = \nu(1-\beta_i)\bar{P}_{i,t-1} + \nu(1-\beta_i)(\zeta_i - 1)P_{t-1}, \tag{8}$$

and then

$$P_{i,t} = \frac{\nu(1-\beta_i)}{1-\gamma_i\beta_i L}\bar{P}_{i,t-1} + \frac{\nu(1-\beta_i)(\zeta_i-1)}{1-\gamma_i\beta_i L}P_{t-1}. \tag{9}$$

By the definition of P_t and (9), we have

$$P_t = \sum_{i=1}^{N} \omega_{i,t} \Big[\frac{\nu(1-\beta_i)}{1-\gamma_i\beta_i L}\bar{P}_{i,t-1} + \frac{\nu(1-\beta_i)(\zeta_i-1)}{1-\gamma_i\beta_i L}P_{t-1} \Big]. \tag{10}$$

Setting the limit as $N \to \infty$ and by the definition of $\bar{P}$, a series expansion gives

$$P_t = \nu \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k} \int_{\mathbf{R}} \int_{\mathbf{R}} (\zeta - 1)(1-\beta)\beta^{k-1}\bar{\gamma}^{k-1} dF(\zeta, \beta). \qquad (11)$$

Since, by hypothesis, β and ζ are mutually independent, with distributions F_1 and F_2 respectively, we have

$$P_t = \nu \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k} \bar{\gamma}^{k-1} \int_{\mathbf{R}} \int_{\mathbf{R}} (\zeta - 1)(1-\beta)\beta^{k-1} dF_1(\zeta) dF_2(\beta) =$$

$$= \nu \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k} \bar{\gamma}^{k-1} \int_{\mathbf{R}} (\zeta - 1) dF_1(\zeta) \int_0^1 (1-\beta)\beta^{k-1} dF_2(\beta) =$$

$$= \nu(\bar{\zeta} - 1) \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k} \bar{\gamma}^{k-1} (M_{k-1} - M_k),$$

where M_k is the k-th moment of a random variable satisfying the condition (7). Since

$$\underline{\omega} \sum_{k=1}^{\infty} P_{t-k}(M_{k-1} - M_k) < \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k}(M_{k-1} - M_k) < \overline{\omega} \sum_{k=1}^{\infty} P_{t-k}(M_{k-1} - M_k)$$

and

$$M_{k-1} - M_k \sim k^{-p-1}, \qquad (12)$$

then, by the hypothesis on the γ_i's, we desume

$$\bar{\gamma}^{k-1}(M_{k-1} - M_k) \sim k^{-q-p-1}. \qquad (13)$$

Therefore we have a long memory model $I(d)$ with $d = p + q + 1$ and thus Hurst exponent $H = p + q + \frac{1}{2}$ ([DG1996a], [DG1996b], [G], [GY], [L]).

Remark 1. We can use the Beta distribution $B(p, q)$ for defining the random variable β. In fact, if X is a random variable such that $X \sim B(p, q)$, with $p, q > 0$, then X satisfies the relation stated in (7).

Remark 2. In the particular case $\gamma_i = 1$, for each $i = 1, \dots, N$, the long term memory is allowed uniquely for persistence processes. In this case it results $q = 0$ and, since $p > 0$ by definition, Theorem 1 assures that $H \in (\frac{1}{2}, 1]$.

Remark 3. Structural changes drive a change of the Hurst's parameter of the time series, and thus the degree of memory of the process. In fact, if the chartist calibrating parameter γ_i or the proportionality factor between chartist and fundamentalist, β_i, vary structurally, then the distribution parameters p and q of the related random variables change as well. Therefore H varies, since it depends on q and p. Furthermore, a drastic change can destroy the stationarity property of the time series. In fact, in order to obtain such stationarity property for P_t, we need that $p + q \in [-1/2, 1/2]$, and modifications of q and/or p must not exceed the range.

Remark 4. The parameters q and p could be calibrated in order to obtain a persistent, antipersistent or uncorrelated time series.

2.2 Model property: introducing the dependence structure

This section aims to describe the long-run equilibrium properties of financial time series, in the case in which the weights of the investors can drive the forecasts' strategies. The approach we propose allows to consider the presence of imitative behaviors among the agents. The phenomena of the herding investors is a regularity of financial markets. Since the empirical evidence of crises of the markets, the interests of a wide part of the economists have been focused on the analysis of the financial systems fragility. A part of the literature emphasized the relations between financial crises and weak fundamentals of the economy ([AG], [BER] and [CPR]). A possible explanation of the reasons for the fact, that asset prices does not reflect the fundamentals, can be found in the spreading of information among investors, and in the consequent decision to follow a common behavior.
We model the dependence structure allowing the size of the order to change the proportion between fundamentalist and chartist forecasts.
Then, for each weight $\omega_{i,t}$, we consider a function

$$f_{\omega_{i,t}} : D[0,1] \to D[0,1] \text{ such that } f_{\omega_{i,t}}(\beta) = \tilde{\beta}, \qquad \forall\, i,t. \tag{14}$$

Analogously to the previous section, we formalize a result on the long-run equilibrium properties of the time series P_t in this setting.

Theorem 2. *Given $i = 1, \ldots, N$, let β_i be a sampling drawn from a random variable $\beta \in D[0,1]$.*
Fixed $\omega_{i,t}$, let $f_{\omega_{i,t}}$ be a random variable transformation defined as in (14) such that

$$\mathbf{E}[\{f_{\omega_{i,t}}(\beta)\}^k] = \mathbf{E}[\tilde{\beta}^k] \sim O(c)k^{-1-\tilde{p}} + o(k^{-1-\tilde{p}}) \text{ as } k \to +\infty. \tag{15}$$

Moreover, given $i = 1, \ldots, N$, let ζ_i be a sampling drawn from a random variable ζ, where $\tilde{\beta}$ and ζ are mutually independent.
Furthermore, suppose that there exists $q > 0$ such that

$$(\mathbf{E}[\gamma_i])^{k-1} = \bar{\gamma}^{k-1} \sim k^{-q}, \qquad \text{as } k \to +\infty.$$

Then, for $N \to +\infty$ and $q + \tilde{p} \in [-\frac{1}{2}, \frac{1}{2}]$, we have that P_t has long memory with Hurst exponent given by $H = \tilde{p} + q + \frac{1}{2}$.

Proof. The proof is similar to the one given for Theorem 1.

Remark 5. Remark 1 guarantees, that the $f_{\omega_{i,t}}$ can transform $X \sim B(p,q)$ in $f_{\omega_{i,t}}(X) \sim B(\tilde{p}, \tilde{q})$. Therefore, the changing of the strategy used by the

investors, driven by the weights ω's, can be attained by calibrating the parameters of a Beta distribution.
We use the $B(p,q)$ distribution because of its statistical properties and of the several different shapes that it can assume depending on its parameters values. In the particular case $p = 1$, $q = 1$ it is the uniform distribution. If β_i are sampled in accord to a uniform distribution then there is no prevailing preference on the strategy, and so between either chartist or fundamentalist approach. If β_i are sampled in accord to a random variable β, $\beta \sim B(p,p), p > 1$ then this means that agents opinion agree on mixture parameter values close to the mean of β. If the distribution is U-shaped, this means that there are two most agreeable strategies.

3 Conclusions

This paper has shown how to consider the weight of a trader on market into a microeconomic model that allows to state theoretically the degree of long memory in data. Since $H = 1/2$ is taken into account in the theoretical model, the long-run equilibrium properties of uncorrelated processes represents a particular case. Therefore, the model encompasses a wide range of processes. This approach allows to avoid time-expensive numerical calibration and allows to use the model also for explaining the weight distribution on high-frequency trading. A further analyses has been carried on a more detailed correspondence between the group size and the trader strategy, in both of cases of dependence and independence.
A bayesian statistic approach, to develop the analysis of the dependence structure between size and strategies, can be used. We leave this topic to future research.

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