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Enlarged Controllability and Optimal Control of Sub-Diffusion Processes with Caputo Fractional Derivatives

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Abstract: The present paper investigates the exact enlarged controllability and optimal control of a fractional diffusion equation in Caputo sense. This is done through a new definition of enlarged controllability that allows us to extend available contributions. Moreover, the problem is explored using two approaches: a reverse Hilbert uniqueness method, generalizing the approach introduced by Lions in 1988, and a penalization method, which allows us to characterize the minimum energy control.

Keywords: Fractional calculus and diffusion, Caputo derivatives and enlarged controllability, RHUM approach and minimum energy, fractional optimal control, zone and pointwise actuators.

1 Introduction

Calculus of fractional order began more than three centuries ago. It was first mentioned by Leibniz, in a reply to l'Hôpital, addressing the question whether the derivative remains valid for a non-integer order. The subject has been developed by several mathematicians, such as Euler, Fourier, Liouville, Grunwald, Letnikov and Riemann. Currently, other authors investigate such kind of operators and propose new fractional derivatives [1,2,3,4,5]. Over the last few decades, fractional calculus has gained more and more attention because of its applications in various fields of science, such as physics, engineering, economics, and biology [6,7,8,9,10,11,12].

In control theory, several authors have been interested in fractional calculus since the sixties of last century. The first contributions generalize classical analytical methods and concepts for fractional order systems, such as the transfer function, frequency response, pole and zero analysis, and so on [13,14]. Nowadays, fractional calculus is used in the field of automatic control to obtain better and more accurate models, to develop new control strategies, and to improve the characteristics of control systems [15,16].

In recent years, fractional order sub-diffusion systems have grabbed the attention of several researchers because they have outperformed the traditional integer order systems. More precisely, they can accurately characterize anomalous diffusion processes in various real-world complex systems [17,18,19,20]. In particular, fractional anomalous diffusion has been used to describe different physical scenarios, most prominently within crowded systems, for example protein diffusion within cells or diffusion through porous media. Time-fractional sub-diffusion has also been proposed as a measure of macromolecular crowding in the cytoplasm [21].

Controllability of a fractional order sub-diffusion system can be reformulated as an infinite dimensional control problem. Moreover, not all states can be reached in case of diffusion systems [22,23,24]. Because mathematical models of real systems are obtained from measures or approximation techniques, affected by perturbations if the solutions for such systems are only approximately known, control problems subject to output fractional constraints are more realistic and adapted for system analysis than the classical ones [12,25].

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Many problems in modern science are addressed with the help of optimization theory. Optimal control, as a branch of Mathematics, aims to improve the state variables of a control system in order to maximize the benefit or minimize the given cost. This is applicable to practical situations, where state variables can be temperature, a velocity field, a measure of information, etc. This is the main reason why optimal control is an attractive research area for many scientists in various disciplines. Efficient optimization and optimal control methods have been developed in order to compute the solution of fractional optimal control problems [12,26,27].

The present paper handles the controllability problem of Caputo fractional diffusion equations in presence of constraints on the state variables. This is related to the notion of enlarged controllability, which was first investigated by Lions in 1988 for hyperbolic systems [28] and later developed for linear and semilinear parabolic systems [29,30,31,32]. Moreover, we create a bridge with optimal control of systems described by fractional order differential equations. For that we prove enlarged controllability by means of a reverse Hilbert Uniqueness Method (HUM) and make use of a penalization method, which allows us to characterize the minimum energy control. We consider the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem [33]. For results associated with the Caputo–Fabrizio operators, we refer the reader to the recent paper [34].

The present paper is organized as follows. Definitions and preliminaries on fractional calculus are presented in Section 2. In Section 3, we characterize the exact enlarged controllability of the system. Section 4 is devoted to the results of the exact enlarged controllability, in two different cases: for zone and pointwise actuators. In Section 5, an optimization problem for a system of fractional order is solved using a penalization method. Sections 6 and 7 involve some examples of the two cases of actuators and conclusion. A preprint of this paper is available in [35].

2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be bounded with a smooth boundary $\partial\Omega$. For $T > 0$, denote $Q = \Omega \times [0,T]$ and $\Sigma = \partial\Omega \times [0,T]$. We consider the following abstract fractional sub-diffusion system of order $\alpha \in (0,1)$:

$$\begin{cases} {}^C D^\alpha y(t) = \mathscr{A} y(t) + \mathscr{B} u(t), & t \in [0,T], \\ y(0) = y_0 & \text{in} \quad D(\mathscr{A}), \end{cases} \tag{1}$$

where ${}^C D^\alpha$ denotes the Caputo fractional order derivative (for details on Caputo fractional operators, see, e.g., [1,2]). The second order operator $\mathscr{A}$ is linear and with dense domain, such that the coefficients do not depend on t and generate a C_0-semi-group $(S(t))_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$. We refer the reader to Engel and Nagel [36] as well as Renardy and Rogers [37] for properties on operator $\mathscr{A}$. In the sequel, we let $\mathscr{D}(A)$ be the domain of the operator $\mathscr{A}$; $y \in L^2(0,T;L^2(\Omega))$ and $u \in U = L^2(0,T;\mathbb{R}^m)$, where m is the number of actuators. The initial datum y_0 is in $L^2(\Omega)$, $\mathscr{B} : \mathbb{R}^m \longrightarrow L^2(\Omega)$ is the control operator, which is linear, possibly unbounded, and depending on the number and structure of actuators.

Several definitions and preliminary results are required to investigate the system (1). We begin with the most important function used in fractional calculus, i.e., *Euler's gamma function*, which is defined as

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt.$$

This function is a generalization of the factorial: if $n \in \mathbb{N}$, then $\Gamma(n) = (n-1)!$.

Definition 1(See, e.g., [1]). *The left-sided Caputo fractional derivative of order $\alpha > 0$ of a function z is given by*

$${}_0^C D_t^\alpha z(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} z(s) ds, & n-1 < \alpha < n, \quad t \geq 0, \quad n \in \mathbb{N}, \\ \dfrac{d^n z(t)}{dt^n}, \quad \alpha = n \in \mathbb{N}. \end{cases} \tag{2}$$

The right-sided is pointwise defined. The Caputo fractional derivative is a sort of regulation in the time origin for the Riemann–Liouville fractional derivative.

Definition 2(See, e.g., [38,39,40]). *Let $z : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^+$ and $\alpha > 0$. Then the expressions*

$${}_0 I_t^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, \quad t > 0, \tag{3}$$

and

$${}_t I_T^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} z(s) ds, \quad t < T, \tag{4}$$

are, respectively, called the left-sided and right-sided Riemann–Liouville integrals of order α.

Definition 3(See, e.g., [38,39,40]). *Let $z : \mathbb{R}^+ \to \mathbb{R}$. The left-sided and right-sided Riemann–Liouville fractional derivatives of order α are defined by*

$$ {}_0D_t^{\alpha} z(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} z(s)ds, \quad t > 0, \tag{5} $$

and

$$ {}_tD_T^{\alpha} z(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^T (s-t)^{n-\alpha-1} z(s)ds, \quad t < T, \tag{6} $$

where $\alpha \in (n-1,n)$, $n \in \mathbb{N}$.

We always consider solutions of (1) in the weak sense. We denote that solution by $y(x,t;u)$ and, when no possible ambiguity, we also use the short notation $y_u(t)$ or $y(u)$. Hence, we denote by $y_u(T)$ the mild solution of system (1) at the final time T.

Definition 4(See [41]). *For $t \in [0,T]$ and any given $u \in U$, a function $y \in L^2(0,T;L^2(\Omega))$ is a mild solution of system (1) if it satisfies*

$$ y_u(t) = \mathscr{R}_{\alpha}(t) y_0 + \int_0^t (t-s)^{\alpha-1} K_{\alpha}(t-s) \mathscr{B} u(s) ds, \tag{7} $$

where

$$ \mathscr{R}_{\alpha}(t) = \int_0^{\infty} \phi_{\alpha}(\theta) S(t^{\alpha}\theta) d\theta \tag{8} $$

and

$$ K_{\alpha}(t) = \alpha \int_0^{\infty} \theta \phi_{\alpha}(\theta) S(t^{\alpha}\theta) d\theta \tag{9} $$

with $\phi_{\alpha}(\theta)$ given by

$$ \phi_{\alpha}(\theta) = \frac{1}{\alpha} \theta^{-1-1/\alpha} \psi_{\alpha}(\theta^{-1/\alpha}), $$

where ψ_{α} is the following probability density function:

$$ \psi_{\alpha}(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty). \tag{10} $$

Remark. The probability density function (10) satisfies the following properties:

$$ \int_0^{\infty} e^{-\lambda\theta} \psi_{\alpha}(\theta) d\theta = e^{-\lambda^{\alpha}}, \qquad \int_0^{\infty} \psi_{\alpha}(\theta) d\theta = 1, \quad \alpha \in (0,1), \tag{11} $$

and

$$ \int_0^{\infty} \theta^{\nu} \phi_{\alpha}(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \quad \nu \geq 0. \tag{12} $$

Let $H : L^2(0,T;\mathbb{R}^m) \to L^2(\Omega)$ be defined as

$$ Hu = \int_0^T (T-s)^{\alpha-1} K_{\alpha}(T-s) \mathscr{B} u(s) ds, \quad \forall u \in L^2(0,T;\mathbb{R}^m), \tag{13} $$

where m is the number of actuators. We assume that $(S^*(t))_{t\geq 0}$ is a strongly continuous semi-group generated by A^* on the state space $L^2(\Omega)$. For $v \in L^2(\Omega)$, one has

$$ \begin{aligned} \langle Hu, v \rangle &= \left\langle \int_0^T (T-s)^{\alpha-1} K_{\alpha}(T-s) \mathscr{B} u(s) ds, v \right\rangle_{L^2(\Omega)} \\ &= \int_0^T \langle (T-s)^{\alpha-1} K_{\alpha}(T-s) \mathscr{B} u(s), v \rangle_{L^2(\Omega)} ds \\ &= \int_0^T \left\langle u(s), \mathscr{B}^* (T-s)^{\alpha-1} K_{\alpha}^*(T-s) v \right\rangle_{L^2(0,T;\mathbb{R}^m)} ds \\ &= \langle u, H^* v \rangle, \end{aligned} \tag{14} $$

where by $\langle \cdot, \cdot \rangle$, we denote the duality pairing of space $L^2(\Omega)$, $\mathscr{B}^*$ is the adjoint operator of $\mathscr{B}$, and

$$K^*_\alpha(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) S^*(t^\alpha \theta) d\theta.$$

In order to prove our main results, the following lemmas are needed.

Lemma 1(See [42]). *Let the reflection operator $\mathscr{Q}$ on the interval $[0,T]$ be defined as follows:*

$$\mathscr{Q}h(t) := h(T-t)$$

for function h that is differentiable and integrable. Then, the following relations hold:

$$\mathscr{Q}_0 I_t^\alpha h(t) = {}_tI_T^\alpha \mathscr{Q}h(t), \qquad \mathscr{Q}_0 D_t^\alpha h(t) = {}_tD_T^\alpha \mathscr{Q}h(t)$$

and

$${}_0I_t^\alpha \mathscr{Q}h(t) = \mathscr{Q}_t I_T^\alpha h(t), \qquad {}_0D_t^\alpha \mathscr{Q}h(t) = \mathscr{Q}_t D_T^\alpha h(t).$$

Lemma 2(See, e.g., [43]). *For $t \in [a,b]$ and $n-1 < \alpha < n$, $n \in \mathbb{N}$, the following integration by parts formula holds:*

$$\int_a^b f(t) {}_0^C D_t^\alpha g(t) dt = \sum_{r=0}^{k-1} (-1)^{k-1-r} \left[g^r(t) {}_tD_b^{\alpha-1-r} f(t) \right]_{t=a}^{t=b} + (-1)^k \int_a^b g(t) {}_tD_b^\alpha f(t) dt.$$

In particular, if $0 < \alpha < 1$, then

$$\int_a^b f(t) {}_0^C D_t^\alpha g(t) dt = \left[g(t) {}_tI_b^{1-\alpha} f(t) \right]_{t=a}^{t=b} + \int_a^b g(t) {}_tD_b^\alpha f(t) dt.$$

We also recall the fractional Green's formula:

Lemma 3(See, e.g., [39,44]). *Let $0 < \alpha \le 1$ and $t \in [0,T]$. Then,*

$$\begin{aligned} &\int_0^T \int_\Omega \left({}_0^C D_t^\alpha y(x,t) + \mathscr{A} y(x,t) \right) \varphi(x,t) dx dt = \int_0^T \int_\Omega y(x,t) \left({}_tD_T^\alpha \varphi(x,t) + \mathscr{A}^* \varphi(x,t) \right) \\ &+ \int_0^T \int_{\partial\Omega} y(x,t) {}_tI_T^{1-\alpha} \varphi(x,t) d\Gamma dt - \int_0^T \int_{\partial\Omega} y(x,t) \frac{\partial \varphi(x,t)}{\partial \nu_{\mathscr{A}}} + \int_0^T \int_{\partial\Omega} \frac{\partial y(x,t)}{\partial \nu_{\mathscr{A}}} \varphi(x,t) d\Gamma dt \end{aligned}$$

for any $\varphi \in C^\infty(\overline{Q})$.

3 Regional enlarged controllability

We extend the definition of controllability first introduced by Lions in [45] to the case of sub-diffusion fractional systems. For that, we consider a nonempty sub-vectorial space $G \subset L^2(\Omega)$, which is supposed to be closed and convex.

Definition 5. *Given a final time $T > 0$, we say that system (1) is exactly enlarged controllable (i.e., G-controllable) if, for every y_0 in a suitable functional space, there exists a control u such that*

$$y(\cdot, T; u) \in G. \tag{15}$$

Remark. Obviously, the notion of exact enlarged controllability depends on G.

Remark. If $G = \{0\}$, then we get the classical concept of exact controllability from Definition 5.

Remark. Exact controllability implies exact enlarged controllability (EEC) for every set G. The inverse is, however, untrue.

Theorem 1. *System (1) is said to be exactly enlarged controllable if, and only if,*

$$G - \left\{ \mathscr{R}_\alpha(T) y_0 \right\} \cap ImH \neq \emptyset. \tag{16}$$

Proof. Suppose that one has exact enlarged controllability (EEC) of (1) relatively to G, which means that $y_u(T) \in G$. Then, $y_u(T) = \mathscr{R}_\alpha(T)y_0 + Hu$ and, denoting

$$w = y_u(T) - \mathscr{R}_\alpha(T)y_0 = Hu,$$

it follows that $w \in ImH$ and $w \in G - \{\mathscr{R}_\alpha(T)y_0\}$. Thus, (16) holds. Conversely, suppose (16) is true. Then, there exists $z \in G - \{\mathscr{R}_\alpha(T)y_0\}$ such that $z \in ImH$. So, there exists $u \in L^2(0,T;\mathbb{R}^m)$ such that $z = Hu$. Hence, $z = Hu \in G\{\mathscr{R}_\alpha(T)y_0\}$, $Hu + \mathscr{R}_\alpha(T)y_0 = y_u(T) \in G$, and we have EEC relatively to G.

We recall that an actuator is defined by a couple (D, f), where D is a nonempty closed part of $\overline{\Omega}$, which represents the geometric support of the actuator, and $f \in L^2(D)$, which defines the spacial distribution of the action on the support D. In the case of a pointwise actuator, $D = \{b\}$ and $f = \delta(b - \cdot)$, where δ is the Dirac mass concentrated in b. For more details on actuators, we refer the interested reader to [46,47].

Definition 6. *The actuator (D, f) is said to be G-strategic if one has exact enlarged controllability relatively to G.*

4 Extended RHUM approach

Now, we generalize the RHUM introduced by Lions in [48,49] to the fractional-order case. The aim is to find the control steering system (1) from the initial state y_0 into the functional subspace G. Let us denote by G° the polar space of G. Hence,

$$\varphi_0 \in G^\circ \Longleftrightarrow \langle \varphi_0, \phi \rangle = 0 \quad \forall \phi \in G,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\Omega)$. Let us also denote by $\mathscr{A}^*$ the adjoint operator of $\mathscr{A}$ and, for any $\varphi_0 \in G^\circ$, consider the following adjoint system:

$$\begin{cases} {}_tD_T^\alpha Q\varphi(t) = -\mathscr{A}^* Q\varphi(t), \\ \lim\limits_{t \to T^-} {}_tI_T^{1-\alpha} Q\varphi(t) = \varphi_0 \in D(\mathscr{A}^*) \subseteq L^2(\Omega). \end{cases} \tag{17}$$

From Lemma 2, (17) can be rewritten as follows:

$$\begin{cases} {}_0D_t^\alpha \varphi(t) = -\mathscr{A}^* \varphi(t), \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} \varphi(t) = \varphi_0 \in D(\mathscr{A}^*) \subseteq L^2(\Omega), \end{cases} \tag{18}$$

with solution given by $\varphi(t) = -t^{\alpha-1} K_\alpha^*(t) \varphi_0$.

4.1 Excitation of the system with a zone actuator

We consider system (1) excited by a zone actuator $\mathscr{B}u(t) = \chi_D f(x) u(t)$. Then, the system is written as follows:

$$\begin{cases} {}_0^CD_t^\alpha y(t) = \mathscr{A}y(t) + \chi_D f(x) u(t), \quad t \in [0,T], \\ y(0) = y_0 \in D(\mathscr{A}). \end{cases} \tag{19}$$

Let $w_i(x)$ denote the eigenfunctions of operator $\mathscr{A}$ associated with the eigenvalues λ_i. For any $\varphi_0 \in G^\circ$, we define the following semi-norm on G°:

$$\|\varphi_0\|_{G^\circ}^2 := \int_0^T \langle f, \varphi(t) \rangle_{L^2(D)}^2 dt. \tag{20}$$

Theorem 2. *The semi-norm (20) defines a norm on G° if $\langle w_i, f \rangle_{L^2(D)} \neq 0$. In that case, we have exact enlarged controllability relatively to G.*

Proof. We consider the following problem:

$$\begin{cases} {}_0^CD_t^\alpha \Psi(t) = \mathscr{A}\Psi(t) + \chi_D f(x) u(t), \quad t \in [0,T], \\ \Psi(0) = y_0 \in D(\mathscr{A}). \end{cases} \tag{21}$$

The solution $\Psi : [0,T] \to L^2(\Omega)$ of (21) is continuous. If we can find $\varphi_0 \in G^\circ$ such that

$$\Psi(T) \in G, \tag{22}$$

then $u = \langle f, \varphi(t) \rangle_{L^2(D)}$ is the control that ensures the exact enlarged controllability relatively to G and

$$y(u) = \Psi.$$

To explain (22), it is necessary to introduce the orthogonal projection $\mathscr{P}$ on the orthogonal of G, denoted by $G^\perp$. Let us define the affine operator $M : G^\circ \to G^\perp$ such that

$$M\varphi_0 = \mathscr{P}(\Psi(T)). \tag{23}$$

Then, we need to solve equation

$$M\varphi_0 = 0. \tag{24}$$

For that, we decompose M into two parts: a linear and a constant one. Let Ψ_0 be solution of

$$\begin{cases} {}_0^C D_t^\alpha \Psi_0(t) = \mathscr{A}\Psi_0(t) + \chi_D f(x)u(t), \quad t \in [0,T], \\ \Psi_0(0) = 0, \end{cases} \tag{25}$$

and Ψ_1 solution of

$$\begin{cases} {}_0^C D_t^\alpha \Psi_1(t) = \mathscr{A}\Psi_1(t), \quad t \in [0,T], \\ \Psi_1(0) = y_0 \in D(\mathscr{A}). \end{cases} \tag{26}$$

Then,

$$M\varphi_0 = \mathscr{P}(\Psi_0(T)) + \mathscr{P}(\Psi_1(T)), \tag{27}$$

where we set $M_0\varphi_0 = \mathscr{P}(\Psi_0(T))$ with $M_0 \in \mathscr{L}(G^\circ, G^\perp)$. From (24) and (27), we can solve

$$M_0\varphi_0 = -\mathscr{P}(\Psi_1(T)). \tag{28}$$

For that, we compute the scalar product

$$\mu = \langle M_0\varphi_0, \varphi_0 \rangle, \qquad \varphi_0 \in G^\circ, \tag{29}$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing of $G^\perp$ and G°. By definition,

$$\langle \mathscr{P}(\tilde{g}), \bar{g} \rangle = 0, \qquad \forall \bar{g} \in G^\circ. \tag{30}$$

Using (30), we have

$$\mu = \langle \Psi_0, \varphi_0 \rangle \tag{31}$$

for $\bar{g} = \varphi_0$, $\tilde{g} = \Psi_0(T)$. To compute the last expression (31), we multiply system (25) by φ, integrating over $Q = \Omega \times [0,T]$. We obtain that

$$\int_0^T \int_\Omega {}_0^C D_t^\alpha \Psi_0(t)\varphi(t)dxdt - \int_0^T \int_\Omega \mathscr{A}\Psi_0(t)\varphi(t)dxdt = \int_0^T \int_\Omega \chi_D f(x)u(t)\varphi(t)dxdt.$$

Using Lemma 3 (fractional Green's formula), we have

$$\begin{aligned} -\int_0^T \int_\Omega \mathscr{A}\Psi_0(t)\varphi(t)dxdt = -\int_0^T \int_{\partial\Omega} \frac{\partial \Psi_0(t)}{\partial \nu_{\mathscr{A}}}\varphi(t)d\sigma dt \\ + \int_0^T \int_{\partial\Omega} \Psi_0(t)\frac{\partial \varphi(t)}{\partial \nu_{\mathscr{A}}} d\sigma dt - \int_0^T \int_\Omega \Psi_0 \mathscr{A}^* \varphi(t)dxdt \end{aligned} \tag{32}$$

and

$$\begin{aligned} \int_0^T \int_\Omega {}_0^C D_t^\alpha \Psi_0(t)\varphi(t)dxdt = \int_0^T \int_\Omega \Psi_0(t) {}_0^C D_t^\alpha \varphi(t)dxdt + \int_{\partial\Omega} \Psi_0(T) \lim_{t\to T} {}_t I_T^{1-\alpha}\varphi(T)d\sigma \\ - \int_{\partial\Omega} \Psi_0(0) \lim_{t\to T} {}_t I_T^{1-\alpha}\varphi(0)d\sigma. \end{aligned} \tag{33}$$

From (32) and (33), it follows that

$$\langle M_0\varphi_0,\varphi_0\rangle = \int_0^T \left(\langle f(x),\varphi(t)\rangle_{L^2(D)}\right)^2 dt = \|\varphi_0\|^2_{G^\circ}.$$

Hence,

$$\mu = \int_0^T \left(\langle f(x),\varphi(t)\rangle_{L^2(D)}\right)^2 dt. \tag{34}$$

The essential point now is that the previous formula (34) is a semi-norm on G°. We prove that if $\langle w_i, f\rangle_{L^2(D)} \neq 0$, then the mapping (20) is a norm, which is equivalent to the norm of G°. The mapping (20) is a norm on G°:

$$\|\varphi_0\|_{G^\circ} = 0 \Longleftrightarrow \langle f,\varphi(t)\rangle^2_{L^2(D)} = 0,$$

which is equivalent to

$$-\sum_{i=1}^{\infty} t^{\alpha-1}\alpha\int_0^\infty \theta\phi_\alpha(\theta)e^{\lambda_i(t^\alpha\theta)}d\theta\langle f,w_i\rangle\langle\varphi_0,w_i\rangle = 0. \tag{35}$$

Thus, (35) gives

$$\langle f,w_i\rangle\langle\varphi_0,w_i\rangle = 0.$$

Using the assumption that $\langle f,w_i\rangle \neq 0$, we deduce that $\langle\varphi_0,w_i\rangle = 0$. Therefore, $\varphi_0 = 0$, (20) defines a norm on G°, and μ is an isomorphism from G° to $G^\perp$. Moreover, equation (28) admits a unique solution.

4.2 Excitation of the system with a pointwise actuator

Now, we consider system (1) excited by a pointwise actuator. In this case, the control is of type $Bu(t) = \delta(x-b)u(t)$, where $b \in \Omega$ refers to the location of the actuator and $u \in U$. Hence, system (1) is written as follows:

$$\begin{cases} {}_0^C D_t^\alpha y(t) = \mathscr{A}y(t) + \delta(x-b)u(t), & t\in[0,T],\\ y(0) = y_0 \in D(\mathscr{A}). \end{cases} \tag{36}$$

For $\varphi_0 \in G^\circ$, we consider the adjoint system

$$\begin{cases} {}_t D_T^\alpha Q\varphi(t) = -\mathscr{A}^* Q\varphi(t),\\ \lim_{t\to T^-} {}_t I_T^{1-\alpha} Q\varphi(t) = \varphi_0 \in D(\mathscr{A}^*) \subseteq L^2(\Omega), \end{cases} \tag{37}$$

and the mapping

$$\|\varphi_0\|^2_{G^\circ} := \int_0^T \varphi^2(b,t)dt, \tag{38}$$

which defines a semi-norm on G°. Let us consider system

$$\begin{cases} {}_0^C D_t^\alpha \Phi(t) = \mathscr{A}\Phi(t) + \delta(x-b)u(t), & t\in[0,T],\\ \Phi(0) = y_0 \in D(\mathscr{A}), \end{cases} \tag{39}$$

and the operator $M: G^\circ \to G^\perp$ defined by

$$M\varphi_0 = \mathscr{P}(\Phi(T)),$$

where we write M as

$$M\varphi_0 = \mathscr{P}(\Phi_0(T)+\Phi_1(T))$$

with Φ_0 and Φ_1 solutions of systems

$$\begin{cases} {}_0^C D_t^\alpha \Phi_0(t) = \mathscr{A}\Phi_0(t) + \delta(x-b)u(t), & t\in[0,T],\\ \Phi_0(0) = 0, \end{cases} \tag{40}$$

and

$$\begin{cases} {}_0^C D_t^{\alpha} \Phi_1(t) = \mathscr{A}\Phi_1(t), \quad t \in [0,T], \\ \Phi_1(0) = y_0 \in D(\mathscr{A}), \end{cases} \tag{41}$$

respectively. Let us set $M_0 \varphi_0 = \mathscr{P}(\Phi_0(T))$ with $M_0 \in \mathscr{L}(G^\circ, G^\perp)$. Then, all reduces to solve

$$M_0 \varphi_0 = -\mathscr{P}(\Phi_1(T)). \tag{42}$$

Similar arguments as the ones used in Section 4.1 allow us to prove the following result:

Theorem 3. *If $w_i(b) \neq 0$, then the mapping (38) defines a norm on G° and one has exact enlarged controllability (EEC) relatively to G. Moreover, the control*

$$u = \varphi(b,t)$$

ensures the EEC into G.

5 Fractional optimal control

In this section, we are concerned with the following optimization problem:

$$\begin{cases} \inf \mathscr{J}(u), \\ u \in U_{ad}, \end{cases} \tag{43}$$

where

$$\mathscr{J}(u) = \frac{1}{2}\int_0^T \|u\|_U^2 dt$$

and the feasible set $U_{ad} = \{u \in U \mid y_u(T) \in G\}$ is assumed to be non-empty.

Theorem 4. *Assume that one has exact enlarged controllability relatively to G. Then, the optimal control problem (43) has a unique solution given by $u^*(t) = \langle f, \varphi(t)\rangle$ in case of a zone actuator, and $u^*(t) = \varphi(b,t)$ in case of a pointwise actuator. Such control ensures the transfer of system (1) into G with a minimum energy cost, in the sense of $\mathscr{J}$.*

Proof. Suppose that we have exact enlarged controllability relatively to G. Then, we set $\varepsilon > 0$ and we consider the following problem:

$$\mathscr{J}_\varepsilon(u,z) = \frac{1}{2}\int_0^T u^2(t)dt + \frac{1}{2\varepsilon}\int_Q \left({}_0^C D_t^{\alpha} z(t) - \mathscr{A}z(t) - \chi_D f(x)u(t)\right)^2 dQ, \tag{44}$$

where

$$\begin{cases} {}_0^C D_t^{\alpha} z(t) - \mathscr{A}z(t) - \chi_D f(x)u(t) \in L^2(Q), \\ z(0) = z_0 \in D(\mathscr{A}), \\ z_u(T) \in G. \end{cases} \tag{45}$$

The set of pairs (u,z) that verify (45), denoted by W, is nonempty, and we consider problem

$$\begin{cases} \inf \mathscr{J}_\varepsilon(u,z), \\ (u,z) \in W. \end{cases} \tag{46}$$

Let $\{u_\varepsilon, z_\varepsilon\}$ be solution of (46). Then,

$$0 < \mathscr{J}_\varepsilon(u_\varepsilon, z_\varepsilon) = \inf \mathscr{J}_\varepsilon(u,z) < \inf \mathscr{J}_\varepsilon(u) < \infty, \quad u \in U_{ad}, \tag{47}$$

where $\mathscr{J}_\varepsilon(u) = \frac{1}{2}\int_0^T u^2(t)dt$. Tending ε to 0, we conclude that

$$\begin{cases} \|u_\varepsilon\| \leq C, \\ \|{}_0^C D_t^{\alpha} z(x,t) - \mathscr{A}z(x,t) - \chi_D f(x)u(t)\| \leq C\sqrt{\varepsilon}, \end{cases} \tag{48}$$

where C represents different positive constants independent of ε. It follows from (48) that

$$\|{}_0^C D_t^\alpha z(x,t) - \mathscr{A} z(x,t)\| \leq C(1+\sqrt{\varepsilon}).$$

Hence, when $\varepsilon \to 0$, we have that u_ε is bounded and we can extract a sequence such that

$$\begin{array}{ll} u_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in} & U, \\ z_\varepsilon \rightharpoonup z \ \text{weakly in} & L^2(Q). \end{array}$$

By the semi-continuity of $\mathscr{J}$, one has

$$\mathscr{J}(u^*) \leq \liminf \mathscr{J}_\varepsilon(u_\varepsilon) \leq \liminf \mathscr{J}_\varepsilon(u_\varepsilon, z_\varepsilon).$$

Therefore,

$$\mathscr{J}(u^*) = \inf \mathscr{J}(u), \quad u \in U_{ad},$$

and

$$u^* = \tilde{u}.$$

Define

$$p_\varepsilon = -\frac{1}{\varepsilon}\left({}_0^C D_t^\alpha z_\varepsilon(x,t) - \mathscr{A} z_\varepsilon(x,t) - \chi_D f(x) u_\varepsilon(t)\right).$$

The Euler equation relatively to problem (46) is given by

$$\int_0^T u_\varepsilon(t)u(t)dt - \int_0^T \langle p_\varepsilon, {}_0^C D_t^\alpha \eta(t) - \mathscr{A}\eta(t)\rangle dt = \int_0^T \langle p_\varepsilon, f\rangle u(t)dt$$

with $u \in U_{ad}$ and η such that

$$\begin{cases} {}_0^C D_t^\alpha \eta(t) - \mathscr{A}\eta(t) = \chi_D f(x) u(t) & \text{in} \quad Q, \\ \eta(0) = 0 & \text{on} \quad \Omega, \\ \eta(T) \in G. \end{cases}$$

We deduce that p_ε satisfies

$$\begin{cases} {}_0^C D_t^\alpha p_\varepsilon(t) - \mathscr{A} p_\varepsilon(t) = \chi_D f(x) \langle p_\varepsilon, f\rangle_{L^2(D)} & \text{in} \quad Q, \\ p_\varepsilon(0) = 0 & \text{on} \quad \Omega, \end{cases}$$

and $\langle \eta(T), p_\varepsilon(T)\rangle = 0$ for all η with $\eta(T) \in G$. Then, $p_\varepsilon \in G^\circ$. If we suppose that

$$\int_0^T \langle p_\varepsilon, f\rangle^2 dt \geq C\|p_\varepsilon(T)\|^2_{H_0^1(\Omega)},$$

then we can switch to the limit when ε tends to 0. Moreover, because we have exact enlarge controllability relatively to G, we obtain the following optimality problem:

$$\begin{cases} {}_0^C D_t^\alpha z(t) - \mathscr{A} z(t) = \chi_D f(x) u(t) & \text{in} \quad Q, \\ z(0) = z_0(x) & \text{on} \quad \Omega, \\ {}_0^C D_t^\alpha p(t) - \mathscr{A} p(t) = \chi_D f(x) \langle p, f\rangle_{L^2(D)} & \text{in} \quad Q, \\ p(0) = 0 & \text{on} \quad \Omega, \\ p(T) \in G^\circ. \end{cases}$$

Thus, we take $p(T) \in G^\circ$ and we introduce the solution φ of (18). Then, $\psi = z$ if $\psi(T) \in G$, which proves that (28) has a unique solution for $\varphi_0 \in G^\circ$.

6 Examples

Two examples illustrate the obtained results as follows:

6.1 Example 1: case of a zonal actuator

Let us consider the following time fractional differential equation with a zonal actuator: $Bu(t) = \chi_{[a,b]}u(t)$, $0 \le a \le b \le 1$,

$$\begin{cases} {}_0^C D_t^{0.4} z(t) = \Delta z(t) + \chi_{[a,b]}u(t) & [0,1]\times[0,T], \\ z(x,0) = z_0(x) & [0,1], \\ z(0,t) = z(1,t) = 0 & [0,T]. \end{cases} \tag{49}$$

Here, the state space is $L^2(0,1)$. Since the operator $\mathscr{A} = \Delta = \frac{\partial^2 \cdot}{\partial x^2}$ generates a compact, analytic, self-adjoint C_0-semigroup, we have $\mathscr{A} = \Delta = \frac{\partial^2 \cdot}{\partial x^2}$ and

$$S(t)z(x) = \sum_{i=1}^{+\infty} e^{\lambda_i t}(z, w_i)_{L^2(0,1)} w_i(x),$$

where $\lambda_i = -i^2\pi^2$ and $w_i(x) = \sqrt{2}\sin(i\pi x)$. Moreover,

$$\begin{aligned} K_{0.4}(t)z(x) &= 0.4\int_0^\infty \theta\phi_{0.4}(\theta)S(t^{0.4}\theta)zd\theta \\ &= 0.4\int_0^\infty \theta\phi_{0.4}(\theta)\sum_{i=1}^{\infty} e^{\lambda_i t^{0.4}\theta}(z,w_i)_{L^2(0,1)}w_i(x)d\theta \\ &= 0.4\sum_{i=1}^{\infty}(z,w_i)_{L^2(0,1)}w_i(x)\int_0^\infty \theta\phi_{0.4}(\theta)e^{\lambda_i t^{0.4}\theta}d\theta. \end{aligned}$$

It follows from (12) and Taylor's expansion of the exponential that

$$\begin{aligned} K_{0.4}(t)z(x) &= 0.4\sum_{i=1}^{\infty}(z,w_i)_{L^2(0,1)}w_i(x)\sum_{j=0}^{\infty}\int_0^\infty \frac{\left(\lambda_i t^{0.4}\right)^j}{j!}\theta^{j+1}\phi_{0.4}(\theta)d\theta \\ &= \sum_{i=1}^{\infty}(z,w_i)_{L^2(0,1)}w_i(x)\sum_{j=0}^{\infty}\frac{0.4(j+1)\left(\lambda_i t^{0.4}\right)^j}{\Gamma(1+0.4j+0.4)} \\ &= \sum_{i=1}^{\infty}E_{0.4,0.4}(\lambda_i t^{0.4})(z,w_i)_{L^2(0,1)}w_i(x), \end{aligned}$$

where $E_{p,q}(z) := \sum_{i=0} \frac{z^i}{\Gamma(pi+q)}$, $Re(p) > 0$, $q, z \in \mathbb{C}$, is the generalized Mittag–Leffler function (see [50]). Similarly, we have:

$$\begin{aligned} \mathscr{R}_{0.4}(t)z(x) &= \int_0^\infty \phi_{0.4}(\theta)S(t^{0.4}\theta)zd\theta \\ &= \sum_{i=0}^{\infty}(z,w_i)_{L^2(0,1)}E_{0.4,1}(\lambda_i t^{0.4})w_i(x). \end{aligned}$$

Since operator Δ generates a compact, analytic, self-adjoint and continuous semigroup, it follows that

$$\begin{aligned} (H^*z)(t) &= \mathscr{B}^*(T-t)^{-0.6}K_{0.4}^*(T-t)z(t) \\ &= \mathscr{B}^*(T-t)^{-0.6}\sum_{i=1}^{\infty}E_{0.4,0.4}\left(\lambda_i(T-t)^{0.4}\right)(z,w_i)_{L^2(0,1)}w_I(x) \\ &= (T-t)^{-0.6}\sum_{i=1}^{\infty}E_{0.4,0.4}\left(\lambda_i(T-t)^{0.4}\right)(z,w_i)_{L^2(0,1)}\int_a^b w_i(x)dx. \\ &= (T-t)^{-0.6}\sum_{i=1}^{\infty}E_{0.4,0.4}\left(\lambda_i(T-t)^{0.4}\right)(z,w_i)_{L^2(0,1)}\frac{\sqrt{2}}{i\pi}[\cos(i\pi x)]_a^b \\ &= (T-t)^{-0.6}\sum_{i=1}^{\infty}E_{0.4,0.4}\left(\lambda_i(T-t)^{0.4}\right)(z,w_i)_{L^2(0,1)}\frac{\sqrt{2}}{i\pi}\sin\frac{i\pi(a+b)}{2}\sin\frac{i\pi(a-b)}{2}. \end{aligned}$$

Moreover, by Theorem 2, we get that if system (49) is enlarged controllable, then

$$\begin{aligned}\varphi_0 \to \|\varphi_0\|_{(L^2(0,1))^*} &= \sum_{i=1}^{\infty} t^{-0.6} 0.4 \int_0^{\infty} \theta \phi_{0.4}(\theta) e^{\lambda_i (t^{0.4}\theta)} d\theta \langle f, w_i\rangle \langle \varphi_0, w_i\rangle \\ &= t^{-0.6} K_{0.4}(t) \langle \varphi_0, w_i\rangle \\ &= t^{-0.6} \sum_{i=1}^{\infty} E_{0.4,0.4}(\lambda_i t^{0.4})(z, w_i)_{L^2(0,1)} w_i(x) \langle \varphi_0, w_i\rangle\end{aligned}$$

defines a norm on $(L^2(0,1))^*$. We find that the control given by

$$u^*(t) = t^{-0.6} \sum_{i=0}^{+\infty} E_{0.4,0.4}(\lambda_i t^{0.4})(z, w_i)_{L^2(0,1)} \langle \varphi_0, w_i\rangle$$

steers system (49) to $L^2(0,1)$ at time T.

6.2 *Example 2: case of a pointwise actuator*

We now consider the following system with a pointwise control $Bu(t) = u(t)\delta(x-b)$, $0 < b < 1$:

$$\begin{cases} {}_0^C D_t^{0.4} z(t) = \Delta z(t) + u(t)\delta(x-b) & [0,1]\times[0,T], \\ z(x,0) = z_0(x) = 0 & [0,1], \\ z(0,t) = z(1,t) = 0 & [0,T]. \end{cases} \tag{50}$$

Let the position of the actuator be $b = 1/3$. Similar to the first example, we have:

$$\begin{aligned} &\lambda_i = -i^2\pi^2, \qquad w_i(x) = \sqrt{2}\sin(i\pi x), \quad x \in [0,1], \\ &S(t)z(x) = \sum_{i=1}^{+\infty} e^{\lambda_i t} (z, w_i)_{L^2(0,1)} w_i(x), \\ &\text{and} \quad K_{0.4}(t)z(x) = \sum_{i=1}^{+\infty} E_{0.4,0.4}(\lambda_i t^{0.4}) (z, w_i)_{L^2(0,1)} w_i(x). \end{aligned}$$

Moreover, by Theorem 3, we get that if system (50) is enlarged controllable, then

$$\begin{aligned}\varphi_0 \to \|\varphi_0\|_{(L^2(0,1))^*} &= \int_0^T \left\| (T-s)^{-0.6} K_{0.4}^*(T-s)\varphi_0(b) \right\|^2 ds \\ &= \int_0^T \left\| (T-s)^{-0.6} \sum_{i=0}^{+\infty} E_{0.4,0.4}(\lambda_i (T-s)^{0.4})(z, w_i)_{L^2(0,1)} \varphi_0(b) \right\|^2 ds\end{aligned}$$

defines a norm on $(L^2(0,1))^*$. We also have that $M\varphi_0 = \mathscr{P}(\varphi_1(T))$ is an affine operator from $(L^2(0,1))^*$ to $(L^2(0,1))$, where $\varphi_1(T)$ is the solution of system

$$\begin{cases} {}_0^C D_t^{0.4} \varphi_1(t) = \Delta \varphi_1(t) + (T-t)^{-0.6} K_{0.4}^*(T-t)\varphi_0(b), \\ \varphi_1(0) = 0, \\ \varphi_1(0,t) = \varphi_1(1,t) = 0. \end{cases} \tag{51}$$

Then, by Theorem 4, we find that the control given by

$$u^*(t) = (T-t)^{-0.6} \sum_{i=0}^{+\infty} E_{0.4,0.4}(\lambda_i (T-s)^{0.4})(z, w_i)_{L^2(0,1)} \varphi_0(b)$$

steers system (50) to $L^2(0,1)$ at time T, where φ_0 is the solution of

$$M_0 \varphi_0 = -\mathscr{P}(\Phi_1(T)) \tag{52}$$

and $\Phi_1(t)$ solves

$$\begin{cases} {}_0^C D_t^{0.4} \Phi_1(t) = \Delta \Phi_1(t), & t \in [0,T], \\ \Phi_1(0) = z_0(x) = 0 \in D(\mathscr{A}), \\ \Phi_1(0,t) = \Phi_1(1,t) = 0. \end{cases}$$

Moreover, u^* is the solution of the minimum problem (43).

7 Conclusion

The present paper addressed fractional diffusion equations in the sense of Caputo. We investigated exact enlarged controllability for such control systems using an extended Reverse Hilbert Uniqueness Method (RHUM) and a penalization technique covering both zone and pointwise actuators. The optimal control of a minimum energy problem has been explicitly characterized. The two methods complete each other: using the RHUM approach, we computed the control steering the system for both cases of zone and pointwise actuators; and using the penalization method, we proved that such control is unique. We claim that our techniques and results can be adapted to cover boundary conditions of Dirichlet, Neumann or mixed type, and to deal with other classes of controls (e.g., distributed controls). The present results can be extended to more recent notions of derivatives, e.g., to Atangana–Baleanu operators [51,52]. Another research will conduct numerical experiments illustrating theoretical results.

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