Comparing techniques for modelling uncertainty in a maritime inventory routing problem

Filipe Rodrigues[a], Agostinho Agra[a], Marielle Christiansen[b], Lars Magnus Hvattum[c], Cristina Requejo[a]

[a] *Department of Mathematics and Center for Research and Development in Mathematics and Applications, University of Aveiro, Portugal*
[b] *Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology, Norway*
[c] *Faculty of Logistics, Molde University College, Norway*

Abstract

Uncertainty is inherent in many planning situations. One example is in maritime transportation, where weather conditions and port occupancy are typically characterized by high levels of uncertainty. This paper considers a maritime inventory routing problem where travel times are uncertain. Taking into account possible delays in the travel times is of main importance to avoid inventory surplus or shortages at the storages located at ports.

Several techniques to deal with uncertainty, namely deterministic models with inventory buffers; robust optimization; stochastic programming and models incorporating conditional value-at-risk measures, are considered. The different techniques are tested for their ability to deal with uncertain travel times for a single product maritime inventory routing problem with constant production and consumption rates, a fleet of heterogeneous vessels and multiple ports. At the ports, the product is either produced or consumed and stored in storages with limited capacity. We assume two-stages of decisions, where the routing, the visit order of the ports and the quantities to load/unload are first-stage decisions (fixed before the uncertainty is revealed), while the visit time and the inventory levels at ports are second-stage decisions (adjusted to the observed

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Email addresses: `fmgrodrigues@ua.pt` (Filipe Rodrigues), `aagra@ua.pt` (Agostinho Agra), `mc@iot.ntnu.no` (Marielle Christiansen), `hvattum@himolde.no` (Lars Magnus Hvattum), `crequejo@ua.pt` (Cristina Requejo)

travel times).

Several solution approaches resulting from the proposed techniques are considered. A computational comparison of the resulting solution approaches is performed to compare the routing costs, the amount of inventory bounds deviation, the total quantities loaded and unloaded, and the running times. This computational experiment is reported for a set of maritime instances having up to six ports and five ships.

Keywords: transportation, maritime inventory routing, travel times uncertainty, stochastic programming

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1. Introduction

Maritime transportation is conditioned by unpredictable events such as bad weather conditions and queues forming in ports. Such events lead to high levels of uncertainty for the travel times of vessels when transporting goods between ports. Decision makers should take into account such uncertainty when designing the distribution plans to prevent inventory disruptions when delays occur. This paper considers a maritime inventory routing problem (MIRP) where a single product is produced in a set of producer ports and transported by a fleet of heterogeneous vessels to the consumption ports. Both the production and consumption rates are constant over the planning horizon at each port.

Lower and upper limits on the storage capacity at each port are considered. The MIRP consists of designing routes and schedules for a fleet of ships and to determine the quantities to load and unload at each port to maintain the inventory levels between the given limits, while minimizing the transportation and port costs. The travel times include both the sailing times and the berth times of the ships at the destination ports and are considered uncertain. Consequently, assuming deterministic values such as average times for the travel times, may result in frequent inventory disruptions since the schedules do not take into account possible delays. Conversely, designing conservative plans that consider worst case scenarios for the travel times may prevent inventory disruptions (shortages at consumption ports and surplus at production ports) but usually give solutions with high transportation costs. Hence, it is desirable to find optimization methods that are able to generate solutions providing a good tradeoff between the routing costs and inventory disruptions.

The scientific literature on MIRPs is extensive, with surveys provided by Papageorgiou et al. [16], Christiansen et al. [10], and Christiansen and Fagerholt [9, 8]. Surveys on both land and maritime transportation were produced by Coelho et al. [12] and

Andersson et al. [5]. However, relatively few contributions exist for MIRPs under uncertainty, so most of the MIRP models described in the literatur are deterministic. In real life, these models are often resolved when updated information about the values of the parameters appear. For road-based inventory routing problems considering uncertainty, there exists a solid amount of research. A survey on the inventory routing problems with stochastic lead times and demands can be found in [20]. We limit our further literature review to MIRPs under uncertainty.

Different sources of uncertainty have been considered in the MIRP literature, such as uncertainty in sailing time, port time, production rates, and demand. Halvorsen-Weare et al. [14] studied a liquefied natural gas MIRP and developed heuristic strategies for obtaining robust solutions considering uncertain sailing times and production rates. For a crude oil transportation and inventory problem, Cheng and Duran [7] described a decision support system that takes into account uncertainty in sailing times and demands. The problem is formulated as a discrete time Markov decision process and solved by using discrete event simulation and optimal control theory. Recently, Dong et al. [13] propose a rolling-horizon reoptimization framework and policy analysis for a MIRP. They consider different sources of uncertainty such as time charter vessel availability, trip delays, pick-up window information and consumption/production rates. The framework included a MIP model, stochastic simulations to account for uncertainty sources, and an algorithm integrating reoptimization and stochastic simulation results.

Christiansen and Nygreen [11] used soft inventory levels to handle uncertainties in sailing and port time, and these levels were transformed into soft time windows in a single product deterministic MIRP model. Furthermore, Rakke et al. [17] and Sherali and Al-Yakoob [21, 22] introduced penalty functions for deviating from the customer contracts and the storage limits in their deterministic MIRP models, respectively. The problem considered in Rakke et al. [17] is an annual delivery plan problem involving a single producer and multiple customers in the liquefied natural gas business. This problem is also studied by Zhang et al. [24], but with time windows for deliveries and uncertain travel disruptions. They propose a Lagrangian heuristic scheme where soft constraints are used to derive flexible solutions to MIRPs. In Zhang et al. [23], the length and placement of the time windows are decision variables. The problem is formulated as a two-stage stochastic mixed-integer program (MIP), and the authors propose a two-phase solution approach that considers a sample set of disruptions as well as their recovery solutions. However, Agra et al. [2] were the first to use stochastic programming to model uncertain sailing and port times for a MIRP with several products and inventory management at the consumption ports only. A two-stage stochastic programming model with recourse was developed. A similar problem to that considered here was studied by Agra et al. [3, 4]. In [3] the travel times are considered stochastic, and the stochastic program is solved using a matheuristic. Agra et al. [4] investigated the use of adaptable robust optimization where the sailing times are considered unknown and belong to an uncertainty set. A min-max exact approach is proposed to solve the resulting robust optimization problem.

Some attempts have been made in previous literature to compare different techniques to deal with uncertainty, for related problems. Maggioni et al. [15] compare stochastic programming with robust optimization for a real transportation problem where demand and costs are uncertain. Ribas et al. [18] compare a two-stage stochastic model with a robust min-max regret model and a robust max-min model for an integrated oil supply chain problem. Adida and Perakis [1] provide a computational study to compare robust and stochastic models for a dynamic pricing and inventory control problem. These studies do not provide common conclusions regarding the best strategy. However, two observations were made: the computational complexity of robust optimization is lower than for stochastic programming with recourse, and stochastic programming models are good as long as the distributional assumptions are correct, but not necessarily otherwise.

The main goal of this paper is to compare different techniques to handle uncertainty in the MIRP. We consider a deterministic model with inventory buffers, which in the case of consumption ports corresponds to safety stocks, the stochastic programming method proposed in [3], the robust optimization method proposed in [4], and introduce two new models based on the *conditional value-at-risk* (*CVaR*) measure. As in [3, 4], we assume two-stage models where the vessel routes and the quantities to load and unload are fixed before the travelling times are known, while the arrival times to ports and the inventory levels are adjusted to the travelling times. One *CVaR* model can be regarded as an intermediate strategy that combines the neutral position regarding the risk of inventory disruption resulting from the stochastic programming model with the most conservative model resulting from the robust optimization. In this model the *CVaR* is used on top of the stochastic model in order to amplify the penalty assigned to the inventory disruptions when the penalty given in the stochastic program attains a given threshold value (given by the Value-at-Risk measure). The other model uses the value-at-risk to define soft inventory bounds that establish the inventory buffers for each load/unload operation, and the $CVaR$ measure is used to penalize solutions that violate the given soft inventory bounds.

These models are also compared with the deterministic model where the mean values are assumed for the uncertain parameters (travel times). Based on the proposed models, thirteen solution approaches are derived by specifying different values for parameters, as well as combining the models. The computational comparison aims to provide insight into the quality of the solutions resulting from each solution approach. This quality is measured in terms of the routing costs and inventory disruptions. Moreover, the models are also compared regarding the execution running times.

The paper has two main contributions. One is the introduction of new models to the MIRP with uncertain travel times based on the $CVaR$ measure. The other contribution is related to the main motivation for this research, namely to compare the most common models to handle uncertainty in travel times for the MIRP, and to evaluate the gains of using these models compared to solving the deterministic problem. Our computational tests show that efficient solutions can be obtained by utilizing information obtained when solving stochastic programming models, either in the classical form or by incorporating $CVaR$.

The paper is organized as follows. The deterministic model is given in Section 2. General models to deal with uncertainty and the corresponding mathematical formulations are discussed in Section 3. In Section 4 we describe the experimental setup and the solution approaches resulting from the general models using suitable parameter values. In Section 5 we report the computational results, and in Section 6 we draw the main conclusions.

2. Deterministic Model for the MIRP

Here we present the deterministic mathematical model of the MIRP, where the travel times are not subject to uncertainty. This model was introduced in [3]. We consider a set N of ports that can be either consumption or production ports, and a set V of heterogeneous vessels. Since each port can be visited several times, ship paths are defined on a network with nodes (i, m), representing the m^{th} visit to port i, and arcs (i, m, j, n), denoting the ship movements from node (i, m) to node (j, n). The set of possible nodes is denoted by S^A. Each vessel v may visit a specific set of nodes denoted by S^A_v and the set of its feasible movements is denoted by S^X_v.

Since both consumption and production ports are considered, a parameter J_i is used to identify the port nature. That parameter takes value 1 if port i is a production port and -1 otherwise. At each port, the quantity to load/unload is limited by a minimum and a maximum value, $\underline{Q}_i$ and $\overline{Q}_i$, respectively. The time required to load/unload one unit of product at port i is denoted by T_i^Q. The time for a vessel start the loading/unloading operation at each port visit must be within a given time window $[A_{im}, B_{im}]$, and a minimum time T_i^B between two consecutive visits to port i must be respected. Each port i has an initial stock level, S_i^0, and a constant production/consumption rate, R_i. During the planing horizon, T, the inventory levels at each port must be kept between the lower and the upper stock limits, denoted by $\underline{S}_i$ and $\overline{S}_i$, respectively. Each vessel v has a fixed capacity, C_v, and the travel time required to travel from port i to port j is T_{ijv}.

To formulate the problem, we consider the following routing variables: x_{imjnv} is 1 if ship v travels directly from node (i, m) to node (j, n) and 0 otherwise; w_{imv} is 1 if ship v visits node (i, m) and 0 otherwise; z_{imv} is 1 if ship v ends its route at node (i, m) and 0 otherwise; y_{im} is 1 if a ship is making the m^{th} visit to port i and 0 otherwise. We also consider the following continuous variables: q_{imv} indicating the quantity loaded/unloaded from ship v at node (i, m); f_{imjnv} representing the quantity that ship v transports from node (i, m) to node (j, n); s_{im} representing the stock level at the start of the m^{th} visit at port i, and the time variables t_{im} defining the start time of the m^{th} visit to port i.

The objective is to minimize the total routing costs. The travel cost of ship v from port i to port j is denoted by C_{ijv}^T. The deterministic problem can now be formulated as follows:

$$\min \quad C(X) = \sum_{v \in V} \sum_{(i,m,j,n) \in S_v^X} C_{ijv}^T x_{imjnv} \tag{1}$$

$$s.t.\ w_{imv} - \sum_{(j,n) \in S_v^A} x_{jnimv} = 0, \qquad v \in V, (i, m) \in S_v^A, \tag{2}$$

$$w_{imv} - \sum_{(j,n) \in S_v^A} x_{imjnv} - z_{imv} = 0, \qquad v \in V, (i, m) \in S_v^A, \tag{3}$$

$$\sum_{v \in V} w_{imv} = y_{im}, \qquad (i, m) \in S^A, \tag{4}$$

$$y_{i(m-1)} - y_{im} \geq 0, \qquad (i, m) \in S^A, \tag{5}$$

$$\sum_{(j,n)\in S_v^A} f_{jnimv} + J_i q_{imv} = \sum_{(j,n)\in S_v^A} f_{imjnv}, \qquad v \in V, (i,m) \in S_v^A, \tag{6}$$

$$f_{imjnv} \leq C_v x_{imjnv}, \qquad v \in V, (i,m,j,n) \in S_v^X, \tag{7}$$

$$\underline{Q}_i w_{imv} \leq q_{imv} \leq min\{C_v, \overline{Q}_i\} w_{imv}, \qquad v \in V, (i,m) \in S_v^A, \tag{8}$$

$$t_{im} + \sum_{v\in V} T_i^Q q_{imv} - t_{jn} + \sum_{v\in V} \max\{T'_{im} + T_{ijv} - A_{jn}, 0\} x_{imjnv} \leq T'_{im} - A_{jn}$$

$$(i,m),(j,n) \in S^A, \tag{9}$$

$$t_{im} - t_{i,m-1} - \sum_{v\in V} T_i^Q q_{i,m-1,v} - T_i^B y_{im} \geq 0, \qquad (i,m) \in S_A : m > 1, \tag{10}$$

$$A_{im} \leq t_{im} \leq B_{im}, \qquad (i,m) \in S^A, \tag{11}$$

$$s_{i1} = S_i^0 + J_i R_i t_{i1}, \qquad i \in N, \tag{12}$$

$$s_{im} = s_{i,m-1} - J_i \sum_{v\in V} q_{i,m-1,v} + J_i R_i (t_{im} - t_{i,m-1}), \qquad (i,m) \in S^A : m > 1, \tag{13}$$

$$s_{im} + \sum_{v\in V} q_{imv} - R_i \sum_{v\in V} T_i^Q q_{imv} \leq \overline{S}_i, \qquad (i,m) \in S^A | J_i = -1, \tag{14}$$

$$s_{im} - \sum_{v\in V} q_{imv} + R_i \sum_{v\in V} T_i^Q q_{imv} \geq \underline{S}_i, \qquad (i,m) \in S^A | J_i = 1, \tag{15}$$

$$s_{i\overline{\mu}_i} + \sum_{v\in V} q_{i,\overline{\mu}_i,v} - R_i (T - t_{i\overline{\mu}_i}) \geq \underline{S}_i, \qquad i \in N | J_i = -1, \tag{16}$$

$$s_{i\overline{\mu}_i} - \sum_{v\in V} q_{i,\overline{\mu}_i,v} + R_i (T - t_{i\overline{\mu}_i}) \leq \overline{S}_i, \qquad i \in N | J_i = 1, \tag{17}$$

$$s_{im} \geq \underline{S}_i, \qquad (i,m) \in S^A | J_i = -1, \tag{18}$$

$$s_{im} \leq \overline{S}_i, \qquad (i,m) \in S^A | J_i = 1, \tag{19}$$

$$x_{imjnv} \in \{0,1\}, \qquad v \in V, (i,m,j,n) \in S_v^X, \tag{20}$$

$$w_{imv}, z_{imv} \in \{0,1\}, \qquad v \in V, (i,m) \in S_v^A, \tag{21}$$

$$y_{im} \in \{0,1\}, \qquad (i,m) \in S^A, \tag{22}$$

$$f_{imjnv} \geq 0, \qquad v \in V, (i,m,j,n) \in S_v^X, \tag{23}$$

$$q_{imv} \geq 0, \qquad v \in V, (i,m) \in S_v^A. \tag{24}$$

For brevity, we omit the explanation of each constraint, as their description can be found

Table 1: Display of first and second-stage decisions for each approach.

Approach	**First-stage decisions**	**Second-stage decisions**
Inventory buffers	Routing, visit order, flow, (un)load time, inventory	
Stochastic *CVaR* Robust	Routing, visit order, flow, (un)load	Time, inventory

in [4]. The value $T'_{im} = \min\{T, B_{im} + T_i^Q \overline{Q}_i\}$ is an upper bound for the end time of the port visit (i, m).

3. Models to deal with uncertainty

Here we describe four models to deal with uncertainty: the creation of inventory buffers by including soft inventory bounds in the deterministic problem, the use of stochastic programming, the use of a risk measure (*CVaR*) to control the violation of the inventory limits and the use of robust optimization. When the inventory buffers are considered the resulting model is deterministic. For the remaining three models the uncertainty of the travel times is made explicit. In the *CVaR* and stochastic programming models, the travel times are assumed to follow a given probability distribution, while in the robust model travel times are assumed to belong to an uncertainty set. The last three types of models are recourse models where some variables (the routing, the visit order, the flow and the quantities to load/unload) represent first-stage decisions while other variables (the time and inventory) represent second-stage decisions. In Table 1 we summarize these decisions.

3.1. Inventory buffers

A practical approach to deal with uncertainty is to create inventory buffers. These buffers are imposed by adding soft inventory bounds to the deterministic problem and to penalize solutions violating these bounds. For the consumption ports, the inventory buffers are just the well-known safety stocks used to prevent inventory shortages when delays occur. For of production ports, the inventory buffers are imposed to prevent the inventory from exceeding the storage capacity.

For each port visit $(i, m) \in S^A$ a lower bound $\underline{SS}_{im}$ (safety stock) on the inventory level is considered if port i is a consumption port, and an upper bound $\overline{SS}_{im}$ is considered if port i is a production port. These inventory bounds can be violated and, in that case, a per unit penalty cost C_i^{SS} is incurred.

For the m^{th} visit to port i, we define the continuous variables d_{im}^+ and d_{im}^- that measure the amount of stock above the upper bound in the production ports and below the lower safety stock in the consumption ports, respectively, see Figure 1.

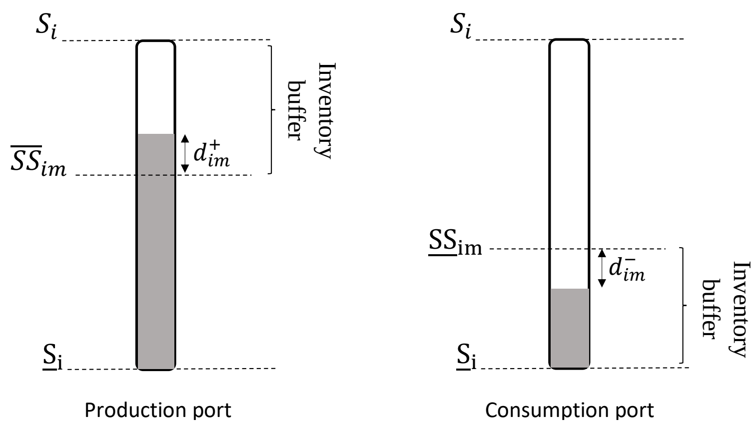

Figure 1: Soft inventory bounds and inventory buffers for the production and consumption ports, respectively.

The mathematical model for the problem with inventory buffers is similar to the model in Section 2 and can be written as follows:

$$\min \; C(X) + \sum_{(i,m)\in S^A|J_i=1} C_i^{SS} d_{im}^+ + \sum_{(i,m)\in S^A|J_i=-1} C_i^{SS} d_{im}^- \tag{25}$$

$$s.t. \quad (2) - (24)$$

$$d_{im}^+ \geq s_{im} - \overline{SS}_{im}, \qquad (i,m) \in S^A|J_i = 1, \tag{26}$$

$$d_{im}^- \geq \underline{SS}_{im} - s_{im}, \qquad (i,m) \in S^A|J_i = -1, \tag{27}$$

$$d_{im}^+ \geq 0, \qquad (i,m) \in S^A|J_i = 1, \tag{28}$$

$$d_{im}^- \geq 0, \qquad (i,m) \in S^A|J_i = -1. \tag{29}$$

The objective function (25) minimizes the total routing cost plus the penalty cost of the stock level deviations from the inventory bounds. Constraints (26) and (27) force variables d_{im}^+ and d_{im}^- to be greater or equal to the deviation of the stock level from the soft inventory bounds for the production and consumption ports, respectively. Constraints (28) and (29) are the nonnegative constraints.

3.2. Stochastic programming

Stochastic programming models for MIRPs were discussed in [2, 3]. Here we follow the approach introduced in [3], where the travel times between ports are assumed to be independent and random, following a known probability distribution. The model is a recourse model with two levels of decisions. The first-stage decisions, those taken before the travel times are known, are the routing decisions, the port visits sequence, the flow and the load/unload quantities. The corresponding first-stage variables are $x_{imjnv}, z_{imv}, w_{imv}, y_{im}, f_{imjnv}$ and q_{imv}. Afterwards, the time of the visits and the inventory level decisions are adjustable to the scenario, that is, are determined after the travel times are known. To ensure feasibility of the second stage problem given a feasible first-stage solution (relatively complete recourse) we assume that the inventory limits can be violated by including a penalty cost P_i for each unit of violation of the inventory limits at each port i, $i \in N$.

To derive the stochastic model, following the sample average approximation method, the true probability distribution for travel times is replaced by a set Ω of discrete scenarios. We denote a scenario by $\xi \in \Omega$, and the variables and parameters that depend on the realization of the uncertainty become functions of ξ. Variables $t_{im}(\xi)$ and $s_{im}(\xi)$ indicate the time and the stock level at node (i, m), when scenario ξ is revealed, and new variables $r_{im}(\xi)$ are introduced to denote the inventory limit violation at node (i, m). When i is a consumption port, $r_{im}(\xi)$ denotes the backlogged consumption, that is the amount of demand satisfied with delay. When i is a production port, $r_{im}(\xi)$ denotes the inventory in excess to the storage capacity. We assume the quantity in excess is not lost but a penalty is incurred, see Figure 2. To facilitate the writing, we name the value of variables $r_{im}(\xi)$ as the backlog at port visit (i, m) independently from i being a consumption or a production port.

The set of time constraints as well as the set of inventory constraints are as follows:

Time constraints

$$t_{im}(\xi) + \sum_{v \in V} T_i^Q q_{imv} - t_{jn}(\xi) + \sum_{v \in V} \max\{T'_{im} + T_{ijv}(\xi) - A_{jn}, 0\} x_{imjnv} \leq T'_{im} - A_{jn} \qquad (i, m), (j, n) \in S^A, \tag{30}$$

$$t_{im}(\xi) - t_{i,m-1}(\xi) - \sum_{v \in V} T_i^Q q_{i,m-1,v} - T_i^B y_{im} \geq 0, \qquad (i, m) \in S_A : m > 1, \xi \in \Omega, \tag{31}$$

$$A_{im} \leq t_{im}(\xi) \leq B_{im}, \qquad (i, m) \in S^A, \xi \in \Omega. \tag{32}$$

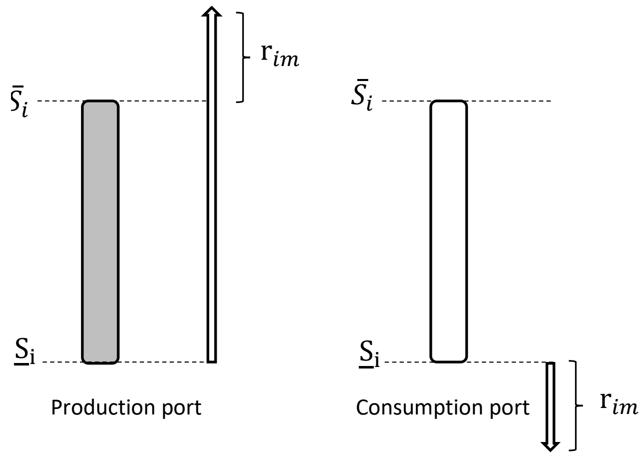

Figure 2: Backlog representation for the production and the consumption ports, respectively.

Inventory constraints

$$s_{i1}(\xi) = S_i^0 + J_i R_i t_{i1}(\xi) - J_i r_{i1}(\xi), \qquad i \in N, \xi \in \Omega, \tag{33}$$

$$s_{im}(\xi) - J_i r_{i,m-1}(\xi) = s_{i,m-1}(\xi) - J_i r_{im}(\xi) - J_i \sum_{v \in V} q_{i,m-1,v} + J_i R_i (t_{im}(\xi) - t_{i,m-1}(\xi)), \qquad (i,m) \in S^A : m > 1, \xi \in \Omega, \tag{34}$$

$$s_{im}(\xi) + \sum_{v \in V} q_{imv} - R_i \sum_{v \in V} T_i^Q q_{imv} \leq \overline{S}_i, \qquad (i,m) \in S^A | J_i = -1, \xi \in \Omega, \tag{35}$$

$$s_{im}(\xi) - \sum_{v \in V} q_{imv} + R_i \sum_{v \in V} T_i^Q q_{imv} \geq \underline{S}_i, \qquad (i,m) \in S^A | J_i = 1, \xi \in \Omega, \tag{36}$$

$$s_{i\overline{\mu}_i}(\xi) + \sum_{v \in V} q_{i,\overline{\mu}_i,v} - R_i(T - t_{i\overline{\mu}_i}(\xi)) + r_{i\overline{\mu}_i}(\xi) \geq \underline{S}_i, \quad i \in N | J_i = -1, \xi \in \Omega, \tag{37}$$

$$s_{i\overline{\mu}_i}(\xi) - \sum_{v \in V} q_{i,\overline{\mu}_i,v} + R_i(T - t_{i\overline{\mu}_i}(\xi)) - r_{i\overline{\mu}_i}(\xi) \leq \overline{S}_i, \quad i \in N | J_i = 1, \xi \in \Omega, \tag{38}$$

$$s_{im}(\xi), r_{im}(\xi) \geq 0, \qquad (i,m) \in S^A, \xi \in \Omega. \tag{39}$$

The objective function that includes the expected value of the penalty cost for the inventory bounds violation is

$$\min \;\; C(X) + \sum_{\xi \in \Omega} \sum_{(i,m) \in S^A} p^{\xi} P_i r_{im}(\xi) \tag{40}$$

where P_i is a penalty for each unit of violation of the inventory limits at port i and p^{ξ} represents the probability that scenario ξ occurs . The stochastic model is given by constraints (2)–(8), (20)–(24), (30)–(39), and objective function (40).

3.3. Conditional Value-at-Risk

The stochastic programming model assumes that the decision maker is neutral regarding the risk of violating inventory limits at the various ports. Moreover, the probability that a given first-stage solution violates the inventory bounds is unknown and can only be indirectly controlled through the variation of the penalty values. The use of risk measures within optimization problems is a recent approach to handle uncertainty. It allow us to generate solutions while keeping control on undesirable events such as inventory limit violations. One of the most common risk measures is the conditional value-at-risk (*CVaR*).

First, we review *CVaR* which uses the Value-at-Risk (*VaR*) concept. Let $f(y, d)$ be a random variable depending on a decision vector $y \in \mathbb{R}^n$ and depending on the realization of a random event d. Let β be a given probability. The $VaR_\beta[f(y,d)]$ is the lowest value of the random variable $f(y, d)$ such that the probability of having a realization lower than this lowest value is higher than the probability $1 - \beta$, i.e.,

$$VaR_\beta[f(y,d)] = \min_\gamma \{\gamma : P[f(y,d) \leq \gamma] \geq 1 - \beta\}$$

The $CVaR_\beta[f(y,d)]$ is the mean value of the random variable $f(y, d)$ having a realization higher than $VaR_\beta[f(y,d)]$ at a probability $1 - \beta$, i.e.,

$$CVaR_\beta[f(y,d)] = E[f(y,d) \mid f(y,d) \geq VaR_\beta[f(y,d)]].$$

To use the *CVaR* in a linear model, the expectation in the definition above can be discretized, see [19], and the *CVaR* can be approximated by

$$CVaR_\beta[f(y,d)] = \min_\gamma \left\{ \gamma + \frac{1}{\beta} \sum_{\xi \in \Omega} p^\xi (f(y, d(\xi)) - \gamma)^+ \right\} \tag{41}$$

where $(a)^+ = \max\{0, a\}$, Ω is the sample set of scenarios and p^ξ is the probability that scenario ξ occurs.

To the best of our knowledge the *CVaR* has not previously been used when solving MIRPs. Using *CVaR*, we create two different models. In the first, the *CVaR* is used to control the global penalty of inventory limit violations (surplus at the production ports and shortages at consumption ports) over the planning horizon. This represents a mix between the stochastic programming technique and the robust optimization technique.

In the second model, the *CVaR* is used to establish an independent inventory buffer for each port visit.

3.3.1. CVaR *based on the total inventory limit violation*

In this approach, function $f(y, d)$ is the total inventory violation penalty cost given by $\sum_{(i,m)\in S^A} P_i r_{im}(\xi)$, with the random event d represented by traveling scenario $\xi \in \Omega$, P_i the penalty cost and $r_{im}(\xi)$ the violation quantity (both defined in Section 3.2). Given a probability β and a set Ω of scenarios, the *VaR*, represented by variable γ, is the minimum value such that the probability of the total penalty cost being lower than this minimum value γ is at least $1 - \beta$. Let $g(\xi) = \left(\sum_{(i,m)\in S^A} P_i r_{im}(\xi) - \gamma\right)^+$. The *CVaR* is given as follows:

$$CVaR = \min_{\gamma} \left\{ \gamma + \frac{1}{\beta} \sum_{\xi\in\Omega} p^{\xi} g(\xi) \right\}.$$

Including the *CVaR* risk measure for the total inventory limit violation in the MIRP, the mathematical formulation becomes

$$\min \quad C(X) + \sum_{\xi\in\Omega} \sum_{(i,m)\in S^A} p^{\xi} P_i r_{im}(\xi) + \epsilon \left(\gamma + \frac{1}{\beta} \sum_{\xi\in\Omega} p^{\xi} g(\xi) \right) \tag{42}$$

$$s.t. \quad (2) - (8), (20) - (24), (30) - (39),$$

$$g(\xi) \geq \sum_{(i,m)\in S^A} P_i r_{im}(\xi) - \gamma, \qquad \forall\, \xi \in \Omega, \tag{43}$$

$$g(\xi) \geq 0, \qquad \forall\, \xi \in \Omega. \tag{44}$$

The objective function minimizes the routing cost plus the average penalty cost for the inventory bounds violation plus the *CVaR* weighted by the nonnegative parameter ϵ. Constraints (43) and (44) define $g(\xi)$ as a set of linear constraints.

Remark 1. *The stochastic programming model for the MIRP can be derived from the model* (42)–(44) *by considering* $\epsilon = 0$*. In that case, constraints* (43) *and* (44) *can be removed from the model.*

In this model the *CVaR* is used to control not only the probability of having a global penalty cost for the inventory bounds violation above the *VaR*, but also the magnitude of those values. The parameter ϵ is used to balance the routing and the average penalty cost against the *CVaR*. Figure 3 compares the penalty cost for the inventory bounds violation for both stochastic and *CVaR* approaches. Assuming the same penalty cost unit for the inventory bounds violation in all ports, i.e, $P{=}P_i$ for all $i \in N$, Figure 3 shows that when the total amount of inventory bounds violation is greater than $\frac{\gamma}{P}$ its unit cost is more penalized in the *CVaR* approach than in the stochastic approach (where the penalty cost per unit is the same independently of the total amount of inventory bounds violation). For β close to one and for a very large ϵ, the model highly penalizes any inventory limit violations, which tends to approximate the model to the robust one. In the case of $\epsilon = 0$, as remarked above we obtain the stochastic model.

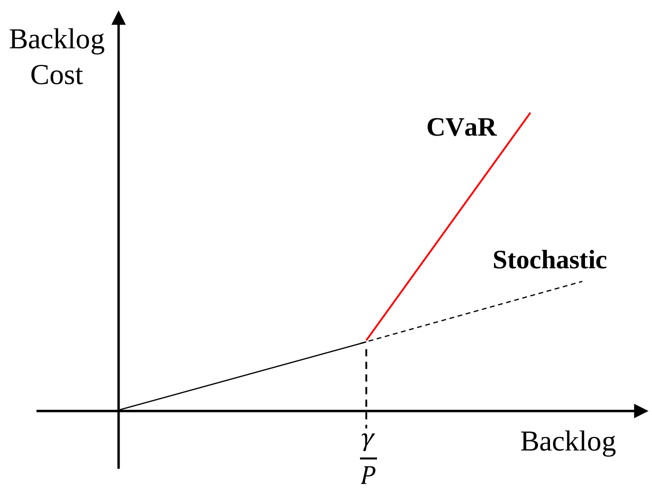

Figure 3: Penalty cost for both stochastic and *CVaR* approaches in terms of the total amount of penalty cost for the inventory bounds violation, assuming the same penalty cost per unit for all the ports.

3.3.2. CVaR *applied to the inventory level at each port*

In the second model using *CVaR*, the inventory buffers are established using the information provided by the *CVaR* risk measure. The idea is motivated by the fact that the inventory buffers are usually defined *a priori*, without any knowledge of the routes and do not take into account the probability of violation of the corresponding soft inventory bounds. Thus, for some visits, the inventory buffer can be large, resulting in a low probability of violation of the soft inventory bounds. For other visits, the inventory buffer can be small, leading to high probabilities of inventory bounds violations. In this model we consider a *VaR* for each port visit (i, m), denoted by γ_{im}, and use such values to define the soft inventory bounds and therefore, the inventory buffers.

For a consumption port $i \in N$, to protect the first-stage solutions against the uncertainty, it is desirable to keep the stock level far from the lower stock limit or, equivalently, keep the stock level close to the upper stock limit. A natural strategy is to consider large safety stocks for each port visit, which is equivalent to creating tight bounds to the gap between the upper stock limit and the stock level, i.e., $L = \bar{S}_i - (s_{im}(\xi) + r_{im}(\xi))$. Note that the amount of backlog in each port visit must be added to the corresponding inventory level since the stock limits can be violated, see Figure 4, at right. Hence, given a probability β, the *VaR* for each port visit (i, m), γ_{im}, is the minimum value such that the probability of gap L in port i and visit m being lower than γ_{im}, is at least $1 - \beta$, for a given set Ω of scenarios.

A similar idea can be followed for the definition of inventory buffers to the production ports. In that case, the aim is to keep the inventory levels as low as possible, that is, to minimize the difference $L = s_{im}(\xi) + r_{im}(\xi) - \underline{S}_i$, see Figure 4, at left.

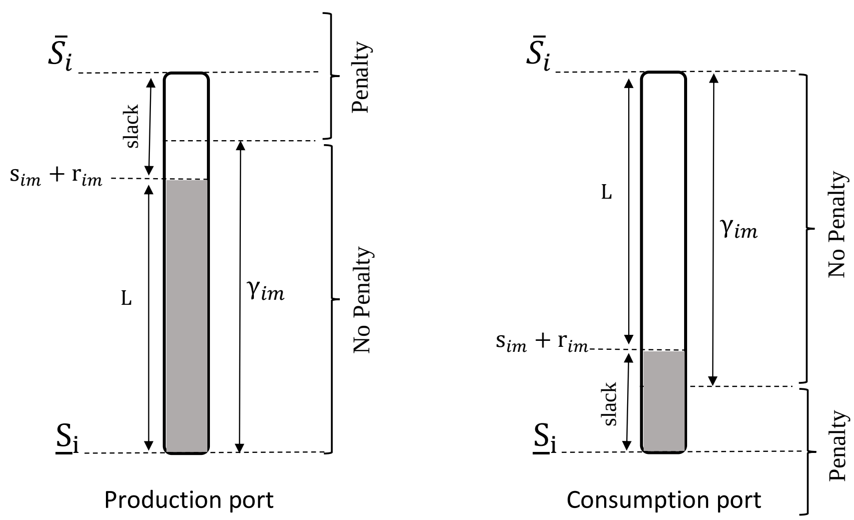

Figure 4: Penalties for the inventory level for the production and the consumption ports.

To define the *CVaR* measure consider the function $g_{im}(\xi) = \left(\bar{S}_i - s_{im}(\xi) - r_{im}(\xi) - \gamma_{im}\right)^+$ for the consumption ports and $g_{im}(\xi) = (s_{im}(\xi) + r_{im}(\xi) - \underline{S}_i - \gamma_{im})^+$ for the production ports. The *CVaR* can be defined by

$$CVaR = \min_{\gamma} \left\{ \gamma + \frac{1}{\beta} \sum_{\xi \in \Omega} p^{\xi} g_{im}(\xi) \right\}.$$

Including the $CVaR$ in the MIRP, the mathematical formulation becomes as follows:

$$\min \ C(X) + \sum_{\xi \in \Omega} \sum_{(i,m) \in S^A} p^\xi P_i r_{im}(\xi) + \epsilon \sum_{(i,m) \in S^A} \left(\gamma_{im} + \frac{1}{\beta} \sum_{\xi \in \Omega} p^\xi g_{im}(\xi) \right) \tag{45}$$

$$s.t. \ (2) - (8), (20) - (24), (30) - (39)$$

$$g_{im}(\xi) \geq \bar{S}_i - s_{im}(\xi) - r_{im}(\xi) - \gamma_{im}, \qquad (i,m) \in S^A : J_i = -1, \xi \in \Omega, \tag{46}$$

$$g_{im}(\xi) \geq s_{im}(\xi) + r_{im}(\xi) - \underline{S}_i - \gamma_{im}, \qquad (i,m) \in S^A : J_i = 1, \xi \in \Omega, \tag{47}$$

$$g_{im}(\xi) \geq 0, \qquad (i,m) \in S^A, \xi \in \Omega. \tag{48}$$

The objective function consists of the routing cost plus the average penalty cost for the inventory bounds violation plus the $CVaR$ associated to each port visit weighted by the nonnegative parameter ϵ. Constraints (46) and (48) define $g_{im}(\xi)$ using a set of linear constraints for the consumption ports and constraints (47) and (48) define $g_{im}(\xi)$ using a set of linear constraints for the production ports.

The model above can be used directly when solving the MIRP. However, one can also use the VaR values (the values of variables γ_{im}) to define soft inventory bounds for each port visit. These soft inventory bounds can be used in the deterministic model with inventory buffers described in Section 3.1. For the consumption ports, the safety stock for the m^{th} visit is defined by $\underline{SS}_{im} := \bar{S}_i - \gamma_{im}$, while for the production ports the soft upper bound for visit m is defined by $\overline{SS}_{im} := \underline{S_i} + \gamma_{im}$.

From preliminary computational experience, we observed that the values obtained for γ_{im} using model (45)–(48) can sometimes be very low, leading to large inventory buffers. To overcome this problem we add the following two additional sets of constraints to model (45)–(48) in order to restrict the values of γ_{im}.

$$\gamma_{im} \geq \delta \times (\bar{S}_i - \underline{S}_i), \qquad \forall (i,m) \in S^A, \tag{49}$$

$$\gamma_{im} \leq \bar{S}_i - \underline{S}_i, \qquad \forall (i,m) \in S^A. \tag{50}$$

The parameter δ is a nonnegative parameter lower than one, $\delta \in [0, 1)$, and is used to impose a lower limit on the VaR values. Constraints (50) impose an upper limit on the VaR values.

3.4. Robust optimization

The robust approach presented here was introduced in [4]. In the robust approach we assume that the travel times T_{ijv} belong to an uncertainty set as introduced by Bertsimas and Sim [6],

$$\Xi^{\Gamma} = \{\xi : \xi_{imjnv} = \bar{T}_{ijv} + \hat{T}_{ijv}\delta_{imjnv},$$
$$0 \le \delta_{imjnv} \le 1, v \in V, (i,m,j,n) \in S_v^X, \sum_{v\in V} \sum_{(i,m,j,n)\in S_v^X} \delta_{imjnv} \le \Gamma\}$$

where $\bar{T}_{ijv}$ is the nominal value corresponding to the expected travel time, $\hat{T}_{ijv}$ is the maximum allowed deviation (delay), δ_{imjnv} is the deviation of parameter T_{imjnv} from its nominal value, and Γ limits the number of deviations.

Similar to the stochastic programming and the *CVaR* models, the robust model is also an adjustable model with the same two levels of decisions. However, backlog is not allowed. Hence, the first-stage solution must ensure that for each travel time vector belonging to the uncertainty set, the stock level at each port i is within the bounds $\underline{S}_i$ and $\overline{S}_i$. For the robust model the time and inventory constraints are replaced by the following constraints:

Time constraints

$$t_{im}(\xi) + \sum_{v\in V} T_i^Q q_{imv} - t_{jn}(\xi) + \sum_{v\in V} \max\{T'_{im} + \xi_{ijv} \; - A_{jn}, 0\} x_{imjnv} \le T'_{im} - A_{jn},$$
$$(i,m),(j,n) \in S^A, \xi \in \Xi^{\Gamma}, \quad (51)$$

$$t_{im}(\xi) - t_{i,m-1}(\xi) - \sum_{v\in V} T_i^Q q_{i,m-1,v} - T_i^B y_{im} \ge 0, \quad (i,m) \in S_A : m > 1, \xi \in \Xi^{\Gamma}, \quad (52)$$

$$A_{im} \le t_{im}(\xi) \le B_{im}, \quad (i,m) \in S^A, \xi \in \Xi^{\Gamma}. \quad (53)$$

Inventory constraints

$$s_{i1}(\xi) = S_i^0 + J_i R_i t_{i1}(\xi), \quad i \in N, \xi \in \Xi^{\Gamma}, \quad (54)$$

$$s_{im}(\xi) = s_{i,m-1}(\xi) - J_i \sum_{v\in V} q_{i,m-1,v} + J_i R_i (t_{im}(\xi) - t_{i,m-1}(\xi)),$$
$$(i,m) \in S^A : m > 1, \xi \in \Xi^{\Gamma}, \quad (55)$$

$$s_{im}(\xi) + \sum_{v\in V} q_{imv} - R_i \sum_{v\in V} T_i^Q q_{imv} \le \overline{S}_i, \quad (i,m) \in S^A | J_i = -1, \xi \in \Xi^{\Gamma}, \quad (56)$$

$$s_{im}(\xi) - \sum_{v\in V} q_{imv} + R_i \sum_{v\in V} T_i^Q q_{imv} \ge \underline{S}_i, \quad (i,m) \in S^A | J_i = 1, \xi \in \Xi^{\Gamma}, \quad (57)$$

$$s_{i\overline{\mu}_i}(\xi) + \sum_{v\in V} q_{i,\overline{\mu}_i,v} - R_i (T - t_{i\overline{\mu}_i}(\xi)) \ge \underline{S}_i, \quad i \in N | J_i = -1, \xi \in \Xi^{\Gamma}, \quad (58)$$

$$s_{i\overline{\mu}_i}(\xi) - \sum_{v \in V} q_{i,\overline{\mu}_i,v} + R_i(T - t_{i\overline{\mu}_i}(\xi)) \leq \overline{S}_i, \qquad i \in N | J_i = 1, \xi \in \Xi^{\Gamma}, \tag{59}$$

$$s_{im}(\xi) \geq \underline{S}_i, \qquad (i,m) \in S^A | J_i = -1, \xi \in \Xi^{\Gamma}, \tag{60}$$

$$s_{im}(\xi) \leq \overline{S}_i, \qquad (i,m) \in S^A | J_i = 1, \xi \in \Xi^{\Gamma}. \tag{61}$$

As in the deterministic case, the objective function of the robust model minimizes the routing cost, so the robust model is defined by (1)–(8), (20)–(24), (51)–(61). See [4] for more details.

4. Experimental setup

Here we describe the set of instances, the solution approaches tested that are based on the general models presented in Section 3, and discuss the corresponding parameter values for the proposed solution approaches.

4.1. Instances description

The instances used are the same as the ones considered in [4]. They are based on real data and come from the short sea shipping segment with long loading and discharge times relative to the travel times. The number of ports of each instance vary between 3 and 6 and the number of ships vary between 1 and 5. We consider 7 combinations of these values giving 7 groups of instances, I, $I \in \{A, B, C, D, E, F, G\}$. Each group has three instances ($\{I1, I2, I3\}$) that differ from each other through the initial inventory levels, giving a total of 21 instances. The time horizon considered is $T = 30$ days. These instances can be found at `http://home.himolde.no/~hvattum/benchmarks/`.

Following [14], we assume that the travel time T_{ijv} follows a three-parameter log-logistic probability distribution whose cumulative probability function can be written as

$$F(T_{ijv}) = \frac{1}{1 + (\frac{1}{t})^{\alpha}},$$

where $t = \frac{T_{ijv} - \eta}{\zeta}$. For this distribution, the minimum travel time is η, and the expected travel time $E[T_{ijv}]$, which is estimated by the average travel time $\overline{T}_{ijv}$, is equal to $\frac{\zeta\pi}{\alpha \sin(\pi/\alpha)} + \eta$. As in [3], we consider $\eta = 0.9\ \overline{T}_{ijv}$, and $\alpha = 2.24$, and ζ is obtained from the equation $\overline{T}_{ijv} = \frac{\zeta\pi}{\alpha \sin(\pi/\alpha)} + \eta$. Samples of the travel times are generated by using the inverse transformation method.

4.2. Calibration

To efficiently approximate the continuous three-parameter log-logistic probability distribution, we have to determine the appropriate number of scenarios that should be considered in both stochastic programming and *CVaR* models. To obtain this number, some out-of-sample stability tests were conducted.

Let Ω be a large set of scenarios sampled from the three-parameter log-logistic distribution. For each number n of scenarios, $n \in \{1, 5, 10, 15, 20, 25, 30, 35\}$,we consider 30 different sampled n-scenario trees. For each one of the 30 n-scenario trees, we solve the corresponding stochastic programming model. Hence, for each number n, we obtain 30 solutions. Let X_i^n be the first-stage solution of the stochastic programming model corresponding to the n-scenario tree i, $i = 1, \ldots, 30$, and let $f(X_i^n, \Omega)$ be the objective function value of solution X_i^n when evaluated using the large set of scenarios, Ω. For every instance the minimum, the average and the maximum objective function values $f(X_i^n, \Omega)$, $i = 1, \ldots, 30$, are obtained.

As an example, for each of the n-scenario trees, the minimum, the average and the maximum objective function values $f(X_i^n, \Omega)$, $i = 1, \ldots, 30$ of the 30 obtained solutions are displayed in Figure 5, for instance $B3$. The figure shows that the gap between the

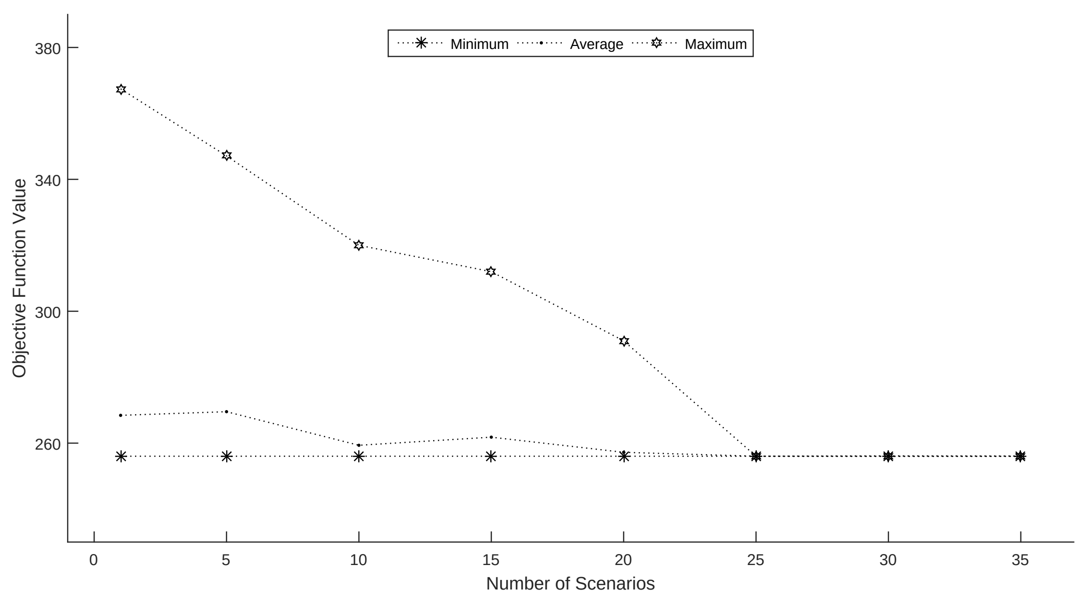

Figure 5: Minimum, average and maximum objective function values of the 30 first-stage solutions for each of the n-scenario trees, $n \in \{1, 5, 10, 15, 20, 25, 30, 35\}$.

three lines tends to decrease as the number of scenarios used increases, and it is very close to zero when at least 25 scenarios are considered for sampling scenario trees. This means that for instance $B3$ we have out-of-sample stability when at least 25 scenarios are used, since the obtained objective function value is the same independently of the sample of 25 scenarios used.

For all instances, the computational tests provide similar results to the ones presented in Figure 5. Therefore, we use scenario trees with $n = 25$ samples whenever the stochastic programming or $CVaR$ models are used.

4.3. Solution approaches

Based on the general models described in Section 3, we derive 13 particular solution approaches by considering different parameters or combining methods. These solution approaches are tested in Section 5.

Deterministic model (D). The deterministic solution approach includes no inventory buffers. This solution approach corresponds to solving the model in Section 2 and is denoted by D.

Deterministic model with inventory buffers (F). The deterministic solution approach with inventory buffers corresponds to solving the model in Section 3.1. For each visit to a production port, the soft upper inventory bound corresponds to 90% of the upper stock limit and the lower inventory bound coincides with the lower stock limit. For all visits to a consumption port the soft lower bounds are set to $\underline{S}_i + 0.1(\overline{S}_i - \underline{S}_i)$, while the soft upper bounds are set to the upper stock limit. In both cases, the obtained soft inventory bounds give rise to inventory buffers of 10% of the difference between the upper and the lower stock limits. The unit penalty considered for soft inventory bounds violations is 5. Results reported in Section 5.3 show the suitability of the values chosen for the parameters of this model. Since the uncertainty is not considered in the definition of the inventory buffers, this is a pure deterministic solution approach and is denoted by F.

Stochastic programming (S). This approach consists of solving model in Section 3.2. We consider two different values, 5 and 25, for the inventory violation penalty, P_i. It is reported in [2, 3] that the use of large penalties make the model hard to solve as the integrality gaps tend to increase. Thus the two chosen values represent the use of a *small* and a *medium* penalty, respectively. Other values for this penalty (1, 10, 15, 20 and 30) were tested, however, the obtained results, not reported here, showed that for the most of the instances the obtained solutions coincide with the ones corresponding either to $P_i = 5$ or $P_i = 25$. These models are solved using the decomposition procedure proposed in [3]. The decomposition follows the idea of the L-shaped algorithm. The problem is decomposed into a master problem and one subproblem for each scenario. The master problem is solved for a subset of scenarios. Then for each disregarded scenario the subproblem checks whether a penalty for inventory limit violations is incurred when the first-stage decision is fixed. If such a scenario is found, additional variables and constraints enforcing that deviation to be penalized in the objective function are added to the master problem. The revised master problem is solved again, and the process is repeated until all the recourse constraints are satisfied. The resulting stochastic optimization solution approaches are denoted by S_5 and S_{25}, respectively.

CVaR *on the global inventory limit violation (C).* This method consists of solving model in Section 3.3.1 using the violation probability $\beta = 0.01$, parameter $\epsilon = 10$ and the same penalties as in the stochastic model, 5 and 25. Since the structure of the *CVaR* model is similar to the structure of the stochastic model, these *CVaR* models are solved using the same decomposition procedure. These solution approaches are denoted by C_5 and C_{25}.

CVaR *on the inventory level at each port visit (V).* This method consists of solving model in Section 3.3.2 using the violation probability $\beta = 0.01$, $\delta = 0.8$ and ϵ taking values 0.01 and 1. Again, the models are solved with the same decomposition procedure used for the stochastic method. These *CVaR* solution approaches are denoted by $V_{0.01}$ and V_1.

Combined CVaR *and deterministic model with inventory buffers (F).* This is an hybrid approach. The *VaR* on the inventory level at each port visit is determined to define soft inventory bounds for each port visit. Then the deterministic model with inventory buffers is solved using the obtained values for the soft inventory bounds. If the solution approach $V_{0.01}$ is used to define the bounds, then the solution approach is denoted by $F_{0.01}$ and if the solution approach V_1 is used, the solution approach is denoted by F_1.

Robust optimization (R). We consider the nominal travel times $\bar{T}_{ijv}$ and a maximum allowed delay, $\hat{T}_{ijv}$. The nominal travel times are defined to be equal to the expected travel times values, that is, $\bar{T}_{ijv} = E[T_{ijv}]$. The maximum allowed delay, $\hat{T}_{ijv}$, is a constant value selected for each instance and obtained as follows. The deterministic model is solved for several values of the travel times $T_{ijv} = \bar{T}_{ijv} + \lambda\bar{T}_{ijv}$, starting with

$\lambda = 0$ and increasing its value by steps of 0.01 until the corresponding deterministic model has no feasible solutions. The last value of λ leading to an instance with a feasible solution is selected, and the maximum allowed delay, $\hat{T}_{ijv}$, is set to $\lambda \bar{T}_{ijv}$.

We consider three robust solution approaches corresponding to $\Gamma = 1, 2, 3$. These values are choseen since for the most of the instances the solution of the box-constrained problem corresponds to $\Gamma \leq 3$. Each robust model is solved by using the decomposition procedure described in [4] that iterates between a master problem that considers a subset of scenarios and an adversarial separation problem that searches for scenarios that make the solution from the master problem infeasible. These robust solution approaches are denoted by R_1, R_2 and R_3.

5. Computational tests

This section reports the computational experiments carried out to compare the performance of the thirteen solution approaches described in Section 4.3. A set of 21 instances of a MIRP described in Section 4.1 is used. All tests were run on a computer with an Intel Core i7-6700HQ processor, having a 2.60GHz CPU and 16GB of RAM, using the Xpress-Optimizer 28.01.04 solver with the default options.

To evaluate the performance of each solution approach and how it reacts to the uncertainty, after the final solution is obtained, the first-stage decision variables are fixed and the two-stages stochastic model defined in Section 3.2 is solved for a large sample of 1000 scenarios. For each scenario, the value of the recourse variables (the second-stage variables) as well as the total amount of backlog are computed. For the large sample, the minimum, average, and maximum amounts of backlog are computed. Additionally, the stock-out probability that corresponds to the empirical probability of having a scenario with a positive amount of backlog is computed. For the first-stage solution, the total routing cost, the total amount of product loaded at all production ports, and the total amount of product unloaded at all consumption ports during the planning horizon are also computed. These values are compared with the corresponding values obtained by the deterministic solution approach D.

Section 5.1 presents the computational results for 18 of the 21 instances. These 18 instances are solved to optimality by all the solution methods tested. The results for the three most difficult instances are reported in Section 5.2.

5.1. *Global results for instances solved to optimality*

Here we present the computational results of the 18 instances solved to optimality by all solution approaches. The instances are $I1$, $I2$ and $I3$ where $I \in \{A, B, C, D, E, F\}$.

Table 2 reports average computational results for the 18 instances obtained by each of the 13 solution approaches. The first column gives information about the solution approach. The second column named "Routing" displays the routing cost when compared to the routing cost obtained by the deterministic solution approach D. The displayed value is the average routing cost for all the 18 instances divided by the routing cost obtained by solution approach D. The third, fourth and fifth columns (named "Min", "Average" and "Max", respectively) report average values of the minimum, average and maximum amounts of backlog over all instances. The sixth column, named "SOut(%)", displays the average of the stock-out probability over all instances. The seventh and eighth columns, named "LQ" and "UQ", respectively, display the average over all instances of the total quantity loaded (LQ) and unloaded (UQ) divided by the total quantity loaded (LQ) and unloaded (UQ) in the corresponding solution obtained by solution approach D. The last column reports the average computational time, in seconds, to solve the problem.

Table 2: Average computational results obtained by the 13 solution approaches for the 18 instances.

Sol. Approach	Routing	Min	Average	Max	SOut(%)	LQ	UQ	Seconds
D	1.00	0.0	2.2	43.3	20.3	1.00	1.00	15
F	1.06	0.0	0.6	24.1	5.0	0.99	1.01	49
$F_{0.01}$	1.07	0.0	0.5	19.1	4.4	1.03	1.05	335
F_1	1.06	0.0	0.9	21.8	6.7	1.01	1.05	1289
$V_{0.01}$	1.00	0.1	0.7	22.1	17.3	1.02	1.05	306
V_1	1.07	0.1	0.8	19.0	16.7	1.02	1.06	1245
C_5	1.08	0.1	0.7	23.5	16.9	1.06	1.07	982
C_{25}	1.11	0.0	0.7	24.9	5.8	1.05	1.03	877
S_5	0.94	0.8	1.9	29.4	28.0	1.03	1.05	352
S_{25}	1.00	0.1	0.8	25.1	17.6	1.00	0.99	326
R_1	1.07	0.0	1.0	23.4	7.4	1.11	1.17	26
R_2	1.10	0.0	0.8	18.4	6.0	1.12	1.16	62
R_3	1.13	0.0	0.8	17.7	6.1	1.12	1.24	139

Table 2 shows that, on average, the worst solutions for the average and maximum values of the backlog correspond to the solutions obtained by the deterministic solution approach D. The stochastic programming approach S_5, with backlog cost equal to five, provides solutions with high amounts of backlog. The average stock-out probability in this solution approach is the highest among all the solution approaches, followed by the deterministic approach D. This behavior can easily be explained by the fact that the cost of the backlog used in solution approach S_5 is not sufficiently high to avoid solutions with a large amount of backlog. However, S_5 has the lowest (average) routing costs. The average routing cost is high in the robust solution approaches R_2 and R_3, and this cost tends to increase as the allowed maximum number of delays increases. The routing cost is also high in the *CVaR* solution approach C_{25}. Comparing the stochastic solution approach S_{25} and the robust solution approaches (R_1, R_2 and R_3), the robust solution approaches show to be less sensitive to high travel times delays, which is reflected by the lower value of the average maximum amount of backlog. Furthermore, the stock-out probability is lower in the robust approaches than in the stochastic ones. A significant difference between the robust solutions and the solutions obtained by the remaining solution approaches is that the quantities loaded and unloaded are higher. When comparing the solutions obtained by the stochastic solution approaches S_5 and S_{25} with the solutions obtained by the *CVaR* approaches C_5 and C_{25} it is possible to see that the *CVaR* approaches have solutions with higher routing cost while having lower amount of backlog. Hence, the average values of the minimum, average and maximum amounts of backlog as well as the stock-out probability are lower in these solution approaches. Another interesting comparison is between the deterministic approach F with inventory buffers and the hybrid approaches $F_{0.01}$ and F_1. All these methods have high (average) routing costs, however the solutions obtained by the solution approaches $F_{0.01}$ are characterized by lower values of the average backlog, the maximum backlog and the stock-out probability. In fact, among all the solution approaches, approach $F_{0.01}$ gives results with lowest average amount of backlog and stock-out probability.

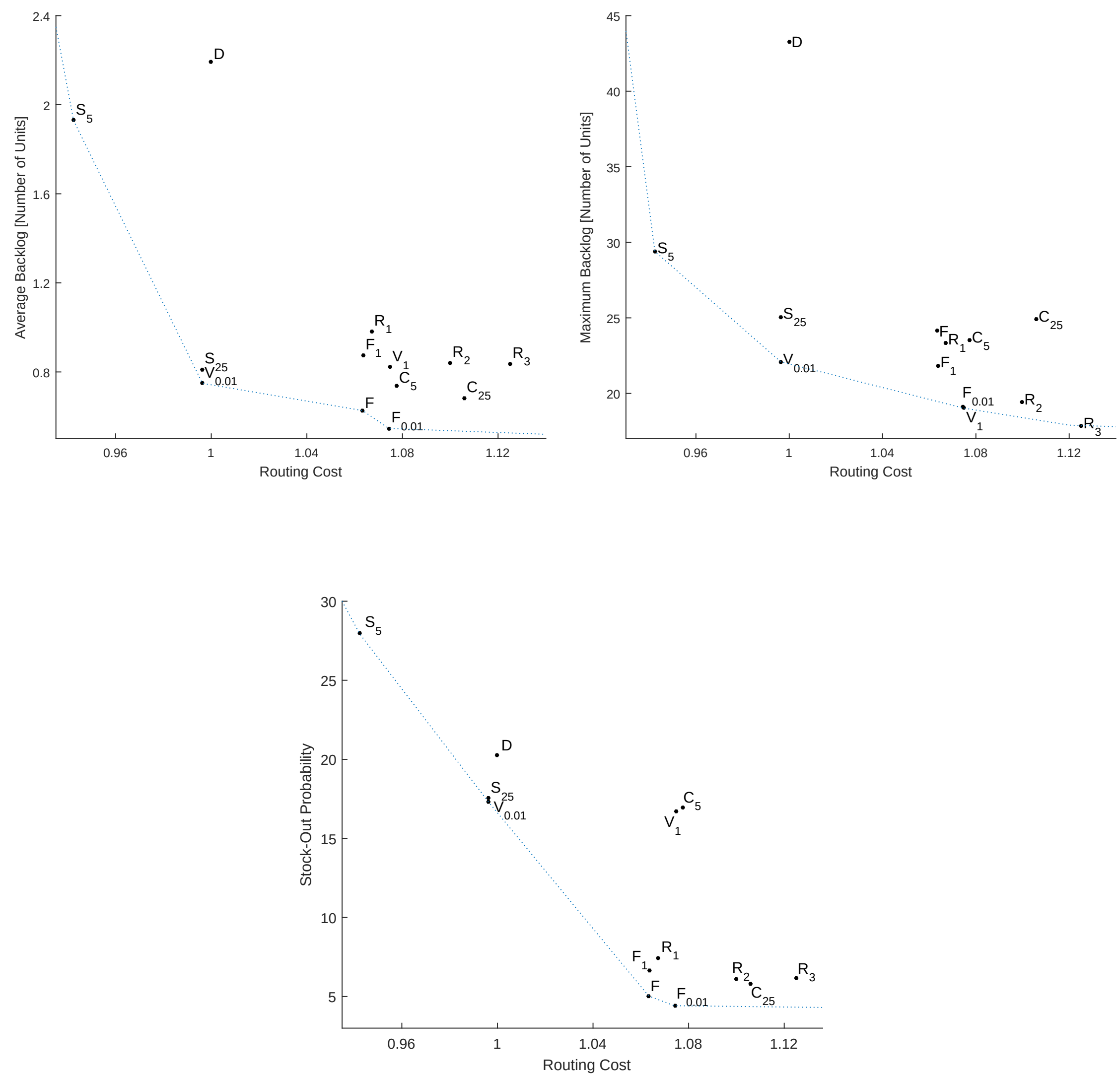

Figure 6: Graphical comparisons between the 13 solution approaches.

Based on the results in Table 2, Figure 6 shows the comparison of the average routing costs with the average backlog, maximum backlog, and stock-out probability. In each of three graphs, a dotted line is drawn, separating the dominated from the non-dominated solution approaches. Analyzing the figure we observe that the stochastic programming approach S_{25} is clearly dominated by approach $V_{0.01}$, and these two solution approaches dominate the deterministic approach D.

5.2. Results for the hard instances

Here we report the computational results for the most difficult instances $G1$, $G2$ and $G3$. These three instances are solved to optimality by the robust solution approaches R_1, R_2 and R_3, and by the deterministic approach D without inventory buffers and approach F with inventory buffers in a reasonable amount of time. Instance $G2$ is also solved to optimality by the stochastic programming approach S_5. The instances are not solved to optimality by the remaining approaches. To solve these instances using these solution approaches, the MIP based local search procedure described in [3] is used. The procedure starts by using the deterministic approach D to obtain an initial solution. Then, the local branching inequality limiting to four the number of variables w_{im}, $(i, m) \in S^A$, allowed to flip their value is added to the stochastic model containing all the scenarios. The resulting model is run for a time limit of 1000 seconds. The process is repeated until no improvement in the objective function is observed. Since a time limit is imposed, the solution obtained in each iteration may not be optimal.

Table 3 displays the computational results for instances $G1$, $G2$ and $G3$, where an asterisk indicates that the instance is solved to optimality. The description of the contents of the columns is the same as in Table 2, however the values are shown for each instance and not as averages of a set of instances. The third column in Table 3 reports the optimality gaps for the instances not solved to optimality by each method. Notice that both the stochastic and the CVaR models are solved by an ILS heuristic, hence, in order to obtain a lower bound to estimate the optimality gap for these methods, we run the corresponding complete models (with all the 25 scenarios) imposing a time limit of two hours. After the inventory bounds are determined by approaches $V_{0.01}$ and V_1, the corresponding deterministic models with inventory buffers are solved to optimality for almost all the instances, except for the instance $G3$ with the inventory bounds given by method V_1.

The comments for Table 2 do not hold here. For instance, solution approach $V_{0.01}$ does not dominate solution approaches S_{25} and D. However, in general, the deterministic approach D produces solutions with a high amount of backlog and a high stock-out probability. Note that the stochastic programming and $CVaR$ approaches have, in general, the highest computational time. These are the solution approaches hardest to solve and, in each iteration, we noticed that the integrality gap is large.

For some instances, the optimality gaps associated to some approaches are too large, as happens for instance G1 in relation to the approaches C_5 and C_{25}. These large gaps

Table 3: Computational results for instances $G1$, $G2$ and $G3$.

Approaches		Gap(%)	Routing	Min	Average	Max	SOut(%)	LQ	UQ	Seconds
	D	0.0	1.00	0.0	2.7	55.0	19.8	1.00	1.00	203
	F	0.0	1.05	0.0	2.4	53.2	18.4	0.99	0.96	241
	$F_{0.01}$	0.0	1.05	0.0	2.4	53.2	18.4	1.00	0.96	3116
	F_1	0.0	1.05	0.0	2.4	53.2	18.4	1.00	0.96	6496
	$V_{0.01}$	25.8	1.07	0.0	2.4	53.2	18.4	1.00	1.00	2586
	V_1	18.9	1.07	0.0	1.9	63.3	10.6	1.00	1.00	6000
G1	C_5	82.6	1.46	0.0	5.2	80.1	27.6	0.94	1.00	2000
	C_{25}	95.6	1.46	0.0	5.2	80.1	17.6	0.94	1.00	2000
	S_5	8.9	1.00	0.0	2.4	53.2	18.4	1.00	1.00	543
	S_{25}	24.2	1.00	0.0	2.4	53.2	18.4	1.00	1.00	1363
	R_1	0.0	1.00	0.0	2.6	54.0	20.8	1.04	1.04	123
	R_2	0.0	1.10	0.0	1.7	64.3	10.4	1.07	1.08	8485
	R_3	0.0	1.10	0.0	1.7	64.3	10.4	1.07	1.08	8485
	D	0.0	1.00	0.0	5.7	86.0	43.2	1.00	1.00	254
	F	0.0	1.00	0.0	2.0	48.4	15.6	0.94	1.02	90
	$F_{0.01}$	0.0	1.00	0.0	1.6	45.2	13.6	0.94	1.02	5807
	F_1	0.0	1.00	0.0	1.2	44.0	10.0	1.00	1.09	5802
	$V_{0.01}$	11.8	1.05	0.0	0.2	23.8	1.8	0.88	1.15	5714
	V_1	6.9	1.05	0.0	0.2	23.8	1.8	0.88	1.15	5714
G2	C_5	14.3	1.13	0.0	0.1	26.9	1.4	0.88	1.15	3585
	C_{25}	19.2	1.13	0.0	0.4	40.9	3.4	0.91	1.15	5271
	S_5	0.0	1.00	0.0	0.2	23.8	1.8	0.91	1.00	1202
	S_{25}	13.3	1.05	0.0	0.2	23.8	1.8	0.88	1.16	1275
	R_1	0.0	1.00	0.0	0.8	39.7	6.8	1.01	1.09	446
	R_2	0.0	1.05	0.0	0.1	10.4	1.0	1.00	1.28	2251
	R_3	0.0	1.05	0.0	0.0	0.0	0.0	1.00	1.28	1984
	D	0.0	1.00	0.0	3.2	58.3	26.2	1.00	1.00	874
	F	0.0	1.05	0.0	2.7	50.5	20.8	0.89	1.05	156
	$F_{0.01}$	0.0	1.05	0.0	3.5	55.5	28.2	0.91	1.05	4238
	F_1	18.9	1.25	0.0	0.3	21.0	3.0	0.89	1.02	9068
	$V_{0.01}$	32.3	1.22	0.0	4.9	79.1	35.6	0.97	1.03	4144
	V_1	12.3	1.41	5.0	7.6	59.5	100.0	0.98	1.00	5468
G3	C_5	40.2	1.57	0.0	0.4	30.0	4.0	0.94	1.00	4589
	C_{25}	40.6	1.61	0.0	0.8	35.3	7.2	0.89	1.00	3625
	S_5	19.7	1.07	4.6	6.0	32.4	100.0	0.99	1.00	1259
	S_{25}	39.2	1.31	0.0	0.8	46.7	8.8	0.97	1.00	4484
	R_1	0.0	1.05	0.0	1.1	44.0	9.0	1.01	1.00	317
	R_2	0.0	1.05	0.0	1.3	46.0	10.0	1.02	1.06	579
	R_3	0.0	1.05	0.0	1.3	46.0	10.0	1.02	1.06	538

can be explained by the fact that for these approaches a running time limit of 1000 seconds was imposed to each iteration. Hence, from the last column of Table 3 we can see that only two iterations were performed in both cases. It means that a feasible solution was founded in the first iteration and it was not possible to obtain a better solution in the second iteration within the imposed time limit.

5.3. Sensitivity analysis of the parameters for the deterministic model with inventory buffers

The performance of the deterministic solution approach with inventory buffers F depends on two parameters: the inventory buffers used and the penalty consider for the violation of the soft inventory bounds. In this section we report average results obtained with model F for 12 combinations of these two parameters. The inventory buffers used correspond to 5%, 10%, 15% and 20% of the difference between the upper and the lower stock limits while the unit values used to penalize soft inventory bounds violations are 1, 5 and 10. Table 4 reports the average results obtained for all the 18 instances that are solved to optimality by all the proposed approaches (the ones used in Section 5.1). The first line identifies the value of the penalty, while the second one identifies the percentage used to define the inventory buffers. The third line reports the average routing cost of the solutions obtained by model F divided by the cost of the solutions obtained by the deterministic model without inventory buffer (model D). The forth and the fifth lines report the average and the maximum number of backlog units, respectively, when the final solution is evaluated in a large sample of 1000 scenarios (as described in Section 5.1). The sixth line reports the average values of the stock-out probability while the last line reports the number of times that each combination of parameters leads to the best solutions based on each of the four quality parameters identified in the previous lines. For example, the solutions obtained with penalty equal to 1 and percentage equal to 5 are the best for all the 18 instances in terms of the routing cost, for 5 instances in terms of the average number of backlog units, for 2 instances in terms of the maximum number of backlog units and also for 2 instances in terms of the stock-out probability, hence, the number presented for this combination of parameters is 27 (18+5+2+2).

Table 4 shows that the results associated with inventory buffers larger than 5% are not sensitive to the penalty used. The same does not hold for inventory buffers equal to 5% where larger differences can be observed for different penalty values. In terms of the routing cost, best results are in general associated to low inventory buffers and low penalties, while the best results in terms of the maximum number of backlog units are

Table 4: Results for the deterministic model with inventory buffers for different values of the inventory buffers and the penalty of the violations.

	Penalty =1				Penalty=5				Penalty=10			
	5%	10%	15%	20%	5%	10%	15%	20%	5%	10%	15%	20%
Routing	1.00	1.06	1.06	1.07	1.02	1.05	1.06	1.06	1.06	1.06	1.06	1.06
Average	1.2	0.6	0.8	0.8	0.9	0.6	0.8	0.8	0.7	0.6	0.8	0.8
Maximum	35.7	23.8	21.5	19.8	32.9	24.1	21.4	20.1	25.8	24.1	21.5	19.9
SOut(%)	10.1	5.1	6.1	6.1	7.2	5.0	6.2	6.1	6.0	4.8	6.1	6.1
# BS	27	38	35	35	30	38	37	32	27	34	36	33

associated to the approaches with larger inventory buffers. The best results in terms of the average number of backlog units, the stock-probability and the total number of best solutions founded are in general obtained when a value of 10% is used for the inventory buffers. Furthermore, for this percentage there are no big differences between the results associated to different penalties. Hence, on average, a percentage of 10% and a penalty equal to 5 are suitable values for the deterministic model with inventory buffers.

5.4. Evaluating the use of CVaR *in the stochastic programming model*

Here we compare the stochastic programming and the *CVaR* approaches over the sample of 25 scenarios used to obtain the first-stage solutions. Figure 7 displays four bars for each of the 18 instances solved to optimality by all the 13 solution approaches. In each set of four bars, the first two (gray) bars represent the average number of backlog units observed in the solutions provided by methods C_5 and S_5, while the next two (black) bars represent the maximum number of backlog units for the same methods. Since no relevant information can be draw from the bars associated to the minimum units of backlog, such bars are not presented here.

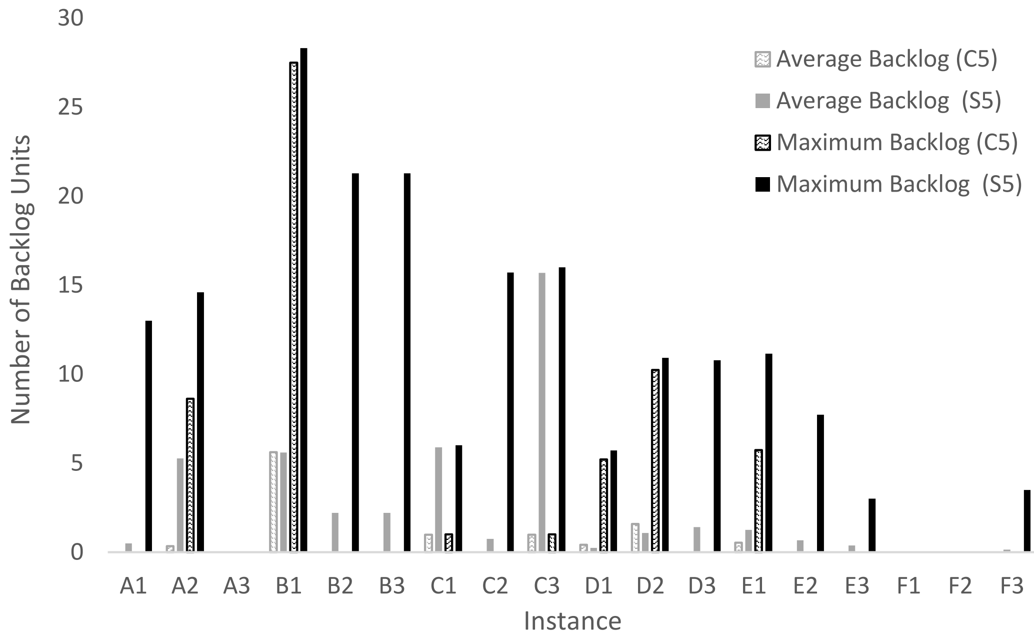

Figure 7: Difference between the solutions obtained by the approaches S_5 and C_5 for the minimum, average and maximum amount of backlog.

When comparing the approaches S_{25} and C_{25}, the results are very similar, so they are not displayed here. However, we note that the differences between approaches S_5 and C_5 are higher than between approaches S_{25} and C_{25}. For all instances, the results displayed in Figure 7 allow us to conclude that the maximum number of backlog units is lower in the solutions obtained by the $CVaR$ approaches than in the solutions obtained by the corresponding stochastic programming approaches. In almost all the instances, the same behavior is observed for the average amount of backlog, except for instances $D1$ and $D2$. However, in these two cases, the differences between the average values are very small. Mann-Whitney hypothesis tests were conducted to realize if there are significant differences between the solutions obtained by both the $CVaR$ and the stochastic approaches, in terms of the average and the maximum number of backlog units. The obtained results reveal that the differences in terms of the average and the maximum number of backlog units between the two approaches are significant for a significance level greater than 0.004 and 0, respectively. Hence, for the confidence levels usually considered (90%, 95% and 99%), the average and the maximum number of backlog units is significantly lower in the solutions provided by the $CVaR$ approaches than in the ones obtained by the stochastic approaches.

Figure 8 shows a pair of bars for each instance. The first bar represents the difference between the stock-out probability in the solution approaches S_5 and C_5. The second bar represents the difference between the stock-out probability in the solution approaches S_{25} and C_{25}.

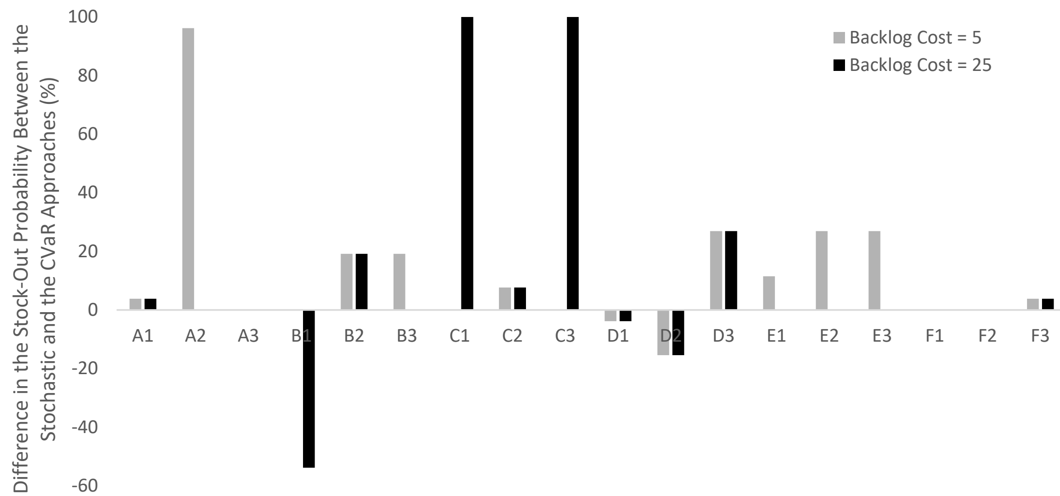

Figure 8: Difference between the stock-out probability in the solutions obtained by the stochastic programming approaches (S_5 and S_{25}) and the corresponding *CVaR* approaches (C_5 and C_{25}).

For almost all the instances, the stock-out probability is less in the solutions from the *CVaR* approaches than in the solutions from the stochastic programming approaches, except for instances $B1$, $D1$ and $D2$.

Observe that in the *CVaR* approaches the amount of backlog tends to be much lower than in the corresponding stochastic approaches and the same holds for the stock-out probability. Furthermore, higher values of the backlog amounts correspond to lower routing costs in the solutions obtained by the stochastic approaches.

5.5. Results for instance $E2$

In this section we report some results for instance $E2$. This instance is selected because its solutions obtained by the hybrid solution approach $F_{0.01}$ have the largest variation of the inventory buffers for each port. Instance $E2$ has 2 production ports and 3 consumption ports, and the maximum number of visits allowed to each port is 4. Table 5 reports the stock limits and the soft inventory bounds (that were used to define the inventory buffers) in both approaches F and $F_{0.01}$, for all port visits. Ports 1 and 2 are the production ports while ports 3, 4 and 5 are the consumption ports. The first column, named “Port”, identifies the port, with P for production and C for consumption. The second column named “Stock limits” presents the lower and the upper stock limits. The lower and upper soft inventory bounds used in method F are presented in the third and forth columns while the values used by approach $F_{0.01}$ are presented in the fifth and sixth columns. Note that, for each port, the soft inventory bound used in approach F

Table 5: Inventory buffers used in the solution approaches F and $F_{0.01}$ for the instance $E2$. The values are presented in terms of the number of units.

		F		$F_{0.01}$	
Port	Stock limits	Lower IB	Upper IB	Lower IB	Upper IB
P_1	[0 , 300]	-	270	-	240 249 252 240
P_2	[0 , 350]	-	315	-	280
C_3	[0 , 250]	25	-	50	-
C_4	[0 , 145]	15	-	7 29 29 29	-
C_5	[0 , 300]	30	-	30 22 54 60	-

is the same for all visits. In approach $F_{0.01}$ the soft inventory bound used is the same for all the visits in ports 2 and 3, while for ports 1, 4 and 5 they differ for each visit. Moreover, the upper soft inventory bounds are defined for the production ports but not for the consumption ports, while the lower soft inventory bounds are defined for the consumption ports but not for the production ports. Instance $E2$ shows the advantage of defining inventory buffers by using a *CVaR* approach for each port visit, because the use of such inventory buffers produced better results than the ones obtained using approach F.

Table 6 presents results for the solutions obtained using solution approaches F and $F_{0.01}$, when evaluated using the large set of 1000 scenarios. This table also displays the results for other solution approaches: the deterministic approach D (as the reference approach), the $CVaR$ approach C_{25}, the stochastic approach S_{25} and the robust approach R_2. These solution approaches were selected because they have the best behavior for this instance and also a global good behavior as shown in Table 2. Notice that the average results for solution approaches R_2 and R_3 are similar but the routing cost is higher for R_3.

Table 6: Results for instance $E2$.

Model	Routing	Min	Average	Max	SOut(%)	LQ	UQ	Seconds
D	1.00	0.0	8.4	137.9	55.6	1.00	1.00	123
F	1.02	0.0	0.5	36.3	2.6	0.99	1.00	47
$F_{0.01}$	1.01	0.0	0.3	22.4	4.0	1.00	1.02	1037
C_{25}	1.01	0.0	0.3	22.4	4.0	1.00	1.02	1051
S_{25}	1.01	0.0	0.3	26.8	3.8	1.00	1.00	1058
R_2	1.01	0.0	0.3	22.4	4.0	1.25	1.02	401

The large set of 1000 scenarios used to evaluate the solutions obtained by each solution approach considered in Table 6 leads to a sample of values where each value corresponds to the amount of backlog in one of the scenarios. The obtained values for each solution approach for instance $E2$ are represented in box-plots in Figure 9. The solution approaches are identified in the x-axis. Each point in the figure corresponds to the backlog amount observed in one scenario.

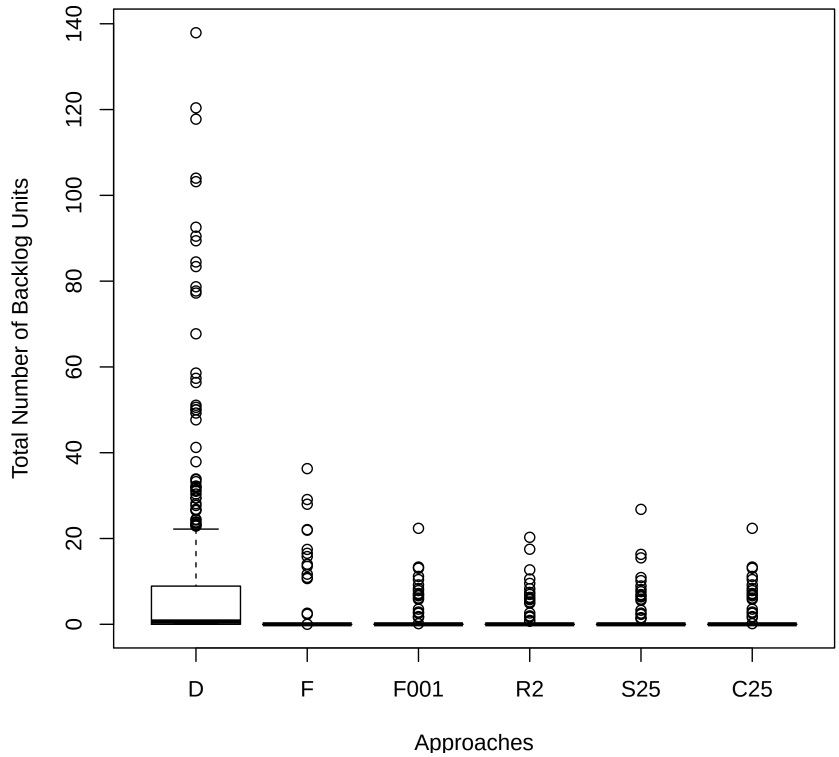

Figure 9: Sample box-plots for instance $E2$.

This box-plot is representative for most of the instances. The highest variance is obtained by solution approach D, where the uncertainty is not considered. A similar behavior can be observed for the results obtained by approach S_5, not presented here. In general, for the remaining solution approaches, there is no stock-out in at least 75% of the scenarios (750 scenarios).

Figure 10 displays the empirical distribution of the backlog for each solution approach considered in the box-plots. Notice that only 5 curves can be observed in this figure because the results for approaches $F_{0.01}$ and C_{25} are exactly the same for this instance.

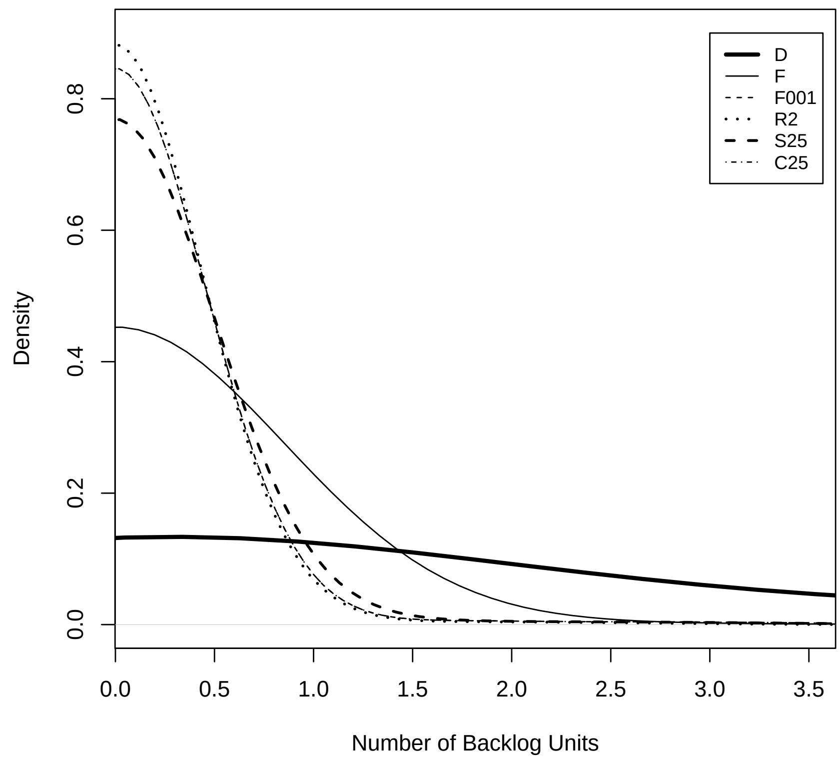

Figure 10: Empirical distribution of the backlog for some solution approaches for instance $E2$.

The results presented show the highest variance of the empirical backlog distribution derived from the deterministic approach D. When the uncertainty is not explicitly considered in the approach, the empirical distribution of the backlog is characterized by heavy tails. In general the tails tend to be smoother in the robust and $CVaR$ approaches than when using stochastic programming.

6. Conclusions

We consider a maritime inventory routing problem where the travel times are uncertain. The impact of the uncertainty is analyzed according to five different general models of the problem: a deterministic model, a deterministic model with inventory buffers, robust optimization, stochastic programming, and stochastic programming models extended using the conditional value-at-risk measure. To the best of our knowledge, models using the conditional value-at-risk have never been used before to solve this problem.

The results obtained for a set of 21 instances show that substantial gains can be achieved when uncertainty is explicitly considered into the problem. In general, the deterministic approach and the stochastic approach with a low penalty value for inventory limit violations generate solutions where both the probability of violation and the total amount of violation are higher. However, these approaches are characterized by lower routing costs. Conversely, the solutions obtained by both the robust and the *CVaR* solution approaches are characterized by high routing costs but provide solutions that are more protected against the uncertainty, in the sense that the inventory limit violation in the obtained solutions is lower. The stochastic programming approach including high penalties for inventory bounds violations, the deterministic approach with inventory buffers and the hybrid approach that solves a deterministic approach with the inventory buffers derived from the *CVaR* approach show a tradeoff between routing costs and the amount and probability of inventory limit violations.

For the planning problem and stochastic programming method described in this paper, it is assumed that the arrival times to ports and the inventory levels may be adjusted due to uncertain traveling times. No flexibility in the route planning is allowed. Therefore, an interesting topic for further research would be to study additional flexibility in the route planning by a multi-stage stochastic model or by a two-stage model solved in a rolling-horizon fashion with updated information.

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