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Electron heating and acceleration for the nonlinear kinetic equation

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Summary. — Time evolution of the isotropic electron distribution function while heating for the nonlinear kinetic equation with the Landau-Fokker-Planck collisional integral is studied. The considered heating sources are mono kinetic distribution, hot ions, and a quasi-linear diffusion operator. The investigation is mainly concentrated on the formation of the distribution function and tail acceleration. The time-dependent solutions allowing the solutions in self-similar variables are examined. Also presented are analytical asymptotic solutions and comparison them with numerical results.

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1. – Preliminaries

In many important cases one should treat plasma transport kinetically. The solutions of the Landau-Fokker-Planck (LFP) equation, which is one of the key ingredients of plasma kinetic equation, have much broader interest ranging from plasma physics to stellar dynamics (*e.g.*, see [1-3] and the references therein).

We consider the isotropic electron distribution function $f_e(v,t)$ and study the electron heating with a simple model:

$$\frac{\partial f_e}{\partial t} = \Gamma \cdot I(f_e) + H(f_e), \qquad 0 \le v < \infty, \quad t \ge 0, \tag{1}$$

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where $I(f_e)$ is the LFP collisional integral for charged particles and $H(f_e)$ is the heating source. We focus on the interplay of the LFP collisional operator with the different heating terms: monokinetic distribution, hot ions $f_i(v,t)$ (two component plasma), and a quasi-linear diffusion operator. Our interest is mainly concentrated on the evolution and formation of the distribution function tails for $t \to \infty$, as $v \to \infty$. At the beginning we shortly review some selected results of our works on the subject (see [4, 5] and the references therein) with a heating source $H(f_e)$ localized in the velocity space. Then a broader class of the heating terms resulting in enhancement of the tail of the distribution function is analytically analyzed. All asymptotic results are confirmed by the numerical computing of nonlinear LFP equation with high accuracy.

The distribution functions are considered normalized to unity and the non-equilibrium electron T_e and ion T_i temperatures are as follows:

$$4\pi \int_0^\infty \mathrm{d}v v^2 f_{e,i}(v,t) = 1, \qquad T_{e,i}(t) = \frac{4\pi m_{e,i}}{3} \int_0^\infty \mathrm{d}x x^4 f_{e,i}(x,t), \tag{2}$$

where the first integral has a sense of the particle density. The system of the LFP collisional integrals for such functions is

$$\begin{aligned} I(f_e) &= \frac{1}{\rho v^2}\frac{\partial}{\partial v}\left[A\frac{\partial f_e}{\partial v} + (B_e + \rho B_i) f_e\right], \\ I(f_i) &= \frac{\rho}{v^2}\frac{\partial}{\partial v}\left[A\frac{\partial f_i}{\partial v} + \left(\frac{1}{\rho}B_e + B_i\right) f_i\right]. \end{aligned} \tag{3}$$

Here, the following notations for the integral operators A, B are used:

$$A = \frac{A_e + A_i}{3v}, \quad A_e = \int_0^v \mathrm{d}x x^4 f_e(x,t) + v^3 \int_v^\infty \mathrm{d}x f_e(x,t), \quad B_e = \int_0^v \mathrm{d}x x^2 f_e(x,t), \tag{4}$$

where $\Gamma = 16\pi^2 e^4 LN/m_e m_i$, L is the Coulomb logarithm and $\varrho = m_e/m_i$. In case of heating, the local Maxwellian distribution depends on time $f_M \sim {v_{\mathrm{th}}}^{-3} \exp[-v^2/{v_{\mathrm{th}}}^2]$, where $v_{\mathrm{th}} = v_{e,i}(t) = \sqrt{2T_{e,i}(t)/m_{e,i}}$ is the corresponding thermal velocity. We also use the energetic variable $\xi = v^2/v_{\mathrm{th}}^2$. For $v, \xi \to \infty$ the nonlinear integro-differential LFP integral becomes linear,

$$I(f) = \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{2T}{m}\frac{1}{v}\frac{\partial f}{\partial v} + f\right) = \frac{1}{\xi^{1/2}}\frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial \xi} + f\right). \tag{5}$$

The Coulomb diffusion influence is utmost in the cold region $0 \le v/v_{\mathrm{th}} < 1$. In the high-velocity region $v/v_{\mathrm{th}} \gg 1$ the LFP parabolic equation degenerates because of known Rutherford cross-section dependence on velocity and acquires more pronounced hyperbolic type. This circumstance leads to inevitable retarding of the distribution tail formation.

2. – Heating localized in the velocity space

At first, we consider the Cauchy problem for (1)-(4) without a heating term dealing with one sort of particles ($\varrho = 1$). The initial distribution function is located in the region $0 \le v \le 1$, the thermal velocity is a velocity scale unit. We are interested in the relaxation of the distribution function tails in the high-energy region $v \to \infty$. Substituting $f(v,t) = \mathcal{G}(v,t) \cdot f_M(v)$ in (5), we consider the period when the relaxation process is practically finished in the thermal velocity region, that is, the period when $\mathcal{G}_M(v) \approx 1$. Hence, we solve the problem for eq. (5) in the superthermal velocity region where the second (transport) term prevails and the slow establishing of the equilibrium solution $\mathcal{G}_M = 1$ occurs, and $\mathcal{G} \to 0$ as $v \gg 1$, $t \gg 1$. From asymptotic analysis the following results are obtained. $\mathcal{G}$ is expressed in terms of the error function

$$\mathcal{G}(v,t) = \Phi\left\{\frac{2}{5}\, v_f \left[\frac{v - v_f(t)}{v_f(t)}\right]^{5/2}\right\}, \qquad \Phi(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{d}y e^{-y^2}, \tag{6}$$

and has a character of a propagating wave, whose front moves under the law $v_f(t) = (3\,t)^{1/3}$ with the constant front width $\Delta_f(t) = \sqrt{\pi}$. Then the distribution function formation for $t \to \infty$, $v \to \infty$ actually means the distribution tail heating. For high energetic region $\xi \gg 1$ the rough approximation can be used $\mathcal{G}(\xi) = \Phi(\xi^{5/2}) \sim \exp[-\xi^{5/2}]$. In other words, the electron distribution slowly tends to its equilibrium solution $f(\xi) \simeq \exp[-\xi] \cdot \exp[-\xi^{5/2}]$ having an underheated tail.

Further, we include into consideration the heating of the electrons by the source of the hot particles which is localized in a high energetic region $\xi_+ \gg 1$. In a cold region the distribution is supposed Maxwellian. Therefore for asymptotic analysis we again use eq. (5). Let the heating source have small intensity $\sigma \ll 1$, so the density and the energy of the system practically do not undergo noticeable changes during the process under consideration. Then a stationary distribution is formed while the source is regarded as still located in a tail distribution region. Under the following conditions: $t \to \infty$, $\sigma \to 0$, $\xi \to \infty$, such that $\Delta t = \sigma \cdot t \cdot \ln(1/\sigma)$ is finite, $\xi_+ = \xi/2T_0 - \ln(1/\sigma)$ is finite, the new quasi–steady-state non-equilibrium distribution is established inside the momentum interval between the energy source ξ_+ and the bulk distribution (or the sink $\xi_- < \xi_+$). If, in particular, the source and the sink are monokinetic distributed functions $H(f) = \sigma \cdot \xi^{-1/2} \cdot [\delta(\xi - \xi_+) - \delta(\xi - \xi_-)]$, with $\xi_- < \xi_+$, then $f(\xi) = Ce^{-\xi} + \sigma \cdot \Delta(\xi)$, where

$$\Delta(\xi) = \eta[\xi_+ - \xi] + \eta[\xi - \xi_+] \exp\left[-(\xi - \xi_+)\right] - \eta[\xi_- - \xi] + \eta[\xi - \xi_-]\, \exp\left[-(\xi - \xi_-)\right]$$

and $\eta[y]$ is a unit function. Numerical results show that the analytical asymptotic estimations obtained for the linear case are fit for a wider class of parameters. The functional dependence of the steady-state non-equilibrium electron distribution is unsensitive to the extent to which the source and sink are located in momentum space. Figure 1 demonstrates the quasi–steady-state distribution function that has plateau essentially exceeding the equilibrium distribution. The size of plateau is shortening with time because of normalization ${v_+}^2/{v_{\mathrm{th}}}^2(t)$. However, the distribution tail formation is described by formulas (6), $f(\xi) \sim \exp[-\xi^{5/2}]$ in the region $\xi \gg \xi_+$.

Now let pass to the classical problem of electron and ion temperature relaxation: $T_e^0 + T_i^0 = T_e(t) + T_i(t) = 2T_{\mathrm{eq}}$. For (1)-(4) with $H(f_e) = 0$ the heating source for electrons are hot ions. For $\rho \to 0$ the usual collision time ordering is the following. During time t_e electrons approach their equilibrium, next ions' maxwellization occurs within time $t_i \gg t_e$. During temperature relaxation that takes time $t_{ei} \gg t_i \gg t_e$, the distribution functions remain Maxwellians $\sim e^{-\xi}$ with time-dependent electron and ion temperatures. Then the formula for temperatures can be obtained from the

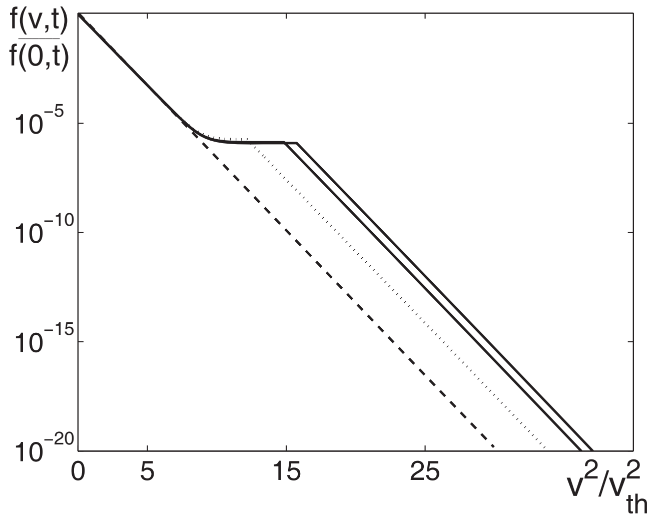

Fig. 1. – Formation of the quasi–steady-state non-equilibrium distribution function *vs.* $v^2/v_{\rm th}^2(t)$ for different time moments and $\xi_+ = 4$, $\xi_- = 0.5$, $\sigma = 10^{-5}$. The dashed-line curve is the Maxwell distribution.

system (1)-(4): $T_e^{3/2}{\rm d}T_e/{\rm d}t = C\,(T_i - T_e)$. Such an approach imposes a severe restriction on the initial temperatures $\rho^{1/3} \ll {T_i}^0/{T_e}^0 \ll \rho^{-1/3}$.

Let us set

$$f_{e,i}(v,t) = \frac{1}{2\pi} v_{e,i}^{-3} \cdot f'\left(v^2/v_{e,i}^2, t'\right), \quad t = t_0 t', \quad t_0 = \frac{2\pi}{\Gamma}\,(2T_{\rm eq}/m_e)^{3/2}, \quad T_{e,i} = T_{\rm eq} {T_{e,i}}',$$

and the squared ratio of thermal speeds $\varepsilon = v_i^2/v_e^2 = \rho T_i/T_e$. We substitute these expressions in formulas (1)-(4) and return to the original notations. Then the equation for the electron function reads

$$T_e{}^{3/2}\frac{\partial f}{\partial t} = \frac{1}{\rho}\frac{1}{\sqrt{\xi}}\frac{\partial}{\partial \xi}\left[(A_e + A_i)\frac{\partial f}{\partial \xi} + (B_e + B_i)\, f + \rho\sqrt{T_e}\frac{\partial T_e}{\partial t}\xi^{3/2} f\right]. \tag{7}$$

Here, coefficients A_e, B_e correspond to the electron-electron collisions

$$A_e = \frac{2}{3}\left[\int_0^{\xi} {\rm d}y\, y^{3/2} f_e(y) + \xi^{3/2}\int_{\xi}^{\infty} {\rm d}y\, f_e(y)\right], \qquad B_e = \int_0^{\xi} {\rm d}y\, y^{1/2} f_e(y), \tag{8}$$

and take a simple form $A_e \simeq 2/3\xi^{3/2}\int_0^{\infty}{\rm d}y\, f_e(y)$, $B_e \simeq 2/3\xi^{3/2} f_e(0)$ for $\xi \ll 1$ and $A_e = B_e = 1$ for $\xi \gg 1$. For the electron-ion collisions

$$A_i = \frac{2}{3}\,\varepsilon\left[\int_0^{\xi/\varepsilon} {\rm d}y y^{3/2} f_i(y) + \left(\frac{\xi}{\varepsilon}\right)^{3/2}\int_{\xi/\varepsilon}^{\infty} {\rm d}y\, f_i(y)\right], \quad B_i = \rho\int_0^{\xi/\varepsilon} {\rm d}y\, y^{1/2} f_i(y). \tag{9}$$

TABLE I. – *Comparison of the electron-electron and electron-ion coefficients.*

$\xi \sim \varepsilon$	$A_e, B_e \sim \varepsilon^{3/2}$	$A_i \sim \varepsilon,\ B_i \sim \rho$
$\varepsilon \ll \xi \ll \varepsilon^{2/3}$	$A_e, B_e \ll \varepsilon$	$A_i \sim \varepsilon,\ B_i \sim \rho$
$\xi \sim \varepsilon^{2/3}$	$A_e, B_e \sim \varepsilon$	$A_i \sim \varepsilon,\ B_i \sim \rho$
$\xi \gg \varepsilon^{2/3}$	$A_e, B_e \gg \varepsilon$	$A_i \sim \varepsilon,\ B_i \sim \rho$

The normalization conditions can now be rewritten as follows:

$$\int_0^\infty \mathrm{d}x\,\sqrt{x}\,f_{e,i}(x,t) = 1, \qquad \int_0^\infty \mathrm{d}x x^{3/2} f_{e,i}(x,t) = \frac{3}{2} \tag{10}$$

and $T_e(t)+T_i(t)=2$. Now we replace eq. (7)-(9) by approximate equations for the small mass ratio $\rho \ll 1$. For fixed $\rho, \varepsilon \ll 1$ and different magnitudes of the variable ξ, the comparison by the order of coefficients (8) and (9) yields the result given in table I. As can be seen, the influence of the coefficients A_i, B_i around $\xi \sim 1$ is small but it enhances sensibly in the vicinity of $0 \le \xi \le \varepsilon^{2/3}$. Thus in the vicinity of the point $\xi \simeq 0$, there exists a composed boundary layer with a width of $\delta \sim \varepsilon^{2/3}$. Within its internal domain $0 \le \xi < \varepsilon$ coefficients A_i, B_i depend essentially on the ion distribution f_i but for $\xi \gg \varepsilon$ this dependence disappears. We obtain for $\rho \to 0$, $\varepsilon \to 0$ and $\xi \gg \varepsilon$ from (7)-(9)

$${T_e}^{3/2}\frac{\partial f_e}{\partial t} = \frac{1}{\xi^{1/2}}\frac{\partial}{\partial \xi}\left[\frac{1}{\rho}\left(A_e\frac{\partial f}{\partial \xi} + B_e f\right) + \left(\frac{T_i}{T_e}\frac{\partial f_e}{\partial \xi} + f_e\right) + {T_e}^{1/2}\frac{\partial T_e}{\partial t}\xi^{3/2} f_e\right], \tag{11}$$

and we obtain the boundary condition and the temperature equation from (10), (11)

$$\left(\frac{T_i}{T_e}\frac{\partial f_e}{\partial \xi} + f_e\right)_{\xi=0} = 0, \qquad {T_e}^{3/2}\frac{\mathrm{d}T_e}{\mathrm{d}t} = \frac{2}{3}\left[T_i f_e(0) - T_e\int_0^\infty \mathrm{d}\xi f_e(\xi)\right]. \tag{12}$$

The main part $\varepsilon \ll \xi \le \varepsilon^{2/3}$ of the boundary layer δ is correctly described by eq. (11). Examination of (7)-(9) shows that the solution can be replaced in the internal domain $\xi \sim \varepsilon$ by the boundary condition (12) with the approximation error of $O(\rho)$. From (10)-(12) for $\rho = 0$, we formally obtain the solution $f^{(0)}(\xi) \simeq \exp[-\xi]$, $\theta^{3/2}\theta_t \simeq (T-\theta)$ that should be "corrected" in the vicinity of $\xi \approx 0$. To adjust (12) we assume

$$f_e(\xi,t) = \frac{2}{\sqrt{\pi}}e^{-\xi}\left[1 + \frac{T_e - T_i}{T_i}\cdot u(\xi,\varepsilon)\right], \tag{13}$$

insert (13) in (10)-(12) and neglect the squared terms of u. After rather lengthy computation one can obtain $u(x) \simeq \delta\psi(\xi/\delta) + O(\varepsilon)$, where $\delta = (3\sqrt{\pi}\varepsilon/4)^{2/3}$ and $\psi(x) = \int_x^\infty \mathrm{d}y/(1+y^{3/2})$. The applicability of the known formula for temperatures $\varepsilon \ll 1$ is hundred times less rigorous than $\rho \ll 1$ and the corrected formula reads

$${T_e}^{3/2}\frac{\mathrm{d}T_e}{\mathrm{d}t} \simeq \frac{4}{3\sqrt{\pi}}\frac{T_i - T_e}{1 + 2.9\varepsilon^{2/3}}. \tag{14}$$

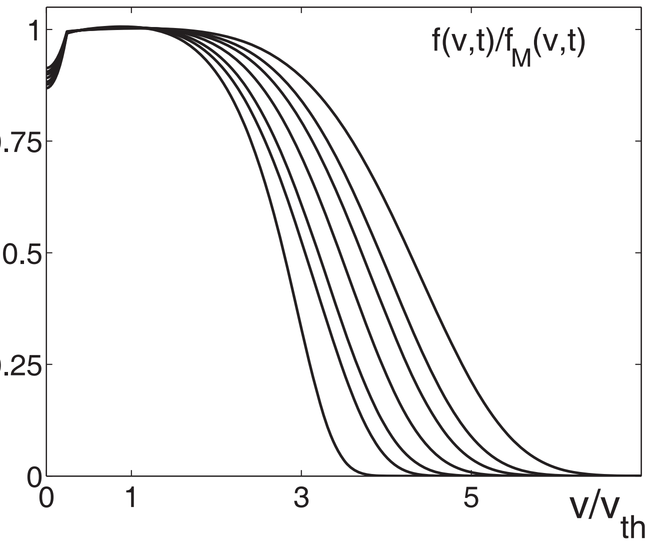

Fig. 2. – Temporal evolution of the electron distribution function tails for the quasi-linear diffusion operator (15) located in $0.05 \geq v \geq 0.75$.

The electron function achieves its maximum absolute deviation from its local Maxwell distribution at $\xi = 0$: $f_e(\xi, t) \simeq 2/\sqrt{\pi} \cdot e^{-\xi}[1+2.9\varepsilon^{2/3}(T_e - T_i)/T_i]$. The relative deviation of the electron distribution from equilibrium can be larger in the tail region than where the tail formation is described by (6). The same behavior of the distribution function tails is valid for any localized heat source having time dependence as $T \sim t^{2/3}$.

An example is the interaction of RF waves with a plasma. It is described by a LFP equation with an added quasi-linear term (usually 2D in velocity space)

$$H(f) = \frac{1}{\xi^{1/2}} \frac{\partial}{\partial \xi} \left[\xi^{3/2} D_{\mathrm{ql}} \frac{\partial f}{\partial \xi} \right], \qquad D_{\mathrm{ql}} = \begin{cases} \text{const}, & \text{if } \xi_1 \leq \xi \leq \xi_2 \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

For this case the numerical result demonstrates in fig. 2 the tail formation. It has a character of a propagating wave with slightly changing slope in time because of heating.

3. – Self-similar solutions with accelerated tails

Unlike the cases considered above the situation changes drastically when in $H(f)$ from (15) the coefficient of the quasi-linear diffusion $D_{\mathrm{ql}}(\xi, t)$ increases with velocity increasing. We consider a special class of functions, for which it is possible to construct a self-similar solution. Thus from (1), (15) instead of (11) we obtain

$$T_e{}^{3/2} \frac{\partial f}{\partial t} = \frac{1}{\xi^{1/2}} \frac{\partial}{\partial \xi} \left[A \frac{\partial f}{\partial \xi} + Bf + \xi^{3/2} T_e{}^{1/2} \overline{D}(\xi, t) \frac{\partial f}{\partial \xi} + T_e{}^{1/2} \frac{\partial T_e}{\partial t} \xi^{3/2} f \right], \tag{16}$$

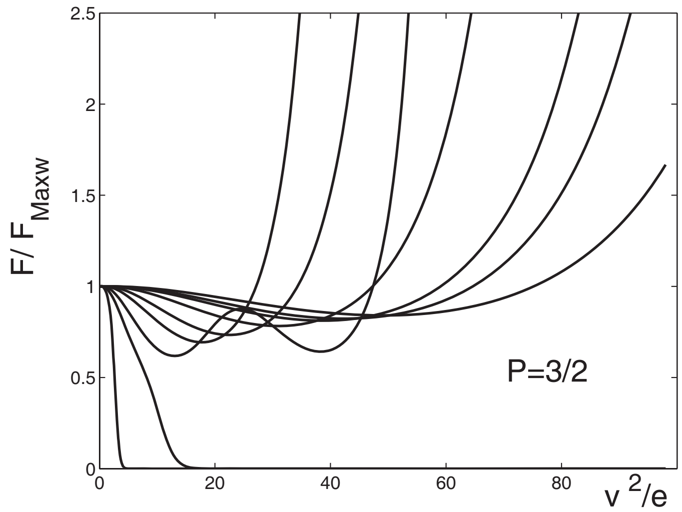

Fig. 3. – Temporal evolution of the electron distribution function tail for $p = 3/2$, $e = 3/2 \cdot v_{\rm th}^2$.

The normalization conditions (10) lead formally to the following equation for $T(t)$:

$$T_t = \frac{\mathrm{d}T}{\mathrm{d}t} = \int_0^\infty \mathrm{d}y f(y) \frac{\partial}{\partial y}[y^{3/2}\overline{D}(y,t)].$$

The only case, when eq. (16) formally admits stationary solutions, $\partial f/\partial t = 0$, corresponds to $\overline{D}_{\rm ql}(\xi, t) = D(\xi)/\sqrt{T(t)}$. Note, we have the same law $T(t) \simeq t^{2/3}$. Taking $D(\xi)\xi^{3/2} = D_0\xi^{p-3/2}$, where D_0 is the normalization constant and $5/2 > p > 0$ is an adjustable parameter, we find the distribution function in the high-velocity region

$$f(\xi \to \infty) \propto \exp\left[-\frac{\xi^{5/2-p}}{5/2-p}\right], \quad 5/2 > p \geq 0. \tag{17}$$

For the case $p = 5/2$ we obtain a power-law tail $f(\xi \to \infty) \propto \xi^{-5/2}$. For $p = 3/2$ the solution of (1), (15) with $D_{\rm ql}(\xi, t) = D_0$ is Maxwellian. Figure 3 shows the distribution function tail formation for different time moments. At the beginning the tail has a Coulombian character and starting from the time $\tau \sim 1/D_0$ it spreads into superthermal velocity region following the character of diffusion action. The formation of the solution (17) is presented for $p = 2$ in fig. 4. Numerical results also show that in the region of $0 \leq v \sim 4v_{\rm th}$ the distribution is mostly close to Maxwellian and display the transition region between the Maxwellian part and the enhanced tail.

4. – Conclusions

Obtained results are consistent with the preceding results. The distribution function is close to Maxwellian in the thermal velocity region. The perturbation of the electron distribution function in case of heavy hot ions, which has a character of a boundary layer, gives small correction in the temperature exchange but left the distribution tail

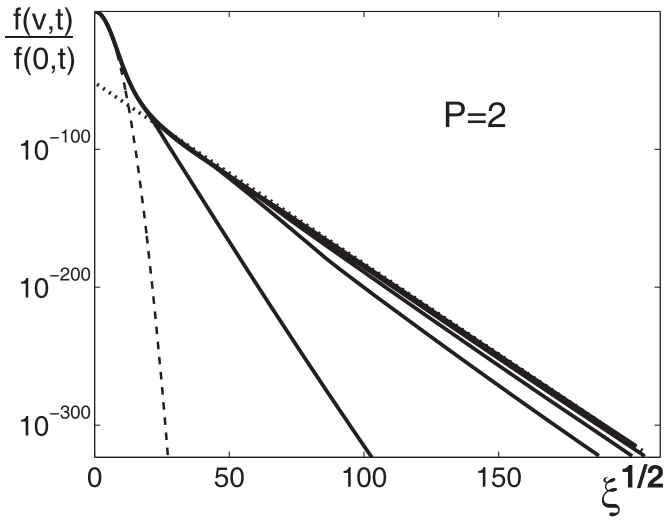

Fig. 4. – Temporal evolution of the electron distribution function for $p = 2$. The dashed-line curve is the corresponding Maxwellian distribution function. In agreement with analytic results, the tail of the distribution function follows expression (17) (dotted line).

underheated as well as any source localized in the velocity space. A broader class of heating terms resulting in an enhancement of the distribution function tail is analyzed analytically. Numerical simulation of nonlinear LFP equation confirms asymptotic results with high accuracy and reveal additional details that can help to test and benchmark complex kinetic codes.

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