IL NUOVO CIMENTO Vol. 33 C, N. 1 Gennaio-Febbraio 2010
DOI 10.1393/ncc/i2010-10583-3

Colloquia: ICTT2009

From Vlasov fluctuations to the BGL kinetic equation

C. Lancellotti(*)

City University of New York - CSI, Staten Island NY 10314, USA

(ricevuto il 2 Ottobre 2009; approvato il 25 Gennaio 2010; pubblicato online il 16 Marzo 2010)

Summary. — This paper shows that the spatially homogeneous Balescu-Guernsey-Lenard kinetic equation is associated, at least formally, with a stochastic process that arises naturally as the $N \to \infty$ limit of a certain N-particle Hamiltonian system. The process describes the long-time motion of a particle traveling in a Vlasov fluctuation field. The Fokker-Planck equation for the process coincides with the Balescu-Guernsey-Lenard equation whenever the solution is analytic in the velocity variables, but should also be considered as a model in its own right.

PACS 05.20.Dd – Kinetic theory.
PACS 05.60.Cd – Classical transport.
PACS 52.25.Dg – Plasma kinetic equations.

1. – Introduction

At the time of its discovery in 1960 [1-3], the Balescu-Guernsey-Lenard (BGL) equation could reasonably have been expected to play in the theory of classical plasmas the same central role that the Boltzmann equation has for (non-ionized) rarefied gases. Fifty years later, it is fair to say that this has not been the case. Part of the reason lies in the intrinsic limitations of the model, which neglects some important physical effects that arise, for instance, in fusion plasmas [4]. But even in its natural field of application—the transport theory of classical, non-turbulent plasmas—the BGL collision operator has largely been sidelined [4] in favor of the simpler Landau operator [5]. Moreover, the mathematical theory of the BGL equation is practically non-existant, even in the spatially homogeneous case; only recently a rigorous study of the *linearized* BGL equation has appeared [6].

Given the BGL collision term's complicated, highly nonlinear structure, it seems likely that progress will require not just studying the kinetic equation *per se*, but also understanding better the deterministic and/or stochastic N-particle models that lie "behind" the BGL kernel. The present work outlines a novel derivation of the spatially homogeneous BGL equation from a N-particle dynamical system. The textbook derivations of

(*) Supported by NSF-DMS Grant 0604946 and also by PSC-CUNY Award 62387-00 40.

the equation from microscopic dynamics are mathematically hard to control and physically not very transparent, based as they are on formal truncations of the BBGKY hierarchy of non-equilibrium statistical mechanics. While still largely formal, the new derivation shows (in the author's opinion!) more potential than the traditional arguments to lead, in the long run, to a rigorous justification of the BGL limit. It also suggests a relatively simple probabilistic interpretation of the BGL equation as the (nonlinear) Fokker-Planck equation for a "BGL stochastic process," to be introduced in the first section of the paper. The BGL process describes, loosely speaking, the random motion of a particle under the long-time effects of the fluctuating force field about the Vlasov dynamics (the Vlasov mean field itself being zero in the spatially homogeneous case). The corresponding Fokker-Planck equation, presented in the second section, is actually more general than the BGL equation; as shown in the third and last section, it takes the BGL form if one can prove very strong regularity for the solutions (analiticity).

One should mention that the connection between the BGL equation and Vlasov fluctuations has been long known to plasma physicists [7], but only in terms of formal manipulations of the BBGKY hierarchy. Here, the link is given a more precise meaning by the BGL stochastic process, whose definition relies on recent rigorous results on the Central Limit Theorem for Vlasov fluctuations [8]. The idea of studying the long-time dynamics in a fluctuating force field around a vanishing Vlasov mean value also has a precedent in the physical literature, in a work by Piasecki and Szamel [9] who, however, considered only the case of a "test" particle, not a system of N particles.

2. – Weak *vs.* mean scaling limits. The BGL process

Let Ω denote the box $[-1,1]^{\otimes 3}$. Each N-particle microstate is a *random vector* $(\mathfrak{p}_1,\dots,\mathfrak{p}_N,\mathfrak{q}_1,\dots,\mathfrak{q}_N)\equiv(\mathfrak{P},\mathfrak{Q})$ in $\mathbb{R}^{3N}\times\Omega^N$ with permutation-symmetric probability measure $\mu_\tau^{(N)}$. To $(\mathfrak{P},\mathfrak{Q})$ corresponds(1) an "empirical density" $\mathcal{E}_N(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,|\mathfrak{P},\mathfrak{Q})=\frac{1}{N}\sum_{j=1}^N\delta(\boldsymbol{p}-\mathfrak{p}_j)\delta(\boldsymbol{q}-\mathfrak{q}_j)$, a normalized random measure on $\mathbb{R}^3\times\Omega$. Let the dynamics of the microstate be determined by the "weak" Newton equations

$$\frac{\mathrm{d}\mathfrak{q}_i}{\mathrm{d}\tau}=\mathfrak{p}_i, \tag{1a}$$

$$\frac{\mathrm{d}\mathfrak{p}_i}{\mathrm{d}\tau}=N\int_{\mathbb{R}^3\times\Omega}\boldsymbol{F}(N(\mathfrak{q}_i-\boldsymbol{q}))\ \mathcal{E}_N(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,|\mathfrak{P},N\mathfrak{Q}), \tag{1b}$$

where $\boldsymbol{F}=-\nabla U$ is the (rotationally symmetric) force between two particles. For the sake of simplicity, we assume a periodic boundary and $\boldsymbol{F}\in\mathcal{C}_b^1(\mathbb{R}^3)$. Then, $\boldsymbol{F}$ is globally Lipschitz and the initial value problem is well-posed(2); the assumptions imply $F(\mathbf{0})=\mathbf{0}$.

If $\Omega_N=[-N,N]^{\otimes 3}$, consider also an auxiliary system of N particles with coordinates $(\mathsf{p}_1,\dots,\mathsf{p}_N,\mathsf{q}_1,\dots,\mathsf{q}_N)\equiv(\mathsf{P},\mathsf{Q})$ in $\mathbb{R}^{3N}\times\Omega_N^N$ that satisfy the "mean" Newton equations

$$\frac{\mathrm{d}\mathsf{q}_i}{\mathrm{d}t}=\mathsf{p}_i, \tag{2a}$$

$$\frac{\mathrm{d}\mathsf{p}_i}{\mathrm{d}t}=\int_{\mathbb{R}^3\times\Omega_N}\boldsymbol{F}(\mathsf{q}_i-\boldsymbol{q})\ \mathcal{E}_N(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,|\mathsf{P},\mathsf{Q}). \tag{2b}$$

---

(1) The correspondence is one-to-one modulo permutations in the particles' labels.

(2) Newton and Coulomb forces, of course, are not included unless they are suitably regularized.

For a given N the two sets of ODEs are related by the scaling transformations $t = N\tau$, $\mathsf{Q} = N\mathfrak{Q}$. Thus, if the time evolution of $(\mathsf{q}_i(t), \mathsf{p}_i(t))$ is known, then $(\mathfrak{q}_i(\tau), \mathfrak{p}_i(\tau)) = (N^{-1}\mathsf{q}_i(N\tau), \mathsf{p}_i(N\tau))$. Similarly, the ensemble probability measure $\nu_t^{(N)}$ for the variables (P, Q) is related to $\mu_\tau^{(N)}$ by $\mu_\tau^{(N)} = \nu_{N\tau}^{(N)}$. One can write eq. ($1b$) in integral form

$$\mathfrak{p}_i(\tau + \delta\tau) - \mathfrak{p}_i(\tau) = N\int_\tau^{\tau+\delta\tau} \mathrm{d}\tau' \int_{\mathbb{R}^3\times\Omega} \boldsymbol{F}(N\mathfrak{q}_i(\tau') - N\boldsymbol{q})\, \mathcal{E}_N(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,|\mathfrak{P}(\tau'), N\mathfrak{Q}(\tau'))$$

and change integration variables, $\tau' \to t = N(\tau' - \tau)$ and $\boldsymbol{q} \to N\boldsymbol{q}$. Exploiting the relationship $[\mathfrak{P}(\tau'), N\mathfrak{Q}(\tau')] = [\mathsf{P}(t), \mathsf{Q}(t)]$—where the mean-field quantities $[\mathsf{P}, \mathsf{Q}]$ still satisfy eq. (2), but now with initial distribution $\nu_0^{(N)} = \mu_\tau^{(N)}$—yields

$$\mathfrak{p}_i(\tau + \delta\tau) - \mathfrak{p}_i(\tau) = \int_0^{N\delta\tau} \mathrm{d}t \int_{\mathbb{R}^3\times\Omega_N} \boldsymbol{F}(\mathsf{q}_i(t) - \boldsymbol{q})\ \mathcal{E}_N(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,|\mathsf{P}(t), \mathsf{Q}(t)). \tag{3}$$

Now the right-hand side of eq. ($2b$) appears inside the t-integration: the change in momentum $\mathfrak{p}_i$ w.r.t. the "slow" variable τ is simply obtained by integrating the mean-field acceleration over a large time scale w.r.t. the "fast" variable t.

The $N \to \infty$ scaling limit for the mean-field equations, eqs. (2), also known as *Vlasov limit*, is mathematically well-understood [10-12], at least in the case of finite total mass. Under our assumptions on $\boldsymbol{F}$ and suitable "chaoticity" hypotheses on $\nu_0^{(N)} = \mu_\tau^{(N)}$ [10], a law of large numbers holds and

$$\mathcal{E}_N(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,|\mathsf{P}(t), \mathsf{Q}(t)) \xrightarrow{w} \nu_t(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}|\mu_\tau) \tag{4}$$

in probability, for $t \in [0, T_0]$, where ν_t is the (weak) solution to the Vlasov equation determined by $\boldsymbol{F}$ with $\nu_0 = \mu_\tau$. Here we will focus on the spatially homogeneous case, which unfortunately is not covered by the cited mathematical results because as the domains Ω_N expand to $\mathbb{R}^3$ the total mass goes to infinity. Clearly, it would very desirable to prove rigorously that the Vlasov limit holds also in this case. We expect that the validity of eq. (4) will require that $\boldsymbol{F}$ decay at infinity at a suitable rate. Leaving that as a problem for future study, for now we shall simply take eq. (4) as an assumption. Then, spatial homogeneity ensures that at every t the average value of the mean-field force in eq. (3) approaches zero as $N \to \infty$ (hence the name "weak" for eqs. (1)). Since in the same limit the t integration extends to infinity, the right-hand side of eq. (3) may still tend to a finite value. "Next-order" information needs to be provided in the form of a Central Limit Theorem for the fluctuations from mean field, which is also well established for finite total mass [8, 10]. Again, it will be assumed (but should be proved!) that a similar result holds for the spatially uniform, infinitely extended density ν_t, giving

$$\sqrt{N}[\mathcal{E}_N(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,|\mathsf{P}(t), \mathsf{Q}(t)) - \nu_t(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}|\mu_\tau)] \xrightarrow{w} \zeta_t(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}|\mu_\tau) \tag{5}$$

in the sense of finite-dimensional distributions. By analogy with the finite-mass case [8], ζ_t will be taken to be a continuous functional on $\mathcal{C}^1_{c,0}(\mathbb{R}^6) \equiv \{g \in \mathcal{C}^1_c(\mathbb{R}^6) : \langle g, \mu_\tau\rangle = 0\}$ determined by the so-called Braun-Hepp integral equations [10]; under the hypothesis of *strong* μ_τ*-chaoticity* [8] on $\mu_\tau^{(N)}$, ζ_0 will be the Gaussian field on $\mathcal{C}^1_c(\mathbb{R}^6)$ with mean zero and covariance $\langle \mu_\tau, g\,h\rangle$ for $g, h \in \mathcal{C}^1_{c,0}(\mathbb{R}^6)$.

In order to exploit eq. (5), we write the right side of eq. (3) as

$$\sqrt{N}\delta\tau\,\frac{1}{N\delta\tau}\int_0^{N\delta\tau}\mathrm{d}t\int_{\mathbb{R}^3\times\Omega_N}\boldsymbol{F}(\mathbf{q}_i(t)-\boldsymbol{q})\,\sqrt{N}\,[\mathcal{E}_N(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,|\mathbf{P}(t),\mathbf{Q}(t))-\nu_t(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}|\mu_\tau)]$$

(where, of course, ν_t contributes nothing to the integral). This formula is still exact; what one would like to do is to take the double limit $N\to\infty$, $\delta t\to 0$ in order to obtain an infinite-particle approximation to eqs. (1). Clearly, it is necessary that $N\delta t\to\infty$, because otherwise the limit is trivially zero. Heuristically, one possible way to proceed is by keeping the t-integration fixed at first and replacing the integrand with its limit as $N\to\infty$, *i.e.* the random force associated with the fluctuations from mean field

$$\boldsymbol{Z}_t(\boldsymbol{q}\,|\,\mu_\tau)=\int_{\mathbb{R}^6}\boldsymbol{F}(\boldsymbol{q}-\boldsymbol{q}')\,\zeta_t(\mathrm{d}\boldsymbol{q}'\mathrm{d}\boldsymbol{p}'|\mu_\tau). \tag{6}$$

Here, the action of the functional ζ_t has been written as an integral to help intuition(3). One can then pass to the limit in the time-average $\frac{1}{N\delta\tau}\int_0^{N\delta\tau}\mathrm{d}t$, and finally consider the scaling limit as $N\to\infty$ of the resulting ODEs. Obviously this is not the only plausible infinite-particle approximation to eqs. (1)—for instance, the time-averaging could be done last. However, the important (and difficult) problem of establishing rigorously the correct limit will not be addressed here. Our more modest goal is to show that the limiting procedure just described leads, at least formally, to the BGL equation.

With this in mind, let us now introduce a $\mathbb{R}^6$-valued "BGL process" $[\mathfrak{p}(\tau),\mathfrak{q}(\tau)]$ with law μ_τ, $\tau\in[0,T]$, as follows. Let $\zeta_0(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}|\mu_\tau)$ (to be written also more compactly as $\zeta_0(\tau)$) be the Gaussian process with index set $\mathcal{C}^1_{c,0}(\mathbb{R}^6)\times[0,T]$ and

$$\mathrm{E}\,(\langle\zeta_0(\tau),g\rangle)=0, \tag{7a}$$

$$\mathrm{E}\,(\langle\zeta_0(\sigma),g\rangle\,,\langle\zeta_0(\tau),h\rangle)=\langle\mu_\tau,g\,h\rangle\,\delta(\sigma-\tau) \tag{7b}$$

for all $g,h\in\mathcal{C}^1_{c,0}(\mathbb{R}^6)$, $\sigma,\tau\in[0,T]$. Let $\zeta_t(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}|\mu_\tau)$ be the Vlasov fluctuation field at time t evolved from $\zeta_0(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}|\mu_\tau)$, and $\boldsymbol{Z}_t(\boldsymbol{q}\,|\,\mu_\tau)$ the corresponding force field as in eq. (6). Let $\epsilon\equiv 1/\sqrt{N}$ and let $[\mathfrak{p}^\epsilon(\tau),\mathfrak{q}^\epsilon(\tau)]$ be the family of processes that satisfy

$$\frac{\mathrm{d}\mathfrak{q}^\epsilon}{\mathrm{d}\tau}=\mathfrak{p}^\epsilon, \tag{8a}$$

$$\frac{\mathrm{d}\mathfrak{p}^\epsilon}{\mathrm{d}\tau}=\frac{1}{\epsilon}\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathrm{d}t\,\boldsymbol{Z}_t(\epsilon^{-2}\mathfrak{q}^\epsilon+\mathfrak{p}^\epsilon t\,|\,\mu_\tau) \tag{8b}$$

with initial law μ_0. Finally, let $[\mathfrak{p}(\tau),\mathfrak{q}(\tau)]=\lim_{\epsilon\to 0}[\mathfrak{p}^\epsilon(\tau),\mathfrak{q}^\epsilon(\tau)]$. Assuming that this limit is well defined, as $N\to\infty$ each particle is affected by the other particles only through the law μ_τ, via $\boldsymbol{Z}_t(\boldsymbol{q}\,|\mu_\tau)$, which suggests that the dynamics approaches a Markovian limit. Of course, one needs to *prove* that the limiting process is mathematically well defined. A crucial question is whether μ_τ remains "linearly stable" in some suitable sense that ensures that the time-average of the fluctuating force field in eq. (8*b*)

(3) This quantity will be well defined only for a suitable subspace of all the functionals on $\mathcal{C}^1_c(\mathbb{R}^6)$ if $\boldsymbol{F}$ is not compactly supported.

exists. If so, the weak-field limit $\epsilon \to 0$ could probably be established, *e.g.*, along the lines in [13] (with the complication that the right-hand side in eq. (8*b*) also depends on μ_τ). Here, however, the well-posedness of the BGL process will be simply taken for granted, and the focus will be on the partial differential equation formally satisfied by μ_τ.

3. – The transport equation

According to well-known formulas [13-15] the BGL process is formally associated with the nonlinear Fokker-Planck equation (in a weak sense)

$$\partial_\tau \mu_\tau = \partial_{\mathfrak{p}} \cdot [\mathbb{D}(\mu_\tau)\partial_{\mathfrak{p}}\mu_\tau + \boldsymbol{V}(\mu_\tau)\mu_\tau] \tag{9}$$

with diffusion matrix

$$\mathbb{D}(\mu_\tau) = \int_0^\tau \mathrm{d}\sigma \; \left[[\mathrm{E}\left(\boldsymbol{Z}_{t_1}(\mathfrak{q}+\mathfrak{p}t_1 \,|\, \mu_\tau) \otimes \boldsymbol{Z}_{t_2}(\mathfrak{q}+\mathfrak{p}t_2 \,|\, \mu_\sigma)\right)]_{t_1}\right]_{t_2} \tag{10}$$

and drift vector

$$\boldsymbol{V}(\mu_\tau) = \int_0^\tau \mathrm{d}\sigma \; \Big[[\mathrm{E}\left(\partial_{\mathfrak{p}} \cdot \boldsymbol{Z}_{t_1}(\mathfrak{q}+\mathfrak{p}t_1 \,|\, \mu_\tau)\, \boldsymbol{Z}_{t_2}(\mathfrak{q}+\mathfrak{p}t_2 \,|\, \mu_\sigma)\right)]_{t_1}\Big]_{t_2}, \tag{11}$$

where we used the notation $[f(t)]_t = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t$.

At each given point $\boldsymbol{q}$ we can regard the fluctuating force field $\boldsymbol{Z}_t(\boldsymbol{q} \,|\, \mu_\tau)$, eq. (6), as the action $\langle \zeta_t, \boldsymbol{g}_{\boldsymbol{q}} \rangle$ of the functional ζ_t over a test function $\boldsymbol{g}_{\boldsymbol{q}}(\boldsymbol{p}', \boldsymbol{q}') \equiv \boldsymbol{F}\big(\boldsymbol{q}-\boldsymbol{q}'\big)$. Under the assumption that the Central Limit Theorem for Vlasov fluctuations [8, 10] can be extended to an infinitely extended uniform distribution of particles, one has

$$\boldsymbol{Z}_t(\boldsymbol{q} \,|\, \mu_\tau) = \int_{\mathbb{R}^6} \mathcal{T}_t \boldsymbol{F}\big(\boldsymbol{q}-\boldsymbol{q}'\big)\, \zeta_0(\mathrm{d}\boldsymbol{q}'\mathrm{d}\boldsymbol{p}' | \mu_\tau). \tag{12}$$

Here ζ_0 is the "initial" (with respect to t) fluctuation field determined by μ_τ and $\mathcal{T}_t$ is an appropriate propagation operator [8, 10], which in our case (where $\boldsymbol{F}(\boldsymbol{q}-\boldsymbol{q}')$ is just a constant function of $\boldsymbol{p}'$) takes the form

$$\begin{aligned} \mathcal{T}_t \boldsymbol{F}(\boldsymbol{q}-\boldsymbol{q}') &= \boldsymbol{F}(\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{p}'t) \\ &\quad - \int_{\mathbb{R}^6} \mu_\tau(\mathrm{d}\boldsymbol{p}''\mathrm{d}\boldsymbol{q}'')\boldsymbol{K}(t, \boldsymbol{p}''\boldsymbol{q}'', \boldsymbol{p}', \boldsymbol{q}' | \mu_\tau) \cdot \nabla \boldsymbol{F}(\boldsymbol{q}-\boldsymbol{q}''-\boldsymbol{p}''t). \end{aligned} \tag{13}$$

The kernel $\boldsymbol{K}$ is determined by the Braun-Hepp [10] linearized Newton equation

$$\begin{aligned} \ddot{\boldsymbol{K}}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}', \boldsymbol{q}' | \mu_\tau) &= \boldsymbol{F}(\boldsymbol{q}+\boldsymbol{p}t-\boldsymbol{q}'-\boldsymbol{p}'t) \\ &\quad - \int_{\mathbb{R}^6} \mu_\tau(\mathrm{d}\boldsymbol{p}''\mathrm{d}\boldsymbol{q}'')\nabla \boldsymbol{F}(\boldsymbol{q}+\boldsymbol{p}t-\boldsymbol{q}''-\boldsymbol{p}''t) \cdot \boldsymbol{K}(t, \boldsymbol{p}''\boldsymbol{q}'', \boldsymbol{p}', \boldsymbol{q}' | \mu_\tau) \end{aligned} \tag{14}$$

whose unique solution can be easily obtained in series form.

Comparing the last three equations reveals an interesting identity:

$$\boldsymbol{Z}_t(\boldsymbol{q}+\boldsymbol{p}t \,|\, \mu_\tau) = \int_{\mathbb{R}^6} \ddot{\boldsymbol{K}}(t, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}', \boldsymbol{q}' | \mu_\tau)\, \zeta_0(\mathrm{d}\boldsymbol{q}'\mathrm{d}\boldsymbol{p}', \mu_\tau). \tag{15}$$

Substituting into eqs. (10), (11) and using eq. (7b) gives

$$\mathbb{D}(\mu_\tau) = \left[\left[\int_{\mathbb{R}^6} \mu_\tau(\mathrm{d}\boldsymbol{p}'\mathrm{d}\boldsymbol{q}')\ddot{\boldsymbol{K}}(t_1,\mathfrak{p},\mathfrak{q},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau)\otimes\ddot{\boldsymbol{K}}(t_2,\mathfrak{p},\mathfrak{q},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau)\right]_{t_1}\right]_{t_2}, \tag{16}$$

$$\boldsymbol{V}(\mu_\tau) = \left[\left[\int_{\mathbb{R}^6} \mu_\tau(\mathrm{d}\boldsymbol{p}'\mathrm{d}\boldsymbol{q}')\,\partial_{\mathfrak{p}}\cdot\ddot{\boldsymbol{K}}(t_1,\mathfrak{p},\mathfrak{q},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau)\,\ddot{\boldsymbol{K}}(t_2,\mathfrak{p},\mathfrak{q},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau)\right]_{t_1}\right]_{t_2}. \tag{17}$$

Equation (14) can be reinterpreted as an integral equation for $\ddot{\boldsymbol{K}}$:

$$\begin{aligned}\ddot{\boldsymbol{K}}(t,\boldsymbol{p},\boldsymbol{q},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau) &= \boldsymbol{F}(\boldsymbol{q}+\boldsymbol{p}t-\boldsymbol{q}'-\boldsymbol{p}'t)\\ &-\int_{\mathbb{R}^6}\mu_\tau(\mathrm{d}\boldsymbol{p}''\mathrm{d}\boldsymbol{q}'')\nabla\boldsymbol{F}(\boldsymbol{q}+\boldsymbol{p}t-\boldsymbol{q}''-\boldsymbol{p}''t)\cdot\int_0^t\mathrm{d}t'\,(t-t')\ddot{\boldsymbol{K}}(t',\boldsymbol{p}''\boldsymbol{q}'',\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau).\end{aligned} \tag{18}$$

Solving for $\ddot{\boldsymbol{K}}$ in series form and substituting into eqs. (16) and (17) determines the diffusion and drift coefficients for the nonlinear Fokker-Planck equation (9).

4. – Balescu-Guernsey-Lenard form

We are now going to show that the transport equation, eq. (9) together with eqs. (16), (17) and (18), coincides with the BGL equation *in the special case* when μ_τ is so regular that the diffusion and drift coefficients can be obtained by solving eq. (18) not in series form but rather in Fourier-Laplace variables. From now on, it will be assumed that μ_τ is absolutely continuous w.r.t. Lebesgue with density $f_\tau(\boldsymbol{p})$, *i.e.* $\mu_\tau(\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}) = f_\tau(\boldsymbol{p})\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}$. Further, f_τ will be taken to be *analytic*.

4·1. *Fourier-Laplace solution.* – Abusing the notation we use $\ddot{\boldsymbol{K}}$ to indicate also its integral transforms, changing only the arguments. Fourier transforming eq. (18) w.r.t. $\boldsymbol{q}$

$$\begin{aligned}\ddot{\boldsymbol{K}}(t,\boldsymbol{p},\boldsymbol{k},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau) &= \hat{\boldsymbol{F}}(\boldsymbol{k})\,e^{i\boldsymbol{k}\cdot\boldsymbol{q}'-i\boldsymbol{k}\cdot(\boldsymbol{p}-\boldsymbol{p}')t}\\ &+\hat{\boldsymbol{F}}(\boldsymbol{k})\int_{\mathbb{R}^3}\mathrm{d}\boldsymbol{p}''\,f_\tau(\boldsymbol{p}'')\,e^{-i\boldsymbol{k}\cdot(\boldsymbol{p}-\boldsymbol{p}'')t}\,i\boldsymbol{k}\cdot\int_0^t\mathrm{d}t'\,(t-t')\ddot{\boldsymbol{K}}(t',\boldsymbol{p}'',\boldsymbol{k},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau).\end{aligned} \tag{19}$$

Then, multiplying by $e^{i\boldsymbol{k}\cdot\boldsymbol{p}t}$ and taking the Laplace transform with respect to t yields

$$\begin{aligned}\ddot{\boldsymbol{K}}(s-i\boldsymbol{k}\cdot\boldsymbol{p},\boldsymbol{p},\boldsymbol{k},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau) &= \frac{\hat{\boldsymbol{F}}(\boldsymbol{k})\,e^{i\boldsymbol{k}\cdot\boldsymbol{q}'}}{s-i\boldsymbol{k}\cdot\boldsymbol{p}'}\\ &+\hat{\boldsymbol{F}}(\boldsymbol{k})\int_{\mathbb{R}^3}\frac{f_\tau(\boldsymbol{p}'')\mathrm{d}\boldsymbol{p}''}{(s-i\boldsymbol{k}\cdot\boldsymbol{p}'')^2}\,i\boldsymbol{k}\cdot\ddot{\boldsymbol{K}}(s-i\boldsymbol{k}\cdot\boldsymbol{p}'',\boldsymbol{p}'',\boldsymbol{k},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau)\end{aligned} \tag{20}$$

from which it follows easily that

$$\int_{\mathbb{R}^3}\frac{f_\tau(\boldsymbol{p}'')\mathrm{d}\boldsymbol{p}''}{(s-i\boldsymbol{k}\cdot\boldsymbol{p}'')^2}\,i\boldsymbol{k}\cdot\ddot{\boldsymbol{K}}(s-i\boldsymbol{k}\cdot\boldsymbol{p}'',\boldsymbol{p}'',\boldsymbol{k},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau) = \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{q}'}}{s-i\boldsymbol{k}\cdot\boldsymbol{p}'}\,\frac{1-\varepsilon(s,\boldsymbol{k}|\mu_\tau)}{\varepsilon(s,\boldsymbol{k}|\mu_\tau)}, \tag{21}$$

where

$$\varepsilon(s,\boldsymbol{k}|\mu_\tau) = 1 - i\boldsymbol{k}\cdot\hat{\boldsymbol{F}}(\boldsymbol{k})\int_{\mathbb{R}^3}\frac{f_\tau(\boldsymbol{p}'')\mathrm{d}\boldsymbol{p}''}{(s-i\boldsymbol{k}\cdot\boldsymbol{p})^2}. \tag{22}$$

Substituting back into eq. (20) gives the solution

$$\ddot{\boldsymbol{K}}(s,\boldsymbol{p},\boldsymbol{k},\boldsymbol{p}',\boldsymbol{q}'|\mu_\tau) = \frac{1}{\varepsilon(s,\boldsymbol{k}|\mu_\tau)}\,\frac{\hat{\boldsymbol{F}}(\boldsymbol{k})\,e^{i\boldsymbol{k}\cdot\boldsymbol{q}'}}{s+i\boldsymbol{k}\cdot\boldsymbol{p}-i\boldsymbol{k}\cdot\boldsymbol{p}'}\,. \tag{23}$$

4˙2. *Diffusion matrix.* – We use eq. (23) in eq. (16) and write

$$\begin{aligned}\mathbb{D}(\mu_\tau) = \Bigg[\Bigg[\int_{\mathbb{R}^3}\frac{\mathrm{d}\boldsymbol{k}_1}{8\pi^3}\,e^{-i\boldsymbol{k}_1\cdot\mathfrak{q}}\int_{\mathbb{R}^3}\frac{\mathrm{d}\boldsymbol{k}_2}{8\pi^3}\,e^{-i\boldsymbol{k}_2\cdot\mathfrak{q}}\int_{B_1}\frac{\mathrm{d}s_1}{2\pi i}\,e^{(s_1-i\boldsymbol{k}_1\cdot\mathfrak{p})t_1}\int_{B_2}\frac{\mathrm{d}s_2}{2\pi i}\,e^{(s_2-i\boldsymbol{k}_2\cdot\mathfrak{p})t_2}\\ \times\int_{\mathbb{R}^6}\mathrm{d}\boldsymbol{p}'\mathrm{d}\boldsymbol{q}'\,f_\tau(\boldsymbol{p}')\bigotimes_{j=1}^{2}\frac{1}{\varepsilon(s_j,\boldsymbol{k}_j|\mu_\tau)}\,\frac{\hat{\boldsymbol{F}}(\boldsymbol{k}_j)\,e^{i\boldsymbol{k}_j\cdot\boldsymbol{q}'}}{s_j-i\boldsymbol{k}_j\cdot\boldsymbol{p}'}\Bigg]_{t_1}\Bigg]_{t_2},\end{aligned} \tag{24}$$

where B_1 and B_2 are suitable integration contours to be specified shortly. The $\boldsymbol{q}'$-integration produces a factor $8\pi^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2)$, which in turn makes it possible to carry out another integration. A standard manipulation leads to

$$\begin{aligned}\mathbb{D}(\mu_\tau) = \Bigg[\Bigg[\int_{\mathbb{R}^3}\frac{\mathrm{d}\boldsymbol{k}}{8\pi^3}\,e^{-i\boldsymbol{k}.\mathfrak{p}(t_1-t_2)}\int_{B_1}\frac{\mathrm{d}s_1}{2\pi i}\,e^{s_1t_1}\int_{B_2}\frac{\mathrm{d}s_2}{2\pi i}\,e^{s_2t_2}\,\frac{\hat{\boldsymbol{F}}(\boldsymbol{k})\otimes\hat{\boldsymbol{F}}(-\boldsymbol{k})}{\varepsilon(s_1,\boldsymbol{k}\,|\,\mu_\tau)\varepsilon(s_2,-\boldsymbol{k}\,|\,\mu_\tau)}\\ \cdot\frac{1}{s_1+s_2}\int_{\mathbb{R}^3}\mathrm{d}\boldsymbol{p}\,f_\tau(\boldsymbol{p})\Bigg(\mathrm{PV}\,\frac{1}{s_1-i\boldsymbol{k}\cdot\boldsymbol{p}}+\pi i\delta(s_1-i\boldsymbol{k}\cdot\boldsymbol{p})\\ +\mathrm{PV}\,\frac{1}{s_2+i\boldsymbol{k}\cdot\boldsymbol{p}}+\pi i\delta(s_2+i\boldsymbol{k}\cdot\boldsymbol{p})\Bigg)\Bigg]_{t_1}\Bigg]_{t_2}.\end{aligned} \tag{25}$$

Here, we abused the notation by using $\varepsilon(s,\boldsymbol{k}\,|\,\mu_\tau)$ to indicate not the function in eq. (22) but rather its analytic continuation *à la* Landau. Under the traditional assumption [1-3] that $f_\tau(\boldsymbol{p})$ is "linearly stable", all zeroes of the Landau dielectric function $\varepsilon(s,\boldsymbol{k}\,|\,\mu_\tau)$ have negative real parts, and the integration contours B_1 and B_2 for the inverse Laplace transforms can be taken to coincide with the imaginary axis. Accordingly, the integrals in $\mathrm{d}\boldsymbol{p}$ also have been analytically continued by adding the necessary delta-functions and interpreting the singular integrals as principal values. In calculating the inverse Laplace transforms, all the contributions from the poles with negative real parts cancel under the action of the time averages. Hence, for the integral in $\mathrm{d}s_2$ one only need consider the imaginary pole $s_2=-s_1$. The residue calculation is trivial and leads to the cancellation of the principal values, whereas the delta terms add leaving

$$\begin{aligned}\mathbb{D}(\mu_\tau) = \Bigg[\Bigg[\int_{\mathbb{R}^3}\frac{\mathrm{d}\boldsymbol{k}}{8\pi^3}\,e^{-i\boldsymbol{k}\cdot\mathfrak{p}(t_1-t_2)}\int_{B_1}\mathrm{d}s_1\,e^{s_1(t_1-t_2)}\,\frac{\hat{\boldsymbol{F}}(\boldsymbol{k})\otimes\hat{\boldsymbol{F}}(-\boldsymbol{k})}{\varepsilon(s_1,\boldsymbol{k}\,|\,\mu_\tau)\,\varepsilon(-s_1,-\boldsymbol{k}\,|\,\mu_\tau)}\\ \int_{\mathbb{R}^3}\mathrm{d}\boldsymbol{p}\,f_\tau(\boldsymbol{p})\,\delta(s_1-i\boldsymbol{k}\cdot\boldsymbol{p})\Bigg]_{t_1}\Bigg]_{t_2}.\end{aligned} \tag{26}$$

Exchanging orders of integration, integrating in $\mathrm{d}s_1$ and using the symmetries of the functions $\boldsymbol{F}$ and ϵ yields

$$\mathbb{D}(\mu_\tau) = \Bigg[\Bigg[\int_{\mathbb{R}^3}\frac{\mathrm{d}\boldsymbol{k}}{8\pi^3}\int_{\mathbb{R}^3}\mathrm{d}\boldsymbol{p}\,f_\tau(\boldsymbol{p})\,e^{i\boldsymbol{k}\cdot(\boldsymbol{p}-\mathfrak{p})(t_1-t_2)}\,\frac{\hat{\boldsymbol{F}}(\boldsymbol{k})\otimes\hat{\boldsymbol{F}}(\boldsymbol{k})}{|\varepsilon(i\boldsymbol{k}\cdot\boldsymbol{p},\boldsymbol{k}\,|\,\mu_\tau)|^2}\Bigg]_{t_1}\Bigg]_{t_2}. \tag{27}$$

Finally, by calculating the time-averages we get the BGL expression

$$\mathbb{D}(\mu_\tau) = \int_{\mathbb{R}^3} \frac{\mathrm{d}\boldsymbol{k}}{8\pi^3} \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{p}\, f_\tau(\boldsymbol{p}) \frac{\hat{\boldsymbol{F}}(\boldsymbol{k}) \otimes \hat{\boldsymbol{F}}(\boldsymbol{k})}{|\varepsilon(i\boldsymbol{k}\cdot\mathfrak{p}, \boldsymbol{k} \,|\, \mu_\tau)|^2}\, \delta(i\boldsymbol{k}\cdot\boldsymbol{p} - i\boldsymbol{k}\cdot\mathfrak{p}). \tag{28}$$

4·3. *Drift vector*. – Following the exact same steps as in the calculation of the diffusion matrix leads from eq. (17) to

$$\boldsymbol{V}(\mu_\tau) = \left[\left[\int_{\mathbb{R}^3} \frac{\mathrm{d}\boldsymbol{k}}{8\pi^3} \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{p}\, f_\tau(\boldsymbol{p}) (-i\boldsymbol{k}t_1) e^{i\boldsymbol{k}\cdot(\boldsymbol{p}-\mathfrak{p})(t_1-t_2)} \cdot \frac{\hat{\boldsymbol{F}}(\boldsymbol{k}) \otimes \hat{\boldsymbol{F}}(\boldsymbol{k})}{|\varepsilon(i\boldsymbol{k}\cdot\boldsymbol{p}, \boldsymbol{k} \,|\, \mu_\tau)|^2}\right]_{t_1}\right]_{t_2}. \tag{29}$$

Carrying out the average in t_2 shows that also in this case for each $\boldsymbol{k}$ the d$\boldsymbol{p}$ integration must be restricted to the plane $\mathbb{L} = \{\boldsymbol{p} \in \mathbb{R}^3 : i\boldsymbol{k}\cdot\boldsymbol{p} - i\boldsymbol{k}\cdot\mathfrak{p} = 0\}$, leaving

$$\boldsymbol{V}(\mu_\tau) = \left[\int_{\mathbb{R}^3} \frac{\mathrm{d}\boldsymbol{k}}{8\pi^3} \int_{\mathbb{L}} \mathrm{d}\boldsymbol{p}\, f_\tau(\boldsymbol{p}) \frac{\mathrm{d}e^{i\boldsymbol{k}\cdot(\boldsymbol{p}-\mathfrak{p})t_1}}{\mathrm{d}\boldsymbol{p}} \cdot \frac{\hat{\boldsymbol{F}}(\boldsymbol{k}) \otimes \hat{\boldsymbol{F}}(\boldsymbol{k})}{|\varepsilon(i\boldsymbol{k}\cdot\mathfrak{p}, \boldsymbol{k} \,|\, \mu_\tau)|^2}\right]_{t_1}. \tag{30}$$

Then, an integration by parts (on $\mathbb{L}$) gives the BGL drift vector

$$\boldsymbol{V}(\mu_\tau) = -\int_{\mathbb{R}^3} \frac{\mathrm{d}\boldsymbol{k}}{8\pi^3} \int_{\mathbb{R}^3} \mathrm{d}\boldsymbol{p}\, \frac{\partial f_\tau}{\partial \boldsymbol{p}} \cdot \frac{\hat{\boldsymbol{F}}(\boldsymbol{k}) \otimes \hat{\boldsymbol{F}}(\boldsymbol{k})}{|\varepsilon(i\boldsymbol{k}\cdot\mathfrak{p}, \boldsymbol{k} \,|\, \mu_\tau)|^2}\, \delta(i\boldsymbol{k}\cdot\boldsymbol{p} - i\boldsymbol{k}\cdot\mathfrak{p}). \tag{31}$$

5. – Concluding remarks

Even though the BGL equation is commonly described [4] as a "collisional" model, the present discussion shows rather clearly that there is nothing "binary" about the BGL kernel, since it describes diffusion in phase space due to the long-term *collective* effect of fluctuations around the Vlasov limit. Only time will tell if it is feasible to turn this formal link into a rigorous justification of the BGL equation from N-particle dynamics. For the time being, it should be pointed out that eq. (9)—together with eqs. (16), (17) and (18)—poses an interesting mathematical problem in its own right. In a sense, this "generalized" model takes logical precedence on the BGL equation itself—since the latter has been shown to be a special case associated with highly regular (analytic) solutions. A detailed study of its properties will be a worthy project for the future.

REFERENCES

[1] Balescu R., *Phys. Fluids*, **3** (1960) 52.
[2] Guernsey R. L., *The Kinetic Theory of Fully Ionized Gases*. Off. Nav. Res. Contract No. 1224 (15), July 1960.
[3] Lenard A., *Ann. Phys.*, **3** (1960) 390.
[4] Balescu R., *Transport Processes in Plasmas*, Vol. **1** (North Holland, Amsterdam) 1988.
[5] Landau L. D., *Phys. Z. Sov. Union*, **10** (1936) 154.
[6] Strain R. M., *Commun. Partial Differ. Equ.*, **32** (2007) 1551.
[7] Oberman C. R. and Williams E. A., *Theory of Fluctuations in Plasmas* in *Handbook of Plasma Physics*, edited by Rosenbluth M. N. and Sagdeev R. Z. (Elsevier, New York) 1983, p. 279.
[8] Lancellotti C., *J. Stat. Phys.*, **136** (2009) 643.

[9] Piasecki J. and Szamel G., *Physica A*, **143** (1987) 114.
[10] Braun W. and Hepp K., *Commun. Math. Phys.*, **56** (1977) 101.
[11] Neunzert H., *Fluid Dyn. Trans.*, **9** (1978) 229.
[12] Dobrushin R. L., *Funct. Anal. Appl.*, **13** (115) 1979.
[13] Poupaud F. and Vasseur A., *J. Math. Pures Appl.*, **82** (2003) 711.
[14] Van Kampen N. G., *Phys. Rep.*, **24** (1976) 171.
[15] Kesten H. and Papanicolau G. C., *Commun. Math. Phys.*, **78** (1980) 19.