IL NUOVO CIMENTO Vol. 33 C, N. 1 Gennaio-Febbraio 2010
DOI 10.1393/ncc/i2010-10580-6

Colloquia: ICTT2009

An analytical solver for the multi-group two-dimensional neutron-diffusion equation by integral transform techniques

B. E. J. Bodmann, M. T. de Vilhena, L. S. Ferreira and J. B. Bardaji
Universidade Federal do Rio Grande do Sul - Porto Alegre, RS, Brazil

(ricevuto il 23 Ottobre 2009; approvato il 27 Gennaio 2010; pubblicato online il 16 Marzo 2010)

Summary. — In this work, we present an analytical solver for neutron diffusion in a rectangular two-dimensional geometry by a two-step integral transform procedure. To this end, we consider a regionwise homogeneous problem for two energy groups, *i.e.* fast and thermal neutrons, respectively. Each region has its specific physical properties, specified by cross-sections and diffusion constants. The problem is set up by two coupled bi-dimensional diffusion equations in agreement with general perturbation theory. These are solved by integral transforms Laplace transform and generalized integral transform technique yielding analytical expressions for the scalar neutron fluxes. The solutions for neutron fluxes are presented for fast and thermal neutrons in the four regions.

PACS 28.20.Gd – Neutron transport: diffusion and moderation.
PACS 82.56.Lz – Diffusion.

1. – Introduction

The renaissance of nuclear power brought up innovative reactor conceptions, which in its design state need to be tested by means of simulations. In the present work we focus on the specific task of neutron transport calculations. Although those shall start from a genuine transport equation, many approaches reduce the problem to diffusion equations, since the Boltzmann equation for neutron transport is still considered a challenge (see for instance [1, 2] and references therein). A detailed motivation for the use of the diffusion equation is given in ref. [2]. Our principal concern here is an effective analytical method for the neutron diffusion equation by an integral transform technique, that may be directly cast into a symbolic manipulation code in order to determine the scalar neutron fluxes for any number of energy groups and any rectangular segmentation of the volume into regions with different physical properties (cross-sections and diffusion coefficients). The project, that is behind the present discussion, has as one of its aims to prepare a symbolic solution library for coupled diffusion equations, which needs as input only the physical and geometrical parameters and returns an analytical expression as solution with negligible error. The strong convergence in the whole domain (space) is controlled and proven by the Cardinal Theorem of Interpolation Theory [3]. To this end we present a procedure that allows to construct an analytical solution of the multi-group neutron diffusion equation in segmented Cartesian geometry using an integral

transform procedure [4]. Once the general structure of the solution is determined one may calculate directly the scalar neutron flux as an analytical expression, which we apply to a simple example and show that the symbolic procedure works. Due to the fact that the geometrical extension of the reactor core is typically very much larger in one dimension if compared to the other two length scales we simplify the calculation and cast the diffusion problem into a two-dimensional description.

2. – The multi-group diffusion equation with constraints

The general multi-group diffusion problem in two dimensions is given by $\mathbb{L}\mathbf{\Phi} = \mathbf{S}$, where $\mathbb{L}$ represents the local non-homogeneous diffusion operator including particle multiplication from fission, $\mathbf{\Phi} = (\phi_1(x,y), \ldots, \phi_G(x,y))^T$ signifies the local multi-group neutron flux (in vector representation) and $\mathbf{S} = (S_1(x,y), \ldots, S_G(x,y))^T$ a local multi-group neutron source, for energy groups $g \in \{1, \ldots, G\}$. Note that the scalar flux signifies here the flux integrated over the third coordinate. This does not alter the original equation together with its boundary conditions, since they are scale invariant. The diffusion operator may be decomposed further into group-preserving and group-mixing terms $\mathbb{L} = \mathbb{L}_P + \mathbb{L}_M$, respectively. The diagonal elements contain a local diffusion operator, absorption and fission of the same energy group.

$$\mathbb{L}_P = \operatorname{diag}\left(\nabla\cdot(D_1\nabla) + \Sigma_{c1} - \frac{\chi_1}{k_{\text{eff}}}\nu\Sigma_{f1}, \ldots, \nabla\cdot(D_G\nabla) + \Sigma_{cG} - \frac{\chi_G}{k_{\text{eff}}}\nu\Sigma_{fG}\right). \tag{1}$$

The non-diagonal elements of $\mathbb{L}_M$ contain fission and scattering terms.

$$(\mathbb{L}_M)_{gg'} = -\frac{\chi_g}{k_{\text{eff}}}\nu\Sigma_{fg'} + \Sigma_{gg'}. \tag{2}$$

Here $D_g = D_g(x,y)$ represents the local diffusion coefficient for energy group g, $\Sigma_{cg}(x,y)$, $\Sigma_{fg'}(x,y)$ and $\Sigma_{gg'}(x,y)$ the macroscopic position-dependent capture, fission and scattering cross-section, respectively. The weight factor ν is due to neutron multiplication from the microscopic fission process, χ_g is the integrated neutron spectrum from fission of group g and k_{eff} is the effective multiplication factor, which for a stationary case shall be unity by definition.

The solutions shall obey the piecewise open surface boundary conditions defined by the neutron current density and scalar flux at the contours of the sheet. If Γ denotes the (2D) volume and $\partial\Gamma$ the boundary, then the conditions read with $\partial\Gamma_{xy}$ the boundary pieces according to fig. 1.

$$\left.\frac{\partial\phi_g}{\partial x}\right|_{\partial\Gamma_{0y}} = \left.\frac{\partial\phi_g}{\partial y}\right|_{\partial\Gamma_{x0}} = 0 \qquad \text{and} \qquad \phi_g|_{\partial\Gamma_{\bar{x}y}} = \phi_g|_{\partial\Gamma_{x\bar{y}}} = 0. \tag{3}$$

Since the problem has mirror symmetry with respect to the coordinate axes with either $x = 0$ or $y = 0$ it is sufficient to determine only the solution in the section with $x, y \geq 0$ and may be completed using the afore mentioned reflection operations. A further constraint breaks scale invariance of the non-homogeneous diffusion equation upon introduction of the energy release E_{Rg} per fission induced by group g of the sheet, which relates the power to the total neutron flux $P = \int_\Gamma \sum_g E_{Rg}\Sigma_{fg}\phi_g \, \mathrm{d}\Gamma$.

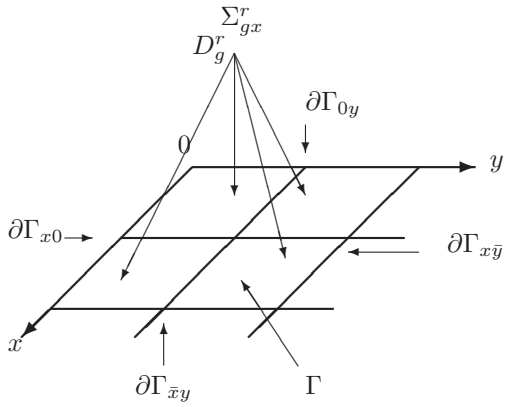

Fig. 1. – The two-dimensional sheet with boundaries, internal interfaces and locally homogeneous physical coefficients.

The diffusion equation system as it stands is unlikely to be solved exactly in closed and finite analytical form. In order to introduce a simplification which nevertheless permits to control convergence in a strict mathematical sense, one may make use of the physical resolution scale set by the inverse of the largest macroscopic cross-section value of the problem in consideration and segment the sheet into several regions $r \in \{1, \ldots, R\}$, with linear dimensions smaller than the mean free path.

We suppose that in a stationary regime the only neutron source is that from fission and consequently ignore the source term ($\mathbf{S}(x,y) \to \mathbf{0}$). Besides the dependence on the specific energy group, the physical coefficients are now "locally" homogeneous, *i.e.* constant in a specific region r ($D_g(x,y) \to D^r_g$,$\Sigma_{cg}(x,y) \to \Sigma^r_{cg}$, $\Sigma_{fg'}(x,y) \to \Sigma^r_{fg'}$ and $\Sigma_{gg'}(x,y) \to \Sigma^r_{gg'}$). The only quantity that preserves its original dependence on the coordinates is the scalar neutron flux, which is determined in its analytical form for each region ($\mathbf{\Phi}(x,y) \to \mathbf{\Phi}^r(x,y)$). Taking into account the modifications from above and upon multiplication of the operator $\mathbb{L}$ from the left by the matrix $(\mathbb{D}^r)^{-1}$ that contains the inverse multi-group diffusion constants of a respective region, allows to rewrite the equation system in a more compact form for each homogeneous region, *i.e.* a multi-component Poisson equation. Here $\mathbb{W}^r$ signifies the cross-section terms divided by the respective diffusion constant of region r.

$$(\mathbb{D}^r)^{-1}\left(\mathbb{L}\mathbf{\Phi}^r\right) = \left(\Delta - \mathbb{W}^r\right)\mathbf{\Phi}^r = \left(\Delta - \mathbb{W}^r_P + \mathbb{W}^r_M\right)\mathbf{\Phi}^r = 0. \tag{4}$$

In addition to the boundary conditions, there appear the piecewise open interface conditions, that combine the solutions of adjacent regions (r and r', respectively) to one unique solution of the whole problem.

$$\mathbb{D}^r \nabla \mathbf{\Phi}^r|_{\partial\Gamma_{xy}} = \mathbb{D}^{r'} \nabla \mathbf{\Phi}^{r'}\Big|_{\partial\Gamma_{xy}} \qquad \text{and} \qquad \mathbf{\Phi}^r|_{\partial\Gamma_{xy}} = \mathbf{\Phi}^{r'}\Big|_{\partial\Gamma_{xy}}. \tag{5}$$

Except for the interface conditions one may consider the total problem divided into smaller similar rescaled problems, each of them having the same solution structure but with different coefficients. Equation (4) together with the boundary and interface conditions (5) define the problem to be solved analytically.

3. – An analytical solution

The constant approximation for the physical parameters of each region together with a combination of a limited Laplace transform and a method denoted as Generalised Integral Transform Technique (GITT) [5, 6], which cast the differential operator into eigenvalues and polynomials, allow to apply standard methods of linear algebra and determine the analytical structure of the solution. For convenience we expand the scalar flux $\phi_g^r = \sum_{i=0}^{\infty} \xi_{gi}^r(x)\eta_{gi}^r(y)$ where without restrictions we apply the limited Laplace transform along the y-axis and use Sturm-Liouville theory for the x degree of freedom to decouple the diffusion equation into a simpler equation system and determine the ξ_{gi} and η_{gi}. Henceforth η_i denotes the vector containing the elements η_{gi}.

If one had only one energy group, and the problem was one dimensional, then eq. (4) assumes the form of a Sturm-Liouville problem. Hence one may think of the terms $\xi_{gi}^r(x)$ representing a linear independent functional base which because of similarity of the structure of the equations may be determined from the auxiliary problem, *i.e.* Sturm-Liouville. The principal idea of GITT is then to substitute differential operators by eigenvalues of that auxiliary problem with known analytical solutions. This auxiliary problem shall obey the same boundary conditions as the original problem in order to minimise the dimension (truncation) of the functional base (the eigenfunctions it supplies for each eigenvalue). More specifically the solutions of the Sturm-Liouville with non-zero eigenvalues $\lambda_i = \frac{(2i-1)\pi}{2(a_{r'}-a_r)} \neq 0$ obey the same boundary conditions as the total problem $(\partial_x)^2\xi_i^r + \lambda_i^2\xi_i^r = 0$ with $\partial_x\xi_i^r|_{a_r} = 0$ and $\xi_i^r|_{a_{r'}} = 0$. In order to adjust the solutions at the interface and take into account deviations of the interface conditions from the boundary conditions of the total problem, a non-orthogonal but linear independent solution of the Sturm-Liouville problem with zero eigenvalue ($\lambda_0 = 0$) is added ($\xi_0^{(r)} = \alpha_r(a_{r'} - x) + \beta_r$). Thus the structure of the solution is the same as the one for a totally homogenised problem except for an additional linear function with coefficients to be determined from the boundary or interface conditions. Note that by this procedure interfaces and boundaries are determined with the same procedure. The orthogonality property of the base of the subspace (with non-zero eigenvalue) opens a pathway to decouple the equation into a set of independent equations. Further, the orthogonal base is the same for all energy groups, so that the coefficients that differentiate the solutions for each energy group are absorbed in the η functions. In other words, the $\eta_i^{(r)}$ functions contain all the physics of the problem whereas $\xi_i^{(r)}$ contains geometrical information.

The differential operator with respect to y may be eliminated by the use of the limited Laplace transform $\mathcal{L}^r[\eta(y)] = \tilde{\eta}^r(s)$, defined within the limits of each region. The derivative term reads then $\mathcal{L}^r[(\partial_y)^2\eta(y)] = s^2\tilde{\eta}^r + \Upsilon^r$, where Υ^r are the boundary terms from partial integration and play an analogue role to the linear functions of the Sturm-Liouville problem, *i.e.* they take care of the matching of the solutions at the boundaries and interfaces. Upon insertion of the expansion and application of Laplace one gets the following equation:

$$\sum_{i=0}^{\infty} \left(\left(\left(s^2 - \left(\lambda_i^r\right)^2 \right) \mathbb{I} + \mathbb{W}^r \right) \tilde{\boldsymbol{\eta}}_i^r + \boldsymbol{\Upsilon}_i^r \right) \xi_i^r = 0. \tag{6}$$

One may use now the projection operator $\int_{a_r}^{a_{r'}} \mathrm{d}x[\xi_j^r]$ to decompose eq. (4) into a set of linear independent equations, which depend only on the y-dual variable s. For conve-

nience we introduce the following shorthand notation $\bar{\delta}_{ij} = (1 - \delta_{ij})$ and

$$\int_{a_r}^{a_{r'}} \xi_j^r \xi_i^r \mathrm{d}x = \bar{\delta}_{i0}\bar{\delta}_{j0}\delta_{ij}\mathcal{I}_\delta^{(r)} + \bar{\delta}_{j0}\delta_{i0}\mathcal{I}_{i0}^{(r)} + \delta_{i0}\delta_{j0}\mathcal{I}_{00}^{(r)}, \tag{7}$$

where all the integrals are known in symbolic form. Note that except for $\mathcal{I}_\delta = \frac{a_{r'}-a_r}{2} = \frac{\Delta a_r}{2}$ the remaining integrals depend on the unknown coefficients of the $\lambda_0 = 0$ terms $\mathcal{I}_{i0}^{(r)} = \frac{\alpha_r}{\lambda_i^{(r)2}} + (-1)^{i+1}\frac{\beta_r}{\lambda_i^{(r)}}$ and $\mathcal{I}_{00}^{(r)} = \frac{\alpha_r^2}{3}\Delta a_r^3 + \alpha_r\beta_r\Delta a_r^2 + \beta_r^2\Delta a_r$. The decoupled equation system reads then with $\mathbb{M}_i^{(r)} = (-\lambda_i^{(r)} + s^2)\mathbb{I} + \mathbb{W}^{(r)}$

$$\mathcal{I}_{i0}^{(r)}\left(\mathbb{M}_0^{(r)}\tilde{\boldsymbol{\eta}}_0^{(r)} + \boldsymbol{\Upsilon}_0^{(r)}\right) + \mathcal{I}_\delta^{(r)}\left(\mathbb{M}_i^{(r)}\tilde{\boldsymbol{\eta}}_i^{(r)} + \boldsymbol{\Upsilon}_i^{(r)}\right) = \mathbf{0}, \tag{8}$$

$$\mathcal{I}_{00}^{(r)}\left(\mathbb{M}_0^{(r)}\tilde{\boldsymbol{\eta}}_0^{(r)} + \boldsymbol{\Upsilon}_0^{(r)}\right) + \sum_{i=1}^{\infty}\mathcal{I}_{i0}^{(r)}\left(\mathbb{M}_i^{(r)}\tilde{\boldsymbol{\eta}}_i^{(r)} + \boldsymbol{\Upsilon}_i^{(r)}\right) = \mathbf{0}. \tag{9}$$

Upon reshuffling terms in such a way that expressions with $\mathbb{M}_0^{(r)}\tilde{\boldsymbol{\eta}}_0^{(r)}$, $\boldsymbol{\Upsilon}_0^{(r)}$ and $\boldsymbol{\Upsilon}_i^{(r)}$ constitute an inhomogeneity, eq. (8) may be solved by Cramer's rule which yields

$$\tilde{\boldsymbol{\eta}}_i^{(r)} = \left|\mathbb{M}_i^{(r)}\right|^{-1}\left(\mathbf{F}_i^{(r)}\tilde{\boldsymbol{\eta}}_0^{(r)} + \mathbf{P}_i^{(r)}\right), \tag{10}$$

where the matrix $\mathbf{F}_i^{(r)}$ contains the factors that go with $\tilde{\boldsymbol{\eta}}_0^{(r)}$ and $\mathbf{P}_i^{(r)}$ the respective expressions with $\Upsilon_0^{(r)}$. One may verify the following identities $\frac{\mathbb{M}_i^{(r)}}{|\mathbb{M}_i^{(r)}|}\mathbf{F}_i^{(r)} = -\frac{\mathcal{I}_{i0}^{(r)}}{\mathcal{I}_\delta^{(r)}}\mathbb{M}_0^{(r)}$ and $\frac{\mathbb{M}_i^{(r)}}{|\mathbb{M}_i^{(r)}|}\mathbf{P}_i^{(r)} = -\frac{\mathcal{I}_{i0}^{(r)}}{\mathcal{I}_\delta^{(r)}}\boldsymbol{\Upsilon}_0^{(r)} - \boldsymbol{\Upsilon}_i^{(r)}$, so that eq. (9) reduces to $\mathbb{M}_0^{(r)}\tilde{\boldsymbol{\eta}}_0^{(r)} + \boldsymbol{\Upsilon}_0^{(r)} = \mathbf{0}$, which can be solved symbolically. In order to solve the equations for all $\eta_{gi}^{(r)}$, one may make use of the inversion procedure by Heaviside, where one needs the zeroes of the respective determinants in the denominator. It is noteworthy that the eigenvalue of the Sturm-Liouville problem shifts the pole of the Laplace transformed solution that determines the solution of the second dimension $\eta_{gi}^{(r)}$ for $i \in \{0, 1, \ldots, n\}$. This is probably a manifestation of linking the geometric information of the problem taken care of by $\xi_i^{(r)}$ to the physics which was contemplated in $\eta_{gi}^{(r)}$.

The explicit solution for an example considering two energy groups is

$$\eta_{10}^{(r)} = \sum_{j=1}^{4} \left. \frac{\Upsilon_{10}^{(r)}\left(s^2+\mathbb{W}_{22}^{(r)}\right)-\mathbb{W}_{12}^{(r)}\Upsilon_{20}^{(r)}}{4s^3+2\left(\mathcal{T}r[\mathbb{W}^{(r)}]\right)s} e^{sy} \right|_{s=s_j^r}, \tag{11}$$

$$\eta_{20}^{(r)} = \sum_{j=1}^{4} \left. \frac{\left(s^2+\mathbb{W}_{11}^{(r)}\right)\Upsilon_{20}^{(r)}-\mathbb{W}_{21}^{(r)}\Upsilon_{10}^{(r)}}{4s^3+2\left(\mathcal{T}r[\mathbb{W}^{(r)}]\right)s} e^{sy} \right|_{s=s_j^r},$$

$$\eta_{1i}^{(r)} = \sum_{j=1}^{4} \left. \frac{\left(-s^2+{\lambda_i^{(r)}}^2-\mathbb{W}_{22}^{(r)}\right)\left(2\frac{\mathcal{I}_{i0}^{(r)}}{\mathcal{I}_{\delta}^{(r)}}\Upsilon_{10}^{(r)}+\Upsilon_{1i}^{(r)}\right)+\mathbb{W}_{12}^{(r)}\left(2\frac{\mathcal{I}_{i0}^{(r)}}{\mathcal{I}_{\delta}^{(r)}}\Upsilon_{20}^{(r)}+\Upsilon_{2i}^{(r)}\right)}{4s^3+2\left(-2{\lambda_i^{(r)}}^2+\mathcal{T}r[\mathbb{W}^{(r)}]\right)s} e^{sy} \right|_{s=s_j^r},$$

$$\eta_{2i}^{(r)} = \sum_{j=1}^{4} \left. \frac{\mathbb{W}_{21}^{(r)}\left(2\frac{\mathcal{I}_{i0}^{(r)}}{\mathcal{I}_{\delta}^{(r)}}\Upsilon_{10}^{(r)}+\Upsilon_{1i}^{(r)}\right)+\left(-s^2+{\lambda_i^{(r)}}^2-\mathbb{W}_{11}^{(r)}\right)\left(2\frac{\mathcal{I}_{i0}^{(r)}}{\mathcal{I}_{\delta}^{(r)}}\Upsilon_{20}^{(r)}+\Upsilon_{2i}^{(r)}\right)}{4s^3+2\left(-2{\lambda_i^{(r)}}^2+\mathcal{T}r[\mathbb{W}^{(r)}]\right)s} e^{sy} \right|_{s=s_j^r},$$

where s_j^r are the zeroes of the determinant $|M_i^{(r)}|$ and

$$s_j^r = \pm\sqrt{{\lambda_i^{(r)}}^2 - \frac{1}{2}\mathcal{T}r[\mathbb{W}^{(r)}] \pm \sqrt{\frac{1}{4}(\mathcal{T}r[\mathbb{W}^{(r)}])^2 - |\mathbb{W}^{(r)}|}}. \tag{12}$$

Recalling that $\boldsymbol{\Upsilon}_i^{(r)} = \left(s\boldsymbol{\eta}_i^{(r)}(b_{r'}) + \partial_y\boldsymbol{\eta}_i^{(r)}(b_{r'})\right)e^{-sb_{r'}} - \left(s\boldsymbol{\eta}_i^{(r)}(b_r) + \partial_y\boldsymbol{\eta}_i^{(r)}(b_r)\right)e^{-sb_r}$ contain the solution $\boldsymbol{\eta}_i^{(r)}$ and derivative at the boundaries, respectively, the previously determined solutions may be used to set up an equation system to determine these coefficients from eqs. (5). By virtue of eq. (11) the analytical expressions for $\eta_i^{(r)}$ contain the still unknowns α_r and β_r in a linear fashion, so that the following equation system combining all boundary matches suffices to determine these coefficients:

$$\beta_r\eta_{g0}^{(r)} = \left(\alpha_{r'}\Delta a_{r'} + \beta_{r'}\right)\eta_{g0}^{(r')} + \sum_{j=1}^{n}\eta_{gj}^{(r')} \qquad \alpha_r\eta_{g0}^{(r)} = \alpha_{r'}\eta_{g0}^{(r')} + \sum_{j=1}^{n}\lambda_i^{(r')}\eta_{gi}^{(r')}. \tag{13}$$

Note that the boundary with the same index as the region refers to the lower boundary. For instance in region r the lower boundary along the x-axis is a_r, the higher boundary is $a_{r'}$.

In fig. 2 we show the numerical solution of the scalar neutron flux for the parameters shown in table I. One clearly sees the expected maximum for thermal neutrons in the transition region between the nuclear fuel-loaded segment and the outer one without fuel. The fast neutron scalar flux has its maximum in the region where fission occurs.

4. – Conclusion

This paper presented a new method, which generates analytical solutions for the globally heterogeneous problem of neutron diffusion in two dimensions. These solutions are in future incorporated into a program library and serve as generic solutions for multi-group neutron flux in a multi-region geometry. The principal steps employed are Laplace transform and the generalised integral transform technique. Motivated by recent

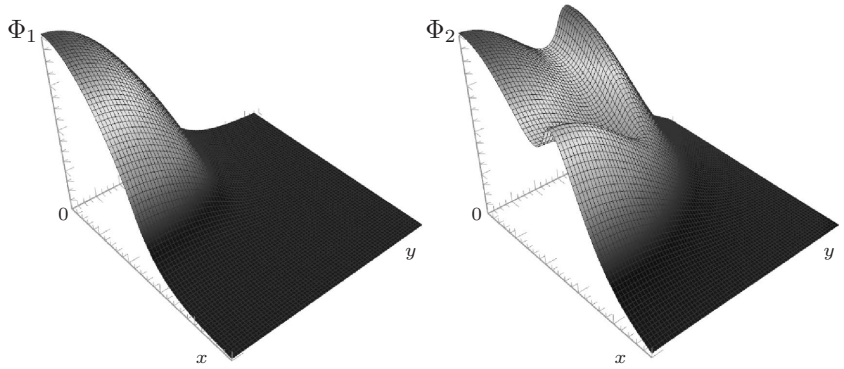

Fig. 2. – Fast (left) and thermal (right) scalar neutron flux for a reactor core sheet 17.5 cm × 14 cm and a reflector with outer border 37.5 cm × 34 cm.

developments in reactor concepts we developed an effective procedure which permits to analyse in an analytical way the neutron flux behaviour for a variety of core geometries and fuel compositions in order to optimise the set-up.

The quality of the solution is controlled by a genuine mathematical convergence criterion. Note that for the y coordinate the Laplace inversion considers only differentiable functions in y-space as well as in the dual (s-space), which defines then a unique relation between the original function and its Laplace transform. This makes the transform procedure manifest exact and the only numerical error comes from truncation, which is determined from the Sturm-Liouville problem. Recalling that the structure of the scalar flux is essentially determined by neutron interaction, present in form of cross-sections, means that between two successive neutron interactions neutrons behave in the average like free particles compatible with a homogeneous flux. Thus one may conclude that with decreasing length ($(m\Sigma_t)^{-1}$ and m increasing) variations in the solution become spurious. Here Σ_t is the total cross-section and m an integer number. One may now employ

TABLE I. – *Physical parameters used to calculate the neutron fluxes, where D_g are the diffusion constants for the two energy groups ($g = 1$: fast, $g = 2$: thermal), Σ_{cg} the respective capture cross-sections, $\nu\Sigma_{fg}$ the microscopic reproduction factor weighted fission cross-sections and $\Sigma_{sgg'}$ the scattering cross-sections.*

Symbol	Units	Core	Reflector
D_1	cm	1.614	1.522
D_2	cm	0.282	0.210
Σ_{c1}	cm^{-1}	0.002	0.001
Σ_{c2}	cm^{-1}	0.050	0.017
$\nu\Sigma_{f1}$	cm^{-1}	0.002	0.000
$\nu\Sigma_{f2}$	cm^{-1}	0.074	0.000
Σ_{s12}	cm^{-1}	0.035	0.056
Σ_{s21}	cm^{-1}	0.000	0.000

the Cardinal Theorem of Interpolation Theory [3] in order to find the truncation n in the series $\mathbf{\Phi} = \sum_{i=0}^{n} \eta_i \xi_i$ that leaves the analytical solution almost exact, *i.e.* introduces only functions that vary significantly in length scales beyond the mentioned limit.

The square integrable function $\chi = \int_r \Phi \, \mathrm{d}y \in \boldsymbol{L}^2$ with spectrum $\{\lambda_i\}$ which is bounded by $m\Sigma_t$ has an exact solution for a finite expansion. This statement expresses the *Cardinal Theorem of Interpolation Theory* for our problem. Since the cut-off $m\Sigma_t$ defines some sort of sampling density, its introduction is an approximation and is related to convergence of the approach and Parseval's theorem may be used to estimate the error. For example, if $m = 10$, then the Scattering-Matrix Formalism [7] tells us that the error in the spectral integral is less than $10^{-2}\%$. In order to keep the solution error within the same order of magnitude, the expansion in the region of interest has to contain $n + 1$ terms, with $n = \mathrm{int}\{\frac{m\Sigma_t \Delta_r}{2\pi} + \frac{1}{2}\}$. For the bounded spectrum and according to the theorem the solution is then exact. In our approximation, if m is properly chosen such that the cut-off part of the spectrum is negligible, then the found solution is almost exact.

Although algebraic manipulations are typically slower in execution than numerical procedures, the symbolic computation has to be executed only once. Moreover, in the present approach the fact that the homogenised global problem has the same solution as the rescaled smaller problem, restricted to a specific region except for the differences imposed by the interface conditions, which are taken care of by a linear correction of the solution. The structure of the solution is the same for all regions and can also be applied to the outer regions which are limited partially by the outer boundary $\partial\Gamma$. Once the number of energy groups and regions are defined, further the truncation of the expansion is determined, then one may prepare a library of solutions using the proposed method. The only task to be executed for applications is to determine numerically the GITT eigenvalues and substitute the physical parameters into the stored solutions which may be calculated directly. In order to get a comparable precision with numerical or stochastic procedures, they will be more time consuming, because they have to execute a numerical algorithm in comparison to function calls for each energy group and region, respectively. This is a clear advantage, if modifications in geometry and material composition are to be examined.

REFERENCES

[1] Leppänen J., *A new assembly-level Monte Carlo neutron transport code for reactor physics calculations*, in *Mathematics and Computation, Supercomputing, Reactor Physics and Nuclear and Biological Applications, Palais des Papes, Avignon, France, September 12-15, 2005*, on CD-ROM, American Nuclear Society, LaGrange Park, IL (2005).

[2] Sekimoto H., *Nuclear Reactor Theory*, COE-INES Tokyo Institute of Technology, ISBN978-4-903054-11-7 C3058 Part II (2007).

[3] Torres R. H., *Stud. Math.*, **100** (1991) 51.

[4] de Vilhena M. T., Heinen I. R. and Bodmann B. E. J., *Ann. Nucl. Energy*, **35** (2008) 2410.

[5] Cotta R. M., *Integral Transforms in Computational Heat and Flow* (CRC Press, McGraw-Hill, Boca Raton, FL, USA) 1993.

[6] Cotta R. M. and Mikhailov M. D., *Heat Conduction—Lumped Analysis, Integral Transforms, Symbolic Computation* (McGraw-Hill, Chichester) 1997.

[7] Barut A. O., *The Theory of the Scattering Matrix for the Interactions of Fundamental Particles* (Macmillan, New York, USA) 1967.