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Chapman-Enskog asymptotic procedure in structured population dynamics

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Summary. — Complexity of many biological models often makes impossible their robust theoretical and numerical analysis and thus requires a systematical method of reducing the number of variables in the system in such a way that the dynamics of the simplified model approximates the original way in a reasonable way. Such an aggregation of variables is often done by *ad hoc* methods. In turns out that in many biological systems the well-known Chapman-Enskog asymptotic procedure is well suited for aggregation which then can be viewed as passing from a microscopic (kinetic) to a macroscopic (hydrodynamic) description of the system. We demonstrate this approach by applying it to an age- and space-structured population model.

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1. – Introduction

Given a complex model in micro-variables it is of interest to have a systematic way of building a simpler aggregated macro-model satisfying the conditions: i) *the set of macro-variables provides a reasonable approximation of the dynamics of the micro-model*, ii) *there is an explicit relation between the macro-variables and the original micro-variables.*

One of the most natural ways of aggregation is based on the existence of different time scales in the system. For instance, in population models individual processes happen faster than the demographic ones and one can reduce the dimension of the system so that the aggregated evolution happens on the slow time scale with the micro-variables appearing in the macro scale only as "averaged" parameters.

Different time scales in the model are accounted for by placing a large parameter in front of the "fast" part of the model, usually making the system singularly perturbed. On the other hand, the ansatz that there is a well-defined "slow dynamics" in the system is tantamount to the existence of a hydrodynamic space (that is, collision invariants) in kinetic models. Thus, equations of structured population dynamics are similar to equations of kinetic theory and the methods of aggregation often are analogous to the methods of asymptotic analysis and, in particular, to the Chapman-Enskog method,

which have been used since the beginning of the XX century to find hydrodynamic approximations to kinetic equations when the mean free time between interactions is small. In fact, the whole theory of asymptotic analysis of kinetic equations is about the "aggregation" of micro variables, such as the individual particles' velocity and position, into macro variables, which are directly measurable, such as density, temperature, etc. of the whole ensemble.

In this paper we provide the Chapman-Enskog analysis of a structured population model introduced in detail in the next section. We construct a formal asymptotic expansion up to the first level and provide numerical evidence that it converges to the solution of the macroscopic equation at the rate predicted by the theory. The rigorous proof of convergence is quite technical and lengthy and is referred to another paper [1].

2. – A structured population dynamics model

Consider a population subject to usual demographic processes such as births, aging and deaths, subdivided in addition into discrete classes according to the geographical location the individuals in a particular class occupy, see, *e.g.* [2]

$$\begin{aligned} \mathbf{n}_t(t,a) &= -\mathbf{n}_a(t,a) - \mathcal{M}(a)\mathbf{n}(t,a) + \frac{1}{\epsilon}\mathcal{C}(a)\mathbf{n}(t,a), \\ \mathbf{n}(t,0) &= \int_0^\infty B(a)\mathbf{n}(t,a)\mathrm{d}a, \qquad \mathbf{n}(0,a) = \mathring{\mathbf{n}}(a), \end{aligned} \tag{1}$$

where $\mathbf{n}(t,a) = (n_1(t,a),\dots,n_N(t,a))$ and $n_i(t,a)$ is the density of the population in patch i at age a, $\mathcal{M}(a) = \mathrm{diag}\{\mu_1(a),\dots,\mu_N(a)\}$ describes the age- and patch-specific mortality rate, $\mathcal{C}(a) = \{c_{ij}(a)\}_{1\le i,j\le N}$ is the matrix of migration rates between patches, and $B(a) = \mathrm{diag}\{\beta_1(a),\dots,\beta_N(a)\}$ describes the fertility rates. The small parameter ϵ in front of the transition matrix accounts for that fact that the migratory processes occur on a much faster time scale that the demographical ones. System (1) is considered as a dynamical system in the space $\mathbf{X} = L_1(\mathbb{R}_+,\mathbb{R}^N)$. The main assumption is that the migration process is conservative with respect to the life dynamics; that is, $c_{ii}(a) = -\sum_{j=1,j\neq i}^N c_{ji}(a)$. Further, the matrix $\mathcal{C}$ is differentiable and irreducible for $a > 0$; as a consequence $\lambda = 0$ is a simple dominant eigenvalue having $\mathbf{1} := (1,\dots,1)$ as its left eigenvector while the positive right eigenvector $\mathbf{k}(a) := (k_1(a),\dots,k_N(a))$ is uniquely chosen subject to the normalizing condition $\mathbf{1}\cdot\mathbf{k}(a) = 1$ for any a. This properties yield the decomposition of $\mathbb{R}^N$ as

$$\mathbb{R}^N = Span\{\mathbf{k}(a)\} \oplus W, \tag{2}$$

where $Span\{\mathbf{k}(a)\}$ is the "hydrodynamic" subspace (manifold) and $W = \{\mathbf{w}\in\mathbb{R}^N; \mathbf{1}\cdot\mathbf{w} = 0\}$, independent of a, is the "kinetic" subspace.

3. – Abstract asymptotic analysis

Assumptions on $\mathcal{C}$ allow to consider (1) as a singularly perturbed abstract kinetic problem in an appropriate Banach space X

$$\begin{cases} \partial_t f_\epsilon = -\mathcal{M}f_\epsilon + \mathcal{S}f_\epsilon + \dfrac{1}{\epsilon}\mathcal{C}f_\epsilon, \\ \mathcal{B}f_\epsilon = f_b, \qquad f_\epsilon(0) = \mathring{f}, \end{cases} \tag{3}$$

where $\mathcal{B}$ is a boundary operator, f is the one-particle density function, the transition operator $\mathcal{C}$ plays the role of the collision operator, and the aging operator $\mathcal{S}$ and the mortality operator $-\mathcal{M}$ correspond, respectively, to the transport part and attenuation terms in the classical linear transport equation.

Singularly perturbed equations of kinetic type (which our population model (1) belongs to) are characterized by the existence of nontrivial solutions to the equation $\mathcal{C}f_0(t) = 0$, which form the so-called "hydrodynamic space" where the evolution of the macroscopic quantities should take place. The Chapman-Enskog procedure splits the solution into the hydrodynamic and kinetic parts and performs a proper asymptotic expansion only on the kinetic part leaving the hydrodynamic part unexpanded. In this way it takes advantage of the structure of the problem to deliver the asymptotic solution in the most accurate way since the hydrodynamic, that is, the macroscopic, part of the solution remains unexpanded and the error comes only from approximating the kinetic part.

To phrase this idea in a mathematical language, we follow [3]. For the hydrodynamic space V to exist, $\lambda = 0$ must be an eigenvalue of the operator $\mathcal{C}$. In general, we assume that $\sigma(\mathcal{C}) = \{0\} \cup \{\lambda \in \sigma(\mathcal{C}); \Re\lambda \leq \lambda_0 < 0\}$. Let $\mathcal{P}$ be the spectral projection onto $V = \mathcal{P}X$. Since typically $\mathcal{C}$ acts only on one group of variables, often the kinetic component of f_0 is of the form $\phi\mathbf{k}$, where $\mathbf{k}$ is the eigenvector of $\mathcal{C}$ with respect to one group of variables while ϕ is a scalar function of the other group, which is to be determined. The operator $\mathcal{Q} = I - \mathcal{P}$ is the projection onto the kinetic space $W = \mathcal{Q}X$, such that $X = V \oplus W$ (compare (2)). Accordingly, by $\mathcal{P}f = v$ and $\mathcal{Q}f = w$ we denote, respectively, the hydrodynamic and the kinetic part of f. Applying these projections to (3) we get

$$\begin{aligned} \partial_t v_\epsilon &= -\mathcal{P}\mathcal{M}\mathcal{P}v_\epsilon + \mathcal{P}\mathcal{S}\mathcal{P}v_\epsilon - \mathcal{P}\mathcal{M}\mathcal{Q}w_\epsilon + \mathcal{P}\mathcal{S}\mathcal{Q}w_\epsilon \\ \partial_t w_\epsilon &= -\mathcal{Q}\mathcal{M}\mathcal{Q}w_\epsilon + \mathcal{Q}\mathcal{S}\mathcal{Q}w_\epsilon - \frac{1}{\epsilon}\mathcal{Q}\mathcal{C}\mathcal{Q}w_\epsilon - \mathcal{Q}\mathcal{M}\mathcal{P}v_\epsilon + \mathcal{Q}\mathcal{S}\mathcal{P}v_\epsilon, \end{aligned} \tag{4}$$

with analogous projections of the boundary and initial conditions. We seek an approximate solution to (4) by expanding w_ϵ into a power series in ϵ and leaving $v_\epsilon = \phi\mathbf{k}$ unexpanded. This yields, at the 0th level, the macroscopic equation for ϕ

$$\partial_t \phi = -\mathcal{P}\mathcal{M}\mathcal{P}\phi + \mathcal{P}\mathcal{S}\mathcal{P}\phi, \tag{5}$$

which, subject to appropriate boundary and initial conditions and possibly complemented with corrections w_{corr} coming from the expansion of w_ϵ, should provide a reasonable approximation to v_ϵ. If we define the error of the approximation by

$$E_\epsilon(t) = f_\epsilon(t) - f_0(t) = (v_\epsilon, w_\epsilon) - (\phi\mathbf{k}, w_{\text{corr}}), \tag{6}$$

then it is easy to see that it satisfies the original, but nonhomogeneous, equation

$$\partial_t E_\epsilon = -\mathcal{M}E_\epsilon + \mathcal{S}E_\epsilon + \epsilon^{-1}\mathcal{C}E_\epsilon + F_\epsilon(t), \tag{7}$$

supplemented by the side conditions $\mathcal{B}E = f_b - \mathcal{B}f_0$ and $E_\epsilon(0) = \overset{\circ}{f} - f_0(0)$. If $F_\epsilon(t)$, $f_b - \mathcal{B}f_0$ and $\overset{\circ}{f} - f_0(0)$ are $O(\epsilon)$, then standard semigroup estimates yield $E_\epsilon = O(\epsilon)$ and the procedure can be stopped. Typically, however, the limit equation (5) involves less independent variables than (3) and the solution of the limit equation cannot satisfy all

the original side conditions. If the error E_ϵ is not small close to $t = 0$, then it is necessary to find an *initial layer* correction by repeating the above procedure with a rescaled time to improve the estimates for small t. The original approximation f_0, which typically is valid only away from $t = 0$, is then referred to as the *bulk approximation*. If introduction of the initial layer does not solve the problem and the approximation is poor close to the spatial boundary and the corners where the spatial and temporal boundaries meet, we iteratively introduce further corrections such as *boundary* and *corner layers* by rescaling appropriate variables till the inhomogeneities in the error equation are of required order.

4. – Chapman-Enskog asymptotic expansion for (1)

Performing the procedure outlined above on (1) we find that the spectral projection $\mathcal{P}\mathbf{n} = (\mathbf{1}\cdot\mathbf{n})\mathbf{k}(a)$, where $\mathbf{1}\cdot\mathbf{n} = n_1 + \ldots + n_N$ is the total population, and the asymptotic expansion at the zeroth level is given as the sum of the bulk part $\bar{\mathbf{n}}$ and initial $\tilde{\mathbf{n}}$, boundary $\hat{\mathbf{n}}$ and corner $\breve{\mathbf{n}}$, layers defined as follows:

1) $\bar{\mathbf{n}}(t,a) = (\bar{n}(t,a), \epsilon\overline{\mathbf{w}}_1(t,a))$, where eq. (5) takes the form

$$\overline{n}_t = -\overline{n}_a - \mu^*\overline{n}, \qquad \overline{n}(t,0) = \int_0^\infty \beta^*(a)\overline{n}(t,a)\mathrm{d}a, \qquad \overline{n}(0,a) = \mathring{n}, \tag{8}$$

and $w_{\mathrm{corr}} = \epsilon\overline{\mathbf{w}}_1 = \epsilon\mathcal{C}|_W^{-1}[\mathbf{k}' + \mathcal{M}\mathbf{k} - \mu^*\mathbf{k}]\bar{n}$. Here, for any matrix $\mathcal{X}$ we denote $x^* := \mathbf{1}\cdot\mathcal{X}\mathbf{k}$.

2) $\tilde{\mathbf{n}}(t/\epsilon, a) = (0, \widetilde{\mathbf{w}}_0(\tau, a))$ where $\tau = t/\epsilon$ and

$$\partial_\tau\widetilde{\mathbf{w}}_0(\tau,a) = \mathcal{C}|_W(a)\widetilde{\mathbf{w}}_0(\tau,a), \qquad \widetilde{\mathbf{w}}_0(0,a) = \mathring{\mathbf{w}}(a). \tag{9}$$

3) $\hat{\mathbf{n}}(t, a/\epsilon) = (0, \widehat{\mathbf{w}}_0(t,\alpha))$, where $\alpha = a/\epsilon$ and

$$\partial_\alpha\widehat{\mathbf{w}}_0(t,\alpha) = \mathcal{C}_W(0)\widehat{\mathbf{w}}_0(t,\alpha), \qquad \widehat{\mathbf{w}}_0(t,0) = \int_0^\infty B(a)\bar{n}(a,t)\mathbf{k}(a)\mathrm{d}a - [\bar{n}\mathbf{k}](t,0). \tag{10}$$

4) $\breve{\mathbf{n}}(\tau,\alpha) = (\breve{n}_0(\tau,\alpha)\mathbf{k}(\alpha), \breve{\mathbf{w}}_0(\tau,\alpha))$, where $\partial_\tau\breve{\mathbf{n}} = -\partial_\alpha\breve{\mathbf{n}} + \mathcal{C}(0)\breve{\mathbf{n}}$, subject to

$$\begin{aligned}
\breve{n}_0(\tau,0) &= \int_0^\infty \mathbf{1}\cdot B(a)\widetilde{\mathbf{w}}_0(\tau,a)\mathrm{d}a,\\
\breve{\mathbf{w}}_0(\tau,0) &= \int_0^\infty B(a)\widetilde{\mathbf{w}}_0(\tau,a)\mathrm{d}a - \widetilde{\mathbf{w}}_0(\tau,0) - [\breve{n}_0\mathbf{k}](\tau,0).
\end{aligned}$$

We note that (8) is the so-called aggregated, macroscopic, equation which can be derived from first principles if one considers the population as a whole, [2] and thus (8) provides the link between the microscopic and macroscopic description of the population.

The proof that $\bar{\mathbf{n}} + \tilde{\mathbf{n}} + \hat{\mathbf{n}} + \breve{\mathbf{n}}$ provides an $O(\epsilon)$ approximation to $\mathbf{n}$ is quite lengthy and involved, requiring recasting (1) as an integral system due to weak regularity of solutions, see [1]. However, the proof shows that $\hat{\mathbf{n}} + \breve{\mathbf{n}}$ as well as the term $\epsilon\bar{w}_1$, while useful in the estimates, are $O(\epsilon)$ themselves and thus the following result is true.

Theorem 1. *Let $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{M}$ be as in sect.* **2** *and $\mathbf{n}_\epsilon(t,a) = n_\epsilon(t,a)\mathbf{k}(a) + \mathbf{w}_\epsilon(t,a)$ be a solution to* (1)*. Then, for each $0 < T < \infty$ there exists a constant C, depending on T,*

$\mathcal{M}$, B and $\mathcal{C}$, such that for any $\overset{\circ}{\mathbf{n}} \in W_1^1(\mathbb{R}_+, \mathbb{R}^N)$ and uniformly on $[0,T]$

$$\|n(t,\cdot) - \bar{n}(t,\cdot)\|_{L_1(\mathbb{R}_+)} \leq \epsilon C \big\|\overset{\circ}{\mathbf{n}}\big\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^N)},$$
$$\Big\|\mathbf{w}(t,\cdot) - e^{\mathcal{C}(\cdot)\frac{t}{\epsilon}}\overset{\circ}{\mathbf{w}}(\cdot)\Big\|_{L_1(\mathbb{R}_+,\mathbb{R}^N)} \leq \epsilon C \big\|\overset{\circ}{\mathbf{n}}\big\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^N)}.$$

Here W_1^1 denotes the standard Sobolev space. Thus, the solution to the aggregated problem (8) provides an $O(\epsilon)$ approximation to the true total population n away from $t = 0$, while to get a uniform estimate of the same order on $[0,T]$ only the initial layer corrector is needed. This is an improvement upon results of [2, 4] where, in the former, the authors required the solution of the whole system in the kinetic space to achieve the uniform convergence and provided only formal expansion in the latter. We also note that applying the Banach-Steinhaus theorem one can prove that the convergence can be extended to all initial values since, however, this approach requires an abstract density argument, in general it is impossible to control the order of convergence.

5. – Numerical results

In this section we illustrate the theory developed above. First, we discuss a numerical approximation of the complete singularly perturbed model (1). Second, we explain how to improve computational performance by using the asymptotic expansion of sect. **4**. Finally, we illustrate the results of sects. **4** and **5** using simple equation of the form (1).

5˙1. *Numerical approximation of* (1). – Let $X_{a_0}(a)$ be a fundamental solution to $X' = -\mathcal{M}(a)X + \dfrac{1}{\epsilon}\mathcal{C}(a)X$, $X_{a_0}(a_0) = I$, then the boundary condition $\mathbf{n}(t,0)$ satisfies

$$\mathbf{n}(t,0) = \int_0^t B(a)X_0(a)\mathbf{n}(t-a,0)\mathrm{d}a + \int_0^\infty B(a+t)X_a(a+t)\overset{\circ}{\mathbf{n}}(a)\mathrm{d}a, \tag{11a}$$

and the solution to (1) is given explicitly by

$$\mathbf{n}(t,a) = \begin{cases} X_0(a)\mathbf{n}(t-a,0), & a < t; \\ X_{a-t}(a)\overset{\circ}{\mathbf{n}}(a-t), & a \geq t. \end{cases} \tag{11b}$$

The numerical approximation to (1) is obtained in two steps: first, we solve the Volterra integral equation (11a) for $\mathbf{n}(t,0)$, second, we recover $\mathbf{n}(t,a)$ by integrating the resulting linear ODEs along the characteristic lines using (11b).

To solve (11a) in $[0,T]$ we set $F(t) = \int_0^\infty B(a+t)X_a(a+t)\overset{\circ}{\mathbf{n}}(a)\mathrm{d}a$, introduce a computational grid $\{t_k\}_{k=1}^M$ and apply $A(\alpha)$-stable, 4-step, order-4 backward differential formula (BDF) to

$$\mathbf{u}(t,s) = F(t) + \int_0^s B(t-a)X_0(t-a)\mathbf{n}(a,0)\mathrm{d}a, \quad \mathbf{n}(t,0) = \mathbf{u}(t,t).$$

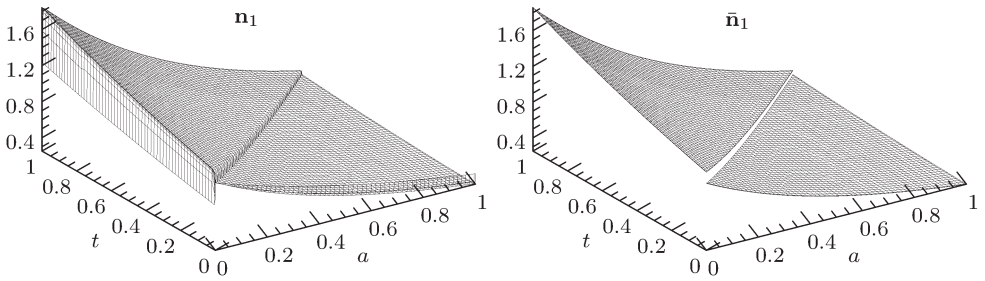

Fig. 1. – Numerical approximation of (1) and the bulk approximation, $\epsilon = 10^{-3}$.

This yields the algorithm

$$\text{(12a)}\quad \sum_{j=0}^{4} a_j \mathbf{u}(\cdot, t_{k-j}) = \tau_{k-1} B(\cdot - t_{k-j}) \mathbf{v}(\cdot - t_{k-j}, t_{k-j}), \quad \mathbf{n}(t_k, 0) = \mathbf{u}(t_k, t_k),$$

$$\text{(12b)}\quad \mathbf{u}(\cdot, t_k) = F(\cdot) + \tau_0 \sum_{j=-4}^{-1} w_{0,j}(t) B(\cdot - t_j) \mathbf{v}(\cdot, t_j), \quad -4 \le k \le 0,$$

$$\text{(12c)}\quad \mathbf{v}_s(s, t_k) = -\mathcal{M}(a)\mathbf{v}(s, t_k) + \frac{1}{\epsilon}\mathcal{C}(a)\mathbf{v}(s, t_k), \quad \mathbf{v}(0, t_k) = \mathbf{n}(t_k, 0),$$

where $\tau_k = t_k - t_{k-1}$. Note that (12) require one evaluation of $F(t)$ per integration step. Here this is done by the three-points, composite Gauss-Lobatto quadrature rule.

It can be shown that under the assumptions on $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{M}$ of sect. **2** the method (12) converges with order four to $\mathbf{n}(t, 0)$ (*i.e.* the global error is $O(\max_k \tau_k^4)$) in any finite interval $[0, T]$, moreover, the convergence is uniform for all $\epsilon > 0$.

5˙2. *Numerical approximation based on the asymptotic expansion.* – The computational cost of the algorithm (12) increases with the dimension N. This is a serious drawback, especially if we take into account the fact that for small ϵ the gross dynamics of the model (1) belongs to the one-dimensional "hydrodynamic" manifold. The anal-

TABLE I. – *The largest pointwise numerical error and the maximal stepsize vs.* ϵ.

ϵ	10^{-1}	10^{-2}	10^{-3}	10^{-4}
τ	1.940^{-2}	$6.106 \cdot 10^{-3}$	$3.505 \cdot 10^{-3}$	$3.249 \cdot 10^{-3}$
error	3.216^{-4}	$3.636 \cdot 10^{-4}$	$1.879 \cdot 10^{-3}$	$2.772 \cdot 10^{-3}$

TABLE II. – *The largest* $L_1(\mathbb{R}_+, \mathbb{R}^N)$ *errors vs.* ϵ.

ϵ	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$\max_t \|\mathbf{n} - \bar{\mathbf{n}}\|_{L_1}$	$9.989 \cdot 10^{-2}$	$9.989 \cdot 10^{-2}$	$9.989 \cdot 10^{-2}$	$9.989 \cdot 10^{-2}$
$\max_t \|\mathbf{n} - \bar{\mathbf{n}} - \tilde{\mathbf{n}}\|_{L_1}$	$2.073 \cdot 10^{-2}$	$2.098 \cdot 10^{-3}$	$8.519 \cdot 10^{-4}$	$4.775 \cdot 10^{-4}$
$\max_t \|\mathbf{n} - \bar{\mathbf{n}} - \tilde{\mathbf{n}} - \hat{\mathbf{n}}\|_{L_1}$	$1.875 \cdot 10^{-2}$	$1.886 \cdot 10^{-3}$	$7.471 \cdot 10^{-4}$	$4.587 \cdot 10^{-4}$
$\max_t \|\mathbf{n} - \bar{\mathbf{n}} - \tilde{\mathbf{n}} - \hat{\mathbf{n}} - \check{\mathbf{n}}\|_{L_1}$	$1.105 \cdot 10^{-2}$	$1.721 \cdot 10^{-3}$	$7.471 \cdot 10^{-4}$	$4.587 \cdot 10^{-4}$

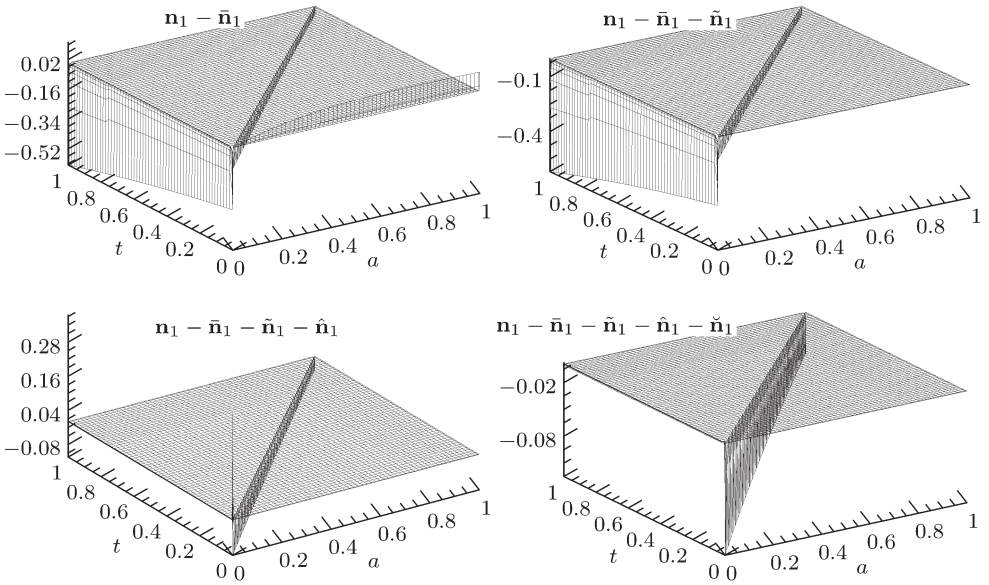

Fig. 2. – The bulk approximation error and the effect of the initial, boundary and corner layer corrections, $\epsilon = 10^{-3}$.

ysis of sect. **4** suggests another numerical algorithm. Instead of solving (1) directly it is better to solve for $\bar{\mathbf{n}}$, $\tilde{\mathbf{n}}$, $\hat{\mathbf{n}}$ and $\check{\mathbf{n}}$. The bulk approximation $\bar{\mathbf{n}}$ requires an integration of the scalar equation (8). This can be accomplished in the same way as for the complete model (1). The initial and the boundary layer corrections involve solution of linear ODEs with constant coefficients. The numerical approximations in these cases are trivial, moreover, both corrections must be computed only inside an $O(\epsilon)$ neighborhood of the boundary. The equation for the corner layer correction is of the same form as (1) but with simpler boundary conditions and does not involve ϵ. Thus, a simplified technique (12) is applicable. Again, $\check{\mathbf{n}}$ vanishes outside an $O(\epsilon)$ neighborhood of the corner and only local approximation is needed.

Hence the proposed discretization is of order four. The analysis of the numerical method (12) together with the results of sect. **4** yield a convergence of order $O(\max_k \tau_k^4 + \epsilon)$ in a finite interval $[0, T]$.

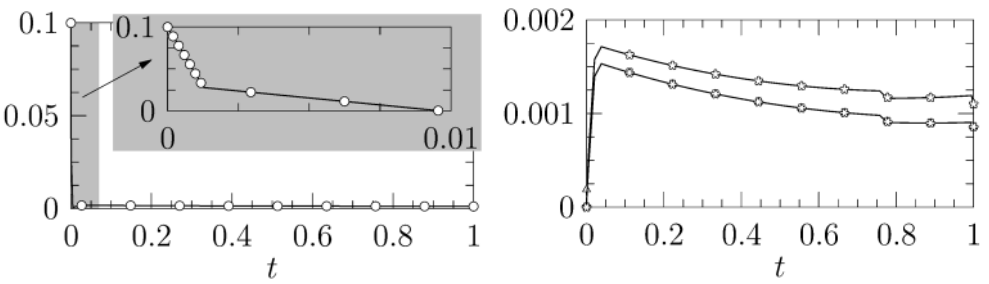

Fig. 3. – The $L_1(\mathbb{R}_+)$ errors before and after layers correction, $\epsilon = 10^{-3}$.

5˙3. *Illustrative example.* – As a reference problem we take (1) with $\mathcal{M} = \mathrm{diag}\{1,1\}$, $B = \mathrm{diag}\{1,2\}$ and $\mathcal{C} = \{c_{ij}\}_{1\le i,j\le 2}$ where $c_{11} = c_{22} = -1$ and $c_{12} = c_{21} = 1$, the initial condition is $\mathbf{n}(0,a) = (e^{-a}, e^{-2a})$. The exact solution is given by (11b) where

$$X_0(t) = \frac{e^{-t}}{2}\begin{bmatrix} 1+e^{-\frac{2t}{\epsilon}} & 1-e^{-\frac{2t}{\epsilon}} \\ 1-e^{-\frac{2t}{\epsilon}} & 1+e^{-\frac{2t}{\epsilon}} \end{bmatrix}, \quad \mathbf{n}(t,0) = \begin{bmatrix} \frac{1+\lambda_2}{\lambda_2-\lambda_1} & \frac{1+\lambda_1}{\lambda_2-\lambda_1} \\ \frac{(2+\lambda_1)(1+\lambda_2)}{\lambda_2-\lambda_1} & \frac{(1+\lambda_1)(2+\lambda_2)}{\lambda_2-\lambda_1} \end{bmatrix} \begin{bmatrix} e^{\frac{\lambda_1 t}{2\epsilon}} \\ -e^{\frac{\lambda_2 t}{2\epsilon}} \end{bmatrix}$$

and $\lambda_{1,2} = \epsilon - 2 \pm \sqrt{4+\epsilon^2}$.

First, we use (12) to integrate (1) numerically for $\epsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$. As expected (see table I) the convergence rate of (12) is uniform with respect to ϵ. The first component of the numerical solution is shown in fig. 1.

Second, we test the numerical approximation based on the asymptotic expansion. The aggregated equation takes the form $n_t + n_a = -n$, where the initial and boundary conditions are $n(0,a) = e^{-a} + e^{-2a}$ and $n(t,0) = \frac{3}{2}\int_0^\infty n(t,a)\mathrm{d}a$. Note that in this example the initial and the boundary conditions of the aggregated problem do not match at the corner, hence, the bulk approximation is discontinuous along the characteristic line $a = t$. We set $\epsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ and solve for $\bar{\mathbf{n}}$, $\tilde{\mathbf{n}}$, $\hat{\mathbf{n}}$ and $\check{\mathbf{n}}$ numerically as explained in subsect. **5**˙2. The results of table II are in a good agreement with Theorem 1, it is clearly visible that the asymptotic expansion approximates the exact solution with order $O(\epsilon)$.

Figures 1, 2 provide further illustration to the asymptotic expansion theory of sect. **4**. The right diagram of fig. 1 shows the bulk approximation $\bar{\mathbf{n}}$, its error $\mathbf{n} - \bar{\mathbf{n}}$ is given in the upper-left diagram of fig. 2. One can clearly see that $\bar{\mathbf{n}}$ provides a good uniform approximation to $\mathbf{n}$ everywhere except at the boundaries and at the characteristic line $a = t$. The upper-right diagram of fig. 2 shows how the initial layer correction improves the accuracy of the approximation. The effects of boundary and the corner layers corrections are depicted in the lower-left and the lower-right diagrams of fig. 2, respectively.

Figure 3 illustrates Theorem 1. The left part of the figure plots the bulk approximation error $\|\mathbf{n}(t,\cdot) - \bar{\mathbf{n}}(t,\cdot)\|_{L_1(\mathbb{R}_+,\mathbb{R}^N)}$ as a function of time. The error is large in a $O(\epsilon)$ neighborhood of the origin and is of magnitude $O(\epsilon)$ away from it. The errors obtained after layers corrections are shown in the right diagram. As predicted in Theorem 1 the initial layer correction is sufficient to reduce the error to $O(\epsilon)$. Use of the boundary and the corner layers improves the error slightly but do not change its order.

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