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An intrinsic approach to forces in magnetoelectric media

R. W. Tucker(*) and T. J. Walton(**)

Department of Physics, Lancaster University - Lancaster, UK
The Cockcroft Institute - Keckwick Lane, Daresbury, WA44AD, UK

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Summary. — This paper offers a conceptually straightforward method for the calculation of stresses in polarisable media based on the notion of a *drive form* and its property of being closed in spacetimes with symmetry. After an outline of the notation required to exploit the powerful exterior calculus of differential forms, a discussion of the relation between Killing isometries and conservation laws for smooth and distributional drive forms is given. Instantaneous forces on isolated spacetime domains and regions with interfaces are defined, based on manifestly covariant equations of motion. The remaining sections apply these notions to media that sustain electromagnetic stresses, with emphasis on homogeneous magnetoelectric material. An explicit calculation of the average pressure exerted by a monochromatic wave normally incident on a homogeneous, magnetoelectric slab *in vacuo* is presented and the concluding section summarizes how this pressure depends on the parameters in the magnetoelectric tensors for the medium.

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1. – Introduction

The calculation of stresses in material media has extensive application in modern science. The balance laws of continuum mechanics offer an established framework for such calculations for matter subject to a wide class of constitutive properties that attempt to accommodate interaction with the environment in terms of phenomenological relations [1, 2]. Such relations are not always easily accessible via experiment, since the response of matter to internal and external interactions can be very complex. If

(*) E-mail: r.tucker@lancaster.ac.uk
(**) E-mail: t.walton2@lancaster.ac.uk

one formulates these interactions in the language of forces derived from stress-energy-momentum tensors, then it is sometimes non-trivial to determine experimentally an appropriate tensor that can be associated with a particular class of interactions on a macroscopic scale [3-8]. This problem has led to numerous debates over the last century about how best to formulate the transmission of electromagnetic forces in polarisable media. Since the electromagnetic interaction is fundamentally relativistic in nature, the problem is compounded if one insists on a relativistically (covariant) theoretical formulation to compare with experiment in the laboratory. Judged by the large literature on this subject, there is no universal consensus on how best to calculate forces in polarisable media and hence the needed experimental input into the subject has been of uncertain value in the past. However, modern technology—with the refined experimental procedures now available—offers the possibility that the appropriate constitutive relations for certain classes of polarisable matter can be determined experimentally [9] over a broad range of field intensities, frequencies and geometric configurations. Furthermore, new materials with novel constitutive properties are being fabricated [10] and their response to time-varying electromagnetic fields also offers new potential for technological advances. With these points in mind, this paper offers a conceptually straightforward method for the calculation of stresses in polarisable media, based on the notion of a *drive form* and its property of being closed in spacetimes with symmetry. Section **2** outlines the notation required to exploit the powerful exterior calculus of differential forms that is used throughout the article. Sections **3** and **4** relate the isometries to conservation laws for smooth and distributional drive forms. Sections **5** and **6** discuss equations of motion in spacetime and how they may be used to define instantaneous forces on isolated domains, while sect. **7** deals with forces on domains with interfaces. The remaining sections apply these notions to media that sustain electromagnetic stresses, with emphasis in sect. **11** on homogeneous, magnetoelectric material. In sects. **12–14**, an explicit calculation of the average pressure exerted by a monochromatic, electromagnetic wave on a homogeneous, magnetoelectric slab *in vacuo* is presented and the discussion in sect. **15** summarises how this pressure depends on the parameters of the magnetoelectric tensors for the medium.

2. – Notation

The formulation below exploits the geometric language of exterior differential forms and vector fields on a manifold M [11]. Such a language is ideally suited to accommodate *local* changes of coordinates that can be used to simplify the description of boundary value problems and naturally encapsulates intrinsic *global* properties of domains with different physical properties. It also makes available the powerful exterior calculus that facilitates the integration of forms over domains described as the images of chain maps and permits a clear formulation of notions such as energy, momentum, angular momentum, force and torque, by fundamentally relating them to isometries of spacetime. In this framework, a p-form α belongs to $\mathcal{S}\Lambda^p M$, the space of sections of the bundle of exterior p-forms over M, while vector fields X belong to $\mathcal{S}TM$, the space of sections of the tangent bundle over M. On a manifold with metric tensor g, we denote $g(X,-)$ by $\widetilde{X} \in \mathcal{S}\Lambda^1 M$ and conversely set $\widetilde{\widetilde{X}} = X$ for all X. In the following, a notational distinction between smooth (C^∞) forms on some regular domain and those with possible singularities or discontinuities is useful. Smooth forms with compact support on spacetime will be referred to as *test forms* [12] and distinguished below by a superposed hat. Manifolds with dimension n will be assumed orientable and endowed with a preferred n-form induced from the metric tensor field g. One then has [11] the linear Hodge operator $\star$ that maps p-forms to $(n-p)$-forms on M. If g has signature t_g, one may write

$$g = \sum_{i=1}^{n} e^i \otimes e^j \, \eta_{ij}, \tag{2.1}$$

where $\eta_{ij} = \mathrm{diag}(\pm 1, \pm 1, \ldots \pm 1)$ and

$$\star 1 = e^1 \wedge e^2 \wedge \ldots \wedge e^n, \tag{2.2}$$

with $t_g = \det(\eta_{ij})$ and $\{e^i\}$ a set of basis 1-forms in $\mathcal{S}\Lambda^1 M$. The natural dual basis $\{X_i\}$ is defined so that $e^i(X_j) = \delta^i_j$ and the contraction operator with respect to X is denoted i_X. Covariant differentiation is performed with respect to the metric compatible Levi-Civita connection ∇, whilst Lie differentiation is denoted $\mathcal{L}$.

3. – Isometries and drive forms

The notion of a *drive form* arises from the theory of gravitation in spacetimes M with isometries. In Einstein's theory of gravitation, the metric g of spacetime is determined by the tensor field equation

$$Ein = \mathcal{T},$$

where $Ein \in \mathcal{S}T^2 M$ denotes the degree 2 symmetric divergence-free Einstein tensor field. Hence $\mathcal{T}$ must be a symmetric divergence-free degree 2 tensor field:

$$\nabla \cdot \mathcal{T} = 0.$$

The tensor $\mathcal{T}$ is regarded as a source of gravitational curvature(1). If K is a Killing vector field generating a spacetime symmetry and $\star$ is the Lorentzian Hodge operator associated with g, then by definition

$$\mathcal{L}_K g = 0$$

and it follows that the *drive 3-form*

$$\tau_K \equiv \star(\mathcal{T}(K, -))$$

is closed on some domain I_j of M:

$$d\,\tau_K = 0.$$

If the spacetime admits a set of Killing vector fields $\{K_i \in \mathcal{S}T\, I_j\}$, one has a conservation law for each K_i in every regular spacetime domain I_j [11,13]. These may be supplemented with (tensor or spinor) field equations

$$\mathcal{E}^{I_j}\left(g, \Phi^{I_j}_\alpha\right) = 0,$$

(1) The tensor $\mathcal{T}$ has dimensions of $[MLT^{-2}]$ (force) constructed from the SI dimensions $[M]$, $[L]$, $[T]$, $[Q]$, where $[Q]$ has the unit of the Coulomb in the MKS system.

for all piecewise smooth (tensor or spinor) fields $\Phi_\alpha^{I_j}$ that interact with each other and gravity. These field equations may induce compatibility conditions and further (non-Killing) conservation laws

$$d\,\mathcal{J}^{I_j}(\Phi_\alpha^{I_j}) = 0$$

(*e.g.*, electric charge-current conservation). In phenomenological models, some of the field equations may be replaced by fixed background fields and source currents, together with consistent constitutive relations between these fields and currents.

An observer field is associated with an arbitrary *unit* future-pointing timelike 4-velocity vector field $U \in \mathcal{S}TM$. The field U may be used to describe an *observer frame* on spacetime and its integral curves model *idealized observers*. The drive form τ_K associated with any K admits a unique orthogonal decomposition with respect to any observer frame U:

$$\tau_K = J_K^U \wedge \widetilde{U} + \rho_K^U,$$

where the spatial forms $\rho_K^U \in \mathcal{S}\Lambda^3 M$ and $J_K^U \in \mathcal{S}\Lambda^2 M$ satisfy $i_U \rho_K^U = i_U J_K^U = 0$. In a local region, the conservation law $d\,\tau_K = 0$ implies, in terms of the K-current J_K^U, the continuity relation in the frame U:

$$d\,J_K^U + \mathcal{L}_U \tau_K = 0.$$

If K is a *spacelike translational* Killing vector field and U a unit time-like (future-pointing) 4-vector observer field([2]), then

$$J_K^U \equiv -i_U \tau_K$$

is the linear momentum current (stress) 2-form in the frame U and

$$\rho_K^U \equiv -(i_U \star \tau_K) \star \widetilde{U}$$

is the associated linear momentum density 3-form in the frame U. If K is a *spacelike rotational* Killing vector field generating $SO(3)$ group isometries, then J_K^U is an angular-momentum current (torque stress) 2-form and ρ_K^U is the associated angular-momentum density 3-form in the frame U. If K is a *timelike translational* Killing vector field, then J_K^U is an energy current (power) 2-form and ρ_K^U is the associated energy density 3-form in the frame U. In the following, attention is restricted to translational spacelike Killing vectors of flat spacetime and the computation of integrals of J_K^U for a particular contribution to τ_K associated with electromagnetic fields in homogeneous but anisotropic media of a particular kind. It will be argued that this formulation leads to a natural definition of integrated static forces in media with discontinuous material behavior and highlights the need for care in giving a practical definition of integrated force in media in the presence of time-varying fields.

([2]) The frame is inertial if $\nabla U = 0$.

4. – Distributional drive forms

To accommodate media with singular time-dependent sources of stress (*e.g*, at surface interfaces or lines in space), introduce the distributional Killing 3-form ${\tau_K}^D$ on spacetime and its distributional source ${\mathcal{K}_K}^D$ satisfying the distributional equation

$$d\,{\tau_K}^D[\hat{\beta}] = {\mathcal{K}_K}^D[\hat{\beta}], \tag{4.1}$$

for all test 4-forms $\hat{\beta}$ [12] on spacetime. Consider a compact medium at time t, with spatial volume determined by the image of the spacelike t-parameterised immersion ${\Sigma^3}_t : W_3 \subset \mathbb{R}^3 \to M$, evolving for a finite interval of time. Denote its immersed history in spacetime by the region I_1. Let I_2 be a compact region of spacetime outside this medium history. It follows from (4.1) that if ${\tau_K}^{I_1}$ is the regular drive form in region I_1 and ${\tau_K}^{I_2}$ is the regular drive form in region I_2, then

$$d\,{\tau_K}^{I_1} = 0 \qquad \text{in} \quad I_1, \tag{4.2}$$

$$d\,{\tau_K}^{I_2} = 0 \qquad \text{in} \quad I_2 \tag{4.3}$$

$$\text{and} \qquad {\Sigma^3}_s^{\,\star}\left({\tau_K}^{I_1} - {\tau_K}^{I_2} + \mathcal{K}_K\right) = 0, \tag{4.4}$$

at an evolving interface defined by the timelike, t-parameterised immersion ${\Sigma^3}_s : S_2 \subset \mathbb{R}^3 \to M$ between I_1 and I_2 with a smooth interface drive form $\mathcal{K}_K$ on its image. The history of these images in spacetime is indicated schematically in fig. 1.

5. – Equation of motion for a smooth domain

The notion of force (and torque) is implicit in the balance laws of classical Newtonian continuum mechanics. In the presence of time-varying fields, it is natural to associate energy, momentum and angular momentum with such fields in order to maintain the conservation of these quantities for closed systems. The only sensible approach to defining force (and torque) density in such circumstances, where the balance law arises from the divergence of a total drive form for the system, is with respect to a particular splitting of this divergence. For systems without mechanical constraint, one assigns a smooth 4-velocity V (and angular velocity) field to each smooth domain to describe the motion. The jumps in these fields at interfaces between domains must be computed from (4.2)–(4.4) above. The 4-acceleration field A of each domain (and possibly its rate of change) will appear in one or more components of the split and the remaining terms in the divergence are often identified with total force (or torque) densities for the domain. However, unless one prescribes how to practically identify component contributions to the total force (for example by cancelling some of them by externally applied mechanical constraints), there is no natural way to identify a canonical split of the divergence of the total drive form. In those situations where the interaction of matter and fields is stationary or static, one can appeal to static experiments with non-moving media to try and give an unambiguous definition to material body forces. For electromagnetic interactions with polarisable media, comparison with experiment is difficult, since the choice of drive form is very model dependent for many materials. However modern technology—with the refined experimental procedures now available—offers the possibility that the appropriate constitutive relation for certain types of matter can be determined experimentally over an extended parameter range [10].

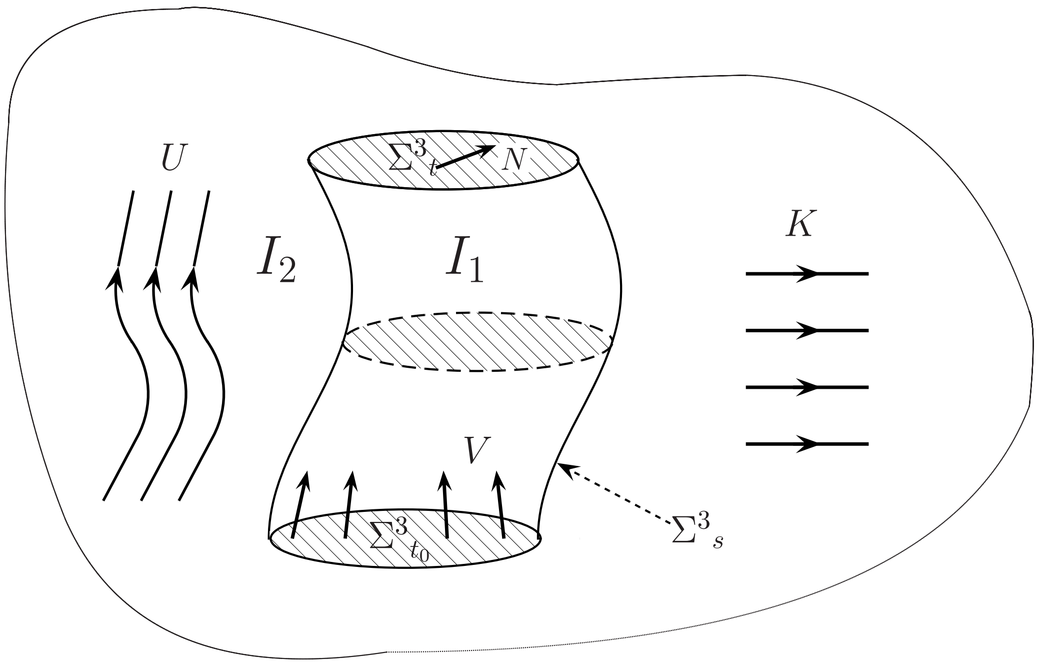

Fig. 1. – The partition of spacetime M by the history of a compact medium (with boundary ${\Sigma^3}_s \cup {\Sigma^3}_{t_0} \cup {\Sigma^3}_t$), evolving with 4-velocity V. The timelike vector field U defines a frame, N is a unit, spacelike vector field and K is a Killing vector field.

To illustrate these general remarks, consider an uncharged (unbounded) medium containing a fixed number of constituents, with number density $\mathcal{N} \in \mathcal{S}\Lambda^0 M$ and mass density $\rho = m_0\mathcal{N}$, $m_0 > 0$, in Minkowski spacetime with mass conservation $d\,(\rho \star \widetilde{V}) = 0$. Write

$$\tau_K = {\tau_K}^V + {\tau_K}^{\text{field}},$$

where V is the (future-pointing) unit, time-like 4-velocity field of the medium and K a Killing vector field. For a simple medium with a smooth mass density, suppose

$${\tau_K}^V \equiv {c_0}^2\, \rho g(V, K) \star \widetilde{V},$$

with c_0 the speed of light *in vacuo*, then

$$d\,\tau_K = 0$$

yields the local equation of motion [11] for the field V:

$${c_0}^2\, \rho \widetilde{A}(K) = f_K,$$

where $f_K \equiv \star d\,{\tau_K}^{\text{field}}$ and the 4-acceleration 1-form $\widetilde{A} \equiv \nabla_V \widetilde{V}$. If $\nabla V = 0$, then $\widetilde{A} = 0$ and the motion of the medium is geodesic. The medium is then static in the frame where

$U = V$. Contracting the local equation of motion

$$c_0{}^2\,\rho\widetilde{A}(K)\star 1 + d\,\tau_K{}^{\text{field}} = 0$$

with the observer field U and integrating over the volume $\Sigma^3{}_t$ yields

$$\dot{P}_K^{\text{mech}\,U}\left[\Sigma^3{}_t\right] = f_K^U\left[\Sigma^3{}_t\right],$$

where

$$\dot{P}_K^{\text{mech}\,U}\left[\Sigma^3{}_t\right] \equiv \int_{\Sigma^3{}_t} \mu_U\,\widetilde{A}(K).$$

Here the mass 3-form

$$\mu_U \equiv -c_0{}^2\rho\star\widetilde{U}$$

and the total instantaneous *integrated* K-drive component on $\Sigma^3{}_t$ at time t in the U frame is

$$f_K^U\left[\Sigma^3{}_t\right] \equiv \int_{\Sigma^3{}_t} i_U\,d\,\tau_K{}^{\text{field}}.$$

6. – The general integrated force form on a regular domain $I_j \subset M$

In Minkowski spacetime, one has a global basis of parallel unit space-like translational Killing vector fields (K_1, K_2, K_3) on $\Sigma^3{}_t$. In local Cartesian coordinates $\{t, x, y, z\}$, with $g = -c_0{}^2 dt\otimes dt + dx\otimes dx + dy\otimes dy + dz\otimes dz$:

$$K_1 = \frac{\partial}{\partial x}, \qquad K_2 = \frac{\partial}{\partial y}, \qquad K_3 = \frac{\partial}{\partial z}.$$

One can then define the instantaneous *integrated* force 1-form on $\Sigma^3{}_t$ at time t in the U frame to be

$$f^U\left[\Sigma^3{}_t\right] \equiv \sum_{j=1}^{3} f_{K_j}^U\left[\Sigma^3{}_t\right]\ \widetilde{K_j}. \tag{6.1}$$

Then, if N is any unit space-like vector field on $\Sigma^3{}_t$, the instantaneous integrated force component in the direction N acting on $\Sigma^3{}_t$ is

$$f^U\left[\Sigma^3{}_t\right](N) = \sum_{j=1}^{3} f_{K_j}^U\left[\Sigma^3{}_t\right]\ \widetilde{K_j}(N). \tag{6.2}$$

In an arbitrary (possibly non-inertial) frame U and domain $I_j \subset M$

$$i_U d\,\tau_K{}^{\text{field}\ I_j} = d\,\sigma_K{}^{U\ \text{field}\ I_j} + \mathcal{L}_U \tau_K{}^{\text{field}\ I_j},$$

where

$$\sigma_K{}^{U\ \mathrm{field}\ I_j} \equiv -i_U \tau_K{}^{\mathrm{field}\ I_j} \tag{6.3}$$

is the total Cauchy stress 2-form([3]). If one identifies an electromagnetic K-drive $\tau_K{}^{\mathrm{EM}\,I_j}$ in $\tau_K{}^{\mathrm{field}\ I_j}$, such that

$$\tau_K{}^{\mathrm{field}\ I_j} = \tau_K{}^{\mathrm{EM}\,I_j} + \tau_K{}^{\mathrm{rem}\ I_j}$$

and

$$\sigma_K{}^{U\ \mathrm{field}\ I_j} = \sigma_K{}^{U\ \mathrm{EM}\,I_j} + \sigma_K{}^{U\ \mathrm{rem}\ I_j},$$

one then has

$$\dot{P}_K^{\mathrm{mech}\,U I_j}\left[\Sigma^3{}_t\right] + \dot{P}_K^{\mathrm{EM}\,U I_j}\left[\Sigma^3{}_t\right] + \dot{P}_K^{\mathrm{rem}\,U I_j}\left[\Sigma^3{}_t\right] = f_K^{\mathrm{EM}\ U I_j}\left[\Sigma^3{}_t\right] + f_K^{\mathrm{rem}\ U I_j}\left[\Sigma^3{}_t\right],$$

where

$$\dot{P}_K^{\mathrm{rem}\,U I_j}\left[\Sigma^3{}_t\right] \equiv -\int_{\Sigma^3{}_t} \mathcal{L}_U \tau_K{}^{\mathrm{rem}\ I_j},$$

$$\dot{P}_K^{\mathrm{EM}\,U I_j}\left[\Sigma^3{}_t\right] \equiv -\int_{\Sigma^3{}_t} \mathcal{L}_U \tau_K{}^{\mathrm{EM}\,I_j}$$

denote integrated rates of change associated with field momenta in $\tau_K{}^{\mathrm{rem}\ I_j}$ and $\tau_K{}^{\mathrm{EM}\,I_j}$ respectively, and

$$f_K^{\mathrm{rem}\ U I_j}\left[\Sigma^3{}_t\right] \equiv \int_{\Sigma^3{}_t} d\,\sigma_K{}^{U\ \mathrm{rem}\ I_j},$$

$$f_K^{\mathrm{EM}\ U I_j}\left[\Sigma^3{}_t\right] \equiv \int_{\Sigma^3{}_t} d\,\sigma_K{}^{U\ \mathrm{EM}\,I_j}$$

denote integrated forces associated with stresses in $\tau_K{}^{\mathrm{rem}\ I_j}$ and $\tau_K{}^{\mathrm{EM}\,I_j}$, respectively.

7. – The general integrated force form in an irregular static domain composed of different media

Suppose $\Sigma^3{}_t = \sum_j I_j$ with $\sigma_K{}^{U\ \mathrm{field}\ I_j}$ the Cauchy stress 2-form for domain I_j in a Minkowski spacetime with frame $U = \frac{1}{c_0}\frac{\partial}{\partial t}$. In general, $\tau_K{}^{\mathrm{field}}$ must contain (time dependent) constraining forces to maintain the overall equilibrium condition

$$i_{\frac{\partial}{\partial t}} d\,\tau_K^{\mathrm{field}} = 0$$

from stresses in each sub-domain I_j of $\Sigma^3{}_t$. In the (possibly constrained) static case,

([3]) This follows from the Cartan identity: $\mathcal{L}_X = i_X d + d i_X$ for any $X \in \mathcal{S}TM$.

$\mathcal{L}_U \tau_K = 0$ and each 2-form

$$i_{\frac{\partial}{\partial t}} d\, {\tau_K}^{\text{field } I_j} = d{\sigma_K}^{U \text{ field } I_j}$$

contributes an integrated reaction force on ${\Sigma^3}_t$ from domain I_j.

In general, each 4-velocity $V^{I_j} \in \mathcal{S}T I_j$ must be determined from the jump conditions for ${\tau_K}^{\text{field } I_j}$. In the static case, one has all $\nabla V^{I_j} = 0$ with $V^{I_j} = U$, and one may define the net integrated K-force for ${\Sigma^3}_t$ in the frame U:

$$f_K^U \left[{\Sigma^3}_t\right] \equiv \sum_j \int_{\Sigma^3_{I_j}} i_{\frac{\partial}{\partial t}} d\, \tau_K^{\text{field } I_j} = \sum_j \int_{\Sigma^3_{I_j}} d\, \sigma_K^{U \text{ field} I_j} = \sum_j \int_{\partial \Sigma^3_{I_j}} \sigma_K^{U \text{ field} I_j}.$$

There may be additional sources of stress with support on submanifolds of M. Singular sources of stress in the electromagnetic field include charges, currents and their multipoles, with support on points, lines Σ^1 or surfaces Σ^2 in space [7]. If the integrals on the right below are finite, the most general integrated force can then be written so as to include such distributional sources:

$$f_K^U \left[{\Sigma^3}_t\right] \equiv \sum_j \int_{\partial \Sigma^3_{I_j}} \sigma_K^{U \text{ field} I_j} + \sum_j \int_{\Sigma^2_{I_j}} \kappa_K^{U \text{ field} I_j} + \sum_j \int_{\Sigma^1_{I_j}} \gamma_K^{U \text{ field} I_j},$$

in terms of line stress 1-forms γ_K and surface stress 2-forms κ_K.

A number of sources of interfacial stress depend on the local mean curvature normal of the interface. For example, if the history of the interface ∂I_j is the spacetime hypersurface $f = 0$ with unit spacelike normal $N = \frac{\widetilde{d f}}{|d f|}$, then the scalar $(\mathrm{Tr}\ H)$ is defined by

$$d\, i_N i_U \star 1 = (\mathrm{Tr}\ H)\ i_U \star 1$$

and $\eta \equiv (\mathrm{Tr}\ H)\, N$ is the mean curvature normal. *Surface tension* at an arbitrary interface depends on η and the local surface tension scalar field γ, yielding the particular interface forces:

$$\int_{\Sigma^2_{I_j} = \partial \Sigma^3_{I_j}} \kappa_K^{U \text{ field} I_j} = \int_{\Sigma^2_{I_j}} (\gamma\, i_K\, \widetilde{\eta} + i_K\, d\gamma)\, i_N\, i_U\, \star 1$$

$$\int_{\Sigma^1_{I_j}} \gamma_K^{U \text{ field} I_j} = \int_{\Sigma^1_{I_j} = \partial \Sigma^2_{I_j}} \gamma\, i_N\, i_U\, i_K\, \star 1.$$

8. – Electromagnetic fields in spacetime

Maxwell's equations for an electromagnetic field in an arbitrary medium can be written as

$$dF = 0 \qquad \text{and} \qquad d \star G = j, \tag{8.1}$$

where $F \in \mathcal{S}\Lambda^2 M$ is the Maxwell 2-form, $G \in \mathcal{S}\Lambda^2 M$ is the excitation 2-form and $j \in \mathcal{S}\Lambda^3 M$ is the 3-form electric current source([4]). To close this system, "electromagnetic constitutive relations" relating G and j to F are necessary. The functional tensor relations

$$G = \mathcal{Z}[F]$$

and

$$j = \mathcal{Z}_1[F]$$

are typical for idealized material without electrostriction losses.

The electric 4-current j describes both (mobile) electric charge and effective (Ohmic) currents in a conducting medium. The *electric field* $\boldsymbol{e} \in \mathcal{S}\Lambda^1 M$ and *magnetic induction field* $\boldsymbol{b} \in \mathcal{S}\Lambda^1 M$ associated with F are defined with respect to an observer field U by

$$\boldsymbol{e} = i_U F \qquad \text{and} \qquad c_0\, \boldsymbol{b} = i_U \star F. \tag{8.2}$$

Thus, $i_U \boldsymbol{e} = i_U \boldsymbol{b} = 0$ and with $g(U,U) = -1$,

$$F = \boldsymbol{e} \wedge \widetilde{U} - \star \left(c_0\, \boldsymbol{b} \wedge \widetilde{U} \right). \tag{8.3}$$

Likewise the *displacement field* $\boldsymbol{d} \in \mathcal{S}\Lambda^1 M$ and the *magnetic field* $\boldsymbol{h} \in \mathcal{S}\Lambda^1 M$ associated with G are defined with respect to U by

$$\boldsymbol{d} = i_U G \qquad \text{and} \qquad \frac{\boldsymbol{h}}{c_0} = i_U \star G. \tag{8.4}$$

Thus,

$$G = \boldsymbol{d} \wedge \widetilde{U} - \star \left(\frac{\boldsymbol{h}}{c_0} \wedge \widetilde{U} \right), \tag{8.5}$$

with $i_U \boldsymbol{d} = i_U \boldsymbol{h} = 0$. The spatial 1-forms $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{h}$ are fields on a general spacetime defined with respect to the frame U, which may be non-inertial if $d\widetilde{U} \neq 0$.

9. – Time-dependent Maxwell systems in space

In the following, attention is restricted to fields on Minkowski spacetime. This can be globally foliated by 3-dimensional spacelike hyperplanes. The Minkowski metric on spacetime induces a metric with Euclidean signature on each spacetime hyperplane. Furthermore, each hyperplane contains events that are deemed simultaneous with respect to a clock attached to any integral curve of a future-pointing, unit, time-like vector field

([4]) All electromagnetic tensors in this article have dimensions constructed from the SI dimensions $[M]$, $[L]$, $[T]$, $[Q]$ where $[Q]$ has the unit of the Coulomb in the MKS system. We adopt $[g] = [L^2]$, $[G] = [j] = [Q]$, $[F] = [Q]/[\epsilon_0]$ where the permittivity of free space ϵ_0 has the dimensions $[Q^2\, T^2 M^{-1}\, L^{-3}]$ and $c_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ denotes the speed of light *in vacuo*. Note that, with $[g] = [L^2]$, for p-forms α in n dimensions one has $[\star\alpha] = [\alpha][L^{n-2p}]$.

$U = \frac{1}{c_0}\frac{\partial}{\partial t}$ defining an inertial observer on Minkowski spacetime and the spacetime Hodge map $\star$ induces a Euclidean Hodge map $\#$ on each hyperplane by the relation

$$\star 1 = c_0 \; dt \wedge \# 1 = \# 1 \wedge \widetilde{U}.$$

The spacetime Maxwell system can now be reduced to a family of parameterised exterior systems on $\mathbb{R}^3$. Each member is an exterior system involving forms on $\mathbb{R}^3$ depending parametrically on some time coordinate t associated with U. Let the $(3+1)$ split of the 4-current 3-form with respect to the foliation be

$$j = -\boldsymbol{J} \wedge dt + \rho \# 1,$$

with $i_{\frac{\partial}{\partial t}} \boldsymbol{J} = 0$. Then, from (8.1),

$$d\, j = 0 \tag{9.1}$$

yields

$$\widehat{d}\boldsymbol{J} + \dot{\rho} \# 1 = 0. \tag{9.2}$$

Here, and below, an over-dot denotes (Lie) differentiation with respect to the parameter t ($\dot{\alpha} \equiv \mathcal{L}_{\frac{\partial}{\partial t}} \alpha$ for all α) and $\widehat{d}$ denotes exterior differentiation on $\mathbb{R}^3$ such that

$$d \equiv \widehat{d} + dt \wedge \mathcal{L}_{\frac{\partial}{\partial t}}.$$

It is convenient to introduce on each spacetime hyperplane the (Euclidean Hodge) dual forms:

$$\boldsymbol{E} \equiv \# \boldsymbol{e}, \qquad \boldsymbol{D} \equiv \# \boldsymbol{d}$$
$$\boldsymbol{B} \equiv \# \boldsymbol{b}, \qquad \boldsymbol{H} \equiv \# \boldsymbol{h}, \qquad \boldsymbol{j} \equiv \# \boldsymbol{J},$$

so that the $(3+1)$ split of the spacetime covariant Maxwell equations (8.1) with respect to $\widetilde{U} = -c_0\, dt$ becomes

$$\widehat{d}\boldsymbol{e} = -\dot{\boldsymbol{B}}, \tag{9.3}$$
$$\widehat{d}\boldsymbol{B} = 0, \tag{9.4}$$
$$\widehat{d}\boldsymbol{h} = \boldsymbol{J} + \dot{\boldsymbol{D}}, \tag{9.5}$$
$$\widehat{d}\boldsymbol{D} = \rho \# 1. \tag{9.6}$$

All p-forms ($p \geq 0$) in these equations are independent of dt, but have components that may depend parametrically on t.

10. – Electromagnetic constitutive tensors for linear media

Attention will now be turned to integrated electromagnetic forces on a class of polarisable media. This requires a discussion of a class of electromagnetic constitutive tensors for linear media. In general, the excitation tensor G is a functional of the Maxwell field tensor F and properties of the medium

$$G = \mathcal{Z}[F, \ldots].$$

Such a functional induces, in general, non-linear and non-local relations between $\boldsymbol{d}$, $\boldsymbol{h}$ and $\boldsymbol{e}$, $\boldsymbol{b}$. Electrostriction and magnetostriction arise from the dependence of $\mathcal{Z}$ on the elastic deformation tensor of the medium. For general *linear continua*, one may define a collection of *constitutive tensor fields* $Z^{(r)}$ on spacetime by the relation

$$G = \Sigma_{r=0}^{N} Z^{(r)}[\nabla^{r} F, \ldots],$$

in terms of the spacetime connection (covariant derivative) ∇.

In idealized (non-dispersive) *simple media*, one adopts the simplified *local* relation

$$G = Z(F),$$

for some degree 4 constitutive tensor field Z and in the vacuum $G = \epsilon_0 F$. Regular *linear isotropic media* are described by a bulk 4-velocity field V, a relative permittivity scalar field ϵ_r and a non-vanishing relative permeability scalar field μ_r. In this case, the structure of Z follows from

$$\frac{G}{\epsilon_0} = \epsilon_r\, i_V F \wedge \widetilde{V} - \mu_r^{-1} \star \left(i_V \star F \wedge \widetilde{V} \right) = \left(\epsilon_r - \frac{1}{\mu_r} \right) i_V F \wedge \widetilde{V} + \frac{1}{\mu_r} F.$$

In a comoving frame with $U = V$, this becomes

$$\boldsymbol{d} = \epsilon_0 \epsilon_r\, \boldsymbol{e} \qquad \text{and} \qquad \boldsymbol{h} = (\mu_0 \mu_r)^{-1} \boldsymbol{b}.$$

To discuss linear (non-dispersive, lossless), inhomogeneous, *anisotropic media*, it is convenient to describe Z in a particular basis associated with the medium. Since Z is a tensor that maps 2-forms to 2-forms, in any spacetime local frame $\{e^0, e^1, e^2, e^3\}$, one may write

$$\frac{1}{2} G_{ab} e^a \wedge e^b = \frac{1}{4} Z^{cd}{}_{ab} F_{cd} e^a \wedge e^b,$$

where

$$Z^{cd}{}_{ab} = -Z^{cd}{}_{ba} = -Z^{dc}{}_{ab} = Z^{dc}{}_{ba}.$$

Thus, Z can be described in terms of spatial rank 3 tensors on spacetime, relating observed electric and magnetic fields in some frame U, with

$$\boldsymbol{d} = \zeta^{de}(\boldsymbol{e}) + \zeta^{db}(\boldsymbol{b}),$$
$$\boldsymbol{h} = \zeta^{he}(\boldsymbol{e}) + \zeta^{hb}(\boldsymbol{b}).$$

In such a frame, the medium is said to exhibit *magneto-electric* properties in general. If ζ^{db} and ζ^{he} are non-zero in the co-moving frame of the medium, it is called magnetoelectric. If ζ^{db} and ζ^{he} are zero in the co-moving frame of the medium, it is called non-magnetoelectric. The spatial tensors ζ^{db} and ζ^{he} may be non-zero in a non-comoving frame for a non-magnetoelectric medium. Due to the behaviour of electric and magnetic fields under Lorentz transformations, all materials exhibit magnetoelectric properties in some frame. Thermodynamic and time symmetry conditions impose the relation $Z = Z^\dagger$ [14] or

$$\zeta^{de\dagger} = \zeta^{de}, \quad \zeta^{hb\dagger} = \zeta^{hb} \quad \text{and} \quad \zeta^{db\dagger} = -\zeta^{he}$$

in all spacetime frames, where the adjoint $T^\dagger$ of a tensor T which maps p-forms to p-forms is defined by

$$\alpha \wedge \star T(\beta) = \beta \wedge \star T^\dagger(\alpha) \quad \text{for all } \alpha, \beta \in \mathcal{S}\Lambda^p M.$$

11. – Homogeneous dispersive magnetoelectric media

In dispersive media, constitutive relations between the spatial fields $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{h}$ are non-local in spacetime. If the medium is *spatially homogenous*, so that it has no preferred spatial origin, then it is possible to Fourier transform the fields with respect to space and time, and work with transformed local constitutive relations.

For any real valued p-form α, define its complex valued Fourier transform $\check{\alpha}_{\boldsymbol{k},\omega}$ by

$$\alpha = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\boldsymbol{k}\ \check{\alpha}_{\boldsymbol{k},\omega}\ \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)], \tag{11.1}$$

where $\boldsymbol{k} \in \mathbb{R}^3$. Then the source free Maxwell system reduces to

$$\boldsymbol{K} \wedge \check{\boldsymbol{e}}_{\boldsymbol{k},\omega} = \omega \check{\boldsymbol{B}}_{\boldsymbol{k},\omega} \tag{11.2}$$

$$\boldsymbol{K} \wedge \check{\boldsymbol{h}}_{\boldsymbol{k},\omega} = -\omega \check{\boldsymbol{D}}_{\boldsymbol{k},\omega}, \tag{11.3}$$

where the real propagation wave 1-form $\boldsymbol{K} \equiv \boldsymbol{k}\cdot d\boldsymbol{r} \in \mathcal{S}\Lambda^1 M$. The remaining transformed Maxwell equations $\boldsymbol{K}\wedge\check{\boldsymbol{B}}_{\boldsymbol{k},\omega} = 0$ and $\boldsymbol{K}\wedge\check{\boldsymbol{D}}_{\boldsymbol{k},\omega} = 0$ follow trivially from (11.2) and (11.3) when $\omega \neq 0$. It also follows trivially that $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega} \wedge \check{\boldsymbol{B}}_{\boldsymbol{k},\omega} = 0$ (*i.e.* $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}$ is perpendicular to $\check{\boldsymbol{b}}_{\boldsymbol{k},\omega}$). Similarly, $\check{\boldsymbol{B}}_{\boldsymbol{k},\omega} \wedge \boldsymbol{K} = 0$ and $\check{\boldsymbol{D}}_{\boldsymbol{k},\omega} \wedge \boldsymbol{K} = 0$.

We assume that the magnetoelectric constitutive relations take the form

$$\check{\boldsymbol{d}}_{\boldsymbol{k},\omega} = \check{\boldsymbol{\zeta}}^{de}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}) + \check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}(\check{\boldsymbol{b}}_{\boldsymbol{k},\omega}), \tag{11.4}$$

$$\check{\boldsymbol{h}}_{\boldsymbol{k},\omega} = \check{\boldsymbol{\zeta}}^{he}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}) + \check{\boldsymbol{\zeta}}^{hb}_{\boldsymbol{k},\omega}(\check{\boldsymbol{b}}_{\boldsymbol{k},\omega}). \tag{11.5}$$

These will (by convolution) give rise to non-local spacetime constitutive relations. We also maintain the above symmetry properties on the magnetoelectric tensors $\check{\boldsymbol{\zeta}}^{de}_{\boldsymbol{k},\omega}$, $\check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}$, $\check{\boldsymbol{\zeta}}^{he}_{\boldsymbol{k},\omega}$, $\check{\boldsymbol{\zeta}}^{hb}_{\boldsymbol{k},\omega}$. Substituting (11.4) and (11.5) in (11.2) and (11.3) yields a degenerate 1-form linear eigen-equation for $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}$:

$$\omega^2 \check{\boldsymbol{\zeta}}^{de}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}) + \omega \check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}\left(\#\left(\boldsymbol{K} \wedge \check{\boldsymbol{e}}_{\boldsymbol{k},\omega}\right)\right) + \omega\#\left(\boldsymbol{K} \wedge \check{\boldsymbol{\zeta}}^{he}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega})\right) + \#\left(\boldsymbol{K} \wedge \check{\boldsymbol{\zeta}}^{hb}_{\boldsymbol{k},\omega}\left(\#\left(\boldsymbol{K} \wedge \check{\boldsymbol{e}}_{\boldsymbol{k},\omega}\right)\right)\right) = 0. \tag{11.6}$$

The field $\check{\boldsymbol{b}}_{\boldsymbol{k},\omega}$ then follows from (11.2), (up to a scaling) and $\check{\boldsymbol{d}}_{\boldsymbol{k},\omega}$, $\check{\boldsymbol{h}}_{\boldsymbol{k},\omega}$ from (11.4), (11.5), respectively. Equation (11.6) may be written as

$$\check{\boldsymbol{\mathcal{D}}}_{\boldsymbol{k},\omega}(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}) = 0, \tag{11.7}$$

defining the $1-1$ tensor $\check{\boldsymbol{\mathcal{D}}}_{\boldsymbol{k},\omega}$. For non-trivial solutions $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}$, the determinant of the matrix $\check{\mathcal{D}}_{\boldsymbol{k},\omega}$ representing $\check{\boldsymbol{\mathcal{D}}}_{\boldsymbol{k},\omega}$ must vanish:

$$\det(\check{\mathcal{D}}_{\boldsymbol{k},\omega}) = 0. \tag{11.8}$$

Note that, in general, the roots of this dispersion relation are not invariant under the transformation $\boldsymbol{K} \to -\boldsymbol{K}$. If one writes $\boldsymbol{k} = \hat{\boldsymbol{k}}|\boldsymbol{k}|$ in terms of the Euclidean norm $|\boldsymbol{k}|$, and introduces the refractive index $\mathcal{N} = |\boldsymbol{k}|\frac{c_0}{\omega} > 0$ and $\hat{\boldsymbol{k}}$ in place of $\boldsymbol{k}$, then solutions propagating in the direction described by $\hat{\boldsymbol{k}}$ with angular frequency $\omega > 0$ correspond to roots of (11.8) (labelled r) that may be expressed in the form $\mathcal{N}_r = \mathcal{F}_r(\hat{\boldsymbol{k}}, \omega)$. Thus, there can be a set of distinct characteristic waves each with its unique refractive index that depends on the propagation direction $\hat{\boldsymbol{k}}$ and frequency ω. When the characteristic equation (11.8) is a quadratic polynomial in $\mathcal{N}^2$ and has two distinct roots that describe two distinct propagating modes for a given ω, the medium is termed *birefringent*. Roots $\mathcal{N}_r^2$ such that $\mathcal{N}_r(\hat{\boldsymbol{k}}, \omega) \neq \mathcal{N}_r(-\hat{\boldsymbol{k}}, \omega)$ imply that harmonic plane waves propagating in the opposite directions $\pm\hat{\boldsymbol{k}}$ have different wave speeds.

Each eigen-wave will have a uniquely defined polarisation obtained by solving the independent equations in (11.7) for $\check{\boldsymbol{e}}^r_{\boldsymbol{k},\omega}$, up to normalisation. Since $\check{\boldsymbol{e}}^r_{\boldsymbol{k},\omega}$ is complex, it is convenient to introduce the eigen-wave normalisation by writing

$$\check{\boldsymbol{e}}^r_{\boldsymbol{k},\omega} = \check{e}^r_{\boldsymbol{k},\omega}\,\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega},$$

in terms of the complex 0-form $\check{e}^r_{\boldsymbol{k},\omega}$ and complex polarisation 1-form $\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega}$, normalised to satisfy

$$\overline{\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega}} \wedge \#\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega} = \#1 \tag{11.9}$$

for each r. If one applies $\#\,\overline{\check{\boldsymbol{e}}^r_{\boldsymbol{k},\omega}} \wedge \#$ to (11.6), making use of the symmetries between the real magnetoelectric tensors $\check{\boldsymbol{\zeta}}^{de}_{\boldsymbol{k},\omega}$, $\check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}$, $\check{\boldsymbol{\zeta}}^{he}_{\boldsymbol{k},\omega}$, $\check{\boldsymbol{\zeta}}^{hb}_{\boldsymbol{k},\omega}$, and evaluates it with the eigen-wave $\check{\boldsymbol{e}}^r_{\boldsymbol{k},\omega}$, one obtains the *real* 0-form dispersion relation for the characteristic mode r:

$$\omega^2\#\left(\overline{\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega}} \wedge \#\check{\boldsymbol{\zeta}}^{de}_{\boldsymbol{k},\omega}(\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega})\right) + \omega\,\#\left(\overline{\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega}} \wedge \#\check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}\left(\#\left(\boldsymbol{K} \wedge \check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega}\right)\right)\right)$$
$$+\omega\,\#\left(\overline{\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega}} \wedge \boldsymbol{K} \wedge \check{\boldsymbol{\zeta}}^{he}_{\boldsymbol{k},\omega}(\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega})\right) + \#\left(\overline{\check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega}} \wedge \boldsymbol{K} \wedge \check{\boldsymbol{\zeta}}^{hb}_{\boldsymbol{k},\omega}\left(\#\left(\boldsymbol{K} \wedge \check{\boldsymbol{n}}^r_{\boldsymbol{k},\omega}\right)\right)\right) = 0,$$

where $\boldsymbol{K} = \frac{\omega}{c_0}\mathcal{N}\hat{\boldsymbol{k}} \cdot d\boldsymbol{r}$ in terms of $\mathcal{N}$ and $\hat{\boldsymbol{k}}$.

12. – Electromagnetic stress-energy-momentum tensors

There has been intense debate over many decades about the appropriate choice of electromagnetic stress-energy-momentum tensor that transmits forces in a (moving) polarisable medium [15]. In 1909, Abraham introduced the *symmetric* electromagnetic stress-energy-momentum tensor $\mathcal{T}^{\mathrm{EM}}$ for a medium with 4-velocity V:

$$2\,\mathcal{T}^{\mathrm{EM}} = -i_a F \otimes i^a G - i_a G \otimes i^a F - \star(F \wedge \star G) g + \widetilde{V} \otimes s + s \otimes \widetilde{V},$$

where $i_a \equiv i_{X_a}$, $i^a \equiv g^{ab}\, i_b$ in any vector basis $\{X_a\}$ and

$$s = \star\left(\frac{1}{c_0}\boldsymbol{e}^V \wedge \boldsymbol{h}^V \wedge \widetilde{V} - c_0\,\boldsymbol{d}^V \wedge \boldsymbol{b}^V \wedge \widetilde{V}\right),$$

where $\quad \boldsymbol{e}^V = i_V F, \quad c_0\,\boldsymbol{b}^V = i_V \star F, \quad \boldsymbol{d}^V = i_V G \quad$ and $\quad \dfrac{\boldsymbol{h}^V}{c_0} = i_V \star G,$

are fields defined relative to the motion of the medium, so that

$$F = \boldsymbol{e}^V \wedge \widetilde{V} - \star\left(c_0\,\boldsymbol{b}^V \wedge \widetilde{V}\right),$$
$$G = \boldsymbol{d}^V \wedge \widetilde{V} - \star\left(\frac{\boldsymbol{h}^V}{c_0} \wedge \widetilde{V}\right)$$

with

$$G = Z(F).$$

For any Killing field K the drive form associated with Abraham's electromagnetic stress-energy-momentum tensor is

$$\tau_K^{\mathrm{EM}} = \frac{1}{2}\left(F \wedge i_K \star G - i_K G \wedge \star F + s(K) \star \widetilde{V} + \widetilde{V}(K) \star s\right). \tag{12.1}$$

It follows from (6.3), (8.3) and (8.5) that

$$\begin{aligned} J_K^U \equiv \sigma_K^U = {} & \frac{1}{2}\left(\boldsymbol{e}(K)\#\boldsymbol{d} + \boldsymbol{d}(K)\#\boldsymbol{e} + \boldsymbol{h}(K)\#\boldsymbol{b} + \boldsymbol{b}(K)\#\boldsymbol{h}\right) \\ & -\frac{1}{2}\#\left(\boldsymbol{e} \wedge \#\boldsymbol{d} + \boldsymbol{b} \wedge \#\boldsymbol{h}\right)\#\widetilde{K} + \frac{1}{2}\widetilde{U}(K)\left(\frac{1}{c_0}\boldsymbol{e} \wedge \boldsymbol{h} + c_0\,\boldsymbol{d} \wedge \boldsymbol{b}\right) \\ & +\frac{1}{2} i_U\left(\widetilde{K} \wedge i_V \star s\right) - \widetilde{V}(K) i_U \star s \end{aligned} \tag{12.2}$$

and

$$\begin{aligned} \rho_K^U = {} & -\frac{1}{2}\widetilde{U}(K)\left(\boldsymbol{b} \wedge \#\boldsymbol{h} + \boldsymbol{e} \wedge \#\boldsymbol{d}\right) + \frac{1}{2}\left(\frac{1}{c_0}\boldsymbol{e} \wedge \boldsymbol{h} + c_0\,\boldsymbol{d} \wedge \boldsymbol{b}\right) \wedge \widetilde{K}^{\perp} \\ & -\frac{1}{2} i_U\left(\widetilde{K} \wedge \widetilde{U} \wedge i_V \star s\right) + \widetilde{V}(K)\, i_U(\star s \wedge \widetilde{U}), \end{aligned} \tag{12.3}$$

where $K^{\perp} \equiv K + \widetilde{U}(K) U$.

By contrast, Minkowski (1908) introduced the *non-symmetric* electromagnetic stress-energy-momentum tensor $\mathcal{T}^{\mathrm{EM}}$ where

$$\mathcal{T}^{\mathrm{EM}} = -i_a F \otimes i^a G - \frac{1}{2}\star(F \wedge \star G) g, \tag{12.4}$$

which exhibits no explicit dependence on the medium 4-velocity V. The corresponding drive form is

$$\tau_K^{\mathrm{EM}} = \frac{1}{2}\left(F \wedge i_K \star G - i_K F \wedge \star G\right)$$

and (6.3), (8.3) and (8.5) yield in this case

$$J_K^U \equiv \sigma_K^U = \boldsymbol{h}(K)\#\boldsymbol{b} + \boldsymbol{e}(K)\#\boldsymbol{d} + \frac{1}{c_0}\widetilde{U}(K)\boldsymbol{e}\wedge\boldsymbol{h} - \frac{1}{2}\#\left(\boldsymbol{e}\wedge\#\boldsymbol{d} + \boldsymbol{b}\wedge\#\boldsymbol{h}\right)\#\widetilde{K} \tag{12.5}$$

and

$$\rho_K^U = c_0\,\boldsymbol{d}\wedge\boldsymbol{b}\wedge\widetilde{K}^{\perp} - \frac{1}{2}\widetilde{U}(K)\left(\boldsymbol{e}\wedge\#\boldsymbol{d} + \boldsymbol{b}\wedge\#\boldsymbol{h}\right). \tag{12.6}$$

More recently other choices for an electromagnetic stress-energy-momentum tensor have been proposed which in themselves simply imply different constitutive relations [16] with respect to a particular *total* stress-energy-momentum tensor. In [17,18], it has been argued that different choices of the electromagnetic stress-energy-momentum tensor for linear polarisable media are equivalent to different choices of Z *and* a different partition of the *total* stress-energy-momentum tensor for the computation of so-called pondermotive forces that arise from the divergence of terms in its decomposition. Furthermore, it was shown how particular choices of the dependence of Z on the gravitational interaction led, via a covariant variational formulation, to either the Abraham tensor or a symmetrized version of that proposed by Minkowski.

In the following, we illustrate how the general theory of drive forms outlined above offers a natural tool to discuss the computation of particular electromagnetic forces for materials that exhibit magnetoelectric properties (at rest) in the laboratory, for a particular choice of electromagnetic drive form. This is an essential step in any program that attempts to confront experimental measurements of such forces with theoretical prediction.

To facilitate this calculation, an *electromagnetic* drive form associated with the tensor obtained by symmetrizing (12.4) will be chosen:

$${\tau_K}^{\mathrm{EM}} = \frac{1}{2}(F\wedge i_K\star G - i_K G\wedge\star F). \tag{12.7}$$

It follows from (6.3), (8.3) and (8.5) that with this drive-form

$$J_K^U \equiv \sigma_K^U = \frac{1}{2}\left(\boldsymbol{e}(K)\#\boldsymbol{d} + \boldsymbol{d}(K)\#\boldsymbol{e} + \boldsymbol{h}(K)\#\boldsymbol{b} + \boldsymbol{b}(K)\#\boldsymbol{h}\right) - \frac{1}{2}\#\left(\boldsymbol{e}\wedge\#\boldsymbol{d} + \boldsymbol{b}\wedge\#\boldsymbol{h}\right)\#\widetilde{K} + \frac{1}{2}\widetilde{U}(K)\left(\frac{1}{c_0}\boldsymbol{e}\wedge\boldsymbol{h} + c_0\,\boldsymbol{d}\wedge\boldsymbol{b}\right) \tag{12.8}$$

and

$$\rho_K^U = \frac{1}{2}\left(\frac{1}{c_0}\boldsymbol{e}\wedge\boldsymbol{h} + c_0\,\boldsymbol{d}\wedge\boldsymbol{b}\right)\wedge\widetilde{K}^{\perp} - \frac{1}{2}\widetilde{U}(K)\left(\boldsymbol{e}\wedge\#\boldsymbol{d} + \boldsymbol{b}\wedge\#\boldsymbol{h}\right). \tag{12.9}$$

For a medium at rest in the laboratory, $U = V = \frac{1}{c_0}\partial_t$. Furthermore, if $\widetilde{U}(K) = 0$, the 2-forms (12.2) and (12.8) coincide, so the following analysis does not discriminate between the choice of tensors (12.1) and (12.7). However, in this case the instantaneous densities (12.3) and (12.9) are different. But, for the polarised monochromatic plane waves discussed below, the time-averaged tensors based on (12.6) and (12.9) also coincide.

If the fields are all differentiable in the medium described by (12.7), one readily obtains

$$\begin{aligned} d\,\tau_K^{\mathrm{EM}} &= \frac{1}{2}\left(i_K dG \wedge \star F + i_K G \wedge d \star F - F \wedge i_K d \star G\right) \\ &= F \wedge \star i_K d\left(\frac{\Pi}{2}\right) - \left(F + \left(\frac{\Pi}{2\epsilon_0}\right)\right) \wedge i_K\, j + G \wedge i_K\, d \star \left(\frac{\Pi}{2\epsilon_0}\right), \end{aligned}$$

where

$$d\,F = 0, \qquad d \star G = j, \qquad G = \epsilon_0 F + \Pi, \qquad \epsilon_0\, d \star F = j - d \star \Pi.$$

Thus, non-zero bulk integrated static electromagnetic forces from such fields require $d\Pi \neq 0$, $d \star \Pi \neq 0$ (magnetisation or electrical polarisation inhomogeneities) or $j \neq 0$ (non-zero local source current or charge density). For a neutral homogeneous material therefore, we consider a medium whose electromagnetic properties change discontinuously at some interface.

13. – The magnetoelectric slab

In terms of the rank 3 identity tensor Id in space, consider an infinitely extended slab(5) of magnetoelectric material with

$$\check{\boldsymbol{\zeta}}^{de}_{\boldsymbol{k},\omega} = \epsilon_{\boldsymbol{k},\omega}\; Id, \tag{13.1}$$

$$\check{\boldsymbol{\zeta}}^{hb}_{\boldsymbol{k},\omega} = \mu^{-1}_{\boldsymbol{k},\omega}\; Id. \tag{13.2}$$

The slab has width L and parallel interfaces (with the vacuum) at $x = 0$ and $x = L$. It is oriented in the laboratory frame $\{\partial_x, \partial_y, \partial_z\}$, so that $\check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}$ takes the particular form

$$\check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega} = \beta_{1,\boldsymbol{k},\omega}\; dz \otimes \partial_y + \beta_{2,\boldsymbol{k},\omega}\; dy \otimes \partial_z, \tag{13.3}$$

In this frame, the modes associated with the branch (13.6) of the dispersion relation below will be polarised in the direction ∂_z and those associated with the branch (13.7) will be polarised in the direction ∂_y. The matrix representing $\check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}$ takes the form

$$\left[\check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}\right] \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_{1,\boldsymbol{k},\omega} \\ 0 & \beta_{2,\boldsymbol{k},\omega} & 0 \end{pmatrix}. \tag{13.4}$$

(5) Such a medium has been considered by Hehl and Obukhov in their classical analysis of the Feigel effect [19].

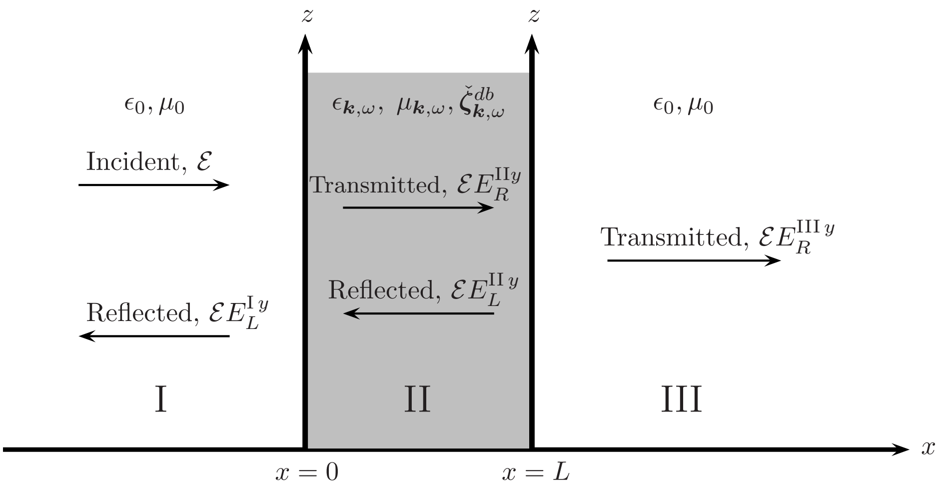

Fig. 2. – Geometry of the magnetoelectric slab and the electric field amplitudes in the three regions.

It follows that

$$\check{\boldsymbol{\zeta}}^{he}_{\boldsymbol{k},\omega} = -\beta_{2,\boldsymbol{k},\omega}\ dz \otimes \partial_y - \beta_{1,\boldsymbol{k},\omega}\ dy \otimes \partial_z. \tag{13.5}$$

With this choice of orientation of the slab, the spatial region $0 < x < L$ will be denoted II and the region with $x > L$ denoted III (see fig. 2). The electromagnetic fields induced in its interior by a plane monochromatic wave normally incident from the right (in region I, $x < 0$) propagating in the direction ∂_x with polarisation in the direction ∂_y can now be readily determined.

From (11.8), the dispersion relation associated with one polarised eigen-mode of $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}$ is

$$\epsilon_{\boldsymbol{k},\omega}\mu_{\boldsymbol{k},\omega}\omega^2 - k^2 - 2\beta_{1,\boldsymbol{k},\omega}k\omega = 0, \tag{13.6}$$

while that associated with the other polarised eigen-mode of $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}$ is

$$\epsilon_{\boldsymbol{k},\omega}\mu_{\boldsymbol{k},\omega}\omega^2 - k^2 + 2\beta_{2,\boldsymbol{k},\omega}k\omega = 0. \tag{13.7}$$

Each relation can describe propagating modes with angular frequency $\omega > 0$ moving in a direction determined by $\mathrm{sign}(k)\,\partial_x$ with phase speed $|\omega/k|$. Since this ratio depends on the values of $\beta_{1,\boldsymbol{k},\omega}$ or $\beta_{2,\boldsymbol{k},\omega}$, it may exceed the speed of light *in vacuo*. In principle, such modes can contribute to the synthesis of wave packets. However, in the following, we restrict to monochromatic incident waves and work with constitutive parameters that inhibit super-luminal waves, with real constants $\epsilon_{\boldsymbol{k},\omega} \equiv \epsilon > 0$, $\mu_{\boldsymbol{k},\omega} \equiv \mu > 0$, $\beta_{1,\boldsymbol{k},\omega} \equiv \beta_1$, $\beta_{2,\boldsymbol{k},\omega} \equiv \beta_2$. For an incident wave with complex amplitude $\mathcal{E}$, no loss of generality arises by taking $\omega > 0$ and writing the solution $\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}$:

$$\check{\boldsymbol{e}}^{\mathrm{I}\,y}_{\boldsymbol{k},\omega} = \mathcal{E}\left(\exp\left[ik_R^{\mathrm{I}y}x - i\omega\,t\right]\ dy + E_L^{\mathrm{I}\,y}\ \exp\left[ik_L^{\mathrm{I}y}x - i\omega\,t\right]\ dy\right), \tag{13.8}$$

$$\check{\boldsymbol{e}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega} = \mathcal{E}\left(E^{\mathrm{II}\,y}_R \exp\left[ik^{\mathrm{II}y}_R x - i\omega\, t\right]\, dy + E^{\mathrm{II}\,y}_L \exp\left[ik^{\mathrm{II}y}_L x - i\omega\, t\right]\, dy\right), \tag{13.9}$$

$$\check{\boldsymbol{e}}^{\mathrm{III}\,y}_{\boldsymbol{k},\omega} = \mathcal{E} E^{\mathrm{III}\,y}_R \exp\left[ik^{\mathrm{III}y}_R x - i\omega\, t\right]\, dy \tag{13.10}$$

where $k^{\mathrm{II}y}_R$ denotes a real root of the dispersion relation (13.7) associated with the polarisation eigenvector ∂_y with $\mathrm{sign}(k^{\mathrm{II}y}_R) > 0$, describing a polarised right-moving wave in the slab (region II). Similarly, $k^{\mathrm{II}y}_L$ denotes a real root of the dispersion relation associated with the polarisation eigenvector ∂_y with $\mathrm{sign}(k^{\mathrm{II}y}_L) < 0$, describing a polarised left-moving wave in the slab (region II). In general, these wave numbers are different. In the vacuum regions, $k^{\mathrm{I}y}_R = -k^{\mathrm{I}y}_L = k^{\mathrm{III}y}_R = \omega/c_0$.

If Ω^*_0 (Ω^*_L) denotes the pull-back of forms to the interface $x = 0$ ($x = L$), the interface boundary conditions [20] are

$$\Omega^*_0\left(\check{\boldsymbol{e}}^{\mathrm{I}\,y}_{\boldsymbol{k},\omega} - \check{\boldsymbol{e}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega}\right) = \Omega^*_L\left(\check{\boldsymbol{e}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega} - \check{\boldsymbol{e}}^{\mathrm{III}\,y}_{\boldsymbol{k},\omega}\right) = 0, \tag{13.11}$$

$$\Omega^*_0\left(\check{\boldsymbol{h}}^{\mathrm{I}\,y}_{\boldsymbol{k},\omega} - \check{\boldsymbol{h}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega}\right) = \Omega^*_L\left(\check{\boldsymbol{h}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega} - \check{\boldsymbol{h}}^{\mathrm{III}\,y}_{\boldsymbol{k},\omega}\right) = 0, \tag{13.12}$$

$$\Omega^*_0\left(\check{\boldsymbol{B}}^{\mathrm{I}\,y}_{\boldsymbol{k},\omega} - \check{\boldsymbol{B}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega}\right) = \Omega^*_L\left(\check{\boldsymbol{B}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega} - \check{\boldsymbol{B}}^{\mathrm{III}\,y}_{\boldsymbol{k},\omega}\right) = 0, \tag{13.13}$$

$$\Omega^*_0\left(\check{\boldsymbol{D}}^{\mathrm{I}\,y}_{\boldsymbol{k},\omega} - \check{\boldsymbol{D}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega}\right) = \Omega^*_L\left(\check{\boldsymbol{D}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega} - \check{\boldsymbol{D}}^{\mathrm{III}\,y}_{\boldsymbol{k},\omega}\right) = 0, \tag{13.14}$$

yielding the linear system for the dimensionless complex amplitudes $E^{\mathrm{I}\,y}_L$, $E^{\mathrm{II}\,y}_R$, $E^{\mathrm{II}\,y}_L$, $E^{\mathrm{III}\,y}_R$:

$$\begin{aligned}
1 + E^{\mathrm{I}\,y}_L &= E^{\mathrm{II}\,y}_R + E^{\mathrm{II}\,y}_L,\\
\left(\frac{k^{\mathrm{II}y}_R}{\mu\omega} - \beta_2\right) E^{\mathrm{II}\,y}_R + \left(\frac{k^{\mathrm{II}y}_L}{\mu\omega} - \beta_2\right) E^{\mathrm{II}\,y}_L &= \frac{1}{\mu_0\omega}\left(k^{\mathrm{I}y}_R + E^{\mathrm{I}\,y}_L k^{\mathrm{I}y}_L\right),\\
E^{\mathrm{II}\,y}_R \exp\left[ik^{\mathrm{II}y}_R L\right] + E^{\mathrm{II}\,y}_L \exp\left[ik^{\mathrm{II}y}_L L\right] &= E^{\mathrm{III}\,y}_R \exp\left[ik^{\mathrm{III}}_R L\right],\\
\left(\frac{k^{\mathrm{II}y}_R}{\mu\omega} - \beta_2\right) E^{\mathrm{II}\,y}_R \exp\left[ik^{\mathrm{II}y}_R L\right] + \left(\frac{k^{\mathrm{II}y}_L}{\mu\omega} - \beta_2\right) E^{\mathrm{II}\,y}_L \exp\left[ik^{\mathrm{II}y}_L L\right] &= \frac{k^{\mathrm{III}y}_R E^{\mathrm{III}\,y}_R}{\mu_0\omega} \exp\left[ik^{\mathrm{III}y}_R L\right].
\end{aligned}$$

This system of equations has the solution

$$\begin{aligned}
E^{\mathrm{I}\,y}_L &= \frac{\Gamma^y_-}{\Gamma^y_+},\\
E^{\mathrm{II}\,y}_L &= \frac{\mu\left(k^{\mathrm{I}y}_L - k^{\mathrm{I}y}_R\right)\left(\mu\mu_0\beta_2\omega + k^{\mathrm{III}}_R\mu - \mu_0 k^{\mathrm{II}y}_R\right)\exp\left[ik^{\mathrm{II}y}_R L\right]}{\Gamma^y_+},\\
E^{\mathrm{II}\,y}_R &= \frac{\mu\left(k^{\mathrm{I}y}_R - k^{\mathrm{I}y}_L\right)\left(\mu\mu_0\beta_2\omega + k^{\mathrm{III}}_R\mu - \mu_0 k^{\mathrm{II}y}_L\right)\exp\left[ik^{\mathrm{II}y}_L L\right]}{\Gamma^y_+},\\
E^{\mathrm{III}\,y}_R &= \frac{\mu\mu_0\left(k^{\mathrm{I}y}_R - k^{\mathrm{I}y}_L\right)\left(k^{\mathrm{II}y}_R - k^{\mathrm{II}y}_L\right)\exp\left[i\left(k^{\mathrm{II}y}_R + k^{\mathrm{II}y}_L - k^{\mathrm{III}}_R\right)L\right]}{\Gamma^y_+},
\end{aligned} \tag{13.15}$$

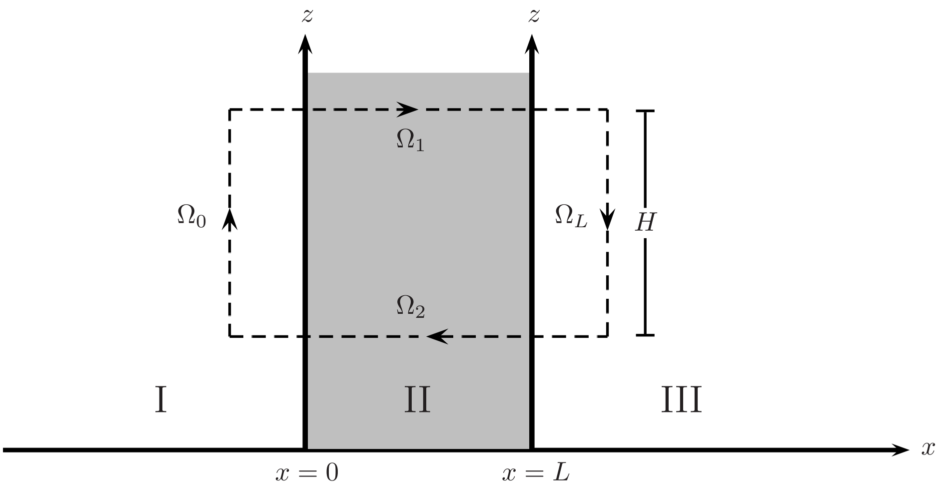

Fig. 3. – Geometry of the 2-chain Ω used to calculate the time-averaged integrated pressure on the magnetoelectric slab.

where it is convenient to introduce

$$\begin{aligned}\Gamma^y_\pm = &\left(\exp\left[ik^{\mathrm{II}y}_R L\right] - \exp\left[ik^{\mathrm{II}y}_L L\right]\right)\Big[k^{\mathrm{I}y}_\pm \mu^2\left(\mu_0\beta_2\omega + k^{\mathrm{III}y}_R\right)\\ &\pm\mu\mu_0^2\beta_2\omega\left(\beta_2\mu\omega - k^{\mathrm{II}y}_R - k^{\mathrm{II}y}_L\right) + \mu_0\left(\beta_2\omega\mu^2 k^{\mathrm{III}y}_R + \mu_0 k^{\mathrm{II}y}_R k^{\mathrm{II}y}_L\right)\Big]\\ &\pm\left(k^{\mathrm{II}y}_R \exp\left[ik^{\mathrm{II}y}_L L\right] - k^{\mathrm{II}y}_L \exp\left[ik^{\mathrm{II}y}_R L\right]\right)\mu\mu_0 k^{\mathrm{III}y}_R\\ &+\left(k^{\mathrm{II}y}_L \exp\left[ik^{\mathrm{II}y}_L L\right] - k^{\mathrm{II}y}_L \exp\left[ik^{\mathrm{II}y}_L L\right]\right)\mu\mu_0 k^{\mathrm{I}y}_\pm\end{aligned}$$

with

$$k^{\mathrm{I}y}_+ = k^{\mathrm{I}y}_R \qquad \text{and} \qquad k^{\mathrm{I}y}_- = k^{\mathrm{I}y}_L.$$

With the electric field amplitudes determined, the complete set of polarised fields $\{\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}, \check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}, \check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}, \check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}\}$ in each region is determined. For completeness, these fields are given in the Appendix.

14. – Average pressure on the magnetoelectric slab

To calculate the average pressure on the sides of the magnetoelectric slab, one integrates the Maxwell-Cauchy stress 2-form over the 2-chain (surface) $\Omega = \Omega_0+\Omega_1+\Omega_L+\Omega_2$ indicated schematically in fig. 3. The image of Ω is the boundary of a box of height H, width W and length L, with faces Ω_0 and Ω_L in regions I and III respectively, parallel to the surfaces of the slab. Integrating over a box with faces wholly within II would give zero total force, since region II is homogeneous. Since the fields are independent of z, contributions to the integral from the oriented chains Ω_1 and Ω_2 cancel.

The above fields yield a net pressure on II that fluctuates with time, with a non-zero average. If $A(\boldsymbol{r},t)$ is a scalar field, its average over any time interval T is

$$\langle A\rangle(\boldsymbol{r}) \equiv \frac{1}{T}\int_0^T A(\boldsymbol{r},t)dt.$$

Hence, if $B(\boldsymbol{r},t)$ is another scalar field,

$$\langle AB\rangle(\boldsymbol{r}) \equiv \frac{1}{T}\int_0^T A(\boldsymbol{r},t)B(\boldsymbol{r},t)dt.$$

Furthermore, if

$$\begin{aligned}\boldsymbol{A} &= \operatorname{Re}\left(\mathcal{A}(\boldsymbol{r})\exp[-i\omega t]\right) \in \mathcal{S}\Lambda^p M,\\ \boldsymbol{B} &= \operatorname{Re}\left(\mathcal{B}(\boldsymbol{r})\exp[-i\omega t]\right) \in \mathcal{S}\Lambda^q M,\end{aligned}$$

where $\mathcal{A}$, $\mathcal{B}$ are complex, then

$$\boldsymbol{A}\wedge\boldsymbol{B} = \frac{1}{2}\operatorname{Re}\left(\mathcal{A}\wedge\mathcal{B}\exp[-2i\omega t]\right) + \frac{1}{2}\operatorname{Re}(\mathcal{A}\wedge\overline{\mathcal{B}}),$$

so

$$\langle \boldsymbol{A}\wedge\boldsymbol{B}\rangle(\boldsymbol{r}) = \frac{1}{2}\operatorname{Re}(\mathcal{A}\wedge\overline{\mathcal{B}}), \tag{14.1}$$

if we take $T=\frac{2\pi}{\omega}$. Thus, (12.8) gives

$$\begin{aligned}\langle\sigma_K^U\rangle(\boldsymbol{r}) = {}& \frac{1}{4}\operatorname{Re}\left(\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}(K)\#\overline{\check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}(K)\#\overline{\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}(K)\#\overline{\check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}(K)\#\overline{\check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}}\right)\\ &-\frac{1}{4}\#\operatorname{Re}\left(\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}\wedge\#\overline{\check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}\wedge\#\overline{\check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}}\right)\#\widetilde{K}\\ &+\frac{1}{4}\widetilde{U}(K)\operatorname{Re}\left(\frac{1}{c_0}\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}\wedge\overline{\check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}} + c_0\,\check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}\wedge\overline{\check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}}\right).\end{aligned}$$

Furthermore, with $U=\frac{1}{c_0}\partial_t$ and $K=\partial_x$, the x-component of the time-averaged Maxwell-Cauchy stress 2-form is

$$\begin{aligned}\langle\sigma_{\partial_x}\rangle(\boldsymbol{r}) = {}& \frac{1}{4}\operatorname{Re}\left(\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}(\partial_x)\#\overline{\check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}(\partial_x)\#\overline{\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}(\partial_x)\#\overline{\check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}(\partial_x)\#\overline{\check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}}\right)\\ &-\frac{1}{4}\#\operatorname{Re}\left(\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}\wedge\#\overline{\check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}\wedge\#\overline{\check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}}\right)\#dx,\end{aligned}$$

which reduces to

$$\langle\sigma_{\partial_x}\rangle(\boldsymbol{r}) = -\frac{1}{4}\#\operatorname{Re}\left(\check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}\wedge\#\overline{\check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega}} + \check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}\wedge\#\overline{\check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega}}\right)dy\wedge dz. \tag{14.2}$$

The time-averaged integrated force is given by[6]

$$\left\langle f_{\partial_x}^{\rm NET}[\Omega]\right\rangle = \int_{\Omega_0}\left\langle \sigma_{\partial_x}^{\rm I}\right\rangle - \int_{\Omega_L}\left\langle \sigma_{\partial_x}^{\rm III}\right\rangle .$$

Denote the time-averaged stress forms due to the fields in regions I and III by

$$\left\langle \sigma_{\partial_x}^{\rm I}\right\rangle = \alpha^{\rm I}\, dy\wedge dz, \qquad \alpha^{\rm I} = -\frac{1}{4}\#\,{\rm Re}\left(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}^{{\rm I}\,y}\wedge\#\overline{\check{\boldsymbol{d}}_{\boldsymbol{k},\omega}^{{\rm I}\,y}} + \check{\boldsymbol{b}}_{\boldsymbol{k},\omega}^{{\rm I}\,y}\wedge\#\overline{\check{\boldsymbol{h}}_{\boldsymbol{k},\omega}^{{\rm I}\,y}}\right)$$

$$\left\langle \sigma_{\partial_x}^{\rm III}\right\rangle = \alpha^{\rm III}\, dy\wedge dz, \qquad \alpha^{\rm III} = -\frac{1}{4}\#\,{\rm Re}\left(\check{\boldsymbol{e}}_{\boldsymbol{k},\omega}^{{\rm III}\,y}\wedge\#\overline{\check{\boldsymbol{d}}_{\boldsymbol{k},\omega}^{{\rm III}\,y}} + \check{\boldsymbol{b}}_{\boldsymbol{k},\omega}^{{\rm III}\,y}\wedge\#\overline{\check{\boldsymbol{h}}_{\boldsymbol{k},\omega}^{{\rm III}\,y}}\right)$$

Thus, the time-averaged net force on the magnetoelectric medium contained in the region bounded by Ω is

$$\left\langle f_{\partial_x}^{\rm NET}[\Omega]\right\rangle = \left(\Omega_0^*\alpha^{\rm I} - \Omega_L^*\alpha^{\rm III}\right)\int_0^W\int_0^H dydz = \left(\Omega_0^*\alpha^{\rm I} - \Omega_L^*\alpha^{\rm III}\right)A,$$

where $A = WH$ and the time-average integrated pressure $\langle p_x[\Omega]\rangle \equiv \frac{\langle f_{\partial_x}^{\rm NET}[\Omega]\rangle}{A}$. Calculating the pull-backs of

$$\begin{aligned}\alpha^{\rm I} = &-\frac{|\mathcal{E}|^2}{4}\Bigg[\epsilon_0\left(1 + 2\,{\rm Re}\left(E_L^{{\rm I}\,y}\exp\left[i\left[k_R^{{\rm I}y}-k_L^{{\rm I}y}\right]x\right]\right) + \left|E_L^{{\rm I}\,y}\right|^2\right)\\ &+\frac{1}{\mu_0\omega^2}\left(\left(k_R^{{\rm I}y}\right)^2 + 2k_R^{{\rm I}y}k_L^{{\rm I}y}\ {\rm Re}\left(E_L^{{\rm I}\,y}\exp\left[i\left[k_R^{{\rm I}y}-k_L^{{\rm I}y}\right]x\right]\right) + (k_L^{{\rm I}y})^2\left|E_L^{{\rm I}\,y}\right|^2\right)\Bigg],\\ \alpha^{\rm III} = &-\frac{\left|\mathcal{E}E_R^{{\rm III}\,y}\right|^2}{4}\left(\epsilon_0 + \frac{\left(k_R^{{\rm III}y}\right)^2}{\mu_0\omega^2}\right)\end{aligned}$$

yields

$$\Omega_0^*\alpha^{\rm I} = -\frac{\epsilon_0|\mathcal{E}|^2}{2}\left(1+\left|E_L^{{\rm I}\,y}\right|^2\right), \qquad \Omega_L^*\alpha^{\rm III} = -\frac{\epsilon_0\left|\mathcal{E}E_R^{{\rm III}\,y}\right|^2}{2},$$

since $k_R^{{\rm I}y} = -k_L^{{\rm I}y} = k_R^{{\rm III}y} = \frac{\omega}{c_0}$. Since the time-averaged body force $\langle\mathcal{L}_U\rho_K^U\rangle = 0$ for ρ_K^U given by (12.3), (12.6) and (12.9), it follows that the average pressure on the magnetoelectric slab is given in terms of the solution (13.15) by

$$\langle p_x[\Omega]\rangle = \frac{\epsilon_0|\mathcal{E}|^2}{2}\left(\left|E_R^{{\rm III}\,y}\right|^2 - \left|E_L^{{\rm I}\,y}\right|^2 - 1\right). \tag{14.3}$$

---

(6) The minus sign occurs due to the opposite orientation of the opposite faces of Ω.

15. – Conclusion

The magnitude and sign of $\langle p_x[\Omega]\rangle$ depends on $\epsilon \equiv \epsilon_r\epsilon_0$, $\mu \equiv \mu_r\mu_0$, β_1 and β_2, where $c_0 = \frac{1}{\sqrt{\epsilon_0\mu_0}}$. As noted above, the wave numbers $k_L^{\mathrm{II}y}$, $k_R^{\mathrm{II}y}$ that follow from the dispersion relation determine the nature of the propagating wave in region II. For the case under discussion here, where the parameters ϵ_r, μ_r, β_1, β_2 are constant, it is of interest to write the dispersion relations in terms of the dimensionless ratio of the wave speeds $w \equiv \frac{v}{v_0}$, where $v = \frac{\omega}{k}$, $v_0 = \frac{1}{\sqrt{\epsilon\mu}}$ and the dimensionless parameters $b_1 \equiv -\beta_1/\sqrt{\epsilon\mu}$, $b_2 \equiv \beta_2/\sqrt{\epsilon\mu}$:

$$\begin{aligned} w^2 + 2b_1 w - 1 = 0,\\ w^2 + 2b_2 w - 1 = 0. \end{aligned}$$

Then the sub-luminal condition $|\frac{v}{c_0}| < 1$ implies $|w| < \sqrt{\epsilon_r\mu_r}$. The relation between w and either b_1 and b_2 can then be seen from the relation of the two branches of the loci where the expression $w^2 + 2bw - 1$ vanishes in the (w, b)-plane. For $\omega > 0$, values of w in the upper (lower) half plane correspond to left (right) moving waves. Furthermore, propagating sub-luminal monochromatic waves will only occur in II for real b, yielding real values of w in the range $-\sqrt{\epsilon_r\mu_r} < w < \sqrt{\epsilon_r\mu_r}$. It is clear from these considerations that the relative sign between β_1 and β_2 can have a significant effect on the behavior of the propagating modes in the region II and hence on the nature of the force on the magnetoelectric slab.

The authors feel that the approach adopted in this paper for the calculation of static, time-averaged and instantaneous forces, offers a conceptually unambiguous method of considerable generality. Once one decides on the drive form appropriate for any subsystem in interaction with external fields, it has immediate application to moving media (in arbitrary relativistic or non-relativistic motion) and can be extended to matter with material losses. Work is in progress to extend the methodology to inhomogeneous media with more general constitutive properties and this will be reported elsewhere.

* * *

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Appendix

Electromagnetic fields in the three regions

For a y-polarised harmonic electromagnetic wave with angular frequency $\omega > 0$, incident normally from the left on a fixed magnetoelectric slab, the electric field solutions in the three regions are given by (13.8)–(13.10). For completeness, the remaining fields in these three regions are given here. The magnetic induction fields follow from (11.2):

$$\begin{aligned} \check{\boldsymbol{b}}^{\mathrm{I}\,y}_{\boldsymbol{k},\omega} &= \frac{\mathcal{E}\,k_R^{\mathrm{I}y}}{\omega}\exp\left[ik_R^{\mathrm{I}y}x - i\omega\,t\right]\,dz + \frac{\mathcal{E}\,k_L^{\mathrm{I}y}E_L^{\mathrm{I}\,y}}{\omega}\exp\left[ik_L^{\mathrm{I}y}x - i\omega\,t\right]\,dz,\\ \check{\boldsymbol{b}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega} &= \frac{\mathcal{E}\,k_R^{\mathrm{II}y}E_R^{\mathrm{II}\,y}}{\omega}\exp\left[ik_R^{\mathrm{II}y}x - i\omega\,t\right]\,dz + \frac{\mathcal{E}\,k_L^{\mathrm{II}y}E_L^{\mathrm{II}\,y}}{\omega}\exp\left[ik_L^{\mathrm{II}y}x - i\omega\,t\right]\,dz,\\ \check{\boldsymbol{b}}^{\mathrm{III}\,y}_{\boldsymbol{k},\omega} &= \frac{\mathcal{E}\,k_R^{\mathrm{III}y}E_R^{\mathrm{III}\,y}}{\omega}\exp\left[ik_R^{\mathrm{III}y}x - i\omega\,t\right]\,dz. \end{aligned}$$

The electric displacement 1-forms in regions I and III are given by the vacuum constitutive relation $\check{\boldsymbol{d}}^y_{\boldsymbol{k},\omega} = \epsilon_0 \check{\boldsymbol{e}}^y_{\boldsymbol{k},\omega}$, whereas the electric displacement 1-form in region II is given by the constitutive relation (11.4), with the spatial tensors $\check{\boldsymbol{\zeta}}^{de}_{\boldsymbol{k},\omega}$ and $\check{\boldsymbol{\zeta}}^{db}_{\boldsymbol{k},\omega}$ given by (13.1) and (13.3), respectively:

$$\begin{aligned}
\check{\boldsymbol{d}}^{\mathrm{I}\,y}_{\boldsymbol{k},\omega} &= \mathcal{E}\,\epsilon_0 \exp\left[ik^{\mathrm{I}y}_R x - i\omega\, t\right]\, dy + \mathcal{E}\,\epsilon_0 E^{\mathrm{I}\,y}_L \exp\left[ik^{\mathrm{I}y}_L x - i\omega\, t\right]\, dy,\\
\check{\boldsymbol{d}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega} &= \mathcal{E}\left(\epsilon + \frac{\beta_2 k^{\mathrm{II}y}_R}{\omega}\right) E^{\mathrm{II}\,y}_R \exp\left[ik^{\mathrm{II}y}_R x - i\omega\, t\right]\, dy\\
&\quad + \mathcal{E}\left(\epsilon + \frac{\beta_2 k^{\mathrm{II}y}_L}{\omega}\right) E^{\mathrm{II}\,y}_L \exp\left[ik^{\mathrm{II}y}_L x - i\omega\, t\right]\, dy\\
\check{\boldsymbol{d}}^{\mathrm{III}\,y}_{\boldsymbol{k},\omega} &= \mathcal{E}\,\epsilon_0 E^{\mathrm{III}\,y}_R \exp\left[ik^{\mathrm{III}y}_R x - i\omega\, t\right]\, dy.
\end{aligned}$$

Similarly, the magnetic 1-forms in the regions I and III are given by the vacuum constitutive relation $\check{\boldsymbol{h}}^y_{\boldsymbol{k},\omega} = \mu_0^{-1} \check{\boldsymbol{b}}^y_{\boldsymbol{k},\omega}$, whereas in region II, the magnetoelectric constitutive relation (11.5), with the spatial tensors $\check{\boldsymbol{\zeta}}^{hb}_{\boldsymbol{k},\omega}$ and $\check{\boldsymbol{\zeta}}^{he}_{\boldsymbol{k},\omega}$ given by (13.2) and (13.5) respectively yield

$$\begin{aligned}
\check{\boldsymbol{h}}^{\mathrm{I}\,y}_{\boldsymbol{k},\omega} &= \frac{\mathcal{E}\, k^{\mathrm{I}y}_R}{\mu_0\omega} \exp\left[ik^{\mathrm{I}y}_R x - i\omega\, t\right]\, dz + \frac{\mathcal{E}\, k^{\mathrm{I}y}_L E^{\mathrm{I}\,y}_L}{\mu_0\omega} \exp\left[ik^{\mathrm{I}y}_L x - i\omega\, t\right]\, dz,\\
\check{\boldsymbol{h}}^{\mathrm{II}\,y}_{\boldsymbol{k},\omega} &= \mathcal{E}\left(\frac{k^{\mathrm{II}y}_R}{\mu\omega} - \beta_2\right) E^{\mathrm{II}\,y}_R \exp\left[ik^{\mathrm{II}y}_R x - i\omega\, t\right]\, dz\\
&\quad + \mathcal{E}\left(\frac{k^{\mathrm{II}y}_L}{\mu\omega} - \beta_2\right) E^{\mathrm{II}\,y}_L \exp\left[ik^{\mathrm{II}y}_L x - i\omega\, t\right]\, dz,\\
\check{\boldsymbol{h}}^{\mathrm{III}\,y}_{\boldsymbol{k},\omega} &= \frac{\mathcal{E}\, k^{\mathrm{III}y}_R E^{\mathrm{III}\,y}_R}{\mu_0\omega} \exp\left[ik^{\mathrm{III}y}_R x - i\omega\, t\right]\, dz.
\end{aligned}$$

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