IL NUOVO CIMENTO Vol. 32 C, N. 1 Gennaio-Febbraio 2009
DOI 10.1393/ncc/i2009-10341-8

Colloquia: GCM8

Noncommuting variations in the Lagrangian field theory and the dissipation

S. Preston

Department of Mathematics and Statistics, Portland State University - Portland, OR, USA

(ricevuto il 3 Aprile 2009; pubblicato online il 15 Giugno 2009)

Summary. — In this work we study the geometrical structure underlying the notion of noncommuting variations of dynamical variables in the Lagrangian field theory. We introduce the class of (twisted) prolongations of variations of dynamical fields to their derivatives that includes, as the special examples, the constructions of T. Levi-Civita, U. Amaldi, B. Vyjanovich and T. Atanaskovic and, finally, those of H. Kleinert and his coauthors. Usage of this variations procedure allows one to obtain non-potential forces terms in the corresponding Euler-Lagrange equations and the source terms in the energy-momentum balance laws. Obstructions for conservation of the brackets of vector fields of variations under twisted prolongation are found. As a special class of such twisted prolongations we introduce those defined by the connections on the bundles of vertical vector fields of the configurational bundle. As an application of this procedure, we get the entropy balance in the 4-dim geometrical model of material aging as the Euler-Lagrange equation for thermacy (thermical displacement) and show that it coincides with the entropy equation obtained for the Lagrangian written in the *proper material* space-time coordinates, using conventional flow prolongation of variations of dynamical fields.

PACS `02.30.Xx` – Calculus of variations.
PACS `02.40.Vh` – Global analysis and analysis on manifolds.

1. – Introduction

Variations of dynamical fields y^i used in Lagrangian field theory as well as in other domains of variational calculus, have the property that in the case of one independent variable t (mechanics) is expressed by the relation $\delta\dot{y}^i = \dot{\delta y^i}$, *i.e.* that operation of taking *derivative of* y^i *by* t *and variation of* y^i *commute* [1]. The same property is basic in the field theory [2,3].

Yet, different authors, starting with V. Volterra, T. Levi-Civita, U. Amaldi (see references in [4] and in [1], sect. 6.1) introduced noncommuting variations for generalized velocities of a *mechanical system with nonholonomic constraints* as a way to adopt conventional Lagrangian formalism to such a system. Later on, A. Lurie and Ju. Neimark studied the noncommutativity rules, as they name them, in details, and, in particular, determined that the early controversies between the viewpoints of G. Hamel and T. Levi-Civita (see [5]) were due to the absence of correct definition of variations of the velocities *outside of real trajectories*.

In 1970th, in a series of works B. Vujanovic and T. Atanackovic have suggested other rules of commutation of time derivative and variations of unknowns y^i as a tool to present some mechanical systems with a non-potential force and the equations of the heat propagation in solids as Euler-Lagrange equations of a variational principle, see [1], Chapt. 6 and references therein.

On the other side in the works of H. Kleinert, P. Fiziev and A. Pelster on the motion of a spinless point in space-times with curvature and torsion [6-8] a new variational principle was suggested. This principle was also based on the modified rule for the commuting of time derivative and the variations of generalized coordinates. Their considerations were geometrical by their nature, employing a non-holonomic transformation of the (flat) configurational space. After such a transformation, equations of motion of a particle gain a *torsion* force defined by the (nonmetric) connection in the Cartan space-time.

Our point of view, supported by the results of geometrical (bundle) form of variational calculus, is that the conventional rules of taking variations of the derivatives of dynamical variables (fields) have important mathematical advantages (preservation of Cartan distribution, preservation of Lie bracket, see below) making them more fundamental. Yet, a more general approach allows us to include into the framework of variational calculus the physical systems that cannot be described by conventional Lagrangian formalism. In that we adopt the point of view of A. Lurie who stated [9] that in difference to the (kinematical) variations of fields y^i, variations of their derivatives $y^i_{,\mu}$ *are not kinematical, but dynamical notions* and should be dealt with as such. That, in particular, allows to introduce into the definition of variations of derivatives of dynamical fields some *geometrical factors* having a dynamical meaning, including, as shown below, the dissipation.

The goal of this work is to study the geometrical structure allowing to introduce noncommuting variations of dynamical variables into the Lagrangian formalism.

2. – Notations

Below we will use the following notations standard in the variational calculus, see [3, 10]:

1. X^n is the n-dim smooth paracompact manifold (physical or material space-time is the basic example).
2. $\pi : Y^{n+m} \to X^n$—*configurational bundle*—the fiber bundle of smooth connected paracompact manifolds, fibers of π are spanned by values of physical fields y^i, $i = 1, \dots, m$.
3. G—pseudo-Riemannian metric in X, η corresponding volume form in X.
4. $\pi^1_0 : J^1(\pi) \to Y$—1-jet bundle of the bundle π.
5. $(x^\mu, y^i; \mu \in \overline{1,n};\ i \in \overline{1,m})$—fibred chart in the bundle π.
6. (x^μ, y^i, z^i_μ)—induced chart in the 1-jet bundle space $J^1(\pi)$.
7. $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\partial_i = \frac{\partial}{\partial y^i}$, $\partial_{z^i_\mu}$—basis of vector fields defined by the fibred coordinates x^μ, y^i in the 1-jet bundle space $J^1(\pi)$.

8. $\omega^i = \mathrm{d}y^i - z^i_\mu \mathrm{d}x^\mu$—basic 1-forms of Cartan distribution in the 1-jet space $J^1(\pi)$.
9. $V(\pi) \subset T(Y)$—subbundle (over Y) of vertical tangent vectors. In fibred coordinates, $\xi \in V(\pi)$ has the form: $\xi = \sum_i \xi^i(x,y)\partial_{y^i}$.

3. – Conventional variations in Lagrangian field theory

Consider a Lagrangian field theory of the first order with a Lagrangian density $L(x,y,z)\eta$, where $L \in C^\infty(J^1(\pi))$ is a smooth function. Variation of action (here $s : D \to Y$ is a section of the bundle π over a domain $D \subset X$)

$$A_D(s) = \int_D L(j^1 s)\eta$$

along a vertical vector field $\xi = \xi^i\partial_i \in V(\pi)$, *i.e.* along variations $y^i \to y^i + \epsilon\xi^i(x,y)$ of dynamical fields, requires the prolongation of this vector field to the vector field in $J^1(\pi)$. Standard prolongation $\xi \to \xi^1$ is the *flow prolongation* of the form [11,3]

$$\xi^1 = \xi^i\partial_i + d_\mu\xi^i\partial_{z^i_\mu}. \tag{3.1}$$

Here $d_\mu f = \partial_\mu f + z^i_\mu\partial_i f$ is the total derivative of the function f by x^μ. Flow prolongation (3.1) has the following two basic properties:

1. ξ^1 is the Lie vector field in $J^1(\pi)$—flow of vector field ξ^1 preserves the *Cartan distribution* Ca in $J^1(\pi)$ generated by the 1-forms $\omega^i = \mathrm{d}y^i - z^i_\mu \mathrm{d}x^\mu$.
2. Mapping $\xi \to \xi^1$ is the *Lie algebra* monomorphism $\mathcal{V}(\pi) \to \mathcal{V}(\pi_1)$, in other words for all π-vertical vector fields ξ, χ,

$$[\xi,\chi]^1 = [\xi^1,\chi^1]. \tag{3.2}$$

Variations (3.1) have the property that in the variational calculus with one-dimensional base (mechanics) is expressed by the relation $\delta\dot{y}^i = \dot{\overline{\delta y^i}}$, *i.e.* that *derivative by* t *and variation of* y^i *commute* [1]. The same property is basic in the multidimensional variational theory (field theory) [2,3]. In the geometrical form this rule expressed by the formula (3.1) requires that *the variation of derivatives (i.e. of variables* z^i_μ*) are (total) derivatives of variations* ξ^i *of the fields* y^i.

It is this rule that has been challenged in the works cited in Introduction. So, in order to present the methods of B. Vujanovic, H. Kleinert and their coauthors in the geometrical form we have to study natural modifications of the lifting procedure (3.1) of the variational vector fields as well as the lifting of the vector fields on the base X to $J^1(\pi)$. The last thing is imperative for the study of the symmetries of Euler-Lagrange equations and corresponding Noether *balance laws*.

Unfortunately, modifying the rule of variations of jet variables z^i_μ, one has to sacrifice some properties of these variations that are taken for granted in the variational calculus.

As an example let us show that the prolongation $\xi \to \xi^1$ is the only linear prolongation of π-vertical vector fields $\xi \in V(\pi)$ in Y to the π^1-vertical vector fields in $J^1(\pi)$ ($\pi^1 : J^1(\pi) \to X$ is the bundle over X) preserving the Cartan distribution Ca.

An arbitrary *linear* prolongation $\mathcal{V}(\pi) \to \mathcal{V}(\pi^1)$ has the form

$$\widehat{\xi} = \xi + (d_\mu \xi^i + K^i_{\mu j}\xi^j)\partial_{z^i_\mu} = \xi + (d^K_\mu \xi^i)\partial_{z^i_\mu} \tag{3.3}$$

with some functions $K^i_{\mu j} \in C^\infty(Z)$. Here we introduced notation $d^K_\mu \xi^i = d_\mu \xi^i + K^i_{\mu j}\xi^j$, the geometrical meaning of which will be explained later, see sect. **7**.

Phase flow of vector field $\widehat{\xi}$ preserves Cartan distribution if and only if

$$\mathcal{L}_{\widehat{\xi}}\omega^i = \lambda^i_k \omega^k, \quad i \in \overline{1,m}$$

for some smooth functions λ^i_k, see [12], Chapt. III. Since the flow prolongation $\xi \to \xi^1$ satisfies to this property and since this equality is linear by ξ, to prove the previous relation it is sufficient (and necessary) to show that

$$\mathcal{L}_\zeta \omega^i = \lambda^i_k \omega^k, \quad i \in \overline{1,m}, \quad \zeta = K^i_{\mu j}\xi^j \partial_{z^i_\mu}$$

with some (other) functions λ^i_k depending on ξ.

Using Cartan formula and the fact that $i_{\partial_{z^i_\mu}}\omega^i = 0$ we get this condition in the form

$$K^i_{\mu j}\xi^j \mathrm{d}x^\mu = \lambda^i_k(\xi)\omega^k = \lambda^i_k(\xi)(\mathrm{d}y^k - z^k_\mu \mathrm{d}x^\mu), \quad i \in \overline{1,m},$$

which can be fulfilled for arbitrary ξ only when $\lambda^i_k(\xi) = 0$, *i.e.* when $K^i_{\mu j}\xi^j = 0$.

4. – Noncommuting variations

In this section we introduce the modified rule of variations of jet variables, corresponding to the modified lift of the vertical vector fields $\xi \in V(\pi)$ to the vertical vector field in the bundle $\pi^1 : J^1(\pi) \to X$:

$$\xi \to \tilde{\xi} = \xi + (d_\mu \xi^i + K^i_{\mu j}\xi^j)\partial_{z^i_\mu}, \tag{4.1}$$

where $K^i_{\mu j} \in C^\infty(J^1(\pi))$ are some functions in the 1-jet space.

Form the variation of the action $A_D(s)$ using this lift of variational vector field ξ (assuming that is zero on the boundary ∂D of domain D). We will have infinitesimal variations of fields and their derivatives (notice that independent variables are not variated)

$$\begin{cases} y^i \to y^i + \epsilon \xi^i(x,y), \\ z^i_\mu \to z^i_\mu + \epsilon(d_\mu \xi^i + K^i_{\mu j}\xi^j). \end{cases}$$

Here ϵ is a small quantity. Variation of action $A_D(s)$ at a section $s : D \to Y$ has the form

$$\Delta A_D(s)(\epsilon\xi) = A_D(s_\epsilon) - A_D(s) = \tag{4.2}$$
$$= \int_D [L(x, s^i(x) + \epsilon\xi^i(x, s(x)), s^i_{,\mu}(x) + \epsilon(d_\mu\xi^i(x, s(x))$$
$$+K^i_{\mu j}(x, j^1 s(x))\xi^j(x, s(x))) - L(x, s(x), s^i_{,\mu}(x))]\eta =$$
$$= \epsilon \int_D \left[\frac{\partial L}{\partial y^j}\xi^j(x, s(x)) + \frac{\partial L}{\partial z^i_\mu}(d_\mu\xi^i(x, s(x)) + K^i_{\mu j}(x, j^1 s(x))\xi^j(x, s(x))\right]\eta + O(\epsilon^2) =$$
$$= \epsilon \int_D \left[\frac{\partial L}{\partial y^j} - d_\mu\left(\frac{\partial L}{\partial z^j_\mu}\right) + K^i_{\mu j}(x, j^1 s(x))\frac{\partial L}{\partial z^i_\mu}\right]\xi^j(x, s(x))\eta + O(\epsilon^2).$$

As a result the system of Euler-Lagrange equations takes the form

$$\frac{\partial L}{\partial y^j} - d_\mu\left(\frac{\partial L}{\partial z^j_\mu}\right) = K^i_{\mu j}(x, j^1 s(x))\frac{\partial L}{\partial z^i_\mu} = K^i_{\mu j}\pi^\mu_i, \quad j = 1, \dots, m. \tag{4.3}$$

Right side in this equation represents the (generically nonpotential) forces $Q_j = -K^i_{\mu j}\pi^\mu_i$ acting on the system. Here $\pi^\mu_i = L_{,z^i_\mu}$ are the *momenta* conjugated to the jet variables z^i_μ.

4˙1. *Examples—special cases.* – Following special cases of the construction of noncommuting variations appeared in the works of authors cited in the Introduction.

4˙1.1. $K \in C^\infty(x, y)$. In the works of Dj. Djukich and B. Vujanovic ([1], Chapt. 6) examples of construction of nonstandard variations with the coefficients $K^i_{\mu j} \in \pi^*_{10}C^\infty(Y)$ were suggested. Thus, coefficients $K^i_{\mu j}(x, y)$ depend on variables (x^μ, y^i).

4˙1.2. $K^i_{\mu j}$ are linear by jet variables z^i_μ. In the papers by H. Kleinert, P. Fiziev and A. Pelster [6, 7], as well as in those of Dj. Djukich and B. Vujanovic and T. Atanaskovich, this construction was used in the case where $n = 1$ and

$$K^i_{\mu j} = L^{i\nu}_{\mu jk}(x, y)z^k_\nu$$

are linear functions of jet variables. In the notations of papers of H. Kleinert and his coworkers, $K^i_{tj} = 2S^i_{kj}z^k_t$ where S^i_{kj} is the torsion tensor of an affine connection in the Cartan space-time.

4˙2. *Energy-momentum balance law.* – Here we write down the energy-momentum balance law corresponding to the first-order Lagrangian $L \in C^\infty(J^1(\pi))$.

Relation between the variational symmetry vector fields (defined by a vector field ξ and the conservation laws of the associated EL-system (Noether conservation laws) presented, for instance, in [13], Chapt. IV,VI, can be trivially extended to the relation between the arbitrary vector fields and the balance laws that are fulfilled for solutions of EL equations. Let $\hat{\xi}$ be a vector field in Y obtained by the lifting of the vector field $\xi = \xi_\sigma = \partial_{x^\sigma}$ in X by the flat connection in the bundle $\pi : Y \to X$. The conservation law corresponding to the vector field $\hat{\xi}$ (*energy-momentum balance law*) has here the form of *balance equation*

$$d_\mu\left(z^i_\sigma L_{,z^i_\mu} - L\delta^\mu_\sigma\right) = -\frac{\partial L}{\partial x^\sigma}_{\text{expl}} + z^i_\sigma K^j_{\mu i}(x, j^1 s(x))\frac{\partial L}{\partial z^j_\mu}, \quad \sigma = 0, \dots, 3. \tag{4.4}$$

In particular, for $\sigma = 0$ we get the energy balance law

$$(4.5)\quad d_\mu \left(z^i_0 L_{,z^i_\mu} - L\delta^\mu_0 \right) = -\frac{\partial L}{\partial x^0}_{\text{expl}} + z^i_0 K^j_{\mu i}(x, j^1 s(x)) \frac{\partial L}{\partial z^j_\mu} = -\frac{\partial L}{\partial x^0}_{\text{expl}} + \dot{y}^i K^j_{\mu i} \pi^\mu_j .$$

Notice the linear dependence of the second force term on the velocities and on the momenta.

5. – Examples

Example 1. Harmonic oscillator

Here $n = 1$, t-time, $x = x(t)$-coordinate, $L = \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2$. Vertical variations $\xi = \xi^1(t,x)\partial_x$. Its general linear lift to 1-jet space $\tilde{\xi} = \xi^1\partial_x + [d_t\xi^1 + K(t,x,\dot{x})\xi^1]\partial_{\dot{x}}$.

Euler-Lagrange equation is the standard linear oscillator with variable dissipation:

$$(5.1)\qquad L_{,x} - d_t(L_{,\dot{x}}) = KL_{,\dot{x}} \Leftrightarrow \ddot{x} + K\dot{x} + \omega^2 x = 0.$$

Example 2. Maxwell equation

Here $\xi \to M^4$ is the complex line bundle over the pseudo-Riemannian manifold (M, g). $A = A_\mu \mathrm{d}x^\mu$ is a connection form in the bundle ξ, $F = \mathrm{d}A = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$—corresponding curvature form. $*$ is the star operator corresponding to the metric g, $d_g v$—corresponding volume form. Action for the connection A is

$$(5.2)\qquad \mathcal{A}_{D^4}(A) = \int_D \|F\|^2_g \, d_g v = \int_D F \wedge *F.$$

Maxwell equation in empty space is $d * F = 0$.

Let $A_\sigma \to A_\sigma + \epsilon\xi^\sigma$ be a variation of A corresponding to the vertical vector field $\xi = \xi^\sigma \partial_{A_\sigma}$. Let $K^\nu_{\mu\sigma}$ be a tensor field defining the twisted variation of jet variables. Variation of the derivatives $z^\sigma_\mu = A_{\sigma,\mu}$ corresponding to the above variation of A_σ is $z^\sigma_\mu \to z^\sigma_\mu + (d_\mu\xi^\sigma + K^\sigma_{\lambda\mu}\xi^\lambda)$. Since only antisymmetric combinations of derivatives $A_{\sigma,\mu}$ enter Lagrangian, we may assume that tensor K is antisymmetric by corresponding variables.

Then, Euler-Lagrange equations for action (5.2), with the variation rule (4.1) have the form

$$(5.3)\qquad d * F = -K * F \Leftrightarrow, \text{ in components } d_\alpha(*F)^{\alpha\beta} = -K^\beta_{\gamma\delta}(*F)^{\gamma\delta}.$$

The energy-momentum balance law takes the form

$$(5.4)\qquad d_\mu(2F^\mu_\sigma - \|F\|^2_g \, \delta^\mu_\sigma) = K^{\nu\lambda}_\mu F_{\nu\sigma}(*F)^\mu_\lambda$$

with the right side being quadratic by F.

Example 3

Let $n = 4$, $\mu = 0, 1, 2, 3$. Take $K^i_{\mu j} = k\delta^0_\mu \delta^i_{i_0} \delta^{j_0}_j$. Then, the lift of a vertical variation $\xi = \xi^i\partial_i$ has the form

$$\tilde{\xi} = \xi + (d_\mu\xi^i + k\delta^0_\mu \delta^i_{i_0} \xi^{j_0})\partial_{z^i_\mu} = \xi^1 + k\xi^{j_0}\partial_{z^{i_0}_0}.$$

As a result the only noncommuting operation here is the time derivative ∂_t of dynamical field y^{i_0} and the variation of this dynamical field. *Noncommutativity* term is proportional to the variation of y^{j_0}: $\delta\partial_t y^{i_0} - \partial_t \delta y^{i_0} = k\delta y^{j_0}$.

Euler-Lagrange equations have the form

$$L_{,y^j} - d_\mu(L_{,z^j_\mu}) = k\delta^{j_0}_j L_{,z^{i_0}_0} = k\delta^{j_0}_j \pi^0_{i_0}. \tag{5.5}$$

Thus, the force appears only in the j_0-th equation and is proportional to $\pi^0_{i_0}$. The simplest case is, of course, when $i_0 = j_0$ and the force appeared in the equation for y^{i_0} is proportional to π_{i_0}.

6. – Obstruction to Lie algebra morphism

Here we will study the obstruction for the lifting mapping $\xi \to \tilde{\xi} : V(\pi) \to V(\pi_1)$ of the form (4.1) to be Lie algebra morphism. Calculating the difference (see [14] for details)

$$[\widetilde{\xi\partial_i}, \widetilde{\eta\partial_j}] - [\widetilde{\xi\partial_i, \eta\partial_j}] = II - I$$

we get

$$\begin{aligned} II - I &= \eta\xi(\partial_i K^k_{\mu j} - \partial_j K^k_{\mu i})\partial_{z^k_\mu} + (\eta d_\nu \xi \partial_{z^i_\nu} K^k_{\mu j}) - \xi d_\nu \eta \partial_{z^j_\nu} K^k_{\mu i})\partial_{z^k_\mu} + \\ &\quad + (\xi\eta_{,y^l} K^l_{\mu i}\delta^k_j - \eta\xi_{,y^l} K^l_{\mu j}\delta^k_i)\partial_{z^k_\mu} + \xi\eta(K^l_{\nu i}\partial_{z^l_\nu} K^k_{\mu j} - K^l_{\nu j}(\partial_{z^l_\nu} K^k_{\mu i}))\partial_{z^k_\mu} = \\ &= \eta\xi[(\partial_i K^k_{\mu j} + K^l_{\nu i}\partial_{z^l_\nu} K^k_{\mu j}) - (\partial_j K^k_{\mu i} + K^l_{\nu j}(\partial_{z^l_\nu} K^k_{\mu i})]\partial_{z^k_\mu} + \\ &\quad + \xi(\eta_{,y^l} K^l_{\mu i}\delta^k_j - d_\nu \eta \partial_{z^j_\nu} K^k_{\mu i})\partial_{z^k_\mu} - \eta(\xi_{,y^l} K^l_{\mu j}\delta^k_i - d_\nu \xi \partial_{z^i_\nu} K^k_{\mu j}))\partial_{z^k_\mu}. \end{aligned} \tag{6.1}$$

Introduce the tensor

$$\tilde{R}^k_{i\mu j} = (\partial_i K^k_{\mu j} - \partial_j K^k_{\mu i}) + (K^l_{\nu i}\partial_{z^l_\nu} K^k_{\mu j} - K^l_{\nu j}\partial_{z^l_\nu} K^k_{\mu i}) = 0, \quad \forall i, j, k, \mu \tag{6.2}$$

and the antisymmetric bracket-like operation (K-*bracket*) on the vector fields

$$\lceil \xi, \eta \rceil_K = [\xi^i \eta^k_{,y^l} K^l_{\mu i} - \eta^j \xi^k_{,y^l} K^l_{\mu j}] - [\xi^i d_\nu \eta^j \partial_{z^j_\nu} K^k_{\mu i} - \eta^j d_\nu \xi^i \partial_{z^i_\nu} K^k_{\mu j}]. \tag{6.3}$$

Let $\xi = \xi^i \partial_{,i}$, $\eta = \eta^j \partial_{,j}$ be arbitrary vertical vector fields. Apply the previous formula for each couple $\xi^i \partial_i$, $\eta^j \partial_j$ and take there sum. As a result, we get

$$[\widetilde{\xi = \xi^i}\partial_i, \widetilde{\eta = \eta^j}\partial_j] - \widetilde{[\xi, \eta]} = \{\tilde{R}^k_{i\mu j}\xi^i\eta^j + \lceil \xi, \eta \rceil_K\}\partial_{z^k_\mu}. \tag{6.4}$$

As a result, we get the following

Proposition 1. 1) *Lifting $\xi \to \tilde{\xi}$ preserves brackets for basic vertical vector fields ∂_i, i.e. $[\tilde{\partial}_i, \tilde{\partial}_j] = 0$ if and only if the "curvature"*

$$\tilde{R}^k_{i\mu j} = (\partial_i K^k_{\mu j} - \partial_j K^k_{\mu i}) + (K^l_{\nu i}\partial_{z^l_\nu} K^k_{\mu j} - K^l_{\nu j}\partial_{z^l_\nu} K^k_{\mu i}), \quad \forall i, j, k, \mu, \tag{6.5}$$

vanishes.

2) *If* $\tilde{R}^k_{i\mu j} \equiv 0$, *then for two arbitrary vertical vector fields* $\xi = \xi^i\partial_{,i}$, $\eta = \eta^j\partial_{,j}$ *the difference*

$$[\tilde{\xi}, \tilde{\eta}] - \widetilde{[\xi, \eta]} = 0 \tag{6.6}$$

vanishes if and only if the K*-bracket* $\lceil \xi, \eta \rceil_K$ vanishes.

7. – Vertical connection and the covariant flow prolongation

Construction of sect. **4** represents a modification of the flow lift of vertical vector fields on Y and it is interesting to see if we can describe this modification in terms of some natural geometrical construction on the configurational bundle $\pi : Y \to X$.

Consider a linear connection L on the vertical subbundle $V(\pi) \to Y$ of the tangent bundle $T(Y)$ of manifold Y. It defines covariant derivatives of *vertical vector fields*—sections of subbundle $V(\pi) \to Y$: for a vertical vector field $\xi = \xi^i\partial_i$,

$$\begin{cases} D^L_\mu \xi^i = \partial_\mu \xi^i + L^i_{\mu j}\xi^j, \\ D^L_k \xi^i = \partial_k \xi^i + L^i_{kj}\xi^j. \end{cases} \tag{7.1}$$

with the connection coefficients $L^i_{\mu j}, L^i_{kj} \in C^\infty(Y)$.

Using L-covariant derivatives D^L of vertical vector fields instead the partial derivatives we get the *covariant total derivative of a vector field* by x^μ:

$$d^L_\mu \xi^i = (\partial_\mu \xi^i + L^i_{\mu j}\xi^j) + z^j_\mu(\partial_j \xi^i + L^i_{jk}\xi^k) = d_\mu \xi^i + K^i_{\mu j}\xi^j. \tag{7.2}$$

In the notations introduced in sect. **4**,

$$K^i_{\mu k} = L^i_{\mu k} + z^j_\mu L^i_{jk}. \tag{7.3}$$

This allows to introduce the *covariant flow lift of a vertical vector field* $\xi = \xi^i(x, y)\partial_i$:

$$\xi^1_L = \xi + (d^L_\mu \xi^i)\partial_{z^i_\mu} = \xi + [d_\mu \xi^i + (L^i_{\mu k} + z^j_\mu L^i_{jk})\xi^k]\partial_{z^i_\mu} = \xi^1 + (K^i_{\mu k}\xi^k)\partial_{z^i_\mu}. \tag{7.4}$$

This *covariant flow lift* of variations of dynamical fields generalizes constructions of taking variations of derivatives of nonholonomic mechanics, those by H. Kleinert and by D. Vyjanovich.

Calculating curvature of the connection L and comparing it with the "curvature" $\tilde{R}(L)$ defined by (6.2) for the prolongation procedure $\xi \to \xi^1_L$ we get the following

Proposition 2. *If connection* L *is flat, then* $\tilde{R}(L) = 0$.

Remark 1. *Considering the pullback* π^*_V *of the vertical bundle* $\pi_V : V(\pi) \to Y$ *to the 1-jet bundle space* $J^1(\pi)$

$$\begin{array}{ccc} \pi^*_{10}V(\pi) & \longrightarrow & V(\pi) \\ {\scriptstyle \pi^*_V}\Big\downarrow & & {\scriptstyle \pi_V}\Big\downarrow \\ J^1(\pi) & \xrightarrow{\pi_{10}} & Y \end{array} \tag{7.5}$$

and the connections on the bundle π^*_V *we may in the same way construct prolongation procedure of variations to the 1-jet variables* (4.1) *with an* arbitrary dependence of coefficients K on x^μ, y^i, z^i_μ.

More detailed results concerning this construction and its comparison with the nonholonomic transformation of H. Kleinert will be presented elsewhere [14].

8. – Application: Entropy balance, material time and the noncommuting variations

In this section we present an example of a Lagrangian theory with non-commuting variations. This example represents a part of the work of A. Chudnovsky *et al.* on the geometrical modeling of aging phenomena in solids. See [15, 16] for basic construction of the following example.

Material body is modeled by the 3-dim manifold B with the boundary ∂B and (local) coordinates X^A, $A = 1, 2, 3$. Manifold B is the base of the bundle $\pi : P \to B$, where P^4 is the *4-dim material space-time* with coordinates T, X^A, $A = 1, 2, 3$. P is endowed with the Riemannian (material) metric G.

History of deformation ϕ represents the diffeomorphic embedding of P^4 $\phi : P \to \mathbf{E}^4$ into the classical (Newtonian) space-time ($\mathbf{E}^4, t, x^i$, $i = 1, 2, 3$) (endowed with the 4-dim Euclidean metric and the Newtonian slicing by the surfaces of fixed physical time, see [17]) such that $\frac{\partial \phi^0}{\partial T} > 0$.

In the ADM representation metric G has the form

$$G = g_{AB}(\mathrm{d}X^A + N^A \mathrm{d}\phi^0) \circ (\mathrm{d}X^B + N^B \mathrm{d}\phi^0) + S\mathrm{d}\phi^{0\ 2}, \tag{8.1}$$

where ϕ^0 is the zeroth component of history of deformation ϕ. Here $\mathbf{N}$ is the shift vector field along the slices $\phi(T, X) = t$ and S is the lapse function, see [15].

Proper time τ—natural parameter along the material world lines, corresponding to the metric G in P—is defined by the relation

$$\mathrm{d}\tau = S\mathrm{d}\phi^0 = S(\phi^0_{,0}\mathrm{d}T + \phi^0_{,X^A}\mathrm{d}X^A).$$

To compare proper material picture that is obtained in proper coordinates ($\tau, X^A, A = 1, 2, 3$) with one in physical material coordinates ($t, X^A, A = 1, 2, 3$) we will use the relation $\mathrm{d}\tau = S(t, X^A)\mathrm{d}t$. We will be using relations between different time derivatives of a function in P^4. For an arbitrary scalar function $f(T, X)$ we have these relations in the form

$$f_{,\tau} = S^{-1} f_{,t} = S^{-1} \phi^{0\ -1}_{,0} f_{,T}. \tag{8.2}$$

Next we will introduce the new scalar field γ called the *thermacy* (named this way by D. Van Dantzig [18], although its introduction goes back to H. von Helmholtz, see [19], sect. 33 or [20]). Thermacy is the measure of "action" of the kinetic energy of micromolecular motion related to it by the relation

$$\gamma \sim k^{-1}\langle \|\mathbf{v}\|^2 \rangle,$$

where $\mathbf{v}$ is the random velocity of micromolecular motion—stochastic component of the velocity field of continuum. Here k is the Boltzmann constant. As in [19,20], *we identify* $\dot{\gamma}$ *with the absolute temperature* ϑ.

Thus, we take a model Lagrangian as the function of the following variables $L = L(S, \dot{\gamma}, \mathbf{N} \cdot \nabla^g \gamma = N^A \gamma_{,X^A})$. In the overall scheme [15] it should also depend on the Cauchy metric induced by the deformation ϕ from the physical 3-dim metric h in E^4, on 3-dim material metric g (including, possibly, its curvature), etc.

We assume that the Lagrangian depends on γ (only) through its derivatives $\dot{\gamma}$ and $\mathbf{N}\gamma = N^A \gamma_{,X^A}$, the last being the direction derivative of γ in the direction of the shift vector field $\mathbf{N}$.

As the vertical connection tensor K we take the tensor depending on the metric component S and ϕ^0 (on the entropy production, as we will see below)

$$K^j_{\mu i} = -S_{,\tau} \delta^j_0 \delta^0_\mu \delta^0_i. \tag{8.3}$$

In this way, *only variations and time derivative do not commute in the sense of* (4.1).

Euler-Lagrange equations (4.3) for the thermacy γ have the form

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial \gamma_{,t}} \right) + \partial_{X^A} \left(N^A \frac{\partial \mathcal{L}}{\partial (N \cdot \gamma)} \right) = -K \mathcal{L}_{,\gamma_{,t}} = \frac{S_{,t}}{S} \mathcal{L}_{,\gamma_{,t}} = S_{,\tau} \mathcal{L}_{,\gamma_{,t}}. \tag{8.4}$$

Introduce now the following identifications (see [20]):

1. $s = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}}$—entropy density,
2. $S^A = N^A \frac{\partial \mathcal{L}}{\partial (N \cdot \gamma)}$—entropy flux,
3. $\sigma = S_{,\tau} L_{,\gamma_{,t}} = S_{,\tau} s$—entropy production.

Then the last equation takes the standard form of the entropy balance

$$\partial_t s + \partial_{X^A} S^A = S_{,\tau} \mathcal{L}_{,\gamma_{,t}} = \sigma. \tag{8.5}$$

In this form the II law of thermodynamics takes the form of the following condition:

$$S_{,\tau} \mathcal{L}_{,\gamma_{,t}} \geqq 0. \tag{8.6}$$

Remark 2. Identifying $S_{,\tau} s = S^{-1} S_{,t} s = \sigma$ *we get* $\ln(S) = \ln(S(0)) + \int_0^t \frac{\sigma}{s} \mathrm{d}t$ *or*

$$S(t) = S(0) e^{\int_0^t \frac{\sigma}{s} \mathrm{d}t}. \tag{8.7}$$

Comparing this with the expression for the proper time $\mathrm{d}\tau = S \mathrm{d}\phi^0$ *and normalizing* $S(t)$ *by the condition* $S(0) = 1$ *we get the basic relation between the relative entropy production* σ/s *and the rate of change of the proper material time*

$$\mathrm{d}\tau = \left(e^{\int_0^t \frac{\sigma}{s} \mathrm{d}u} \right) \mathrm{d}t. \tag{8.8}$$

When $\phi^0(T, X) = T$ *(internal time* T *is synchronized with the physical time* t*)* $\mathrm{d}\tau = S \mathrm{d}t$ *and we get the formula for the* proper (material) time scale

$$S(t) = e^{\int_0^t \frac{\sigma}{s} \mathrm{d}t} S(0). \tag{8.9}$$

MR-principle and the internal Lagrangian picture. One can look at eq. (8.4) from a different point of view. Write down the action principle and the Euler-Lagrange system for the Lagrangian $\tilde{L}(\gamma_{,\tau}, N \cdot \gamma)$ *in the proper material time* τ *but using conventional commutativity rule*:

$$\frac{\delta \tilde{L}}{\delta \gamma} = 0 \Rightarrow \partial_\tau \left(\frac{\partial \tilde{L}}{\partial \gamma_{,\tau}} \right) + \partial_{X^A} \left(N^A \frac{\partial \tilde{L}}{\partial (N \cdot \gamma)} \right) = 0. \tag{8.10}$$

Rewrite this EL system in coordinates (t, X^A), using: $\frac{\partial \tilde{L}}{\partial \gamma_{,\tau}} = S \frac{\partial \tilde{L}}{\partial \gamma_{,t}}$, $\partial_\tau = \frac{\partial t}{\partial \tau} \partial_t = S^{-1} \partial_t$:

$$\begin{aligned} \frac{\delta \tilde{L}}{\delta \gamma} &= S^{-1} \left(S \frac{\partial \tilde{L}}{\partial \gamma_{,t}} \right)_{,t} + \left(N^A \frac{\partial \tilde{L}}{\partial (N \cdot \gamma)} \right)_{,X^A} = 0 \Leftrightarrow \\ \Leftrightarrow \left(\frac{\partial \tilde{L}}{\partial \gamma_{,t}} \right)_{,t} &+ \left(N^A \frac{\partial \tilde{L}}{\partial (N \cdot \gamma)} \right)_{,X^A} = \frac{S_{,t}}{S} \frac{\partial \tilde{L}}{\partial \gamma_{,t}} = S_{,\tau} \tilde{L}_{,\dot{\gamma}} = \sigma. \end{aligned} \tag{8.11}$$

In this form the Euler-Lagrange system for the action written in the proper time and obtained by using conventional variational rule coincides with the system (8.4) obtained from the action written in coordinates (t, X^A) *but using the nonstandard commutating rule*. This result supports the following methodological principle for modeling of material behavior suggested by Prof. A. Chudnovsky (in numerous private communications with author):

"Principle of material relativity": In the intrinsic material space-time frame of reference, the material appears to be perfectly elastic with no signs of aging such as changes in density, stiffness, creep, plasticity etc. It agrees well with the Hindu philosophy position that "there is no time for the internal being". However, the existence of time and aging is manifested in the evolution of the material space-time matrix that is readily detectable by an external observer.

8$\cdot$1. *Energy balance law.* – We consider γ, N^A, S as the dynamical fields in this model. Yet, only γ enters our *model* Lagrangian with its time derivative. To construct the energy balance law (4.5) we calculate the energy density.

$$\epsilon = \dot{\gamma} \frac{\partial \mathcal{L}}{\partial \dot{\gamma}} - \mathcal{L} = \vartheta s - \mathcal{L},$$

and the energy flux (Pointing vector field)

$$U^A = \dot{\gamma} \frac{\partial L}{\partial \gamma_{,A}} = \dot{\gamma} \frac{\partial L}{\partial N \cdot \gamma} N^A = \vartheta S^A.$$

For the second term on the right side of (4.5) we are using expression (8.3) for coefficients K and get

$$z^i_0 K^j_{\mu i} L_{z^j_\mu} = -\dot{\gamma}(S_{,\tau} \delta^0_\mu \delta^j_0) L_{,z^j_0} = -\dot{\gamma} S_{,\tau} s = -\vartheta \sigma.$$

The energy balance (4.5) now takes the form

$$\partial_t(\vartheta s - \mathcal{L}) + \partial_{X^A}(\vartheta S^A) = -\frac{\partial L}{\partial t}_{\text{expl}} - \vartheta\sigma, \tag{8.12}$$

Terms in the RHS represent the energy dissipation processes.

8˙2. *Heat propagation law.* – To get the heat propagation law we modify the Lagrangian to include terms depending on the field $\mathbf{N}$:

$$L = L\left(\dot{\gamma}, \mathbf{N}\cdot\gamma, \frac{1}{2}\|\mathbf{N}\|_g^2\right).$$

This will not change the form of EL equation for γ (entropy balance).

Euler-Lagrange equation for the vector field $\mathbf{N}$ has the form

$$\frac{\partial L}{\partial \mathbf{N}\cdot\gamma}\cdot\nabla_A\gamma + \frac{\partial L}{\partial\|\mathbf{N}\|_g^2}\cdot N_A = 0 \Rightarrow N_A = -\frac{\frac{\partial L}{\partial N\cdot\gamma}}{\frac{\partial L}{\partial\|\mathbf{N}\|_g^2}}\nabla\gamma. \tag{8.13}$$

From this and the identifications above it follows that the entropy flux is proportional to the gradient of γ:

$$S^A = N^A\frac{\partial\mathcal{L}}{\partial(\mathbf{N}\cdot\gamma)} = -\frac{(\frac{\partial L}{\partial\mathbf{N}\cdot\gamma})^2}{\frac{\partial L}{\partial\|\mathbf{N}\|_g^2}}\nabla^A\gamma. \tag{8.14}$$

If we insist on the classical relation between the entropy and heat flux: $\mathbf{q} = \vartheta\mathbf{S}$, then we get

$$\mathbf{q} = -\vartheta\frac{(\frac{\partial L}{\partial\mathbf{N}\cdot\gamma})^2}{\frac{\partial L}{\partial\|\mathbf{N}\|_g^2}}\nabla\gamma. \tag{8.15}$$

Introduce notation $\mathcal{A} = \frac{(\frac{\partial L}{\partial\mathbf{N}\cdot\gamma})^2}{\frac{\partial L}{\partial\|\mathbf{N}\|_g^2}}$. Take time derivative in the last equation and then use it once more to exclude $\nabla\gamma$. With the identifications

$$\tau^{-1} = (\ln(\theta\mathcal{A}))^{\cdot}, \quad \kappa = \tau(\theta A), \tag{8.16}$$

last equation takes the form of Cattaneo constitutive law

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -\kappa\nabla\theta. \tag{8.17}$$

9. – Conclusion

We introduced the construction of (twisted) prolongations of variations of dynamical variables to the jet variables that generalizes constructions of jet variations introduced by T. Levi-Civita, U. Amaldi, B. Vyjanovich and T. Atanaskovic and, finally, those of H. Kleinert and his coauthors.

This construction allows to obtain non-potential forces terms in the corresponding Euler-Lagrange equations and the source terms in the energy-momentum balance law. We demonstrate that the twisted prolongations can be considered as the covariant flow prolongations defined by the connections on the bundles of vertical vector fields of the configurational bundle. Twisted prolongations lack the properties of conservation of the Lie bracket of variational vector fields and the obstructions for such a conservation are related to the curvature of these connections.

As an application of this procedure, we get the entropy balance in the 4-dim geometrical model of material aging as the Euler Lagrange equation for thermacy (thermical displacement) and show that it coincides with the entropy equation obtained for the Lagrangian written in the *proper material* space-time coordinates, using conventional flow prolongation of variations of dynamical fields.

* * *

I am greatly thankful to Prof. A. Chudnovsky for fruitful discussions concerning the relation between the pictures of material behavior in internal material and laboratory (physical) frames of reference. I also would like to thank Prof. P. Podio-Guidulgi making available the text of his talk at the conference in Erlangen-Nurenberg that, in particular, clarified to me the history of notion of "thermacy—thermal displacement".

REFERENCES

[1] Vujanovic B. and Johns S., *Variational Methods in Nonconservative Phenomena* (Academic Press, Boston) 1989.
[2] Gelfand I. and Fomin S., *Variational Calculus* (Prentice-Hall) 1963.
[3] Binz E., S'niatycki S. and Fisher H., *Geometry of Classical Fields* (Amsterdam, North-Holland) 1988.
[4] Levi-Civita T. and Amaldi U., *Lezioni di Meccanica Razionale*, Vol. **II** (N. Zanichelli, Bologna) 1950.
[5] Neimark Ju. and Fufaev N., *Dynamics of Nonholonomic Systems* (AMS, Providence) 1972.
[6] Fiziev P. and Kleinert H., preprint arXiv:gr-qc/9605046 v1 (1996).
[7] Kleinert H. and Pelster A., *Gen. Relativ. Grav.*, **31** (1999) 1439.
[8] Kleinert H., *MULTIVALUED FIELDS in Condensed Matter, Electromagnetism, and Gravitation*, manuscript (2007).
[9] Lurie A., *Analytic Mechanics* (Kluwer) 1991.
[10] Giachetta G., Mangiarotti L. and Sardanashvily G., *New Lagrangian and Hamiltonian Methods in Field Theory* (World Scientific) 1997.
[11] de Leon M., Martin de Diego D. and Santamaria-Merino A., *Int. J. Geom. Methods Mod. Phys.*, **1** (2004) 651.
[12] Krasil'shchick I. and Vinogradov A. (Editors), *Symmetries and Conservative Laws for Differential Equations of Mathematical Physics* (AMS) 1999.
[13] Olver P., *Applications of Lie Groups to Differential Equations*, 2nd ed. (Springer-Verlag, New York) 1993.
[14] Preston S., in preparation.

[15] Chudnovsky A. and Preston S., *Int. J. Geom. Methods Mod. Phys.*, **3** (2006) 1.
[16] Chudnovsky A. and Preston S., *Configurational Mechanics of Materials*, edited by Maugin G. and Kienzler R. (Springer, Wien) 2001, pp. 273-306.
[17] Marsden J. and Hughes H., *Mathematical Foundations of Elasticity* (Dover) 1994.
[18] van Dantzig D., *Physica*, **6** (1939) 673.
[19] von Laue M., *Die Relativitatstheorie, Erster Band* (Brainschweig) 1921.
[20] Maugin G. and Kalpakides V., in *Variational and Extremum Principles in Macroscopic Systems* (Elsevier) 2005, pp. 187-205.