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Interaction of edge waves with swell on a beach(*)

Kwi Joo Lee([1]), I. Shugan([1])([2]), An Jung Sun([1]) and R. Purini([3])

([1]) *Department of Naval Architecture & Ocean Engineering, Chosun University Gwangju 501-759, Korea*
([2]) *Prochorov General Physics Institute of Russian Academy of Sciences - Moscow, Russia*
([3]) *ISAC, CNR - Rome, Italy*

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Summary. — Excitation of edge waves on a beach by incoming swell is considered on the basis of shallow-water model. Subharmonic resonance mechanism of interaction is analyzed by multi-scaled expansion asymptotic techniques. The generation of edge waves between wave breakers is found to have a dynamic threshold. It is defined by intensity and frequency of incoming swell, geometry of a shore zone. Nonlinear no stationary wave solutions for the envelope of interacting edge waves are described by generalized Sine-Gordon model. An infinite set of exact solutions are received by the Lamb method for the phase synchronism regime of wave's interaction.

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1. – Introduction

Edge waves are of interest in coastal oceanography because of their propagation along the beach with amplitude maximal at the shoreline. They are responsible for the formation of beach cusps, generation of rip currents and periodic circulation cells in the near shore region. Edge waves produce on the beach beautiful run-up patterns although propagation is along the straight shoreline [1].

The generation of edge waves has been intensively studied, both experimentally and theoretically, in the last forty years. Greenspan [2] first demonstrated that large-scale edge waves can be excited by atmospheric forcing due to storms moving along the coastline. The typical period of this kind of edge wave is related to the spatial extent of the storm area and is of the order of several hours. For small-scale edge waves Guza and Davis [3] proposed the nonlinear interaction mechanism of edge waves with incoming swell. Using the shallow-water approximation, they showed that a monochromatic harmonic wave train of frequency 2ω, normally incident and strongly reflected on a

(*) The authors of this paper have agreed to not receive the proofs for correction.

beach, is unstable to subharmonic standing edge-wave perturbations of frequency ω. Experiments [4] on a bounded beach indicated that this subharmonic resonance was the strongest, and a subharmonic standing edge-wave was preferentially excited. Guza and Bowen [5] made a systematic examination of the non-linear mechanism of subharmonic resonance. They used Airys shallow-water approximation as the basis of their theory. In addition to the initial instability of edge waves, the incident and reflected waves were found to leak energy by radiation due to quadratic nonlinearity. In a further development of the theory, leading-order nonlinear corrections to the linear dispersion relation of traveling Stokes edge waves were calculated in [6]. Minzoni and Whitham [7] studied the excitation of standing subharmonic edge waves by a normally incident, strongly reflected wave train. They formulated the problem in the full water-wave theory without making the shallow-water approximation and solved it for beach angles $\phi = \pi/2N$, where N is an integer. Their work confirms the results from the shallow-water theory in the small-beach-angle limit.

The large-scale temporal and spatial modulations of subharmonic edge waves excited by resonant interactions with normally incident, strongly reflected wave trains, were studied in [8]. Modulation stability of a propagating edge-wave train was re-examined and confirmed that the instability, predicted in [4], indeed leads to a series of envelope solitons.

Although many results in edge waves dynamics have been already reached, some important questions are still open. Firstly, the effect of the beach geometry on edge waves has not been analytically studied. All the previous analytical work was done on an open beach. But if the beach is bounded by two sidewalls, which is always the case in experiments, this beach geometry will affect the edge-wave dynamics, sometimes even exclude the excitation of edge waves. This effect was clearly shown in experiments [4]. Secondly, the nonlinear evolution of subharmonic edge waves on a wide beach is still not clear. Since in this situation the spatial large-scale modulation arises as well as the temporal one, the evolution equations of these modulations have been derived in [8]. But what these equations imply about the edge-wave evolution is not known.

This paper analyzes both of the above two problems. Excitation and spatial variations of edge waves along a beach due to its resonance interactions with incoming swell are studied on the basis of shallow-water model. An infinite number of analytical solutions describing slow space and temporal evolution of edge waves are found for the common system of modulation equations for resonance interaction of waves with quadratic nonlinearity.

2. – Derivation of modulation equations

We consider a straight and long beach with constant small slope. Let the mean shoreline coincide with y-axis, and let water be in the region $x > 0$. The bottom is described by

$$z = -sx, \quad x > 0, \quad s = \text{const}, \tag{2.1}$$

where z is the vertical axis, directed upward.

Since the slopes are assumed to be small due to the small inclination of the beach we can use one of the forms of shallow-water equations [5]:

$$\zeta_t + [(sx+\zeta)\Phi_x]_x + [(sx+\zeta)\Phi_y]_y = 0; \tag{2.2}$$

$$\Phi_t + \frac{1}{2}(\Phi_x^2 + \Phi_y^2) + g\zeta = 0, \tag{2.3}$$

here $\varsigma = \varsigma(x, y, t)$ is the sea surface level, $\Phi = \Phi(x, y, t)$ the velocity potential function, g the gravity acceleration.

Eliminating ς from the above equations the single non-linear equation for $\Phi(x, y, t)$ is written as shown below.

$$\begin{aligned} \Im\Phi \equiv -\Phi_{tt} + sg[(x\Phi_x)_x + x\Phi_{yy}] = \\ = 2(\Phi_x\Phi_{xt} + \Phi_y\Phi_{yt}) + \Phi_t(\Phi_{xx} + \Phi_{yy}) + \\ + \frac{1}{2}(\Phi_x^2 + \Phi_y^2)(\Phi_{xx} + \Phi_{yy}) + \Phi_x^2\Phi_{xx} + \Phi_y^2\Phi_{yy} + 2\Phi_x\Phi_y\Phi_{xy}\,. \end{aligned} \tag{2.4}$$

The linearized approximation for this equation admits two types of solutions. The first solution is the normally incident and strongly reflected waves with frequency 2ω:

$$\Phi_0 = A_0 J_0(\beta\sqrt{x})e^{-2i\omega t} + \text{c.c.}, \tag{2.5}$$

where $J_0(\beta\sqrt{x})$ is the Bessel function of zero order, which is finite at the shoreline $x = 0$, A_0 is the wave potential amplitude, $\beta = 2(4\omega^2/gs)^{1/2}$, c.c. denotes the complex conjugate.

The second solution is an edge wave mode of frequency ω, propagating along the beach. Here only the lowest mode is considered for simplicity. The expression of the edge wave is given by

$$\Phi_e = A_e e^{-kx} e^{i(ky \pm \omega t)} + \text{c.c.}, \tag{2.6}$$

corresponding for the pair of opposite edge waves:

$$\begin{aligned} \Phi_{e1} = A_{e1} e^{-kx} e^{i(ky-\omega t)}; \\ \Phi_{e2} = A_{e2} e^{-kx} e^{i(ky+\omega t)}, \end{aligned} \tag{2.7}$$

here A_{e1}, A_{e2} are the edge waves potential amplitudes, k is the wave number of edge waves. We will assume all amplitudes to be small of the order $\varepsilon = a_0 k_0$—the typical steepness for gentle waves, where a_0 is the characteristic free-surface displacement, $2\pi/k_0$ is the typical surface wavelength, multi-scaled asymptotic expansions in ε can be used for unknown functions.

To study subharmonic resonance we will consider the velocity potential Φ as the sum of standing wave (2.5) and the pair of opposite edge waves (2.7):

$$\begin{aligned} \Phi = \Phi_0 + \Phi_{e1} + \Phi_{e2} = \\ = A_0 J_0(\beta\sqrt{x})e^{-2i\omega t} + A_{e1} e^{-kx} e^{i(ky-i\omega t)} + A_{e2} e^{-kx} e^{i(ky+i\omega t)} + \text{c.c.} \end{aligned} \tag{2.8}$$

After substitution of expression (2.8) into eq. (2.4) resonant terms of the second order for each of the interacting harmonics $e^{i(ky-i\omega t)}$, $e^{i(ky+i\omega t)}$, $e^{-2i\omega t}$ will appear in the right side of the equation. The method of slowly varying amplitudes will be used to eliminate these terms and to construct the uniformly valid solution in the second order in wave steepness ε [9, 10]. So, all complex wave amplitudes are assumed to be slowly varied in space and time:

$$
\begin{aligned}
A_0 &= A_0(\varepsilon y, \varepsilon t) = A_0(Y, T),\\
A_{e1} &= A_{e1}(\varepsilon y, \varepsilon t) = A_{e1}(Y, T),\\
A_{e2} &= A_{e2}(\varepsilon y, \varepsilon t) = A_{e2}(Y, T),
\end{aligned} \tag{2.9}
$$

where $Y = \varepsilon y, T = \varepsilon t$ are slow space and time variables.

Interaction of waves takes a place in a narrow stripe of width $\sim 1/k$ along the shoreline. That is why we will use modulation equations, averaged over the x direction with the weight of the eigenfunction e^{-kx} for edge waves [5]. After integration over the x direction and collecting resonant terms for all three harmonics one can receive the following system of modulation equations for wave amplitudes in the second order:

$$
\begin{aligned}
(A_{e1})_T + C_g(A_{e1})_Y &= MA_0A_{e2};\\
(A_{e2})_T - C_g(A_{e2})_Y &= MA_0^*A_{e1};\\
(A_0)_T &= -NA_{e1}A_{e2}^*,
\end{aligned} \tag{2.10}
$$

where $C_g = \Delta\omega/\Delta k = (1/2)\sqrt{gs/k}$ is the group velocity of edge waves, $M = 4e^{-2}\left(\frac{\omega^2}{gs}\right)^2$, $N = \frac{2e^4}{3}\left(\frac{\omega^2}{gs}\right)^2$ are the coefficients of the wave energy exchange, the asterisk (*) denotes the complex conjugate. These are the final modulation equations for resonant interactions between opposite traveling edge waves of frequency ω and normal incident standing wave of frequency 2ω.

3. – Analysis of results

Homogeneous-in-space waves interaction can be described by the system of eqs. (2.10) with time derivatives in its left side. It has three no independent integrals of motion (Manly-Row) [11]:

$$
\begin{aligned}
|A_{e1}|^2 &= |A_{e2}|^2 + \text{const};\\
|A_0|^2 + |A_{e1}|^2 &= \text{const};\\
|A_0|^2 + |A_{e2}|^2 &= \text{const}.
\end{aligned} \tag{3.1}
$$

Integrals show the directions of energy exchange in this wave system: the incoming swell can give energy simultaneously to both of the resonance edge waves, and inversely the pair of interacting edge waves can radiate wave energy out of the beach. Intensity of energy exchange between resonant waves is determined by its amplitudes and the coefficients of energy exchange M and N on the right side of eqs. (3.1). It increases with amplitude, the frequency ω of incoming swell and decreasing with a slope of the sea bottom. Periodic-in-time solutions can be expressed as elliptical functions [11].

Temporal modulations of edge waves can take place on an open beach. But if the beach is bounded by two sidewalls (wave breakers), this beach geometry will affect the edge wave dynamics, sometimes even excluding the excitation of edge waves. Clearly the process of the standing edge waves generation needs the along-shore distance at least equal to the space period of exciting waves. That means that for preventing of edge

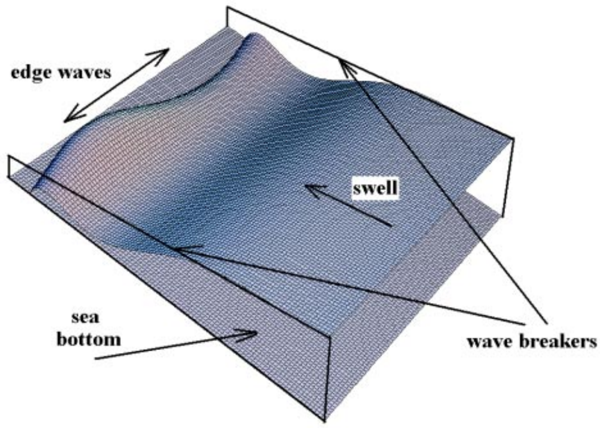

Fig. 1. – Interaction of edge waves and a swell in a shore zone.

waves generation wave breakers have to cut the shore in smaller "pieces". This effect is clearly seen in experiments [4]. The spatial large-scale modulation arises in such a case as well as the temporal one. Most intriguing here are the dynamical conditions, necessary for excitation of edge waves by incoming swell. A sketch of this process is presented in fig. 1.

To study edge wave excitation on a constrained beach zone we will consider stationary space modulations with a constant normal incident wave A_0.

From the first two equations of system (2.10) one can easily obtain the harmonic oscillation equation for the edge waves amplitudes A_{ei}:

$$(A_{ei})_{YY} + \chi^2 A_{ei} = 0, \tag{3.2}$$

where $i = 1, 2$, $\chi = A_0 \frac{8e^{-2}k^{5/2}}{\sqrt{gs}}$.

The general solution of eq. (3.2),

$$A_{ei} = C_{0i} \cos(\chi Y + C_{1i}), \tag{3.3}$$

where C_{0i}, C_{1i} are constants of integration, shows that the edge wave amplitude has slow space oscillations with wave number χ. The length of edge waves increases with the wavelength of the incoming swell and decreases with its amplitude.

Standing edge waves can be excited in between wave breakers spaced at a distance of D, if both amplitudes A_{ei} satisfy the following homogeneous non-propagating boundary conditions:

$$A_{ei}(Y = 0) = A_{ei}(Y = D) = 0. \tag{3.4}$$

To satisfy (3.4) the period of the edge wave envelope oscillation $T = 2\pi/\chi$ has to be not more than $2D$. That means the existence of dynamic space threshold or minimum necessary distance D for the possibility of edge waves excitation by the incoming swell:

$$D = \pi/\chi = \pi \frac{\sqrt{gs}}{8e^{-2}k^{5/2}A_0}\,. \tag{3.5}$$

Less intensity of the edge wave formed between the wave breakers is considered as a favorable condition. Therefore the distance between waves breakers has to be made not more than threshold D (eq. (3.5)) to prevent edge waves excitation.

4. – Nonlinear waves interaction

General dynamics of resonant energy exchange between waves includes the spatial large-scale wave modulation as well as the temporal one. Let us rewrite the common system of resonance interactions (2.10) by the following way:

$$
\begin{aligned}
&(A_{e1})_T + C_g (A_{e1})_Y = L A_0 A_{e2};\\
&(A_{e2})_T - C_g (A_{e2})_Y = L A_0^* A_{e1};\\
&(A_0)_T = -S A_{e1} A_{e2}^*,
\end{aligned}
\tag{4.1}
$$

where $L = 4e^{-2}k^2$, $S = (2/3)k^2 e^4$.

After introducing new moving variables,

$$
\xi = C_g T + Y, \quad \eta = C_g T - Y,
$$
$$
\frac{\partial}{\partial T} = C_g \frac{\partial}{\partial \xi} + C_g \frac{\partial}{\partial \eta}; \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}
$$

we can rewrite the system of eqs. (4.1):

$$
\begin{aligned}
\frac{\partial A_{e1}}{\partial \xi} &= \frac{L}{2C_g} A_0 A_{e2};\\
\frac{\partial A_{e2}}{\partial \eta} &= \frac{L}{2C_g} A_0 A_{e1}^*;\\
\frac{\partial A_0}{\partial \xi} + \frac{\partial A_0}{\partial \eta} &= -\frac{S}{2C_g} A_{e1} A_{e2}^*.
\end{aligned}
$$

Renormalization of unknown functions:

$$
a_0 = \frac{L}{2C_g} A_0; \quad a_{1e} = \sqrt{\frac{LS}{2C_g}} A_{e1}; \quad a_{2e} = \sqrt{\frac{LS}{2C_g}} A_{e2}
$$

gives the final system of modulation equations:

$$
\begin{aligned}
&a_{1e\xi} = a_0 a_{2e};\\
&a_{2e\eta} = a_0^* a_{1e};\\
&a_{0\xi} + a_{0\eta} = -a_{1e} a_{2e}^*.
\end{aligned}
\tag{4.2}
$$

The system of eqs. (4.2) has two first integrals of motion:

$$
\begin{aligned}
&a_{1e} a_{1e}^* + 2a_0 a_0^* = g(\eta);\\
&a_{2e} a_{2e}^* + 2a_0 a_0^* = f(\xi).
\end{aligned}
\tag{4.3}
$$

Phase synchronism of waves is a typical regime for wave generation problems, so let us assume it here, and that means for all wave amplitudes to be real functions of space and time. Following substitution of variables and unknown functions,

$$a_{1e} = \sqrt{g(\eta)}\sin(\varphi), \quad a_{2e} = \sqrt{f(\xi)}\sin(\psi);$$
$$\widehat{\xi} = \frac{1}{2}\int f(\xi)\mathrm{d}\xi, \quad \widehat{\eta} = \frac{1}{2}\int g(\eta)\mathrm{d}\eta,$$

where g and f are arbitrary functions of its arguments, gives the system of equations for new φ and ψ functions [12]:

$$\begin{aligned} \varphi_{\widehat{\xi}} &= \sin(\psi), \\ \psi_{\widehat{\eta}} &= \sin(\varphi). \end{aligned} \tag{4.4}$$

The received system of equations is equivalent to the generalized Sine-Gordon equation [12-14] and describes a wide range of nonlinear wave's phenomena.

System of eqs. (4.4) has a striking property: sum and difference of unknown functions satisfied to standard nonlinear Sine-Gordon [11] equation:

$$(\varphi \pm \psi)_{\widehat{\xi}\,\widehat{\eta}} = \sin(\varphi \pm \psi)\,. \tag{4.5}$$

So, we can easily construct a large (infinite) set of solutions for our model by using different known solutions of Sine-Gordon equation. If $S1$ and $S2$ are two of them, then eqs. (4.4) due to (4.5) will be satisfied by the following functions:

$$\begin{aligned} \varphi &= \frac{S1 + S2}{2}; \\ \psi &= \frac{S1 - S2}{2}. \end{aligned} \tag{4.6}$$

As the zero function is evidently the solution of eq. (4.5), it is possible to take $S2 = 0$, $S1$—one wave soliton solutions of the Sine-Gordon equation [11], and to receive the brightest physical example here—constantly running wave solitons of the both edge wave envelopes:

$$\tan(\varphi) = \tan(\psi) = C\exp(\lambda\,\widehat{\eta} + \widehat{\xi}\,/\lambda), \tag{4.7}$$

where λ and C are free parameters. By taking two different solutions of (4.4) as (φ_1, ψ_1), (φ_2, ψ_2) then the sum and the difference of them will satisfy (4.5):

$$(\varphi_1 + \psi_1), (\varphi_2 + \psi_2), (\varphi_1 - \psi_1), (\varphi_2 - \psi_2). \tag{4.8}$$

And using (4.7) it is possible to construct a subset of new solutions of the system (4.4) as

$$\begin{aligned} &(\varphi_1 + \psi_1 + \varphi_2 + \psi_2)/2; (\varphi_1 + \psi_1 - \varphi_2 - \psi_2)/2, \\ &(\varphi_1 + \psi_1 - \varphi_2 + \psi_2)/2; (\varphi_1 + \psi_1 + \varphi_2 - \psi_2)/2. \end{aligned} \tag{4.9}$$

By repeating this procedure it is possible to construct an infinite number of solutions of eqs. (4.4) and to describe the different regimes of many waves interaction.

5. – Conclusions

Resonant interactions between incoming waves and edge waves significantly depend on factors like wave frequency and amplitude of incoming waves, slope of the beach. Intensity of energy exchange between waves increases with amplitude and frequency of the incoming swell. The normal to the shoreline incident wave can give energy simultaneously to a pair of opposite resonance edge waves, and inversely a pair of interacting edge waves can leak wave energy out of a beach. Analysis of spatial modulation enables us to study the excitation of edge waves in between wave breakers. Less intensity of the edge wave formed between the wave breakers is considered as a favorable condition. The distance between wave breakers can be adjusted for less generation of edge waves. As the incident wave length increases the distance between wave breakers can be increased, and increasing the amplitude of the incoming wave leads to decreasing of the minimum distance between wave breakers. Consideration of fully nonlinear modulation model leads to an infinite number of wave solutions and describes various regimes of the many waves interaction.

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