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Dispersion of Love waves in a stochastic layer

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Summary. — We consider the problem of propagation of Love waves in an elastic layer of uniform thickness overlying a half-space. The layer is assumed to have elastic properties which vary randomly with position. The mean field in the layer is obtained using the smoothing method. Some interesting cases are described graphically.

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Introduction

The existence of horizontally polarized shear waves propagating in a homogeneous elastic layer overlying an elastic half-space was shown by Love [1]. Since then, these surface waves, commonly known as Love waves, have attracted considerable attention due to the applications in earthquake engineering and theoretical seismology. The observed data for Love waves in a seismological station is also used to interpret the internal structure of the earth. However, the homogeneous and isotropic layered model of the earth considered by Love does not describe the real situation very well. The anisotropic nature of the earth has been subject of study of many authors, *e.g.*, Anderson [2, 3] considered the Rayleigh waves in an anisotropic single half-space model while Kelley [4] studied propagation of Love waves in case of some interesting variations in the overlying layer using numerical methods. Others, among them Sato [5] and Paul [6] studied the problem of variations in the shape of the overlying layer using analytical methods.

The problem of a non-homogeneous layer or the half-space due to poor consolidation or presence of inhomogeneities has also attracted interest. Ghosh [7], Chattopadhaya *et al.* [8] have considered Love waves in an inhomogeneous layer overlying a homogeneous half-space. Similar studies in which the density was assumed to be varying were carried out by Zaman *et al.* [9, 10]. All these models assumed the inhomogeneity to be described by a deterministic function of depth.

The earth, in general and the near surface in particular, have a very complex elastic parameter distribution. The deterministic models mentioned above do not account for point-to-point irregularities of the earth. These are best described by assuming the elastic properties to be randomly varying with position. This gives rise to a stochastic model. In this model, the random inhomogeneity is described through a random function of space variables and a random parameter. In realistic media model, both density and rigidity show random variation as from the geophysical point of view, the variation in both density and rigidity occur. Korvin [11] has combined the process by considering the shear velocity to be a random function. In this paper we consider the density to be a random function of depth. However a corresponding change in rigidity may also be considered at the cost of a more complicated model. Our present work is restricted to the case in which only density of the medium is assumed to be a random function of depth. The theory of wave propagation in such a media has been discussed by Sobczyk [12]. Some interesting studies in this direction are by Korvin [11,13] who used perturbation method to calculate the attenuation coefficient due to random inhomogeneities. Chu, Askar and Cakmak [14] used the same method in their work to measure elastic properties of the medium. Li and Hudson [15] used the Born approximation to study the elastic waves in a laterally heterogenous layer. In this paper, we consider propagation of the Love waves in an elastic layer of uniform thickness overlying a homogeneous, isotropic half-space. The overlying layer is assumed to have depth dependent randomly varying properties. Such a model could be used to describe the situation where the overlying layer has a sand-clay mixture constitution. Assuming that the inhomogeneities caused by such a mixture are small, we use the first-order smoothing perturbation method to obtain the mean field in the layer. The randomness in the parameters is taken to be an ergodic and statistically homogeneous stochastic process. The mean field thus obtained is used to numerically present the affects of presence of inhomogeneities in different ways.

Smoothing method

We present here a brief account of the smoothing method. Many practical problems associated with continuous stochastic media lead to differential equations with random coefficients which can be written as

$$L(\gamma) = g\,, \tag{1}$$

where $L(\gamma)$ is a linear stochastic operator depending upon the random parameter γ and g is a determinate function.

Let us represent $L(\gamma)$ and the solution $u(\gamma)$ as

$$L(\gamma) = \langle L \rangle + L_1\,, \tag{2}$$

$$u(\gamma) = \langle u \rangle + u_1\,, \tag{3}$$

where $\langle\,\rangle$ denotes the statistical mean and $\langle L_1 \rangle = 0$, $\langle u_1 \rangle = 0$. $\langle L \rangle = L_0$ and $\langle u \rangle = u_0$ are the mean deterministic operator and mean field respectively. Equation (1) then becomes

$$(L_0 + L_1)u = g\,. \tag{4}$$

If L_0 is an invertible operator, we may write eq. (4) as

$$u = L_0^{-1} g - L_0^{-1} L_1 u \,. \tag{5}$$

Taking average of eq. (5), we get

$$\langle u \rangle = L_0^{-1} g - L_0^{-1} \langle L_1 u \rangle \ . \tag{6}$$

In order to find $\langle L_1 u \rangle$, we refer back to (5) and obtain

$$L_1(u) = L_1(L_0^{-1} g) - L_1 L_0^{-1} L_1(u) \,. \tag{7}$$

Upon averaging, eq. (7) yields

$$\langle L_1 u \rangle = - \left\langle L^{-1} L_0^{-1} L_1 u \right\rangle \ . \tag{8}$$

If we assume the Bourret local independence hypothesis [12], we get

$$\left\langle L_1 L_0^{-1} L_1 u \right\rangle \simeq \left\langle L_1 L_0^{-1} L_1 \right\rangle \langle u \rangle \ . \tag{9}$$

Thus using eqs. (8) and (9), we may write (6) as

$$L_0 \langle u \rangle - \left\langle L_1 L_0^{-1} L_1 \right\rangle \langle u \rangle = g \,. \tag{10}$$

Now, let us be more specific and consider the stochastic Helmholtz equation that can be written as

$$L_0 u(\overrightarrow{x}) + X(\overrightarrow{x}, \gamma) u(\overrightarrow{x}) = g \,, \tag{11}$$

where L_0 is a deterministic Laplace differential operator and $X(\overrightarrow{x}, \gamma)$ is a random field. Assuming L_0 to be invertible, we may write

$$L_0^{-1} f(\overrightarrow{x}) = \int G_0(\overrightarrow{x}, \overrightarrow{x}_0) f(\overrightarrow{x}_0) \mathrm{d}\overrightarrow{x}_0 \,, \tag{12}$$

where $G_0(\overrightarrow{x}, \overrightarrow{x}_0)$ is the Green's function of the operator L_0. Hence eq. (10) leads to the following where L_1 is replaced by $X(\overrightarrow{x}, \gamma)$:

$$L_0 \langle u(\overrightarrow{x}, \gamma) \rangle - \int G_0(\overrightarrow{x}, \overrightarrow{x}_0) K_{XX} \langle u(\overrightarrow{x}_0, \gamma) \rangle \, \mathrm{d}\overrightarrow{x}_0 = g(\overrightarrow{x}) \,, \tag{13}$$

where $K_{XX}(\overrightarrow{x}, \overrightarrow{x}_0)$ is the correlation function of the random field.

Equation (13) is an integro-differential equation in the mean field $\langle u(\overrightarrow{x}, \gamma) \rangle$. We apply these ideas to the Love wave problem.

Formulation of the problem

We consider a layer of uniform thickness h overlying a homogeneous and isotropic half-space. The axes are chosen such that the free surface coincides with $z = 0$ and the interface between the layer and the half-space is the plane $z = h$. The horizontally polarized shear wave is assumed to be propagating in the positive x-axis direction in the upper layer. The solution to this problem is well known if the upper layer is homogeneous and isotropic see, *e.g.*, Aki and Richard [16]. We assume that the overlying upper layer has inhomogeneities which are random functions of position. As a consequence the horizontally polarized shear wave, commonly known as Love wave, will appear as a random field. The equation of motion can be written as

$$\frac{\partial^2 v(\overrightarrow{x},t)}{\partial x^2} + \frac{\partial^2 v(\overrightarrow{x},t)}{\partial z^2} - \frac{1}{\beta^2}\frac{\partial^2 v(\overrightarrow{x},t)}{\partial t^2} = \sigma(\overrightarrow{x},t)\,, \tag{14}$$

where $\sigma(\overrightarrow{x},t)$ is the source function, $\beta = \sqrt{\mu/\rho}$ is the shear wave velocity, μ being rigidity and ρ density of the medium and ω is the angular frequency. Assuming that the source $\sigma(\overrightarrow{x},t)$ and the displacement $v(\overrightarrow{x},t)$ are time-harmonic, we take these in the following form:

$$\begin{cases} v(x,z,t) = V(z)\exp\left[i(kx-\omega t)\right]\,, \\ \sigma(x,z,t) = S(z)\exp\left[i(kx-\omega t)\right]\,, \end{cases} \tag{15}$$

where k is the wave number. The equation of motion (14) thus transforms into the following ordinary differential equation:

$$\frac{\mathrm{d}^2 V(z)}{\mathrm{d}z^2} + \left(\frac{\omega^2\rho}{\mu} - k^2\right)V(z) = S(z)\,. \tag{16}$$

Now, let us assume the elastic parameters to be varying as a random process. Although both ρ and μ can be taken to be randomly varying functions, we assume only the density ρ to be a function of the random variable γ and neglect variations in μ. As mentioned earlier, this does not take into account a realistic but more complicated situation in which both ρ and μ would vary. Thus we may write

$$\rho = \langle\rho\rangle\left[1 + \epsilon\rho'(z,\gamma)\right]\,, \tag{17}$$

where $\langle\rho\rangle$ is the mean density, ϵ a small parameter and $\rho'(z,\gamma)$ is a random field such that $\langle\rho'(z,\gamma)\rangle = 0$.

Let us use the subscripts 1 and 2 to denote the displacement field and density in the upper layer and the half-space, respectively. The equations of motion in the layer and the half-space can be written as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \sigma_1^2\right]V_1(z,\gamma) + \epsilon\left\langle k_1^2\right\rangle\rho'(z,\gamma)V_1(z,\gamma) = S(z) \tag{18}$$

and

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \sigma_2^2\right]V_2(z,\gamma) = 0\,, \tag{19}$$

where

$$\begin{cases} k_1^2 = \dfrac{\omega^2}{\mu_1} \langle \rho \rangle \left(1 + \epsilon \rho'(z,\gamma)\right), \\ \left\langle k_1^2 \right\rangle = \dfrac{\omega^2}{\mu_1} \langle \rho \rangle , \quad k_2^2 = \dfrac{\omega^2}{\mu_2} \rho_2 \\ \sigma_1^2 = \left\langle k_1^2 \right\rangle - k^2, \quad \sigma_2^2 = k^2 - k_2^2 . \end{cases} \tag{20}$$

The following boundary conditions are to be satisfied.

1) Normal stress at the free surface should vanish, *i.e.*

$$\left[\frac{\mathrm{d}V_1}{\mathrm{d}z} \right]_{z=0} = 0 . \tag{21}$$

2) The displacement and the stress at the interface between the layer and the half-space should be continuous,

$$V_1(h) = V_2(h) = q \text{ (say)}, \tag{22}$$

$$\mu_1 \left[\frac{\mathrm{d}V_1}{\mathrm{d}z} \right]_{z=h} = \mu_2 \left[\frac{\mathrm{d}V_2}{\mathrm{d}z} \right]_{z=h} . \tag{23}$$

Solution of the problem

We are interested in finding the field $\langle V_1(z) \rangle$ in the upper layer. To this end, we consider the auxiliary problem consisting of the differential equation (18) subject to the initial conditions

$$\begin{cases} V_1(z)\big|_{z=0} = X_0 , \\ \left[\dfrac{\mathrm{d}V_1}{\mathrm{d}z} \right]_{z=0} = Y_0 . \end{cases} \tag{24}$$

Once the solution to the initial value problem (24) is obtained, the solution of our problem can be found by imposing the boundary conditions (21)-(23). Now, using the integro-differential equation (13) derived from the stochastic Helmholtz equation, we get from eq. (18)

$$\begin{aligned} &\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \sigma_1^2 \right] \langle V_1(z) \rangle = \\ &\epsilon^2 \left\langle k_1^2 \right\rangle^2 \int_0^z G_1(z,z_0) K_{\rho\rho}(z,z_0) \langle V_1(z_0) \rangle \,\mathrm{d}z_0 + S(z), \end{aligned} \tag{25}$$

where $K_{\rho\rho}$ is the correlation function for the process $\rho(z,\gamma)$ and $G_1(z,z_0)$ is the Green's function given by

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}^2 G_1(z,z_0)}{\mathrm{d}z^2} + \sigma_1^2 G_1(z,z_0) = \delta(z-z_0)\,, \\ \left.\dfrac{\mathrm{d}G_1(z,z_0)}{\mathrm{d}z}\right|_{z=0} = 0, \quad G_1(z,z_0)|_{z=0} = 0\,. \end{array}\right. \tag{26}$$

The solution to (26) can be found to be

$$G_1(z,z_0) = \left\{\begin{array}{ll} \dfrac{1}{\sigma_1}\sin\sigma_1(z-z_0), & z < z_0\,, \\ 0, & \text{otherwise.} \end{array}\right. \tag{27}$$

We can choose the correlation function in the form

$$K_{\rho\rho}(z-z_0) = D^2 \exp\left[-\left|z-z_0\right| b\right]\,,$$

where D^2 is the variance and b is the inverse of the correlation length. Equation (25) then gives the following integro-differential equation in the mean field $\langle V_1(z)\rangle$:

$$\begin{aligned} &\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \sigma_1^2\right]\langle V_1(z)\rangle = \\ &\frac{\epsilon^2 \left\langle k_1^2\right\rangle^2 D^2}{\sigma_1}\int_0^z \sin\sigma_1(z-z_0) e^{-|z-z_0|b}\langle V_1(z_0)\rangle\,\mathrm{d}z_0 + S(z)\,. \end{aligned} \tag{28}$$

Case I

Let us first take the source $S(z)=0$, *i.e.* we assume that the wave is coming from a far away source. The solution can be found by taking the Laplace transform of (28) as the kernel of the integral is of the convolution type. The transformed integro-differential equation (28) together with the initial conditions (24) lead to the solution

$$F_1(s) = X_0 \frac{s\left[(s+b)^2+\sigma_1^2\right]}{P(s)} + Y_0\frac{\left[(s+b)^2+\sigma_1^2\right]}{P(s)}, \tag{29}$$

where $F_1(s)$ is the Laplace transform of $\langle V_1(z)\rangle$ and

$$P(s) = (s^2+\sigma_1^2)\left[(s+b)^2+\sigma_1^2\right] - \epsilon^2\left\langle k_1^2\right\rangle^2 D^2. \tag{30}$$

We use the Cauchy residue integral formula to find the inverse Laplace transform as

$$\langle V_1(z)\rangle = X_0\sum_{k=1}^4 A_k e^{s_k z} + Y_0\sum_{k=1}^4 B_k e^{s_k z}, \tag{31}$$

where

$$A_k = \left. \frac{s\left[(s+b)^2+\sigma_1^2\right]}{(\mathrm{d}/\mathrm{d}s)P(s)} \right|_{s=s_k} \tag{32}$$

and

$$B_k = \left. \frac{\left[(s+b)^2+\sigma_1^2\right]}{(\mathrm{d}/\mathrm{d}s)P(s)} \right|_{s=s_k} . \tag{33}$$

Here s_k, $k = 1, 2, 3, 4$ are the zeros of $P(s)$. We can now eliminate the auxiliary values X_0 and Y_0 using (21) and (22) to get

$$\begin{cases} Y_0 = 0, \\ X_0 = \dfrac{q}{\sum_{k=1}^{4} A_k e^{s_k h}}. \end{cases} \tag{34}$$

Further, in the half-space, the displacement field will be of the form

$$V_2(z) = q e^{-\sigma_2 z} , \tag{35}$$

so that using continuity condition (22) at $z = h$, we get

$$\langle V_1(z) \rangle = \frac{q \sum_{k=1}^{4} A_k e^{s_k z}}{\sum_{k=1}^{4} A_k e^{s_k h}} . \tag{36}$$

Equation (36) gives the mean field in the layer $0 < z < h$. Using the continuity of the stress at the interface (eq. (23)), we obtain the so-called dispersion relation for the Love wave in the stochastic medium as

$$\frac{\sum_{k=1}^{4} s_k A_k e^{s_k h}}{\sum_{k=1}^{4} A_k e^{s_k h}} = -\frac{\mu_2 \sigma_2}{\mu_1}. \tag{37}$$

Case II

Let us now assume that an energy source $S(z) = 4\pi\delta(z-h)$ is present at the interface between the two media. Following the same procedure as in case I, the integro-differential equation (25) leads to

$$F_1(s) = X_0 \frac{s\left[(s+b)^2+\sigma_1^2\right]}{P(s)} + Y_0 \frac{\left[(s+b)^2+\sigma_1^2\right]}{P(s)} + \\ + \frac{e^{-sh}\left[(s+b)^2+\sigma_1^2\right]}{P(s)} - \epsilon^2 \left\langle k_1^2 \right\rangle^2 , \tag{38}$$

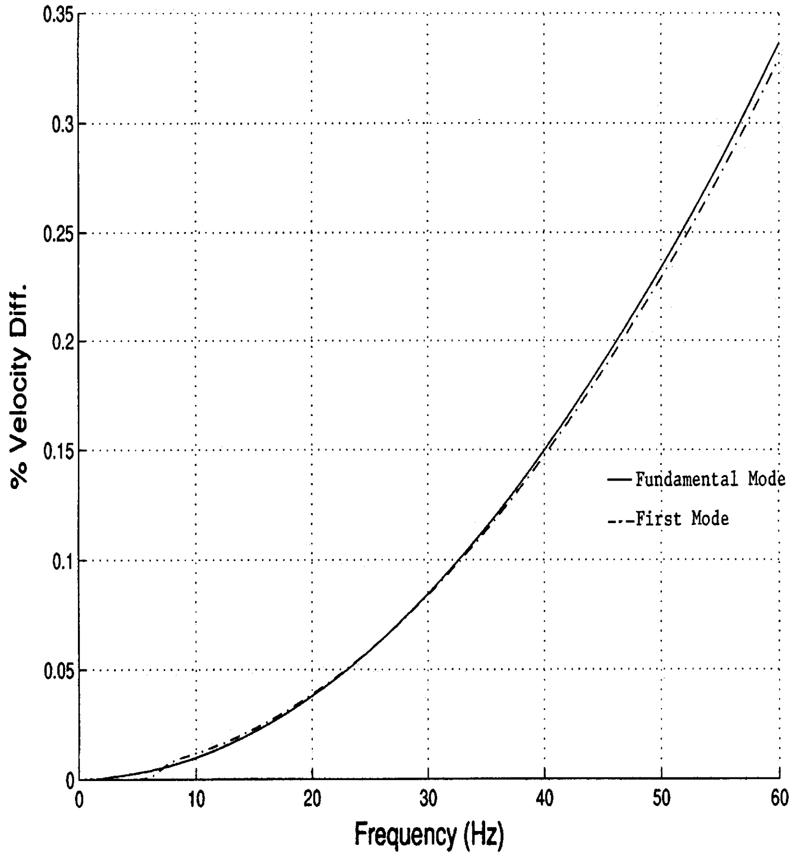

Fig. 1. – Velocity difference *vs.* frequency.

where $D(s)$ is the same as in eq. (30). The inverse transform now yields

$$\langle V_1(z)\rangle = X_0 \sum_{k=1}^{4} A_k e^{s_k z} + Y_0 \sum_{k=1}^{4} B_k e^{s_k z} + \sum_{k=1}^{4} C_k e^{s_k z}, \tag{39}$$

where A_k and B_k are given by eqs. (28) and (29), while

$$C_k = \left. \frac{e^{-sh}\left[(s+b)^2 + \sigma_1^2\right]}{(\mathrm{d}/\mathrm{d}s)P(s)} \right|_{s=s_k}, \tag{40}$$

s_k, $k = 1, \ldots.., 4$ being zeros of $P(s)$.

Using the boundary conditions (21) through (23), we can again determine X_0 and Y_0 to get

$$\langle V_1(z)\rangle = \frac{qe^{-\sigma_2 h} - \sum_{k=1}^{4} C_k e^{s_k h}}{\sum_{k=1}^{4} A_k e^{s_k h}} \sum_{k=1}^{4} A_k e^{s_k z} + \sum_{k=1}^{4} C_k e^{s_k z}. \tag{41}$$

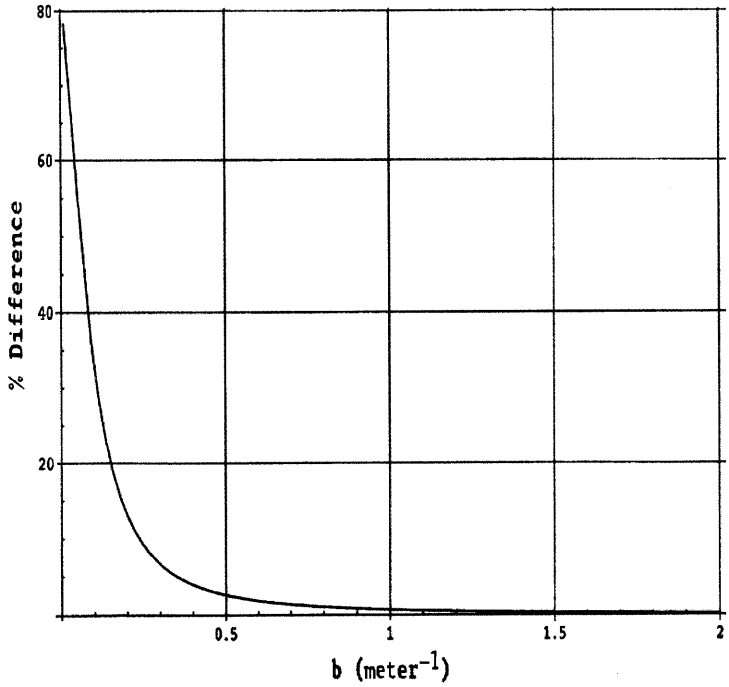

Fig. 2. – Effect of correlation length.

Discussion and numerical results

In order to test our results and in particular the dispersion relation of Love waves given by eq. (37), we compare it to the case of a homogeneous and isotropic layer overlying a half-space. For this case $c = 0$ and $b = 0$. The zeros of the $P(s)$ are now given by $s_1 = s_2 = \sigma_1 i$, $s_3 = s_4 = -\sigma_1 i$. Using these values we determine the coefficients A_k in eq. (32). This gives $A_k = \dfrac{s(s^2+\sigma_1^2)}{4s(s^2+\sigma_1^2)}$, $k = 1, .., 4$.

So that $\sum_{k=1}^{k=4} A_k e^{s_k h} = \dfrac{1}{2} e^{\sigma_1 ih} + \dfrac{1}{2} e^{-\sigma_1 ih}$, and $\sum_{k=1}^{k=4} A_k s_k e^{s_k h} = \frac{\sigma_1 i}{2} e^{\sigma_1 ih} - \frac{\sigma_1 i}{2} e^{-\sigma_1 ih}$.

Using these results in the dispersion relation derived by us in (37) we obtain

$$\frac{\mu_2\sigma_2}{\mu_1} = \alpha_1 i e^{\sigma_1 ih} - e^{-\sigma_1 ih} e^{\sigma_1 ih} + e^{-\sigma_1 ih}. \tag{42}$$

This gives

$$\frac{\mu_2\sigma_2}{\mu_1} = \tan\sigma_1 h, \tag{43}$$

which is the dispersion relation for Love wave propagating in a homogeneous and isotropic layer overlying a half-space [16].

The model considered by us takes into account the change in density of the medium only. This does not adequately model the rock behavior in which density and rigidity

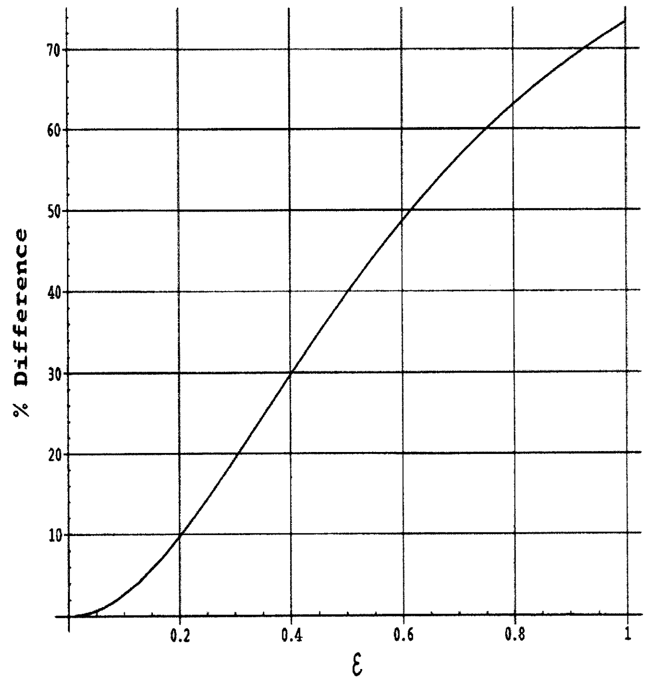

Fig. 3. – Effect of inhomogeneity.

are proportional. Keeping in mind this limitation, we compute the difference in the phase velocity of Love wave in the presence of random variation in density with that of homogeneous case. Using $\varepsilon = 0.1$ and the correlation length $b = 2$, the difference in the two phase velocities is plotted against frequency in fig. 1. It shows an increase with the frequency. The bold line corresponds to the fundamental mode while the dotted line shows the first mode. This illustrate the need to consider the stochastic effect. In order to study the effect of randomness of the medium, the inverse correlation length b has been chosen. To confirm the established fact that randomness causes attenuation, we plot the percentage attenuation as the correlation inverse length b increases. Figure 2 shows that difference between the homogeneous case and the stochastic case (corresponding to $\varepsilon = 0.1$) decreases as b increases and goes to zero very rapidly after $b = 0.5$. This agrees with elastic waves in a random media [11]. Finally, the effect of increase in inhomogeneity is studied through increase in the parameter ε. By choosing $z = h/2$, and $b = 0.5$, we plot the difference of displacement between the homogeneous and random case. This is presented in fig. 3.

As a conclusion, one can assert the need for a stochastic model whenever the overlying layer does not have a homogeneous constitution. The perturbation procedure outlined here can be extended to include the variation of rigidity of the layer and variation of elastic properties of the underlying half-space. The present paper highlights the significance and effect of the randomness in the layer and shows that this effect dies out as the inverse correlation length increases.

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