Symmetry defect in single-gyre, wind-driven oceanic systems

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(ricevuto il 7 Giugno 1999; revisionato il 7 Febbraio 2000; approvato il 21 Febbraio 2000)

Summary. — We explore some symmetry properties of the leading terms that constitute the solution describing the flow field structure in a wind-driven, bottom-dissipated ocean. Both the weakly non-linear and the highly non-linear regime are investigated. The main result is that the northward displacement and the westward intensification of the current system, which are typical of the subtropical gyres (for instance the North Atlantic Ocean), can be ascribed to an interplay between the symmetries of these terms. Moreover, a duality relationship allows us to relate the conclusions concerning one regime to the other.

PACS 92.10.Fj – Dynamics of the upper ocean.

1. – Introduction

The main reason of the success of the classical barotropic models of wind-driven ocean circulation lies in the overall qualitative agreement between the observed and the predicted transport patterns, with special regard to the subtropical gyre [1]. Due to their strictly adiabatic nature, the atmospheric forcing plays a purely mechanical role through the wind, and it is usually represented by a steady, suitably modulated, wind stress varying on the atmospheric sinoptic scale. As the horizontal length scale of the motion derives essentially from the atmospheric pressure field, this fact poses a lower bound to the extension of the flow field which the models are able to take into account. A special, but very important, application concerns the basin-scale dynamics, at least in its simplest form. In this framework, among the features reproduced by these models in good qualitative accordance with the observations, we recognize the *westward intensification* (WI) and the *northward displacement* (ND) of the gyres in the subtropical basins. A link between geometry and dynamics is thus expected: the present paper aims to focus just this point.

We take into account, as is traditional, a square basin on the beta plane and a latitudinally sinusoidal wind-stress curl negative in the basin interior and vanishing only at the latitudes of the zonal boundaries. The symmetry defect of the gyre, producing the WI and the ND, will be analyzed by means of the *East-West* and

North-South mirror reflections of the fluid domain into itself. On purpose, the forcing field is left unchanged by these transforms.

From the dynamical point of view, in accordance with the climatological character of the motion under investigation, there is no loss of generality by assuming the well-known quasi-geostrophic approximations. On this subject, we recall that the equations describing the potential vorticity conservation tendency can be doubly classified, *i.e.* according to the adopted parametrization of turbulence and to the relative importance of friction with respect to non-linearity. We know that every parametrization of turbulence which increases the order of the differential governing equations demands, at the same time, dynamic boundary conditions whose form is, to a large extent, *a priori* arbitrary. At the same time, each special choice of these boundary conditions influenced deeply the resulting flow field also in the basin interior, that is far from the coastline. Therefore, to avoid an increase of the complexity of the problem without any clear benefit, we will investigate the behaviour of a purely bottom dissipated fluid layer and thus we will deal with boundary conditions as simple as possible, that is only no mass flux across the coastline and we will investigate the behaviour of a purely bottom dissipated fluid layer. In this framework, we will explore two deeply separate regimes by using the weakly non-linear Stommel-Veronis model [2] and the highly non-linear model of Niiler [3]. The reason of this choice is twofold: In the first place, in both regimes a partial or total analytical approach to the circulation problem is feasible; secondly, a surprising duality between the symmetry properties of the solutions in the selected regimes turns out to hold.

We obtain, for each regime, the first two terms of a truncated expansion of the gyre-like solution and each term has a definite symmetry property under the above-quoted mirror reflections. The result allows us to see how the WI and the ND arise from the superposition of these first two terms and to understand their different generation in the framework of two almost opposite dynamical regimes.

2. – Geometry

We consider a non-dimensional square fluid domain D on the beta plane such that

$$D = [0 \leqslant x \leqslant 1] \times [0 \leqslant y \leqslant 1]$$

and assume a wind-stress curl $T \equiv \mathbf{k} \cdot \nabla \times \boldsymbol{\tau}$ vanishing in $y = 0$, and $y = 1$, negative in the basin interior and invariant under the mid-latitude and mid-longitude mirror reflections. This kind of field mimics the forcing that acts on the observed subtropical gyres and, at the same time, allows us to simplify the analysis of the symmetry defect in an idealized gyre. Explicit examples are the following:

$$T \equiv -\pi \sin(\pi y) \tag{2.1}$$

or

$$T \equiv -\pi \sin(\pi x) \sin(\pi y) . \tag{2.2}$$

At this point we introduce the mirror reflection transforms of D into itself defined, respectively, by

$$(x, y) \rightarrow (x, 1 - y) \tag{2.3}$$

and by

$$(x, y) \to (1-x, y) \,. \tag{2.4}$$

It is trivial to check that the fields (2.1) and (2.2) are invariant under transforms (2.3) and (2.4). In general we denote by a tilde every function ψ such that

$$\tilde{\psi}(x, y) \equiv \psi(x, 1-y)$$

and use an overbar to indicate that

$$\overline{\psi}(x, y) = \psi(1-x, y) \,.$$

Transforms (2.3) and (2.4) applied to the Jacobian and Laplacian operators yield, respectively,

$$J = -\tilde{J} = -\overline{J} \qquad \text{and} \qquad \nabla^2 = \tilde{\nabla}^2 = \overline{\nabla}^2 . \tag{2.5}$$

With reference to (2.3), we can express every function as

$$\psi = \psi^{(\mathrm{s})} + \psi^{(\mathrm{a})}, \tag{2.6}$$

where

$$\psi^{(\mathrm{s})} \equiv \frac{1}{2}(\psi + \tilde{\psi}) \qquad \text{and} \qquad \psi^{(\mathrm{a})} \equiv \frac{1}{2}(\psi - \tilde{\psi}) \,.$$

Analogously, with reference to (2.4), the identity

$$\psi = \psi_{\mathrm{s}} + \psi_{\mathrm{a}} \tag{2.7}$$

holds, where

$$\psi_{\mathrm{s}} = \frac{1}{2}(\psi + \overline{\psi}) \qquad \text{and} \qquad \psi_{\mathrm{a}} = \frac{1}{2}(\psi - \overline{\psi}) \,.$$

Obviously,

$$\psi^{(\mathrm{s})} = \tilde{\psi}^{(\mathrm{s})}, \qquad \psi^{(\mathrm{a})} = -\tilde{\psi}^{(\mathrm{a})}, \qquad \psi_{\mathrm{s}} = \overline{\psi}_{\mathrm{s}}, \qquad \psi_{\mathrm{a}} = -\overline{\psi}_{\mathrm{a}} . \tag{2.8}$$

Useful properties of the antisymmetric functions are

$$\psi^{(\mathrm{a})}\left(x, \frac{1}{2}\right) = \psi_{\mathrm{a}}\left(\frac{1}{2}, y\right) = 0 \tag{2.9}$$

and

$$\int_0^1 \psi^{(\mathrm{a})} \, \mathrm{d}y = \int_0^1 \psi_{\mathrm{a}} \, \mathrm{d}x = 0 \,. \tag{2.10}$$

Finally, we underline that, if $\psi = 0$, $\forall (x, y) \in \partial D$, then the same boundary condition holds also for $\psi^{(\mathrm{s})}$, $\psi^{(\mathrm{a})}$, ψ_{s}, ψ_{a}.

These are the geometrical tools we need in the development of our discussion.

3. – Dynamics

We define preliminarily the typical intensity U_S of the Sverdrup current through the equation

$$\tau_0 = \varrho \beta_m D L U_S, \tag{3.1}$$

where τ_0 is the typical value of the wind stress, ϱ is the fluid density, β_m is the planetary vorticity gradient evaluated at the mean latitude of the basin, D is the depth of the motion and finally L is the typical horizontal length of the basin-scale circulation. Then, the basin-scale, quasi-geostrophic steady vorticity equation is written as (see the appendix)

$$\left(\frac{\delta_I}{L}\right)^2 J(\psi, \nabla^2 \psi) + \frac{\partial \psi}{\partial x} = \frac{U_S}{U} T - \frac{\delta_v}{L} \nabla^2 \psi , \tag{3.2}$$

where δ_I is the inertial boundary layer width, U_S/U is the ratio between the Sverdrup current and the actual current of the interior and δ_v is the viscous boundary layer width. The forcing $T = T(x, y)$ given by (2.1) or (2.2) will be investigated in detail. Equation (3.2) contemplates several dynamical configurations and, in particular, we will focus on the following two.

1) Weakly non-linear Stommel-Veronis' model: the interior is in Sverdrup balance and the viscous boundary layer is much greater the that inertial one;

2) Highly non-linear Niiler's model: the almost zonal interior velocity is much greater than U_S defined in (3.1) and the inertial boundary layer is dominant.

In both cases the only boundary condition is

$$\psi = 0 \qquad \forall (x, y) \in \partial D . \tag{3.3}$$

In case (1), eq. (3.2) becomes

$$\left(\frac{\delta_I}{L}\right)^2 J(\psi, \nabla^2 \psi) + \frac{\partial \psi}{\partial x} = T - \frac{\delta_v}{L} \nabla^2 \psi \tag{3.4}$$

and, due to the assumed smallness of the ratio $(\delta_I/\delta_v)^2$, the solution of problem (3.3), (3.4) can be approximated by the sum of the solution ψ_0 of the linear Stommel's model plus the first-order western boundary layer correction ϕ_1. The problem for ψ_0 is

$$\frac{\partial \psi_0}{\partial x} = T - \frac{\delta_v}{L} \nabla^2 \psi_0, \tag{3.5}$$

$$\psi_0 = 0 , \qquad \forall (x, y) \in \partial D , \tag{3.6}$$

where $\psi_0 = \psi_I + \phi_0$ is the sum of the Sverdrup solution ψ_I for the interior plus a zeroth-order western boundary layer correction ϕ_0. For a generic forcing T, the solution of problem (3.5), (3.6) in terms of ϕ_I and ψ_0 takes the form

$$\psi_I(x, y) = -\int_x^1 T(\lambda, y)\, \mathrm{d}\lambda , \tag{3.7}$$

$$\phi_0(\xi, y) = -\psi_I(0, y) \exp[-\xi] . \tag{3.8}$$

The problem for ϕ_1 is given, in terms of the western boundary layer variable $\xi = (L/\delta_{\rm v})\, x$ and the latitude y, by the equation [4]

$$\frac{\partial\phi_0}{\partial\xi}\,\frac{\partial^3\phi_0}{\partial\xi^2\,\partial y} - \left(\frac{\partial\psi_{\rm I}}{\partial y} + \frac{\partial\phi_0}{\partial y}\right)\frac{\partial^3\phi_0}{\partial\xi^3} = -\frac{\partial\phi_1}{\partial\xi} - \frac{\partial^2\phi_1}{\partial\xi^2} \tag{3.9}$$

with the boundary and matching conditions

$$\phi_1(0, y) = 0\,, \tag{3.10}$$

$$\lim_{\xi\to+\infty}\phi_1(\xi, y) = 0\,. \tag{3.11}$$

Thus, the total solution is

$$\psi = \psi_0 + \left(\frac{\delta_{\rm I}}{\delta_{\rm v}}\right)^2\phi_1 + {\rm O}\left(\left(\frac{\delta_{\rm I}}{\delta_{\rm v}}\right)^4\right). \tag{3.12}$$

In case (2), putting for shortness $U_{\rm S}/U \equiv r$, eq. (3.2) becomes (see the appendix)

$$\left(\frac{\delta_{\rm I}}{L}\right)^2 J(\psi, \nabla^2\psi) + \frac{\partial\psi}{\partial x} = r\left(T - \frac{\delta_{\rm I}}{L}\nabla^2\psi\right). \tag{3.13}$$

Equation (3.13) includes the parameters $\delta_{\rm I}/L$ and r whose relative magnitude depends on the dynamical regime we are interested in. As, in the inertial regime, we can assume r smaller and smaller, we must accordingly request a definite behaviour to the equation itself in the limit for $r \to 0$. If $\delta_{\rm I}/L \geqslant r$, r and hence the whole rhs of (3.13) goes to zero without influencing the lhs of the same equation; therefore the dominant vorticity equation turns out to be simply $J(\psi, (\delta_{\rm I}/L)^2\nabla^2\psi + y) = 0$. On the contrary, if $\delta_{\rm I}/L < r$, the dynamical balance between the rhs and the lhs of (3.13) is preserved whatever $r(\ll 1)$ may be. This is just the regime we will investigate. The reason is that a solution, describing both the WI and the ND of the gyre, is not consistent with an interior strictly zonal as that implied by every unforced balance.

The smallness of r allows us to write the total solution in terms of an inertial term and a forced correction, according to the expansion

$$\psi = \psi_0 + r\psi_1 + {\rm O}(r^2)\,, \tag{3.14}$$

so, the substitution of (3.14) into (3.3) and (3.13) states, at zeroth and first order in r, the following problems for ψ_0 and ψ_1, respectively:

$$\left(\frac{\delta_{\rm I}}{L}\right)^2 J(\psi_0, \nabla^2\psi_0) + \frac{\partial\psi_0}{\partial x} = 0\,, \tag{3.15}$$

$$\psi_0 = 0\,, \qquad \forall (x, y) \in \partial D\,, \tag{3.16}$$

$$\left(\frac{\delta_{\rm I}}{L}\right)^2 [J(\psi_0 \nabla^2\psi_1) + J(\psi_1, \nabla^2\psi_0)] + \frac{\partial\psi_1}{\partial x} = T - \frac{\delta_{\rm I}}{L}\nabla^2\psi_0, \tag{3.17}$$

$$\psi_1 = 0\,, \qquad \forall (x, y) \in \partial D\,. \tag{3.18}$$

In the next sections we will analyze the behaviour of the solutions of the above-posed problems under transforms (2.3) and (2.4) in order to see how WI and ND can be explained on the basis of the symmetry properties exhibited by these solutions.

4. – Asymmetries of the weakly non-linear gyre

In this section we will infer some useful properties of the leading terms ψ_0 and ϕ_1 of the solution of problem (3.3), (3.4) by analyzing separately problem (3.5), (3.6) for ψ_0 and problem (3.9), (3.10), (3.11) for ϕ_1.

The first feature we point out is that problem (3.5), (3.6) has a unique solution, given by (3.7), (3.8) within the boundary layer approximation and it is inherently asymmetric under (2.4), in the sense that both $\psi_{0\mathrm{s}}$ and $\psi_{0\mathrm{a}}$ cannot be identically vanishing.

About uniqueness, if ψ_0^{I} and ψ_0^{II} are two solutions of problem (3.5), (3.6), then their difference $\varphi \equiv \psi_0^{\mathrm{I}} - \psi_0^{\mathrm{II}}$ turns out to be solution of the problem

$$\frac{\partial \varphi}{\partial x} = -\frac{\delta_{\mathrm{v}}}{L} \nabla^2 \varphi , \tag{4.1}$$

$$\varphi = 0 , \qquad \forall (x, y) \in \partial D . \tag{4.2}$$

We easily obtain, after multiplication of (4.1) by φ and the subsequent integration on D with the aid of (4.2), the equation

$$\int_D |\nabla \varphi|^2 \mathrm{d}x \, \mathrm{d}y = 0 . \tag{4.3}$$

Finally, from (4.2) and (4.3) we conclude that $\varphi \equiv 0$, that is $\psi_0^{\mathrm{I}} \equiv \psi_0^{\mathrm{II}}$.

About asymmetry, substitution of (2.7) into (3.5) and the application of (2.4) allow us to single out the coupled problems for $\psi_{0\mathrm{s}}$ and $\psi_{0\mathrm{a}}$, *i.e.*

$$\frac{\partial \psi_{0\mathrm{a}}}{\partial x} = T - \frac{\delta_{\mathrm{v}}}{L} \nabla^2 \psi_{0\mathrm{s}}, \tag{4.4}$$

$$\frac{\partial \psi_{0\mathrm{s}}}{\partial x} = -\frac{\delta_{\mathrm{v}}}{L} \nabla^2 \psi_{0\mathrm{a}}, \tag{4.5}$$

$$\psi_{0\mathrm{s}} = 0 , \qquad \forall (x, y) \in \partial D , \tag{4.6}$$

$$\psi_{0\mathrm{a}} = 0 , \qquad \forall (x, y) \in \partial D . \tag{4.7}$$

Both the components $\psi_{0\mathrm{s}}$ and $\psi_{0\mathrm{a}}$ are non-vanishing. In fact, if $\psi_{0\mathrm{s}} \equiv 0$, then (4.5) implies $\nabla^2 \psi_{0\mathrm{a}} = 0$, $\forall (x, y) \in D$, but this last equation, together with (4.7), implies $\psi_{0\mathrm{a}} \equiv 0$ and hence $\psi_0 \equiv 0$, in contrast with the assumption of a non-vanishing forcing T. Moreover, if $\psi_{0\mathrm{a}} \equiv 0$, then (4.5) implies $\psi_{0\mathrm{s}} = \psi_{0\mathrm{s}}(y)$ but, in this case, (4.6) demands $\psi_{0\mathrm{s}} \equiv 0$ so, again, we have $\psi_0 \equiv 0$.

The second feature we point out is the generation of the WI from the superposition of $\psi_{0\mathrm{s}}$ and $\psi_{0\mathrm{a}}$. To this purpose, consider the integration of (4.4) and (4.5) on D. By resorting to (4.6) and (4.7) to integrate the lhs of the equations above, we obtain, respectively,

(4.8) $$\int_D T\,\mathrm{d}x\,\mathrm{d}y = \frac{\delta_\mathrm{v}}{L}\oint_{\partial D} \nabla\psi_{0\mathrm{s}}\cdot\mathbf{n}\,\mathrm{d}s$$

and

(4.9) $$\oint_{\partial D} \nabla\psi_{0\mathrm{a}}\cdot\mathbf{n}\,\mathrm{d}s = 0\,,$$

where $\mathbf{n}$ is the unit normal vector and $\mathrm{d}s$ is the differential line element along ∂D. In terms of the geostrophic current $\mathbf{u}_0 = \mathbf{k}\times\nabla\psi_0$, eqs. (4.8) and (4.9) can be written as

(4.10) $$\int_D T\,\mathrm{d}x\,\mathrm{d}y = \frac{\delta_\mathrm{v}}{L}\oint_{\partial D} \mathbf{u}_{0\mathrm{s}}\cdot\mathbf{t}\,\mathrm{d}s$$

and

(4.11) $$\oint_{\partial D} \mathbf{u}_{0\mathrm{a}}\cdot\mathbf{t}\,\mathrm{d}s = 0\,,$$

where $\mathbf{u}_{0\mathrm{s}} = \mathbf{k}\times\nabla\psi_{0\mathrm{s}}$, $\mathbf{u}_{0\mathrm{s}} = \mathbf{k}\times\nabla\psi_{0\mathrm{a}}$ and $\mathbf{t} = \mathbf{k}\times\mathbf{n}$ is the unit tangent vector along ∂D, positive anticlockwise.

In the interior, (4.4) gives

(4.12) $$\frac{\partial\psi_{0\mathrm{a}}}{\partial x} \approx T(<0)\,.$$

Equation (4.10) states that, in the subtropical gyre, $\oint_{\partial D} \mathbf{u}_{0\mathrm{s}}\cdot\mathbf{t}\,\mathrm{d}s < 0$, that is $\mathbf{u}_{0\mathrm{s}}$ is clockwise along the boundary and, due to the symmetry of $\psi_{0\mathrm{s}}$, we can write

$$0 < v_{0\mathrm{s}} \equiv \left(\frac{\partial\psi_{0\mathrm{s}}}{\partial x}\right)_{x=0} = -\left(\frac{\partial\psi_{0\mathrm{s}}}{\partial x}\right)_{x=1}.$$

A set of streamlines of $\psi_{0\mathrm{s}}$ corresponding to the special case in which T is given by (2.1) and $\delta_\mathrm{v}/L = 0.05$ is displayed in fig. 1a.

Because of (2.9) and (4.12), the current $\mathbf{u}_{0\mathrm{a}}$ has a convergence point in $\left(\frac{1}{2}, 1\right)$ and one of divergence in $\left(\frac{1}{2}, 0\right)$. The southward flow of the interior, fed by the convergence into $\left(\frac{1}{2}, 1\right)$ and the divergence in $\left(\frac{1}{2}, 0\right)$, turns northward along both the meridional walls. Some representative streamlines of $\psi_{0\mathrm{a}}$ are reported in fig. 1b, for the same T and δ_v/L as in fig. 1a. The antisymmetry of $\psi_{0\mathrm{a}}$ yields

$$0 < v_{0\mathrm{a}} \equiv \left(\frac{\partial\psi_{0\mathrm{a}}}{\partial x}\right)_{x=0} = \left(\frac{\partial\psi_{0\mathrm{a}}}{\partial x}\right)_{x=1}.$$

Note that (4.12) implies that $\psi_{0\mathrm{a}}$ vanishes only in $\left(\frac{1}{2}, y\right)$. This is in accordance with

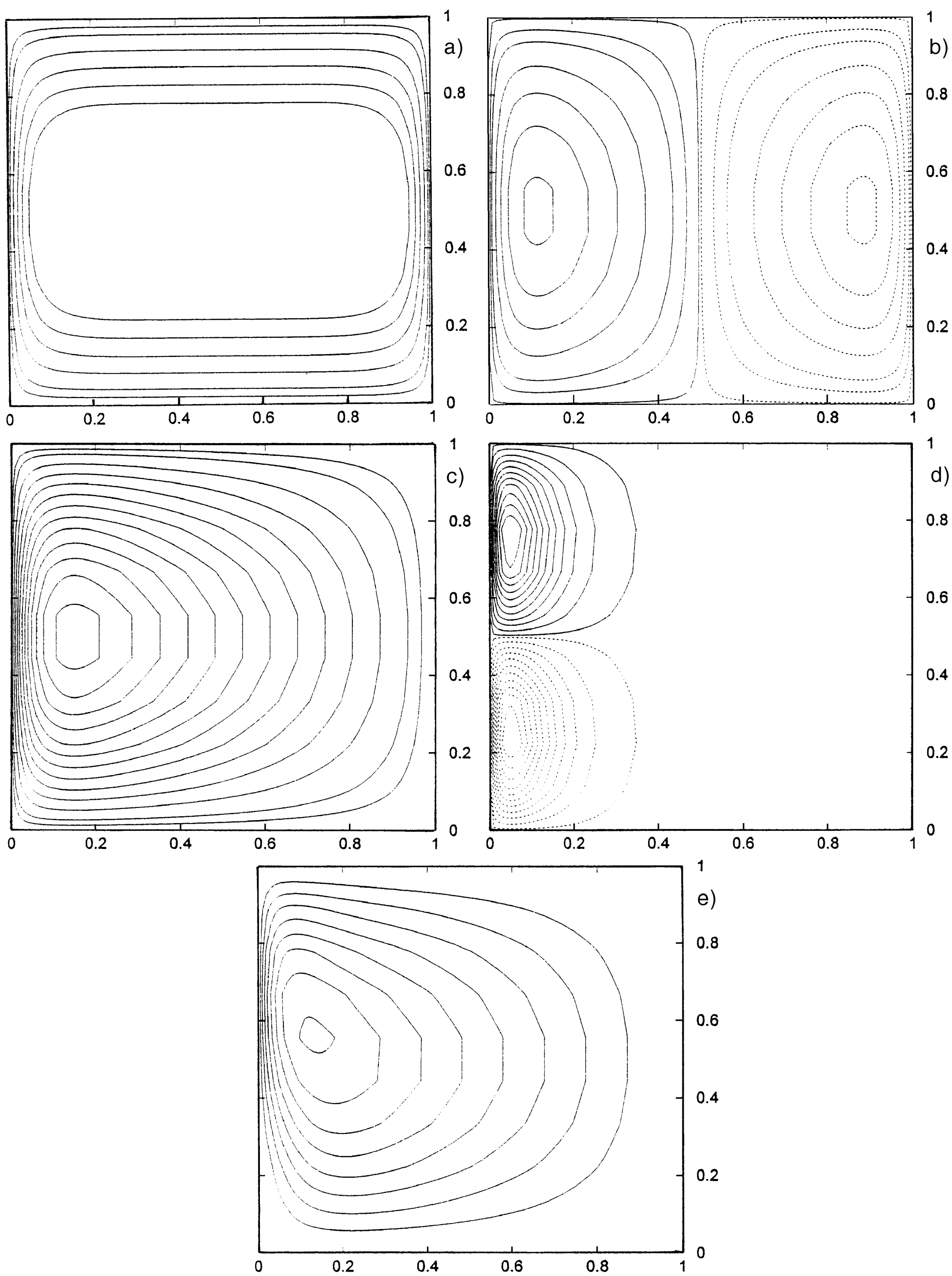

Fig. 1. – a) Streamlines of the East ↔ West invariant component of the zeroth-order solution of problem (3.3), (3.4). b) Streamlines of the East ↔ West antisymmetric component of the zeroth-order solution of problem (3.3), (3.4). c) Stommel's solution: the North ↔ South invariance is apparent. d) Streamlines of the first-order correction streamfunction of the western boundary layer. e) Weakly non-linear Veronis' solution.

(4.11). On the whole, the meridional current v_W along the western boundary is

$$v_W = \left(\frac{\partial \psi_{0s}}{\partial x} \right)_{x=0} + \left(\frac{\partial \psi_{0a}}{\partial x} \right)_{x=0} = v_{0s} + v_{0a}, \tag{4.13}$$

while along the eastern boundary the current v_E is

$$v_E = \left(\frac{\partial \psi_{0s}}{\partial x} \right)_{x=1} + \left(\frac{\partial \psi_{0a}}{\partial x} \right)_{x=1} = - v_{0s} + v_{0a}. \tag{4.14}$$

To determine the sign of v_E, we evaluate the integral of (3.5) on D. The result is $\int_D T \, dx \, dy = (\delta_v / L) \oint_{\partial D} \mathbf{u}_0 \cdot \mathbf{t} \, ds < 0$, so $\mathbf{u}_0$ is clockwise along ∂D and thus, in particular, $v_E < 0$, that is

$$v_{0a} < v_{0s}. \tag{4.15}$$

Equations (4.13), (4.14) and (4.15) imply

$$\left| \frac{v_W}{v_E} \right| = \frac{v_{0s} + v_{0a}}{v_{0s} - v_{0a}} \tag{4.16}$$

and (4.16) explains the generation of the WI on the basis of the sole zeroth-order solution ψ_0.

The zeroth-order solution ψ_0 is not able to generate the ND of the gyre. This fact can be easily verified by using (2.6) into (3.5) and then (2.3) to dissociate the problem for $\psi_0^{(a)}$ from that for $\psi_0^{(s)}$. These problems take, respectively, the following form:

$$\frac{\partial}{\partial x} \psi_0^{(a)} = - \frac{\delta_v}{L} \nabla^2 \psi_0^{(a)}, \tag{4.17}$$

$$\psi_0^{(a)} = 0 \, , \qquad \forall (x, y) \in \partial D \tag{4.18}$$

and

$$\frac{\partial}{\partial x} \psi_0^{(s)} = T - \frac{\delta_v}{L} \nabla^2 \psi_0^{(s)},$$

$$\psi_0^{(s)} = 0 \, , \qquad \forall (x, y) \in \partial D \, . \tag{4.20}$$

Problem (4.17), (4.18) coincides with (4.1), (4.2) and therefore

$$\psi_0^{(a)} \equiv 0 \, . \tag{4.21}$$

On the other hand, problem (4.19), (4.20) coincides with (3.5), (3.6) but we know that this has a unique solution, so we conclude that

$$\psi_0 \equiv \psi_0^{(s)}. \tag{4.22}$$

The streamfunction (4.22) gives the celebrated Stommel's solution, shown in fig. 1c, for the same T and δ_v / L as in figs. 1a and 1b.

The non-existence of the ND in ψ_0 is stated just by eqs. (4.21) and (4.22). This result is in qualitative contraposition to the observed circulation patterns and it leads us to consider the small contribution coming from non-linearity and represented by the first-order correction ϕ_1. To analyse the symmetry properties of ϕ_1 with the aid of transform (2.3), we preliminarily note that, due to the assumed invariance of the wind-stress curl, eqs. (3.7) and (3.8) ensure us that $\psi_{\mathrm{I}} = \tilde{\psi}_{\mathrm{I}}$ and $\phi_0 = \tilde{\phi}_0$, so transform (2.3) applied to (3.9) gives

$$-\frac{\partial \phi_0}{\partial \xi}\frac{\partial^3 \phi_0}{\partial \xi^2 \partial y} + \left(\frac{\partial \psi_{\mathrm{I}}}{\partial y} + \frac{\partial \phi_0}{\partial y}\right)\frac{\partial^3 \phi_0}{\partial \xi^3} = -\frac{\partial \tilde{\phi}_1}{\partial \xi} - \frac{\partial^2 \tilde{\phi}_1}{\partial \xi^2}. \tag{4.23}$$

Adding (3.9) to (4.23) we obtain

$$\frac{\partial \phi_1}{\partial \xi} + \frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{\partial \tilde{\phi}_1}{\partial \xi} + \frac{\partial^2 \tilde{\phi}_1}{\partial \xi^2} = 0. \tag{4.24}$$

If we define $2\phi_1^{(\mathrm{s})}(\xi, y) = \phi_1 + \tilde{\phi}_1$, the problem for $\phi_1^{(\mathrm{s})}(\xi, y)$ coming from (3.10), (3.11) and (4.24) takes the form

$$\left(\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \xi^2}\right)\phi_1^{(\mathrm{s})} = 0,$$

$$\phi_1^{(\mathrm{s})}(0, y) = 0,$$

$$\lim_{\xi \to +\infty} \phi_1^{(\mathrm{s})}(\xi, y) = 0$$

and it has only the null solution $\phi_1^{(\mathrm{s})}(\xi, y) \equiv 0$. Hence we conclude that $\phi_1 = -\tilde{\phi}_1$, that is to say

$$\phi_1 \equiv \phi_1^{(\mathrm{a})}. \tag{4.25}$$

The antisymmetrical structure of ϕ_1 together with its boundary layer character raise the question of what kind of circulation must be associated to ϕ_1 itself. In particular, the cyclonic or anticyclonic nature of ϕ_1, say in the northern half-basin, can be deduced by the sign of the associated zonal current

$$u_1\left(\xi, \frac{1}{2}\right) \equiv \left[-\frac{\partial \phi_1}{\partial y}\right]_{y=\frac{1}{2}}. \tag{4.26}$$

If (4.26) is negative, we expect an anticyclonic circulation in the northern half of the basin and a cyclonic one in the southern half. The opposite must happen if (4.26) is positive. In any case

$$v_1\left(\xi, \frac{1}{2}\right) = \left[\frac{\partial \phi_1}{\partial \xi}\right]_{y=\frac{1}{2}} = 0. \tag{4.27}$$

To evaluate u_1 from ϕ_1 we substitute (3.7) and (3.8) into (3.9) and obtain the following equation for ϕ_1 that holds just along the western boundary:

$$\frac{1}{2}\frac{\partial}{\partial y}[\psi_{\mathrm{I}}(0, y)]^2 \exp[-\xi] = \frac{\partial \phi_1}{\partial \xi} + \frac{\partial^2 \phi_1}{\partial \xi^2}, \tag{4.28}$$

where ϕ_1 must satisfy also the boundary and matching conditions

$$\phi_1(0, y) = 0 , \tag{4.29}$$

$$\lim_{\xi \to +\infty} \phi_1(\xi, y) = 0 . \tag{4.30}$$

The solution of problem (4.28), (4.29) and (4.30) is

$$\phi_1(\xi, y) = -\frac{1}{2}\frac{\partial}{\partial y}[\psi_I(0, y)]^2 \xi \exp[-\xi]. \tag{4.31}$$

We easily check that

$$u_1\left(\xi, \frac{1}{2}\right) < 0 , \tag{4.32}$$

where $u_1\left(\xi, \frac{1}{2}\right) = \left[-\frac{\partial}{\partial y}\phi_1(\xi, y)\right]_{y=\frac{1}{2}}$ and ϕ_1 is given by (4.31), and therefore in the northern half of the basin an anticyclone develops while a symmetrical cyclone forms in the southern half. To prove (4.32), we note preliminarily that we can write, with reference to (4.31),

$$\psi_I(0, y) = \gamma \sin(\pi y) , \tag{4.33}$$

where $\gamma = \pi$ or $\gamma = 2$ in correspondence to the adopted wind-stress field (2.1) or (2.2). At this point a straightforward computation gives, using also (4.33), $u_1(\xi, y) = (\gamma\pi)^2 \cos(2\pi y)\, \xi e^{-\xi}$ and, in particular,

$$u_1\left(\xi, \frac{1}{2}\right) = -(\gamma\pi)^2 \xi e^{-\xi}. \tag{4.34}$$

Equation (4.34) trivially satisfies (4.32) and (4.27) is also verified by (4.31). Close to the western boundary, both the cyclone and the anticyclone go linearly to zero in ξ while they decay exponentially for increasing longitudes. On the whole, the superposition $\psi = \psi_0^{(s)} + \left(\frac{\delta_I}{\delta_v}\right)^2 \phi_1^{(a)}$ coming from (3.12), (4.22) and (4.25) explains the formation of the ND of the weakly non-linear gyre. This superposition reproduces the well-known Veronis' solution, reported in fig. 1e, for ψ_0 as in fig. 1c, ϕ_1 as in fig. 1d and $\left(\frac{\delta_I}{\delta_v}\right)^2 = 4\cdot 10^{-3}$. We can obtain an expression analogous to (4.16) as follows. Problem (4.19), (4.20) implies $\int_D \nabla^2 \psi_0^{(s)}\, dx\, dy < 0$ and thus $\mathbf{k} \times \boldsymbol{\nabla}\psi_0^{(s)}$ is anticyclonic. Therefore

$$0 < u_0^{(s)} \equiv \left(-\frac{\partial \psi_0^{(s)}}{\partial y}\right)_{y=1} = -\left(-\frac{\partial \psi_0^{(s)}}{\partial y}\right)_{y=0}.$$

Moreover, the previous analysis of $\phi_1^{(a)}$ allows us to write

$$0 < u_1^{(a)} \equiv \left(-\frac{\partial \phi_1^{(a)}}{\partial y}\right)_{y=1} = \left(-\frac{\partial \phi_1^{(a)}}{\partial y}\right)_{y=0},$$

so the zonal current u_N along $y = 1$ is

$$u_\mathrm{N} = \left(-\frac{\partial \psi_0^{(s)}}{\partial y}\right)_{y=1} + \left(\frac{\delta_\mathrm{I}}{\partial_\mathrm{v}}\right)^2 \left(-\frac{\partial \phi_1^{(a)}}{\partial y}\right)_{y=1} = u_0^{(s)} + \left(\frac{\delta_\mathrm{I}}{\partial_\mathrm{v}}\right)^2 u_1^{(a)}.$$

In the same way, the zonal current u_S along $y = 0$ is

$$u_\mathrm{S} = -u_0^{(s)} + \left(\frac{\delta_\mathrm{I}}{\delta_\mathrm{v}}\right)^2 u_1^{(a)}.$$

At this point we can write

$$\left|\frac{u_\mathrm{N}}{u_\mathrm{S}}\right| = \frac{u_0^{(s)} + \left(\frac{\delta_\mathrm{I}}{\delta_\mathrm{v}}\right)^2 u_1^{(a)}}{u_0^{(s)} - \left(\frac{\delta_\mathrm{I}}{\delta_\mathrm{v}}\right)^2 u_1^{(a)}}.$$

This last equation shows that, due to the smallness of $\left(\frac{\delta_\mathrm{I}}{\delta_\mathrm{v}}\right)^2$, the intensity of the northern zonal current is only slightly greater than the southern one. In other words, the ND is rather weak in the dissipative regime and it demands the presence of the first-order correction term.

Equation (4.25) plays an important role in the integral vorticity balance extended to the whole basin. In fact, integration of (3.4) on D gives, with the aid of (3.3), the equation

$$\int_D T\,\mathrm{d}x\,\mathrm{d}y = \frac{\delta_\mathrm{v}}{L}\int_D \nabla^2 \psi\,\mathrm{d}x\,\mathrm{d}y\,, \tag{4.35}$$

while the same procedure applied to (3.5) gives

$$\int_D T\,\mathrm{d}x\,\mathrm{d}y = \frac{\delta_\mathrm{v}}{L}\int_D \nabla^2 \psi_0\,\mathrm{d}x\,\mathrm{d}y\,, \tag{4.36}$$

where, in the range of weak non-linearity, expansion (3.12) can be truncated as

$$\psi \approx \psi_0 + \left(\frac{\delta_\mathrm{I}}{\delta_\mathrm{v}}\right)^2 \phi_1. \tag{4.37}$$

Comparison of (4.35), written in terms of (4.37), with (4.36) shows that the vorticity input by the wind is fully dissipated through the bottom friction associated to the zeroth-order field ψ_0 and thus the following constraint must hold on the first-order correction in order that the system be able to achieve a steady state:

$$0 = \int_D \nabla^2 \phi_1\,\mathrm{d}x\,\mathrm{d}y\,. \tag{4.38}$$

This last equation is equivalent to

$$\int_0^{+\infty}\int_0^1 \left[\left(\frac{L}{\delta_{\mathrm{v}}}\right)^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial y^2}\right]\phi_1 \,\mathrm{d}y\,\mathrm{d}\xi = 0\,. \tag{4.39}$$

As the differential operator appearing into the square braket of (4.39) is invariant under (2.3) and $\phi_1 \equiv \phi_1^{(\mathrm{a})}$, we can resort to (2.10) to conclude that (4.38) is verified, thus implying the vorticity balance that ensures the steadiness of the motion.

5. – Asymmetries of the highly non-linear gyre

In this section we will explore some symmetry properties of the leading terms ψ_0 and ψ_1 of the solution of problem (3.3), (3.13) on the basis of problems (3.15), (3.16) and (3.17), (3.18). We wish to anticipate that problem (3.3), (3.13) can be solved only numerically and this has been formerly done in [5], with T given by (2.1), (2.2) and $\frac{\delta_{\mathrm{I}}}{L} = 0.05$, $r = 0.1$. The relative solutions are displayed in figs. 2a and 3a.

The basic difficulty in dealing with problems (3.15), (3.16) and (3.17), (3.18) is that, while (3.15) allows us to write

$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 \nabla^2 \psi_0 + y = F(\psi_0)\,, \tag{5.1}$$

where $F(\psi_0)$ may be, *a priori*, an arbitrary differentiable function of its argument, on the other hand (3.17) presupposes a link between $F(\psi_0)$ and the forcing T. This means that $F(\psi_0)$ can actually be singled out only once T is explicitly given. In general, the evaluation of $F(\psi_0)$ requires numerical procedures, but the cases (2.1) and (2.2) have been formerly investigated in [5], where all the details of the employed method are extensively explained. In both cases the scatter plot of the total vorticity $\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 \nabla^2 \psi_0 + y$ *vs.* ψ_0, reported in figs. 2b and 3b, shows that

$$\frac{\partial F}{\partial \psi_0} > 0 \tag{5.2}$$

and that the unifunctional function $F(\psi_0)$ is *linear* in the interior within a very good approximation (recall that $\psi_0 = 0$ along the boundary, and F is not linear for values close to zero). Due to the smallness of $\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2$ with respect to unity, in the interior (5.1) can be approximated by

$$y \approx F(\psi_0)\,, \tag{5.3}$$

so that the zeroth-order flow is zonal in this region and, because of (5.3),

$$u_0 \approx -\frac{\partial}{\partial y} F^{-1}(y)\,. \tag{5.4}$$

From (5.2) and (5.4) we see that $u_0 < 0$, *i.e.* westward, in the interior.

Inequality (5.2) implies that ψ_0 is invariant under transform (2.4), that is to say

$$\psi_0 \equiv \overline{\psi}_0\,. \tag{5.5}$$

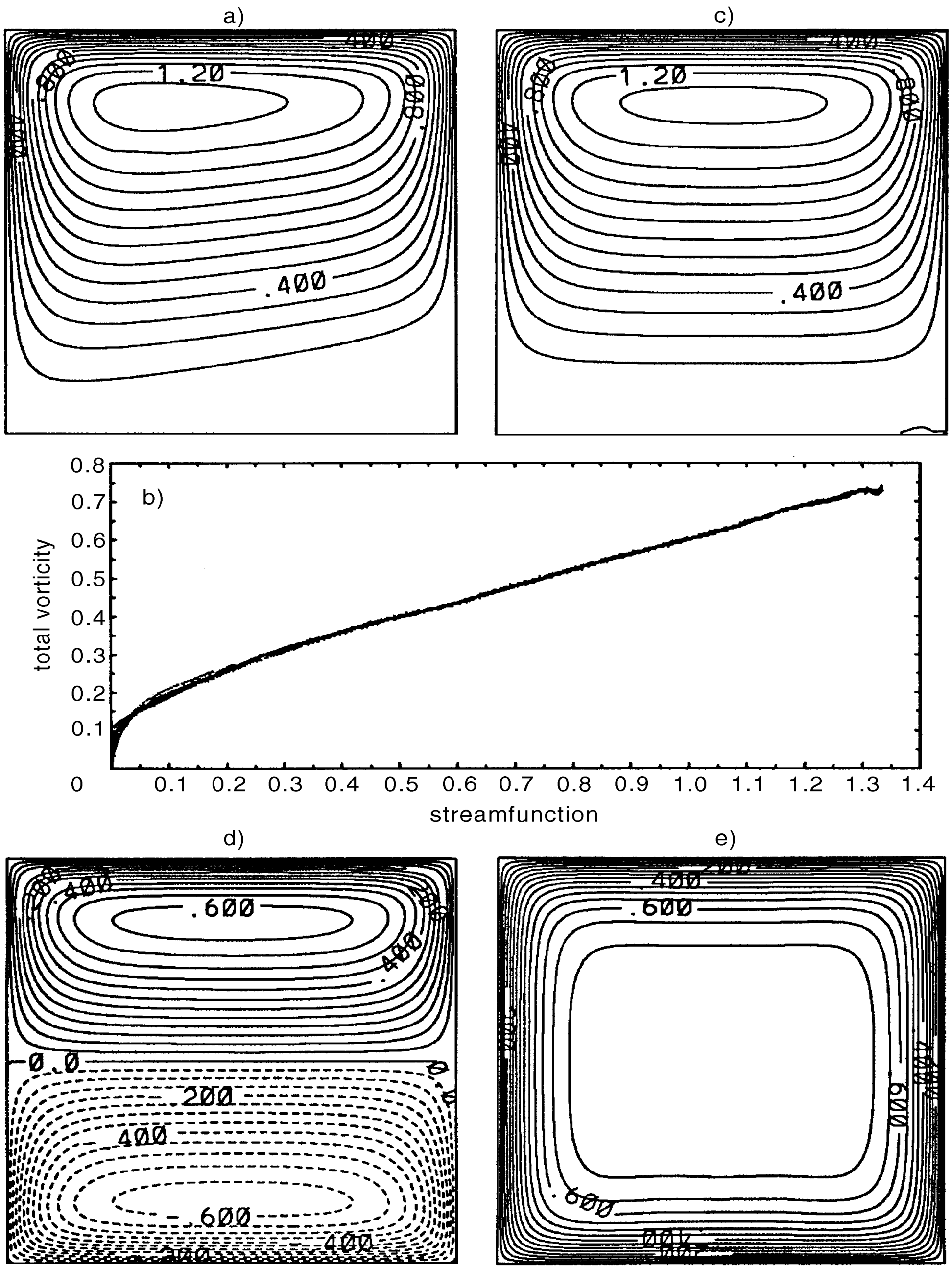

Fig. 2 . – a) Streamlines of the solution of problem (3.3), (3.13) for T given by (2.1). Note the marked ND of the gyre and its weak WI. b) Scatter plot of the total vorticity *vs.* the streamfunction for the inertial problem (3.15), (3.16) where (2.1) has been used to evaluate $F(\psi_0)$. c) Streamlines of the zeroth-order solution, for T given by (2.1). The East-West mirror invariance is evident. d) Streamlines of $\psi_0^{(a)}$ for ψ_0 given in c). e) Streamlines of $\psi_0^{(s)}$ for ψ_0 given in c).

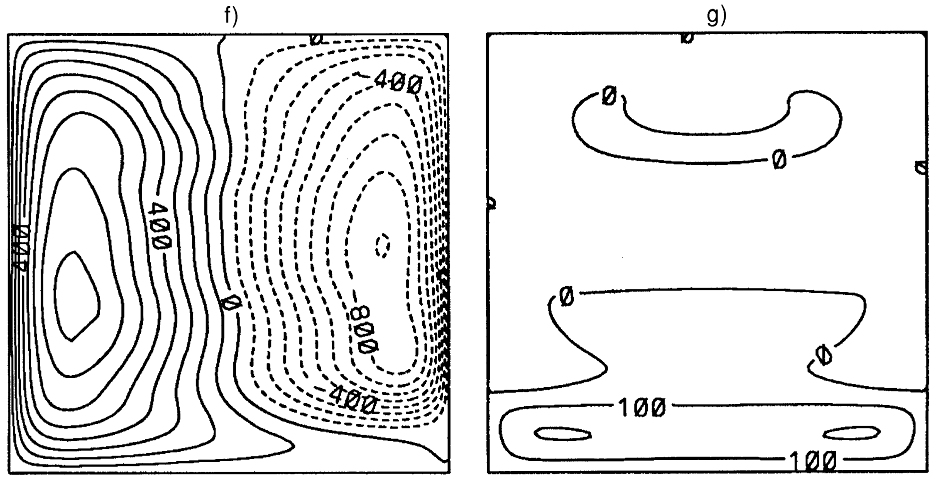

Fig. 2. (*Continued*) – f) Streamlines of the residual streamfunction $\psi - \psi_0 = \delta\psi_1 + O(\delta^2)$, where ψ and ψ_0 are displayed in a) and c), respectively. Within the approximation introduced by the term $O(\delta^2)$, this plot points out the antisymmetry of ψ_1. g) East-West symmetric component of $\psi - \psi_0$ of f). This plot supports statement (5.28).

In fact, eq. (5.1) can be transformed into

$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 \nabla^2 \overline{\psi}_0 + y = F(\overline{\psi}_0), \tag{5.6}$$

so (5.1) and (5.6) yield

$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 \nabla^2 (\psi_0 - \overline{\psi}_0) = F(\psi_0) - F(\overline{\psi}_0). \tag{5.7}$$

Setting $\psi_0 - \overline{\psi}_0 = \varphi$, and hence $\varphi = 0$ along ∂D, from (5.7) we have

$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 \varphi \nabla^2 \varphi = \varphi^2 \frac{F(\psi_0) - F(\overline{\psi}_0)}{\psi_0 - \overline{\psi}_0} \tag{5.8}$$

and the integration of (5.8) on D gives

$$-\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 \int_D |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}y = \int_D \varphi^2 \frac{F(\psi_0) - F(\overline{\psi}_0)}{\psi_0 - \overline{\psi}_0} \,\mathrm{d}x\,\mathrm{d}y\,. \tag{5.9}$$

Because of (5.2), the rhs of (5.9) is non-negative while the lhs of the same equation is, trivially, non-positive. Therefore, the only possibility is $\varphi = 0$ everywhere, that is to say (5.5) or, with an equivalent notation,

$$\psi_0 \equiv \psi_{0\mathrm{s}}. \tag{5.10}$$

Invariance (5.10) is quite evident in figs. 2c and 3c. At this point, a kind of duality begins to emerge between the symmetry properties of the zeroth-order solution of

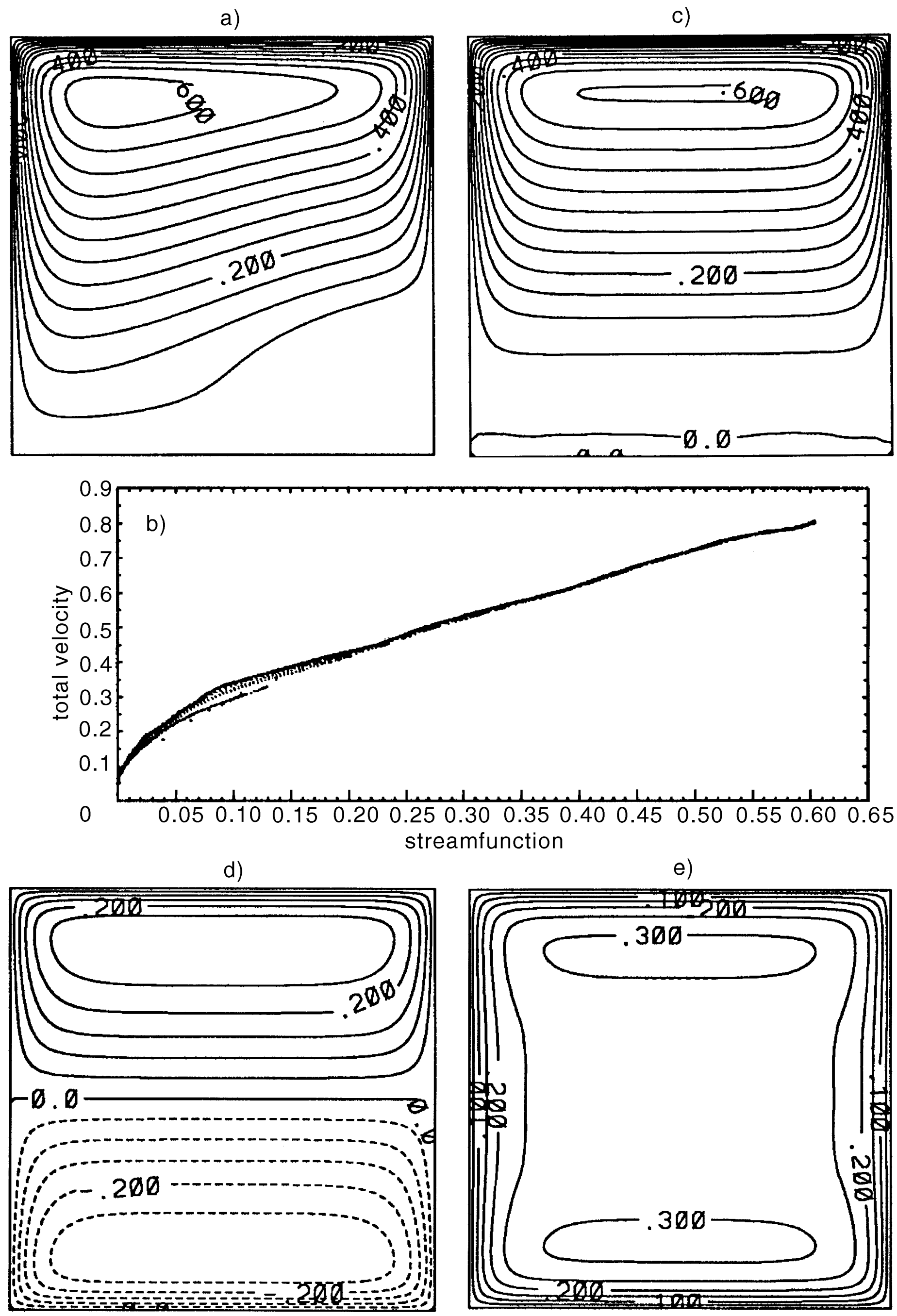

Fig. 3. – a) The same as fig. 2a), but with T given by (2.2). b) The same as fig. 2b), but with T given by (2.2). c) The same as fig. 2c), but with T given by (2.2). d) The same as fig. 2d), but with T given by (2.2). e) The same as fig. 2e), but with T given by (2.2).

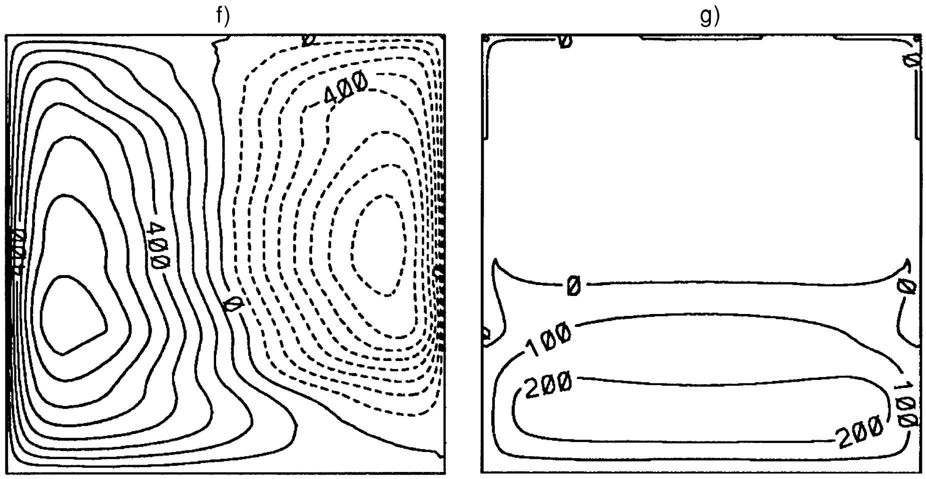

Fig. 3. (*Continued*) – f) The same as fig. 2f), but with T given by (2.2). g) The same as fig. 2g), but with T given by (2.2).

(3.5), (3.6) and that of (3.15), (3.16): to the North-South invariance (4.22) of the weakly non-linear case, the invariance (5.10) of the highly non-linear regime is associated in the duality relationship. In this relationship, the forced meridional current of the interior (4.12) is associated with the inertial zonal current (5.4), in the sense that the first is North-South mirror-symmetric while the second is East-West mirror symmetric. We will see, in the last part of this section, the contribution of (5.10) to the formation of the WI of the highly non-linear gyre, in strict duality with the contribution of (4.22) to the formation of the ND in the weakly non-linear regime.

In analogy with the generation of the WI by means of the superposition of $\psi_{0\mathrm{s}}$ and $\psi_{0\mathrm{a}}$ in the weakly non-linear regime, we show how, in the highly non-linear regime, the superposition of $\psi_0^{(\mathrm{s})}$ with $\psi_0^{(\mathrm{a})}$ generated the ND. The coupled equations for $\psi_0^{(\mathrm{s})}$ and $\psi_0^{(\mathrm{a})}$ are obtained by inserting the identity (2.6), referred to the zeroth-order solution, into (3.15) and then by applying (2.3) to the result. The procedure leads to the following equations:

(5.11)
$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 [J(\psi_0^{(\mathrm{s})}, \nabla^2\psi_0^{(\mathrm{a})}) + J(\psi_0^{(\mathrm{a})}, \nabla^2\psi_0^{(\mathrm{s})})] + \frac{\partial\psi_0^{(\mathrm{s})}}{\partial x} = 0\,,$$

(5.12)
$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 [J(\psi_0^{(\mathrm{s})}, \nabla^2\psi_0^{(\mathrm{s})}) + J(\psi_0^{(\mathrm{a})}, \nabla^2\psi_0^{(\mathrm{a})})] + \frac{\partial\psi_0^{(\mathrm{a})}}{\partial x} = 0\,.$$

Consider first the antisymmetric component $\psi_0^{(\mathrm{a})}$: because of (5.11), it cannot be identically vanishing. In fact, under this circumstance, (5.11) implies that $\psi_0^{(\mathrm{s})}$ is strictly zonal, but the no mass boundary condition, in turn, imposes $\psi_0^{(\mathrm{s})} \equiv 0$, so, on the whole, we would have $\psi_0 \equiv 0$.

Because of (5.2), relation (5.3) can be reverted to give, in the interior,

$$\psi_0^{(s)} + \psi_0^{(a)} \approx F^{-1}(y)\,, \tag{5.13}$$

and hence also

$$\psi_0^{(s)} - \psi_0^{(a)} \approx F^{-1}(1-y)\,. \tag{5.14}$$

From (5.13) and (5.14) we obtain $\psi_0^{(a)} \approx \frac{1}{2}[F^{-1}(y) - F^{-1}(1-y)]$ and, in particular,

$$\psi_0^{(a)}\left(x, \frac{1}{2}\right) = 0\,. \tag{5.15}$$

Because of the unifunctional structure of F and (5.2), $y = \frac{1}{2}$ is the sole latitude where $\psi_0^{(a)}$ vanishes. As we already know from (5.4), the interior current is westward and therefore the field $\psi_0^{(a)}$ represents an anticyclone in the northern half-basin and a cyclone in the southern one, according to figs. 2d and 3d.

Consider now the symmetric component $\psi_0^{(s)}$. Neither (5.11) nor (5.12) seem to prevent from the solution $\psi_0^{(s)} \equiv 0$, however another argument can be invoked to prove that, out of necessity, $\psi_0^{(s)} \neq 0$. In fact, integration of (3.17) on D with the aid of (3.16) and (3.18) gives, recalling also (2.10) with $\nabla^2 \psi_0^{(a)}$ in place of $\psi_0^{(a)}$,

$$\int_D T\,\mathrm{d}x\,\mathrm{d}y = \frac{\delta_{\mathrm{I}}}{L} \int_D \nabla^2 \psi_0^{(s)}\,\mathrm{d}x\,\mathrm{d}y \tag{5.16}$$

and, as the lhs of (5.16) is negative, (5.16) itself would be inconsistent if $\psi_0^{(s)} \equiv 0$. The behaviour of $\psi_0^{(s)}$ in the interior can be deduced from (5.13) and (5.14). From these two equations we evaluate $\psi_0^{(s)} \approx \frac{1}{2}[F^{-1}(y) + F^{-1}(1-y)]$ and therefore we see that $\psi_0^{(s)}$ is zonal and, due to the linearity of $F(y)$ in the interior, $\frac{\partial \psi_0^{(s)}}{\partial y} \equiv 0$. On the whole, in the interior $\psi_0^{(s)}$ is flat while it describes a circulation that takes place inside a "ring" close to the boundary of the fluid domain. Note that (5.16) can be written as $\int_D T\,\mathrm{d}x\,\mathrm{d}y = \frac{\delta_{\mathrm{I}}}{L} \oint_{\partial D} \mathbf{u}_0^{(s)} \cdot \mathrm{d}\mathbf{t}$ and thus $\oint_{\partial D} \mathbf{u}_0^{(s)} \cdot \mathrm{d}\mathbf{t} < 0$. This means that the current associated to $\psi_0^{(s)}$, that is $\mathbf{u}_0^{(s)} = \mathbf{k} \times \boldsymbol{\nabla} \psi_0^{(s)}$, flows clockwise and $\psi_0^{(s)}$ represents an anticyclonic circulation. Some streamlines of $\psi_0^{(s)}$ are shown in figs. 2e and 3e.

The superposition of $\psi_0^{(s)}$ with $\psi_0^{(a)}$ generates the ND of the gyre that turns out to be a feature of the zeroth-order solution (we recall figs. 2c and 3c), in the same manner as the WI is a characteristic of the zeroth-order solution in the weakly non-linear regime. The current $\mathbf{u}_0^{(s)}$ has the same intensity along both the zonal boundaries, but it is amplified in the northern half-basin by the anticyclone corresponding to $\psi_0^{(a)}$ while it is weakened in the southern half-basin by the cyclone generated by the same antisymmetric solution. We see, from this picture, that in the southern half of the basin $\boldsymbol{\nabla} \psi_0^{(s)}$ and $\boldsymbol{\nabla} \psi_0^{(a)}$ are opposite. Why the circulation produced there by $\psi_0 = \psi_0^{(s)} + \psi_0^{(a)}$ is actually anticyclonic on the whole? In general, if R is any region of the fluid domain encircled by a (close) streamline ∂R of ψ_0, then integration of (3.17) written with the aid of (5.1) as

$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 J(\psi_0, \nabla^2\psi_1) + \frac{\partial F}{\partial \psi_0} J(\psi_1, \psi_0) = T - \frac{\delta_{\mathrm{I}}}{L}\nabla^2\psi_0$$

yields

$$\int_R T\,\mathrm{d}x\,\mathrm{d}y = \frac{\delta_{\mathrm{I}}}{L}\oint_{\partial R} \mathbf{u}_0 \cdot \mathrm{d}\mathbf{t}\,, \tag{5.17}$$

where $\mathbf{u}_0 = \mathbf{k} \times (\nabla\psi_0^{(\mathrm{s})} + \nabla\psi_0^{(\mathrm{a})})$. As the lhs of (5.17) is certainly negative, $\mathbf{u}_0 \cdot \mathrm{d}\mathbf{t} \leqslant 0$ whatever the streamline ∂R may be and every cyclonic circulation ascribed to ψ_0 is excluded. The intense ND of the highly non-linear regime can be pointed out by using the symmetry properties of $\psi_0^{(\mathrm{s})}$ and $\psi_0^{(\mathrm{a})}$. We define the westward zonal currents

$$0 < u_0^{(\mathrm{s})} \equiv \left(-\frac{\partial \psi_0^{(\mathrm{s})}}{\partial y}\right)_{y=1} = -\left(-\frac{\partial \psi_0^{(\mathrm{s})}}{\partial y}\right)_{y=0}$$

and

$$0 < u_0^{(\mathrm{a})} \equiv \left(-\frac{\partial \psi_0^{(\mathrm{a})}}{\partial y}\right)_{y=1} = \left(-\frac{\partial \psi_0^{(\mathrm{a})}}{\partial y}\right)_{y=0}$$

and evaluate the zonal currents along the northern and southern boundaries, u_{N} and u_{S}, respectively:

$$u_{\mathrm{N}} = \left(-\frac{\partial \psi_0^{(\mathrm{s})}}{\partial y}\right)_{y=1} + \left(-\frac{\partial \psi_0^{(\mathrm{a})}}{\partial y}\right)_{y=1} = u_0^{(\mathrm{s})} + u_0^{(\mathrm{a})}$$

and

$$u_{\mathrm{S}} = \left(-\frac{\partial \psi_0^{(\mathrm{s})}}{\partial y}\right)_{y=0} + \left(-\frac{\partial \psi_0^{(\mathrm{a})}}{\partial y}\right)_{y=0} = -u_0^{(\mathrm{s})} + u_0^{(\mathrm{a})}.$$

Recalling that $-u_0^{(\mathrm{s})} + u_0^{(\mathrm{a})} < 0$, we evaluate the ratio

$$\left|\frac{u_{\mathrm{N}}}{u_{\mathrm{S}}}\right| = \frac{u_0^{(\mathrm{s})} + u_0^{(\mathrm{a})}}{u_0^{(\mathrm{s})} - u_0^{(\mathrm{a})}}$$

that can reach values $\gg 1$ for $u_0^{(\mathrm{s})}$ and $u_0^{(\mathrm{a})}$ sufficiently close to each other.

The anticyclonic structure of ψ_0 shall reveal itself to be crucial in the generation of the WI, as will be clarified in what follows.

To investigate the generation of the WI , we need to have suitable information about the behaviour of ψ_1 in terms of its components $\psi_{1\mathrm{a}}$ and $\psi_{1\mathrm{s}}$. To this purpose, we substitute the identity $\psi_1 = \psi_{1\mathrm{a}} + \psi_{1\mathrm{s}}$ into (3.17), recalling also (5.10) to obtain the equations

$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 [J(\psi_0, \nabla^2\psi_{1\mathrm{a}}) + J(\psi_{1\mathrm{a}}, \nabla^2\psi_0)] + \frac{\partial \psi_{1\mathrm{a}}}{\partial x} = T - \frac{\delta_{\mathrm{I}}}{L}\nabla^2\psi_0 \tag{5.18}$$

and

$$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2 J(\psi_0, \nabla^2\psi_{1\mathrm{s}}) + J(\psi_{1\mathrm{s}}, F(\psi_0)) = 0 \,, \tag{5.19}$$

with the boundary conditions

$$\psi_{1\mathrm{a}} = \psi_{1\mathrm{s}} = 0 \,, \qquad \forall (x, y) \in \partial D \,.$$

A direct consequence of (5.18) is that $\psi_{1\mathrm{a}}$ cannot be everywhere vanishing in D. In fact, under such a circumstance, we would have, point by point,

$$T = \frac{\delta_{\mathrm{I}}}{L}\nabla^2\psi_0$$

and hence, because of (5.1),

$$\frac{\delta_{\mathrm{I}}}{L} T + y = F(\psi_0) \,. \tag{5.20}$$

In $y = 0$ we have both $T = 0$ and $\psi_0 = 0$ and the same holds in $y = 1$. Therefore (5.20) yields, at the same time, $F(0) = 0$ and $F(0) = 1$. This contradiction demands

$$\psi_{1\mathrm{a}} \neq 0 \quad \text{for some } (x, y) \text{ of } D \,. \tag{5.21}$$

Because of (2.9), it follows that

$$\psi_{1\mathrm{a}}\left(\frac{1}{2}, y\right) = 0 \,, \qquad \forall y \in [0, 1] \tag{5.22}$$

and, in the interior, (5.18) gives

$$\frac{\partial \psi_{1\mathrm{a}}}{\partial x} \approx T(<0) \,. \tag{5.23}$$

Equations (5.22) and (5.23) are representative of a double-gyre flow field, with an anticyclone on the western side and a cyclone on the eastern one. In particular, northward return flows are expected both along the western and the eastern sides of the basin. Numerical experiments [4] actually corroborate this picture, which is reported in figs. 2f and 3f. Therefore

$$0 < v_{1\mathrm{a}} \equiv \left(\frac{\partial \psi_{1\mathrm{a}}}{\partial x}\right)_{x=0} = \left(\frac{\partial \psi_{1\mathrm{a}}}{\partial x}\right)_{x=1} , \tag{5.24}$$

while the anticyclonic structure of $\psi_0 \equiv \psi_{0\mathrm{s}}$ allows us to put

$$0 < v_{0\mathrm{s}} \equiv \left(\frac{\partial \psi_{0\mathrm{s}}}{\partial x}\right)_{x=0} = -\left(\frac{\partial \psi_{0\mathrm{s}}}{\partial x}\right)_{x=1} , \tag{5.25}$$

so the WI can be easily explained in terms of the superposition of $v_{0\mathrm{s}}$ with $v_{1\mathrm{a}}$. In fact, disregarding for the time being the symmetric part of the first-order correction, along the western coast the meridional current v_W can be written as

$$v_W = \left(\frac{\partial \psi_{0s}}{\partial x} \right)_{x=0} + r \left(\frac{\partial \psi_{1a}}{\partial x} \right)_{x=0},$$

while, along the eastern coast, the meridional current v_E is

$$v_E = \left(\frac{\partial \psi_{0s}}{\partial x} \right)_{x=1} + r \left(\frac{\partial \psi_{1a}}{\partial x} \right)_{x=1}.$$

Therefore, in terms of (5.24) and (5.25),

$$v_W = v_{0s} + r v_{1a}$$

and

$$v_E = -v_{0s} + r v_{1a}.$$

The relative intensity of the meridional currents is

$$\left| \frac{v_W}{v_E} \right| = \frac{v_{0s} + r v_{1a}}{v_{0s} - r v_{1a}}. \tag{5.26}$$

The ratio (5.26) shows that, in the regime under investigation, the WI is weak. This unlike (4.16) where, due to the possibility to have a small denominator, $\left| \frac{v_W}{v_E} \right| \gg 1$. On the contrary, in (5.26), because of the smallness of r, $\left| \frac{v_W}{v_E} \right| = O(1)$.

The possible $O(r)$ contribution (if any) of $v_{1s} \equiv \frac{\partial \psi_{1s}}{\partial x}$ does not modify the mechanism of the generation of the WI quoted above, as $v_{0s} + r v_{1s}$ has the same symmetry property as v_{0s}. However, an interesting point arises: what can be inferred about ψ_{1s}?

The first-order correction ψ_1 can be evaluated numerically, within a certain approximation, as the difference between the solution of problem (3.3), (3.13) with $r = 0.1$ and the solution of the same problem, for a very low value of r. In a previous paper [4], $r = 0.001$ was chosen and the so-obtained solution was identified with ψ_0. Then, ψ_1 was evaluated through the approximate equation

$$\psi_1 \approx 10(\psi - \psi_0). \tag{5.27}$$

Finally, starting from (5.27) and applying (2.4) to it, both ψ_{1a} and ψ_{1s} were singled out. The result is that, compatibly with the approximations so introduced, ψ_{1s} turns out to be identically vanishing (figs. 2g and 3g). Actually,

$$\psi_{1s} \equiv 0, \qquad \forall (x, y) \in D \tag{5.28}$$

is a special solution of (5.19) which, however, admits, in general, also non-vanishing solutions (see the following eq. (5.36)). The question is: why *the starting problem* (3.3), (3.13) does prefer (5.28)? The answer resorts to the vorticity balance of the steady circulation, according to the following arguments. Integration of (3.13) on D gives

$$\int_D T \, dx \, dy = \frac{\delta_I}{L} \int_D \nabla^2 \psi \, dx \, dy, \tag{5.29}$$

while integration of (3.17) on D yields

$$\int_D T \, \mathrm{d}x \, \mathrm{d}y = \frac{\delta_{\mathrm{I}}}{L} \int_D \nabla^2 \psi_0 \, \mathrm{d}x \, \mathrm{d}y \,. \tag{5.30}$$

By equating the rhs of (5.29) to the rhs of (5.30) and recalling (3.14), we obtain

$$\int_D \nabla^2 (\psi_0 + r\psi_1 + \ldots) \, \mathrm{d}x \, \mathrm{d}y = \int_D \nabla^2 \psi_0 \, \mathrm{d}x \, \mathrm{d}y \,. \tag{5.31}$$

At the first order in r, (5.31) implies

$$\int_D \nabla^2 \psi_1 \, \mathrm{d}x \, \mathrm{d}y = 0 \,. \tag{5.32}$$

If we set $\psi_1 = \psi_{1\mathrm{a}} + \psi_{1\mathrm{s}}$ and take (2.10) into account, we conclude from (5.32) that

$$\int_D \nabla^2 \psi_{1\mathrm{s}} \, \mathrm{d}x \, \mathrm{d}y = 0 \,. \tag{5.33}$$

Equation (5.33) is a constraint that selects, among all the possible solutions of the problem

$$\left(\frac{\delta_{\mathrm{I}}}{L} \right)^2 J(\psi_0, \nabla^2 \psi_{1\mathrm{s}}) + J(\psi_{1\mathrm{s}}, F(\psi_0)) = 0 \,, \qquad \forall (x, y) \in D \,, \tag{5.34}$$

$$\psi_{1\mathrm{s}} = 0 \,, \forall (x, y) \in \partial D \,, \tag{5.35}$$

the sole solution (5.28). This fact can be *explained* in the framework of boundary layer solutions of (5.34), (5.35) by applying (5.32) to them: the only admissible solution turns out to be just (5.28).

To this purpose, we note preliminarily that (5.34) is equivalent to

$$\left(\frac{\delta_{\mathrm{I}}}{L} \right)^2 \nabla^2 \psi_{1\mathrm{s}} - \frac{\partial F}{\partial \psi_0} \psi_{1\mathrm{s}} = G(\psi_0) \,, \tag{5.36}$$

where $G(\psi_0)$ is any differentiable function of its argument. Then, because of (5.35), (5.36) implies

$$\nabla^2 \psi_{1\mathrm{s}} = \left(\frac{L}{\delta_{\mathrm{I}}} \right)^2 G(0) \,, \qquad \forall (x, y) \in \partial D \,. \tag{5.37}$$

Equation (5.37) will be useful in the following. Consider now (5.34) written as

$$\left(\frac{\delta_{\mathrm{I}}}{L} \right)^2 \mathbf{k} \cdot \boldsymbol{\nabla} \psi_0 \times \boldsymbol{\nabla} (\nabla^2 \psi_{1\mathrm{s}}) - \frac{\partial F}{\partial \psi_0} \mathbf{k} \cdot \boldsymbol{\nabla} \psi_0 \times \boldsymbol{\nabla} \psi_{1\mathrm{s}} = 0 \,, \tag{5.38}$$

and focous attention to the western boundary. We introduce the western boundary layer coordinate

$$\xi = \frac{L}{l} x$$

and define the boundary layer non-dimensional width L/l through the equation

(5.39) $$\left(\frac{\delta_{\mathrm{I}}}{L}\right)^2\left(\frac{L}{l}\right)^2 = F_0^{\mathrm{I}}$$

where F_0^{I} is the constant value that $\partial F/\partial\psi_0$ takes along the boundary. From the value taken by δ_{I}/L and the scatter plots representative of the function $F(\psi_0)$, we see that (5.39) implies

(5.40) $$\frac{L}{l} \gg 1\,.$$

According to the standard technique, we set

(5.41) $$\psi_{\mathrm{W}} = \psi_{\mathrm{I}}(y) + \phi_{\mathrm{W}}(\xi, y)\,,$$

where the zonality of the interior $\psi_{\mathrm{I}}(y)$ of $\psi_{1\mathrm{s}}$ immediately comes from the zonality of ψ_0 and the equation $J(\psi_{1\mathrm{s}}, \psi_0) = 0$ that holds in the interior, this last being the approximate version of (5.34), valid for this area of the fluid domain. From (5.41) we have

$$\nabla\psi_{\mathrm{W}} = \nabla\psi_{\mathrm{I}}(y) + \frac{L}{l}\frac{\partial}{\partial\xi}\phi_{\mathrm{W}}\mathbf{i} + \frac{\partial}{\partial y}\phi_{\mathrm{W}}\mathbf{j}$$

and, because of (5.40), we can introduce the basic approximation

(5.42) $$\nabla\psi_{\mathrm{W}} \approx \frac{L}{l}\frac{\partial}{\partial\xi}\phi_{\mathrm{W}}\mathbf{i}\,.$$

Moreover, from (5.42) we easily evaluate

(5.43) $$\nabla^2\psi_{\mathrm{W}} \approx \left(\frac{L}{l}\right)^2\frac{\partial^2}{\partial\xi^2}\phi_{\mathrm{W}}$$

whence

(5.44) $$\nabla(\nabla^2\psi_{\mathrm{W}}) \approx \left(\frac{L}{l}\right)^3\frac{\partial^3}{\partial\xi^3}\phi_{\mathrm{W}}\mathbf{i}\,.$$

Substitution of (5.42), (5.43) and (5.44) into (5.38) with F_0^{I} instead of $\partial F/\partial\psi_0$ gives, recalling also (5.39),

(5.45) $$\frac{\partial^3}{\partial\xi^3}\phi_{\mathrm{W}} - \frac{\partial}{\partial\xi}\phi_{\mathrm{W}} = 0\,.$$

The unique solution of (5.45) satisfying the asymptotic behaviour

$$\phi_{\mathrm{W}}(+\infty, y) = 0$$

and the boundary condition

$$\psi_{\mathrm{I}}(y) + \phi_{\mathrm{W}}(0, y) = 0$$

is

$$\phi_{\mathrm{W}} = -\psi_{\mathrm{I}}(y)\exp[-\xi]$$

and hence

$$\psi_{\mathrm{W}} = \psi_{\mathrm{I}}(y)[1 - \exp[-\xi]]. \tag{5.46}$$

In particular, from (5.46) we have

$$\nabla^2 \psi_{\mathrm{W}}(x=0) \approx -\psi_{\mathrm{I}}(y)\left(\frac{L}{l}\right)^2 \tag{5.47}$$

and the comparison of (5.47) with (5.37) yields

$$\psi_{\mathrm{I}}(y) = -\frac{G(0)}{F_0^{\mathrm{I}}},$$

so

$$\psi_{\mathrm{W}} = -\frac{G(0)}{F_0^{\mathrm{I}}}[1 - \exp[-\xi]]. \tag{5.49}$$

Therefore, using (5.49), the meridional current along the western boundary is

$$v(0, y) = \left.\frac{\partial \psi_{\mathrm{W}}}{\partial x}\right|_{x=0} = -\frac{L}{\delta_{\mathrm{I}}}\frac{G(0)}{\sqrt{F_0^{\mathrm{I}}}}. \tag{5.50}$$

By repeating analogous calculations for the remaining sides of the boundary, we obtain

$$v(1, y) = \frac{L}{\delta_{\mathrm{I}}}\frac{G(0)}{\sqrt{F_0^{\mathrm{I}}}}, \qquad u(x, 1) = -\frac{L}{\delta_{\mathrm{I}}}\frac{G(0)}{\sqrt{F_0^{\mathrm{I}}}} \text{ and } u(x, 0) = \frac{L}{\delta_{\mathrm{I}}}\frac{G(0)}{\sqrt{F_0^{\mathrm{I}}}}. \tag{5.51}$$

The integral constraint (5.33) can be written as

$$\int_0^1 [v(1, y) - v(0, y)]\,\mathrm{d}y - \int_0^1 [u(x, 1) - u(x, 0)]\,\mathrm{d}x = 0 \tag{5.52}$$

and the substitution of (5.50) and (5.51) into (5.52) yields $4\frac{L}{\delta_{\mathrm{I}}}\frac{G(0)}{\sqrt{F_0^{\mathrm{I}}}} = 0$, that is to say

$$G(0) = 0. \tag{5.53}$$

Equation (5.53) implies, through (5.49), $\psi_{\mathrm{W}} = 0$ and, in the same way, also $\psi_{\mathrm{N}} = \psi_{\mathrm{E}} = \psi_{\mathrm{S}} = 0$, so $\psi_{1\mathrm{s}} \equiv 0$. To summarize, the boundary layer method, which is characterized in the present context by (5.42), *singles out a class of solutions*, each depending on a special value of $G(0)$ (recall, for instance, (5.49)). *Within this class*, the constraint (5.33) implies (5.53) and hence (5.28). This result is consistent with the above-described numerical experiments but, probably, it is not exhaustive with respect to *all the solutions* of problem (5.34), (5.35).

6. – Duality

A duality relationship, between the regimes investigated in the present paper, can be easily drawn from a summary of the basic features derived in sect. **4** and **5**, according to the following list:

Weakly non-linear regime	*Highly non-linear regime*
Truncated streamfunction:	Truncated streamfunction:
$\psi = \psi_0 + \left(\frac{\delta_\mathrm{I}}{\delta_\mathrm{v}}\right)^2 \phi_1$	$\psi = \psi_0 + r\psi_1$
$\psi_0 \equiv \psi_0^{(\mathrm{s})}$, *i.e.* N $\leftrightarrow$ S invariance	$\psi_0 \equiv \psi_{0\mathrm{s}}$, *i.e.* E $\rightarrow$ W invariance
$\psi_0 = \psi_{0\mathrm{s}} + \psi_{0\mathrm{a}}$:	$\psi_0 = \psi_0^{(\mathrm{s})} + \psi_0^{(\mathrm{a})}$:
the superposition generates a marked WI	the superposition generates a marked ND
$\left\lvert \frac{v_\mathrm{W}}{v_\mathrm{E}} \right\rvert = \frac{v_{0\mathrm{s}} + v_{0\mathrm{a}}}{v_{0\mathrm{s}} - v_{0\mathrm{a}}}$	$\left\lvert \frac{u_\mathrm{N}}{u_\mathrm{S}} \right\rvert = \frac{u_0^{(\mathrm{s})} + u_0^{(\mathrm{a})}}{u_0^{(\mathrm{s})} - u_0^{(\mathrm{a})}}$
$\phi_1 \equiv \phi_1^{(\mathrm{a})}$, *i.e.* N $\leftrightarrow$ S antisymmetry	$\psi = \psi_{1\mathrm{a}}$, *i.e.* E $\leftrightarrow$ W antisymmetry
$\psi = \psi_0^{(\mathrm{s})} + \left(\frac{\delta_\mathrm{I}}{\partial_\mathrm{v}}\right)^2 \phi_1^{(\mathrm{a})}$:	$\psi = \psi_{0\mathrm{s}} + r\psi_{1\mathrm{a}}$:
the superposition generates a weak ND	the superposition generates a weak WI
$\left\lvert \frac{u_\mathrm{N}}{u_\mathrm{S}} \right\rvert = \frac{u_0^{(\mathrm{s})} + \left(\frac{\delta_\mathrm{I}}{\delta_\mathrm{v}}\right)^2 u_1^{(\mathrm{a})}}{u_0^{(\mathrm{s})} - \left(\frac{\delta_\mathrm{I}}{\delta_\mathrm{v}}\right)^2 u_1^{(\mathrm{a})}}$	$\left\lvert \frac{v_\mathrm{W}}{v_\mathrm{E}} \right\rvert = \frac{v_{0\mathrm{s}} + rv_{1\mathrm{a}}}{v_{0\mathrm{s}} - rv_{1\mathrm{a}}}$

The situation reported in the above scheme is amenable to a description as follows, where the terms of the kind *A*/*B* mean *A* if they are referred to the weakly non-linear regime and *B* if referred to the highly non-linear one. We recall that both solutions are written as truncated expansions constituted by a suitable zeroth-order solution plus a first-order correction multiplied by a small coupling parameter.

The zeroth-order solution is invariant under the mirror reflection $\frac{\mathrm{N} \leftrightarrow \mathrm{S}}{\mathrm{E} \leftrightarrow \mathrm{W}}$, while both its parts, that are, respectively, symmetric and antisymmetric under the $\frac{\mathrm{E} \leftrightarrow \mathrm{W}}{\mathrm{N} \leftrightarrow \mathrm{S}}$ exchange, are non-vanishing. Their superposition generates a marked $\frac{\mathrm{WI}}{\mathrm{ND}}$, in which $\frac{|v_\mathrm{W}|}{|u_\mathrm{N}|} \gg \frac{|v_\mathrm{E}|}{|u_\mathrm{S}|}$. The first-order correction is antisymmetric under the transform $\frac{\mathrm{N} \leftrightarrow \mathrm{S}}{\mathrm{E} \leftrightarrow \mathrm{W}}$ and the truncated streamfunction, on the whole, generates a weak $\frac{\mathrm{ND}}{\mathrm{WI}}$, in which, because of the smallness of the coupling parameter, $\frac{|u_\mathrm{N}|}{|v_\mathrm{W}|}$ is only slightly greater than $\frac{|u_\mathrm{S}|}{|v_\mathrm{E}|}$.

It is quite apparent that the conclusions concerning one regime hold also for the other, provided that the formal substitutions

$$(\mathrm{N} \leftrightarrow \mathrm{S}) \leftrightarrow (\mathrm{E} \leftrightarrow \mathrm{W}),$$

$$\mathrm{WI} \leftrightarrow \mathrm{ND},$$

$$v_{\mathrm{W}} \leftrightarrow u_{\mathrm{N}},$$

$$v_{\mathrm{E}} \leftrightarrow u_{\mathrm{S}}$$

are carried out in the statements.

The duality relationship pointed out in this section can be ultimately ascribed to the two possible magnitude orders of the ratio $\frac{U_{\mathrm{S}}}{U}$ appearing in the starting equation (3.2). In fact, if $\frac{U_{\mathrm{S}}}{U} = \mathrm{O}(1)$, then the N $\leftrightarrow$S invariance of the forcing T implies, through the Sverdrup balance, the same invariance for the *interior* zeroth-order streamfunction. Further, the validity of this symmetry property also for the boundary layer solution (recall (3.8)) implies its extension to the whole basin. On the contrary, if $\frac{U_{\mathrm{S}}}{U} \ll 1$, the E $\leftrightarrow$ W invariance of the *interior* zeroth-order streamfunction comes from its strict zonality due to the potential vorticity conservation applied to a flat-bottomed, unforced ocean in the beta plane. Moreover, because of the positive definiteness of the derivative (5.2), this symmetry extends to the whole fluid domain.

APPENDIX

We start from the mesoscale (index M) equation, that is eq. (2.37) of [4] with $\frac{\partial}{\partial t} \equiv 0$ and $E_{\mathrm{H}} = 0$:

$$J_{\mathrm{M}}(\psi_{\mathrm{M}}, \nabla^2_{\mathrm{M}} \psi_{\mathrm{M}}) + \beta_{\mathrm{M}} \frac{\partial \psi_{\mathrm{M}}}{\partial x_{\mathrm{M}}} = \frac{\beta_{\mathrm{M}} \tau_0}{\varrho V D \beta_{\mathrm{m}} l} \mathbf{k} \cdot \boldsymbol{\nabla}_{\mathrm{M}} \times \boldsymbol{\tau} - \frac{\sqrt{E_{\mathrm{v}}}}{2 \varepsilon_{\mathrm{M}}} \nabla^2_{\mathrm{M}} \psi_{\mathrm{M}}. \tag{A.1}$$

In (A.1) $\beta_{\mathrm{M}} = \frac{\beta_{\mathrm{m}} l^2}{V}$, where l is the typical horizontal length of the mesoscale motion and V is the related velocity, E_{v} is the vertical Ekman number and $\varepsilon_{\mathrm{M}} = \frac{V}{f_{\mathrm{m}} l}$ is the mesoscale Rossby number, f_{m} being the Coriolis parameter or, better, its "mean" value for the basin under investigation. If we assume that the wind stress depends only on the basin-scale coordinates, *i.e.* $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$ and $UL \approx Vl$, eq. (A.1) can be written in terms of the sole basin-scale quantities (index M dropped) as follows (see ref [4] for details):

$$J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = \beta \frac{U_{\mathrm{S}}}{U} \mathbf{k} \cdot \boldsymbol{\nabla} \times \boldsymbol{\tau} - \frac{\sqrt{E_{\mathrm{v}}}}{2 \varepsilon} \nabla^2 \psi , \tag{A.2}$$

where $\beta = \frac{\beta_{\mathrm{m}} L^2}{U}$ and $\varepsilon = \frac{U}{f_{\mathrm{m}} L}$. In terms of the positions $\frac{1}{\beta} = \left(\frac{\delta_{\mathrm{I}}}{L}\right)^2$, where $\delta_{\mathrm{I}} = \sqrt{\frac{U}{\beta_{\mathrm{m}}}}$,

$\frac{\sqrt{E_v}}{2\varepsilon\beta} = \frac{\delta_v}{L}$, where $\delta_v = \frac{f_m \sqrt{E_v}}{2\beta_m}$ and $\mathbf{k}\cdot\boldsymbol{\nabla}\times\boldsymbol{\tau} = T$, from (A.2) we have

$$\left(\frac{\delta_I}{L}\right)^2 J(\psi, \nabla^2\psi) + \frac{\partial\psi}{\partial x} = \frac{U_S}{U} T - \frac{\delta_v}{L}\nabla^2\psi \tag{A.3}$$

that coincides with eq. (3.2) of the present paper.

Consider now the highly non-linear regime, in which $U \gg U_S$. Along the boundary, the current is amplified by the factor $\frac{L}{\delta_I}$, in the sense that $U \to \frac{L}{\delta_I} U$, so the non-dimensional current grows there from O(1) to $O\left(\frac{L}{\delta_I}\right)$. To find the relation between δ_I and δ_v, consider the integral of (A.3) extended on the whole basin. It takes the form

$$\frac{U_S}{U}\int_D T\,\mathrm{d}x\,\mathrm{d}y - \frac{\delta_v}{L}\oint_{\partial D} \mathbf{u}\cdot\mathrm{d}\mathbf{t} = 0\,. \tag{A.4}$$

Since $\int_D T\,\mathrm{d}x\,\mathrm{d}y = O(1)$ and, as we have just seen, $\oint_{\partial D}\mathbf{u}\cdot\mathrm{d}\mathbf{t} = O\left(\frac{L}{\delta_I}\right)$, eq. (A.4) states that $\frac{U_S}{U} \approx \frac{\delta_v}{\delta_I}$, *i.e.*

$$\frac{\delta_v}{L} \approx \frac{U_S}{U}\frac{\delta_I}{L}\,. \tag{A.5}$$

Using (A.5) into (A.3) and putting $\frac{U_S}{U} = r$, we finally obtain

$$\left(\frac{\delta_I}{L}\right)^2 J(\psi, \nabla^2\psi) + \frac{\partial\psi}{\partial x} = r\left(T - \frac{\delta_I}{L}\nabla^2\psi\right)$$

that coincides with (3.13) of this paper.

* * *

FC is grateful to T. OZGOKMEN who carried out the numerical experiments reported in sect. **5** and to R. PURINI for useful discussions and a critical reading of the manuscript.

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