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Construction of $(n + 1)$-dimensional dual-mode nonlinear equations: multiple shock wave solutions for $(3 + 1)$-dimensional dual-mode Gardner-type and KdV-type

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Abstract

The goal of this study is to offer an exclusive functional conversion to produce $(n + 1)$-dimensional dual-mode nonlinear equations. This transformation has been implemented and new $(3 + 1)$-dimensional dual-mode Gradner-type and KdV-type have been established. Finally, the simplified bilinear method is used to tell the necessary conditions on these new models to have multiple singular-solitons.

Keywords: $(3 + 1)$-dimensional dual-mode Gradner-type; $(3 + 1)$-dimensional dual-mode KdV-type; Simplified bilinear method; Multiple singular-soliton shock wave solutions

1 Introduction

The main upstream of understanding the physical nature of mathematical models arising in different disciplines of science is to extract their traveling wave solutions. Seeking for possible reliable solutions require suggesting and developing mathematical methods with supportive geometric analysis such as conservation laws and symmetry analysis [1–10].

Traveling wave solutions have different types which give a complete understanding of the dynamics of a particular physical model. Solitons, kinks, and periodics are the most popular types that propagate as single-moving-waves as in KdV, mKdV, and Burgers'. But, in the case of Boussinesq equation, its traveling wave solutions propagate as dual-waves with interaction phase velocity.

The phenomenon of dual-waves has been adopted by Korsunsky and developed by Wazwaz [11, 12] when they considered the KdV equation of second order in time which reads

$$\phi_{tt} - s^2\phi_{xx} + \left(\frac{\partial}{\partial t} - \alpha s\frac{\partial}{\partial x}\right)\phi\phi_x + \left(\frac{\partial}{\partial t} - \beta s\frac{\partial}{\partial x}\right)\phi_{xxx} = 0, \tag{1.1}$$

where $\phi = \phi(x, t)$ is a field function, s is the interaction phase velocity, α is the nonlinearity factor, and β is the dispersive factor with $s \geq 0$, $|\alpha| \leq 1$, $|\beta| \leq 1$, and we refer to equation (1.1) as the two-mode KdV equation (TMKdV). The equation given in (1.1) was revisited by Alquran and Jarrah, and new Jacobi elliptic sine-cosine solutions were obtained [13].

Inspired by the form of TMKdV, many new two-mode or dual-mode models have been established. Two-mode Burgers equation (TMBE) and two-mode fifth-order KdV equations (TMFKdV) [14, 15], the two-mode higher-order Boussinesq–Burger system [16], two-mode coupled Burgers equation [17], two-mode coupled modified Korteweg–de Vries [18], two-mode coupled Korteweg–de Vries [19], two-mode Korteweg–de Vries–Burgers equation [20], the weak-dissipative two-mode perturbed Burgers and Ostrovsky models [21], two-mode Kuramoto–Sivashinsky [22], the dual-mode nonlinear Schrodinger's equation and Kerr-law nonlinearity [23], the two-mode second- and third-order dispersive Fisher [24, 25] and the dual-mode Kadomtsev–Petviashvili model with strong-weak surface tension [26]. Single and multiple soliton/kink solutions have been obtained for the aforementioned models by using a simplified bilinear method, tanh method, sine-cosine method, Kudryashov method, and the (G'/G)-expansion method.

The motivation of this work is to introduce for the first time a formulation of $(n + 1)$-dimensional dual-mode equations and to establish new $(3 + 1)$-dimensional dual-mode equations of type Gardner and KdV. Also, we aim to find the necessary constraint conditions that enable such equations possess soliton solutions, singular soliton solutions, multiple soliton solutions, and multiple singular soliton solutions by using the simplified bilinear method.

The forms of single-mode $(3 + 1)$-dimensional Gardner and KdV-type equations are, respectively, read as

$$v_t + 6lvv_x + v_{xxx} - \frac{3}{2}k^2v^2v_x + 3h^2\partial_x^{-1}v_{yy} - 3khv_x\partial_x^{-1}v_y + 3h^2\partial_x^{-1}v_{zz} - 3khv_x\partial_x^{-1}v_z = 0, \quad (1.2)$$

and

$$v_t + 6v_xv_y + v_{xxy} + v_{xxxxz} + 60v_x^2v_z + 10v_{xxx}v_z + 20v_xv_{xxz} = 0. \quad (1.3)$$

The above two equations are widely used in physics and its applications such as quantum field theory, plasma physics, and fluid physics. Also, different types of solutions have been obtained by using many methods such as Hirota's direct method, the Casorati and Grammian determinant solutions, and the inverse scattering method [27–30].

2 Formulation of $(n + 1)$-dimensional dual-mode equations

Wazwaz and Korsunsky [11, 12, 14, 15] established the $(1 + 1)$-dimensional two-mode equation in a scaled form as

$$v_{tt} - c^2v_{xx} + \left(\frac{\partial}{\partial t} - cb\frac{\partial}{\partial x}\right)L(v_{mx}) + \left(\frac{\partial}{\partial t} - cd\frac{\partial}{\partial x}\right)N(v, v_x, \dots) = 0, \quad (2.1)$$

where $m \geq 2$, $L(v_{kx})$ is a linear term, $N(v, v_x, \dots)$ is a nonlinear term, $c > 0$ is the phase velocities, $x \in (-\infty, \infty)$, $t > 0$, $|b| \leq 1$, and $|d| \leq 1$.

In this study we propose a new scale for the $(n + 1)$-dimensional dual-mode equations in the variables $t, x_1, x_2, x_3, \dots, x_n$. The new scale is suggested to have the following form:

$$0 = v_{tt} - \sum_{i=1}^{n} c^2v_{x_ix_i} + \left(\frac{\partial}{\partial t} - \sum_{i=1}^{n} ca_i\frac{\partial}{\partial x_i}\right)L + \left(\frac{\partial}{\partial t} - \sum_{i=1}^{n} cb_i\frac{\partial}{\partial x_i}\right)N, \quad (2.2)$$

where L and N are, respectively, linear and nonlinear, $|a_i| \leq 1$, and $|b_i| \leq 1$, $i = 1, 2, \dots, n$. Note that when $c = 0$ and integrating once with respect to t, (2.2) is reduced to the standard single-mode $(n+1)$-dimensional equation.

3 Analysis of the method

In this section, we give a brief description of the simplified bilinear method to find N-soliton solutions for nonlinear partial differential equations (NPDEs) as follows:

First, we substitute

$$v(x_1, x_2, \dots, x_n, t) = e^{\omega_i(x_1, x_2, \dots, x_n, t)},$$

where

$$\omega_i(x_1, x_2, \dots, x_n, t) = \sum_{j=1}^{n} l_{j_i} x_j - \lambda_i t, \quad i = 1, 2, 3, \dots, N,$$

in the problem under consideration, to find the relation among l_{j_i} and λ_i. To find the soliton solutions, we use an appropriate transformation formula. We often use one of the following formulas:

$$v(x_1, x_2, \dots, x_n, t) = R \ln f(x_1, x_2, \dots, x_n, t)$$
$$v(x_1, x_2, \dots, x_n, t) = R\bigl(\ln f(x_1, x_2, \dots, x_n, t)\bigr)_{x_i},$$
$$v(x_1, x_2, \dots, x_n, t) = R\bigl(\ln f(x_1, x_2, \dots, x_n, t)\bigr)_{x_i x_j},$$

where $i, j \in \{1, 2, \dots, N\}$, $R \in \mathbb{R}$.

For one-soliton solutions, we use the auxiliary function

$$f(x_1, x_2, \dots, x_n, t) = 1 + c_1 e^{\omega_i(x_1, x_2, \dots, x_n, t)}, \quad c_1 = \pm 1.$$

For two-soliton solutions, we use the auxiliary function

$$f(x_1, x_2, \dots, x_n, t) = 1 + c_1 e^{\omega_1} + c_2 e^{\omega_2} + c_1 c_2 v_{12} e^{\omega_1 + \omega_2}, \quad c_1 = c_2 = \pm 1.$$

For three-soliton solutions, we use the auxiliary function

$$\begin{aligned} f(x_1, x_2, \dots, x_n, t) = 1 &+ c_1 e^{\omega_1} + c_2 e^{\omega_2} + c_3 e^{\omega_3} + c_1 c_2 v_{12} e^{\omega_1 + \omega_2} \\ &+ c_1 c_3 v_{13} e^{\omega_1 + \omega_3} + c_2 c_3 v_{23} e^{\omega_2 + \omega_3} + c_1 c_2 c_3 v_{123} e^{\omega_1 + \omega_2 + \omega_3}, \end{aligned}$$

$$c_1 = c_2 = c_3 = \pm 1,$$

provided that three-soliton solutions exist if $v_{123} = v_{12} v_{13} v_{23}$. Moreover, for any nonlinear PDEs that have three-soliton solutions, they also have N-soliton solutions for $N \geq 4$.

4 Applications

The purposes of this section is to apply the above described method to solve new $(3+1)$-dimensional dual-mode nonlinear PDEs.

4.1 Soliton solutions for (3 + 1)-dimensional dual-mode Gardner equation

Applying the suggested prescribed scale (2.2) on (1.2), the (3 + 1)-dimensional dual-mode Gardner equation will have the following form:

$$\begin{aligned}0 &= v_{tt} - c^2 v_{xx} - c^2 v_{yy} - c^2 v_{zz}\\ &\quad + \left(\frac{\partial}{\partial t} - ca_1\frac{\partial}{\partial x} - ca_2\frac{\partial}{\partial y} - ca_3\frac{\partial}{\partial z}\right)\left\{v_{xxx} + 3h^2\partial_x^{-1}v_{yy} + 3h^2\partial_x^{-1}v_{zz}\right\}\\ &\quad + \left(\frac{\partial}{\partial t} - cb_1\frac{\partial}{\partial x} - cb_2\frac{\partial}{\partial y} - cb_3\frac{\partial}{\partial z}\right)\\ &\quad \times \left\{6lvv_x - \frac{3}{2}k^2v^2v_x - 3khv_x\partial_x^{-1}v_y - 3khv_x\partial_x^{-1}v_z\right\}. \end{aligned} \tag{4.1}$$

We aim to find the needed necessary conditions in order to obtain multiple soliton and multiple singular-soliton solutions by using a simplified bilinear method [31–41]. To drop the presence of the operator ∂_x^{-1}, we use the transformation

$$v(x,y,z,t) = w_x(x,y,z,t).$$

Accordingly, a new equivalent version of (3 + 1)-TMGE (4.1) is given by

$$\begin{aligned}0 &= w_{xtt} - c^2 w_{xxx} - c^2 w_{xyy} - c^2 w_{xzz} + w_{xxxxt} - ca_1 w_{xxxxx} - ca_2 w_{xxxxy} - ca_3 w_{xxxxz}\\ &\quad + 3h^2 w_{yyt} - 3h^2 ca_1 w_{yyx} - 3h^2 ca_2 w_{yyy} - 3h^2 ca_3 w_{yyz}\\ &\quad + 3h^2 w_{zzt} - 3h^2 ca_1 w_{zzx} - 3h^2 ca_2 w_{zzy} - 3h^2 ca_3 w_{zzz}\\ &\quad + 6l(w_x w_{xx})_t - 6lcb_1(w_x w_{xx})_x - 6lcb_2(w_x w_{xx})_y - 6lcb_3(w_x w_{xx})_z\\ &\quad - \frac{3}{2}k^2\left(w_x^2 w_{xx}\right)_t + \frac{3}{2}k^2cb_1\left(w_x^2 w_{xx}\right)_x + \frac{3}{2}k^2cb_2\left(w_x^2 w_{xx}\right)_y + \frac{3}{2}k^2cb_3\left(w_x^2 w_{xx}\right)_z\\ &\quad - 3kh(w_{xx}w_y)_t + 3khcb_1(w_{xx}w_y)_x + 3khcb_2(w_{xx}w_y)_y + 3khcb_3(w_{xx}w_y)_z\\ &\quad - 3kh(w_{xx}w_z)_t + 3khcb_1(w_{xx}w_z)_x + 3khcb_2(w_{xx}w_z)_y + 3khcb_3(w_{xx}w_z)_z. \end{aligned} \tag{4.2}$$

Inserting

$$w(x,y,z,t) = e^{\omega_i(x,y,z,t)}$$

with

$$\omega_i(x,y,z,t) = \alpha_i x + \beta_i y + \zeta_i z - \gamma_i t, \quad i = 1,2,3,\dots,N,$$

into the linear terms of (4.2), we get the dispersion relations

$$\gamma_i = \frac{-(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2) \pm \sqrt{(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2)^2 + 4\alpha_i\Delta_i}}{2\alpha_i}, \tag{4.3}$$

where

$$\begin{aligned}\Delta_i &= \big(c^2\alpha_i^3 + ca_1\alpha_i^5 + ca_2\alpha_i^4\beta_i + ca_3\alpha^4\zeta_i + c^2\beta_i^2\alpha_i \\ &\quad + 3h^2ca_1\beta_i^2\alpha_i + 3h^2ca_2\beta_i^3 + 3h^2ca_3\beta_i^2\zeta_i \\ &\quad + c^2\alpha_i\zeta_i^2 + 3h^2ca_1\zeta_i^2\alpha_i + 3h^2ca_2\zeta_i^2\beta_i + 3h^2ca_3\zeta_i^3\big).\end{aligned} \tag{4.4}$$

Now, we consider the Cole–Hopf transformation

$$v(x,y,t) = R\big(\ln f(x,y,z,t)\big)_x \tag{4.5}$$

which leads to

$$w(x,y,t) = R\ln f(x,y,z,t) \tag{4.6}$$

provided that R is a constant and $f(x,y,z,t)$ is an auxiliary function. For the one-soliton solution, we consider

$$f(x,y,z,t) = 1 + c_1 e^{\alpha_1 x+\beta_1 y+\zeta_1 z-\frac{-(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)\pm\sqrt{(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)^2+4\alpha_1\Delta_1}}{2\alpha_1}t}, \tag{4.7}$$

where $c_1 = \pm 1$. Following (4.3), inserting (4.6) and (4.7) into (4.2) and solving for R, the one soliton solution of (4.2) exists if

$$\begin{aligned}R &= \frac{2}{k}, \\ a_1 &= a_2 = a_3 = b_1 = b_2 = b_3, \\ \beta_1 &= \frac{2\alpha_1 l - k\alpha_1^2 - hk\zeta_1}{hk}.\end{aligned} \tag{4.8}$$

By the Cole–Hopf transformation (4.6), we conclude the one-soliton solution of (4.2) as

$$w(x,y,z,t) = \frac{2}{k}\ln\big(1 + c_1 e^{\alpha_1 x+\frac{2\alpha_1 l-k\alpha_1^2-hk\zeta_1}{hk}y+\zeta_1 z-\frac{-(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)\pm\sqrt{(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)^2+4\alpha_1\Delta_1}}{2\alpha_1}t}\big),$$

and then,

$$v(x,y,z,t) = \frac{2}{k}\frac{\alpha_1 c_1 e^{\alpha_1 x+\frac{2\alpha_1 l-k\alpha_1^2-hk\zeta_1}{hk}y+\zeta_1 z-\frac{-(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)\pm\sqrt{(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)^2+4\alpha_1\Delta_1}}{2\alpha_1}t}}{1 + c_1 e^{\alpha_1 x+\frac{2\alpha_1 l-k\alpha_1^2-hk\zeta_1}{hk}y+\zeta_1 z-\frac{-(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)\pm\sqrt{(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)^2+4\alpha_1\Delta_1}}{2\alpha_1}t}}.$$

In the case $c_1 = 1$, we get the single-soliton solution as follows:

$$\begin{aligned}v(x,y,z,t) &= \frac{2}{k}\frac{\alpha_1 e^{\alpha_1 x+\frac{2\alpha_1 l-k\alpha_1^2-hk\zeta_1}{hk}y+\zeta_1 z-\frac{-(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)\pm\sqrt{(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)^2+4\alpha_1\Delta_1}}{2\alpha_1}t}}{1 + e^{\alpha_1 x+\frac{2\alpha_1 l-k\alpha_1^2-hk\zeta_1}{hk}y+\zeta_1 z-\frac{-(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)\pm\sqrt{(\alpha_1^4+3k^2\beta_1^2+3h^2\zeta_1^2)^2+4\alpha_1\Delta_1}}{2\alpha_1}t}} \\ &= \frac{\alpha_1}{k}\left[1 + \tanh\left(\frac{\omega_1(x,y,z,t)}{2}\right)\right].\end{aligned}$$

For the case $c_1 = -1$, we get the singular single-soliton solution as follows:

$$\begin{aligned} v(x,y,z,t) &= \frac{2}{k}\frac{\alpha_1 e^{\alpha_1 x + \frac{2\alpha_1 l - k\alpha_1^2 - hk\zeta_1}{hk}y + \zeta_1 z - \frac{-(\alpha_1^4 + 3k^2\beta_1^2 + 3h^2\zeta_1^2) \pm \sqrt{(\alpha_1^4 + 3k^2\beta_1^2 + 3h^2\zeta_1^2)^2 + 4\alpha_1\Delta_1}}{2\alpha_1}t}}{-1 + e^{\alpha_1 x + \frac{2\alpha_1 l - k\alpha_1^2 - hk\zeta_1}{hk}y + \zeta_1 z - \frac{-(\alpha_1^4 + 3k^2\beta_1^2 + 3h^2\zeta_1^2) \pm \sqrt{(\alpha_1^4 + 3k^2\beta_1^2 + 3h^2\zeta_1^2)^2 + 4\alpha_1\Delta_1}}{2\alpha_1}t}} \\ &= \frac{\alpha_1}{k}\left[1 + \coth\left(\frac{\omega_1(x,y,z,t)}{2}\right)\right], \end{aligned}$$

where

$$\begin{aligned} \omega_1(x,y,z,t) = \alpha_1 x &+ \frac{2\alpha_1 l - k\alpha_1^2 - hk\zeta_1}{hk}y + \zeta_1 z \\ &- \frac{-(\alpha_1^4 + 3h^2\beta_1^2 + 3h^2\zeta_1^2) \pm \sqrt{(\alpha_1^4 + 3h^2\beta_1^2 + 3h^2\zeta_1^2)^2 + 4\alpha_1\Delta_1}}{2\alpha_1}t. \end{aligned}$$

For the two-soliton solutions, we set the auxiliary function

$$f(x,y,z,t) = 1 + c_1 e^{\omega_1(x,y,z,t)} + c_2 e^{\omega_2(x,y,z,t)t} + c_1 c_2 \nu_{12} e^{\omega_1(x,y,z,t)+\omega_2(x,y,z,t)}, \tag{4.9}$$

where $c_i = \pm 1$ and $i = 1, 2$. Inserting (4.6), (4.8), and (4.9) in (4.2), the two-soliton solution of (4.2) exists if

$$a_1 = a_2 = a_3 = b_2 = b_2 = b_3 = \pm 1, \tag{4.10}$$

$$\beta_1 = \frac{2\alpha_1 l - k\alpha_1^2 - hk\zeta_1}{hk}, \tag{4.11}$$

$$\beta_2 = \frac{2\alpha_2 l - k\alpha_2^2 - hk\zeta_2}{hk}, \tag{4.12}$$

$$\zeta_1 = a\alpha_1, \qquad \zeta_2 = a\alpha_2, \tag{4.13}$$

$$\nu_{12} = 0, \tag{4.14}$$

where a is any real number.

Combining (4.9)–(4.14) and (4.6), the two-soliton solution is

$$\begin{aligned} w(x,y,z,t) = \frac{2}{k}\ln\bigl(1 &+ c_1 e^{\alpha_1 x + \frac{2\alpha_1 l - k\alpha_1^2 - ahk\alpha_1}{hk}y + a\alpha_1 z - \frac{-(\alpha_1^4 + 3k^2\beta_1^2 + 3h^2\zeta_1^2) \pm \sqrt{(\alpha_1^4 + 3k^2\beta_1^2 + 3h^2\zeta_1^2)^2 + 4\alpha_1\Delta_1}}{2\alpha_1}t} \\ &+ c_2 e^{\alpha_2 x + \frac{2\alpha_2 l - k\alpha_2^2 - ahk\alpha_2}{hk}y + a\alpha_2 z - \frac{-(\alpha_2^4 + 3k^2\beta_2^2 + 3h^2\zeta_2^2) \pm \sqrt{(\alpha_2^4 + 3k^2\beta_2^2 + 3h^2\zeta_2^2)^2 + 4\alpha_2\Delta_2}}{2\alpha_2}t}\bigr), \end{aligned}$$

and thus,

$$v(x,y,z,t) = \frac{2}{k}\frac{\alpha_1 c_1 e^{\omega_1(x,y,z,t)} + \alpha_2 c_2 e^{\omega_2(x,y,z,t)}}{1 + c_1 e^{\omega_1(x,y,z,t)} + c_2 e^{\omega_2(x,y,z,t)}}. \tag{4.15}$$

By setting $c_1 = c_2 = 1$ in (4.15), the two-soliton solution is

$$v(x,y,z,t) = \frac{2}{k}\frac{\alpha_1 e^{\omega_1(x,y,z,t)} + \alpha_2 e^{\omega_2(x,y,z,t)}}{1 + e^{\omega_1(x,y,z,t)} + e^{\omega_2(x,y,z,t)}}, \tag{4.16}$$

and the singular two-soliton solution by substituting $c_1 = c_2 = -1$ is

$$v(x,y,z,t) = \frac{2}{k}\frac{\alpha_1 e^{\omega_1(x,y,z,t)} + \alpha_2 e^{\omega_2(x,y,z,t)}}{-1 + e^{\omega_1(x,y,z,t)} + e^{\omega_2(x,y,z,t)}}. \tag{4.17}$$

For the three-soliton solution, we use the auxiliary function

$$f(x,y,z,t) = 1 + c_1 e^{\omega_1(x,y,z,t)} + c_2 e^{\omega_2(x,y,z,t)} + c_3 e^{\omega_3(x,y,z,t)}, \tag{4.18}$$

where $c_i = \pm 1$, $i = 1,2,3$. Accordingly, the three-soliton solution is

$$\begin{aligned} w(x,y,z,t) = \frac{2}{k}\ln\Bigl(1 &+ c_1 e^{\alpha_1 x + \frac{2\alpha_1 l - k\alpha_1^2 - ahk\alpha_1}{hk} y + a\alpha_1 z - \frac{-(\alpha_1^4 + 3k^2\beta_1^2 + 3h^2\zeta_1^2) \pm \sqrt{(\alpha_1^4 + 3k^2\beta_1^2 + 3h^2\zeta_1^2)^2 + 4\alpha_1\Delta_1}}{2\alpha_1} t} \\ &+ c_2 e^{\alpha_2 x + \frac{2\alpha_2 l - k\alpha_2^2 - ahk\alpha_2}{hk} y + a\alpha_2 z - \frac{-(\alpha_2^4 + 3k^2\beta_2^2 + 3h^2\zeta_2^2) \pm \sqrt{(\alpha_2^4 + 3k^2\beta_2^2 + 3h^2\zeta_2^2)^2 + 4\alpha_2\Delta_2}}{2\alpha_2} t} \\ &+ c_3 e^{\alpha_3 x + \frac{2\alpha_3 l - k\alpha_3^2 - ahk\alpha_3}{hk} y + a\alpha_3 z - \frac{-(\alpha_3^4 + 3k^2\beta_3^2 + 3h^2\zeta_3^2) \pm \sqrt{(\alpha_3^4 + 3k^2\beta_3^2 + 3h^2\zeta_3^2)^2 + 4\alpha_3\Delta_3}}{2\alpha_3} t}\Bigr) \end{aligned}$$

and then,

$$v(x,y,z,t) = \frac{2}{k}\frac{\alpha_1 c_1 e^{\omega_1(x,y,z,t)} + \alpha_2 c_2 e^{\omega_2(x,y,z,t)} + \alpha_3 c_3 e^{\omega_3(x,y,z,t)}}{1 + c_1 e^{\omega_1(x,y,z,t)} + c_2 e^{\omega_2(x,y,z,t)} + c_3 e^{\omega_3(x,y,z,t)}}. \tag{4.19}$$

By setting $c_i = 1$ for $i = 1,2,3$, we obtain the three-soliton solution

$$v(x,y,z,t) = \frac{2}{k}\frac{\alpha_1 e^{\omega_1(x,y,z,t)} + \alpha_2 e^{\omega_2(x,y,z,t)} + \alpha_3 e^{\omega_3(x,y,z,t)}}{1 + e^{\omega_1(x,y,z,t)} + e^{\omega_2(x,y,z,t)} + e^{\omega_3(x,y,z,t)}}. \tag{4.20}$$

Setting $c_i = -1$ for $i = 1,2,3$, the singular three-soliton solution is

$$v(x,y,t) = \frac{2}{k}\frac{\alpha_1 e^{\omega_1(x,y,z,t)} + \alpha_2 e^{\omega_2(x,y,z,t)} + \alpha_3 e^{\omega_3(x,y,z,t)}}{-1 + e^{\omega_1(x,y,z,t)} + e^{\omega_2(x,y,z,t)} + e^{\omega_3(x,y,z,t)}}. \tag{4.21}$$

Remark For a finite N, where $N \geq 4$ and under the conditions $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = \pm 1$ and $\beta_i = \frac{2\alpha_i l - k\alpha_i^2 - hk\zeta_i}{hk}$, $\zeta_i = a\alpha_i$, $i = 1,2,\ldots,N$, (4.2) has N-soliton solutions and singular N-soliton solutions given by [31, 32]

$$v(x,y,z,t) = \frac{2}{k}\frac{\sum_{i=1}^N \alpha_i e^{\alpha_i x + \frac{2\alpha_i l - k\alpha_i^2 - ahk\alpha_i}{hk} y + a\alpha_i z - \frac{-(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2) \pm \sqrt{(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2)^2 + 4\alpha_i\Delta_i}}{2\alpha_i} t}}{1 + \sum_{i=1}^N e^{\alpha_i x + \frac{2\alpha_i l - k\alpha_i^2 - ahk\alpha_i}{hk} y + a\alpha_i z - \frac{-(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2) \pm \sqrt{(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2)^2 + 4\alpha_i\Delta_i}}{2\alpha_i} t}},$$

and

$$v(x,y,z,t) = \frac{2}{k}\frac{\sum_{i=1}^N \alpha_i e^{\alpha_i x + \frac{2\alpha_i l - k\alpha_i^2 - ahk\alpha_i}{hk} y + a\alpha_i z - \frac{-(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2) \pm \sqrt{(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2)^2 + 4\alpha_i\Delta_i}}{2\alpha_i} t}}{-1 + \sum_{i=1}^N e^{\alpha_i x + \frac{2\alpha_i l - k\alpha_i^2 - ahk\alpha_i}{hk} y + a\alpha_i z - \frac{-(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2) \pm \sqrt{(\alpha_i^4 + 3h^2\beta_i^2 + 3h^2\zeta_i^2)^2 + 4\alpha_i\Delta_i}}{2\alpha_i} t}}.$$

4.2 Soliton solutions for (3 + 1)-dimensional dual-mode KdV-type

Applying the new formulation (2.2), the new $(3+1)$-dimensional dual-mode KdV equation ($(3+1)$-TMKdV) has the following form:

$$\begin{aligned} 0 &= v_{tt} - c^2 v_{xx} - c^2 v_{yy} - c^2 v_{zz} + v_{xxyt} - ca_1 v_{xxyx} - ca_2 v_{xxyy} - ca_3 v_{xxyz} \\ &\quad + v_{xxxxzt} - ca_1 v_{xxxxzx} - ca_2 v_{xxxxzy} - ca_3 v_{xxxxzz} \\ &\quad + 6(v_x v_y)_t - 6cb_1(v_x v_y)_x - 6cb_2(v_x v_y)_y - 6cb_3(v_x v_y)_z \\ &\quad + 60\bigl(v_x^2 v_z\bigr)_t - 60cb_1\bigl(v_x^2 v_z\bigr)_x - 60cb_2\bigl(v_x^2 v_z\bigr)_y - 60cb_3\bigl(v_x^2 v_z\bigr)_z \\ &\quad + 10(v_{xxx} v_z)_t - 10cb_1(v_{xxx} v_z)_x - 10cb_2(v_{xxx} v_z)_y - 10cb_3(v_{xxx} v_z)_z \\ &\quad + 20(v_x v_{xxz})_t - 20cb_1(v_x v_{xxz})_x - 20cb_2(v_x v_{xxz})_y - 20cb_3(v_x v_{xxz})_z. \end{aligned} \tag{4.22}$$

Inserting

$$w(x,y,z,t) = e^{\omega_i(x,y,z,t)}$$

with

$$\omega_i(x,y,z,t) = \alpha_i x + \beta_i y + \zeta_i z - \gamma_i t, \quad i = 1,2,3,\ldots,N,$$

into the linear terms of (4.22), we get the dispersion relations

$$\gamma_i = \frac{(\alpha_i^2\beta_i + \alpha_i^4\zeta_i) \pm \sqrt{(\alpha_i^2\beta_i + \alpha_i^4\zeta_i)^2 + 4\Delta_i}}{2}, \tag{4.23}$$

where

$$\begin{aligned} \Delta_i &= \bigl(c^2\alpha_i^2 + c^2\beta_i^2 + c^2\zeta_i^2 + ca_1\alpha_i^3\beta_i + ca_2\alpha_i^2\beta_i^2 \\ &\quad + ca_3\alpha_i^2\beta_i\zeta_i + ca_1\alpha_i^5\zeta_i + ca_2\alpha_i^4\beta_i\zeta_i + ca_3\alpha_i^4\zeta_i^2\bigr). \end{aligned}$$

Now, we consider the Cole–Hopf transformation

$$v(x,y,t) = R\bigl(\ln f(x,y,z,t)\bigr)_x \tag{4.24}$$

and the auxiliary function

$$f(x,y,z,t) = 1 + c_1 e^{\alpha_1 x + \beta_1 y + \zeta_1 z - \frac{(\alpha_i^2\beta_i + \alpha_i^4\zeta_i) \pm \sqrt{(\alpha_i^2\beta_i + \alpha_i^4\zeta_i)^2 + 4\Delta_i}}{2} t}. \tag{4.25}$$

Using (4.23), inserting (4.24) and (4.25) into the $(3+1)$-TMGE (4.22) and solving for R, the one-soliton solution of (4.22) exists if

$$R = 1,$$

$$a_1 = a_2 = a_3 = b_2 = b_2 = b_3.$$

Therefore, the one-soliton solution of the (3 + 1)-TMGE is

$$v(x,y,z,t) = \frac{\alpha_1 c_1 e^{\omega_1/2}}{1 + c_1 e^{\omega_1/2}}.$$

In case $c_1 = 1$, we get the single-soliton solution

$$\begin{aligned} v(x,y,z,t) &= \frac{\alpha e^{\omega_1/2}}{1 + e^{\omega_1/2}} \\ &= \alpha_1\left[1 + \tanh\left(\frac{\omega_1(x,y,z,t)}{4}\right)\right], \end{aligned}$$

and for the case $c_1 = -1$, we get the singular single-soliton solution

$$\begin{aligned} v(x,y,z,t) &= \frac{\alpha_1 e^{\omega_1/2}}{-1 + e^{\omega_1/2}} \\ &= \alpha_1\left[1 + \coth\left(\frac{\omega_1(x,y,z,t)}{4}\right)\right]. \end{aligned}$$

For the two-soliton solutions, we use the auxiliary function

$$f(x,y,z,t) = 1 + c_1 e^{\omega_1(x,y,z,t)} + c_2 e^{\omega_2(x,y,z,t)t} + c_1 c_2 v_{12} e^{\omega_1(x,y,z,t)+\omega_2(x,y,z,t)}, \tag{4.26}$$

where $c_i = \pm 1$ and $i = 1,2$. Inserting (4.23), (4.24), and (4.26) in (4.22), the two-soliton solution of (4.22) exists if

$$\begin{aligned} &R = 1, \\ &a_1 = a_2 = a_3 = b_2 = b_2 = b_3 = \pm 1, \\ &v_{12} = \frac{(\alpha_1 - \alpha_2)((\alpha_1^2\beta_1 + 2\alpha_1\alpha_2\beta_1 - 2\alpha_1\alpha_2\beta_2 - \alpha_2^2\beta_1) + \theta_1)}{\theta_2} \quad \text{where,} \\ &\theta_1 = \left[\alpha_1^4\zeta_2 + (2\alpha_1\alpha_2\zeta_1 - 2\alpha_1\alpha_2\zeta_2)\left(2\alpha_1^2 - 3\alpha_1\alpha_2 + 2\alpha_2^2\right) - \alpha_2^4\zeta_1\right], \\ &\theta_2 = (\alpha_1 + \alpha_2)\left[\left(\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1 + 2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1\right) + \theta_3\right], \\ &\theta_3 = \left[\alpha_1^4\zeta_1 + (2\alpha_1\alpha_2\zeta_1 + 2\alpha_1\alpha_2\zeta_2)\left(2\alpha_1^2 + 3\alpha_1\alpha_2 + 2\alpha_2^2\right) + \alpha_1^4\zeta_1\right]. \end{aligned}$$

Therefore, the two-soliton solution of the (3 + 1)-TMKDVE is given by

$$v(x,y,z,t) = \frac{\alpha_1 c_1 e^{\omega_1/2} + \alpha_2 c_1 e^{\omega_1/2} + \frac{c_1 c_2 v_{12} e^{(\omega_1+\omega_2)/2}}{(\alpha_1+\alpha_2)}}{1 + \alpha_1 c_1 e^{\omega_1/2} + \alpha_2 c_1 e^{\omega_1/2} + c_1 c_2 v_{12} e^{\omega_1}}.$$

In case $c_1 = 1$, we get the two-soliton solution

$$v(x,y,z,t) = \frac{\alpha_1 e^{\omega_1/2} + \alpha_2 e^{\omega_1/2} + \frac{v_{12} e^{(\omega_1+\omega_2)/2}}{(\alpha_1+\alpha_2)}}{1 + e^{\omega_1/2} + e^{\omega_2/2} + v_{12} e^{(\omega_1+\omega_2)/2}},$$

and for the case $c_1 = -1$, we get the singular two-soliton solution

$$v(x,y,z,t) = \frac{-\alpha_1 e^{\omega_1} - \alpha_2 e^{\omega_1} + \frac{v_{12} e^{(\omega_1+\omega_2)/2}}{(\alpha_1+\alpha_2)}}{1 - \alpha_1 e^{\omega_1/2} - \alpha_2 e^{\omega_2/2} + v_{12} e^{(\omega_1+\omega_2)/2}}.$$

We should remark here that we cannot find three or more soliton solutions for (4.22) because this type of KdV equation is not integrable.

5 Conclusions

In this work, we proposed a functional conversion that produces $(n+1)$-dimensional dual-mode equations of the form

$$0 = v_{tt} - \sum_{i=1}^{n} c^2 v_{x_i x_i} + \left(\frac{\partial}{\partial t} - \sum_{i=1}^{n} cb_i \frac{\partial}{\partial x_i}\right) L + \left(\frac{\partial}{\partial t} - \sum_{i=1}^{n} cd_i \frac{\partial}{\partial x_i}\right) N.$$

These types of equations describe the spreading of dual-waves moving simultaneously with interaction phase velocity. The simplified bilinear method with the aid of some Cole–Hopf transformations is used to study (3 + 1)-dimensional dual-mode Gardner-type and KdV-type. We concluded the following results:

- Kink solutions and singular kink solutions for (3 + 1)-TMGE exist only if $a_1 = a_2 = a_3 = b_1 = b_2 = b_3$ and $\beta_1 = \frac{2\alpha_1 l - k\alpha_1^2 - hk\zeta_1}{hk}$, while the N-soliton and singular N-soliton solutions exist only if $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = \pm 1$, $\zeta_i = a\alpha_i$ and $\beta_i = \frac{2\alpha_i l - k\alpha_i^2 - hk\zeta_i}{hk}$, $i = 1, 2, \ldots, N$.
- One-soliton solutions for (3 + 1)-TMKdV exist only if $a_1 = a_2 = a_3 = b_2 = b_2 = b_3$ and two-soliton solutions exist if $a_1 = a_2 = a_3 = b_2 = b_2 = b_3 = \pm 1$. This equation is non-integrable, it possesses no k-soliton solutions for $k = 3, 4, \ldots$.

As future work, we may consider a fractional version of dual-mode equations and conduct the same analysis as that used in [42–47].

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