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# On the existence of positive solutions for generalized fractional boundary value problems

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## Abstract

The existence of positive solutions is established for boundary value problems defined within generalized Riemann–Liouville and Caputo fractional operators. Our approach is based on utilizing the technique of fixed point theorems. For the sake of converting the proposed problems into integral equations, we construct Green functions and study their properties for three different types of boundary value problems. Examples are presented to demonstrate the validity of theoretical findings.

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## 1 Introduction

The branch of mathematics that deals with the study of noninteger order derivatives and integrals is called fractional calculus (FC). FC is almost 300 years old as the classical calculus. The interesting thing about this subject is that in contrast to the classical derivatives, the fractional derivatives are not a point quantity. Indeed, the fractional derivative of a function of order  $\alpha$  at some point is a local property only for  $\alpha$  being an integer. On the other hand, when  $\alpha$  is not an integer, the derivative does not only depend on the graph of the function very close to the point but it also depends on some history.

FC has not been a famous applied field of interest among scientists and engineers in the previous years. Thus, many researchers have not recognized the rich applications of FC for a long period of time. In recent decades, however, it has been realized that the FC has several potential applications in different areas of engineering and science such as propagation, electrochemistry, finance, and bio engineering. In the literature, one can figure out that there are many definitions of fractional derivatives. For instance, we refer here to the most well-known types such as Caputo derivative, Liouville derivative, Hadamard derivative, Katugampola derivative, and many others. Consequently, this has led to several types of fractional differential equations defined by different fractional operators [11, 22]. The

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best way to deal with such a variety of fractional operators is to accommodate generalized forms of fractional operators that include other operators.

Researchers who are interested in this subject have introduced many generalizations of fractional derivatives such as  $\psi$ -Hilfer fractional derivative, Hilfer–Katugampola fractional derivative, and generalized proportional fractional derivative [1, 2, 12, 14–17, 23, 25–28]. In [1], Abdo and Panchal considered a general form of  $\psi$ -Hilfer fractional derivative with respect to another function of a fractional integro-differential equation. They presented results on existence, uniqueness, and stability of the solutions. In [16], the authors introduced a new generalized derivative involving exponential functions in their kernels which, upon considering limiting cases, converges to classical derivatives. They solved Cauchy linear fractional type problems within this derivative. In [17], however, the author introduced Katugampola fractional derivative. Indeed, he presented a generalized fractional derivative that generalizes the regular Hadamard and Riemann–Liouville fractional derivatives. In [23], the authors considered a class of nonlinear fractional initial value problems, and they proved the existence and uniqueness of solutions. Following this trend, the existence of positive solutions of the regular fractional boundary value problems (FBVPs) have been discussed in many papers such as [5, 6, 9, 10, 21, 31, 33–35]. For the sake of completeness, we refer afterwards to some relevant papers that study the existence of solutions in the frame of the classical Riemann–Liouville and Caputo derivatives. More precisely, the authors in [8] considered the problem

$$\begin{cases} D_{0+}^{\alpha} z(x) + f(x, z(x)) = 0, & 0 < x < 1, 1 < \alpha \leq 2, \\ z(0) = z(1) = 0, \end{cases}$$

where  $D_{0+}^{\alpha}$  is the Riemann–Liouville operator. They investigated the existence and multiplicity of positive solutions for the above problem. In [7], the same author proved the existence and uniqueness of positive solutions for the same problem but under different boundary conditions of the form

$$\begin{cases} D_{0+}^{\alpha} z(x) + f(x, z(x)) = 0, & 0 < x < 1, 1 < \alpha \leq 2, \\ z(0) = 0 = \beta z(\eta) = z(1), & 0 < \beta \eta^{\alpha-1} < 1. \end{cases}$$

The existence and multiplicity of positive solutions of the following problem were discussed in [32]:

$$\begin{cases} {}^c D_{0+}^{\alpha} z(x) + f(x, z(x)) = 0, & 0 < x < 1, 1 < \alpha \leq 2, \\ z(0) + z'(0) = 0, & z(1) + z'(1) = 0, \end{cases}$$

where  ${}^c D_{0+}^{\alpha}$  is the Caputo operator. On the other hand, and to the best of our insight, the existence of positive solutions of FBVPs within  $\varphi$ -Riemann–Liouville and  $\varphi$ -Caputo operators has not been discussed so far.

The results of this paper are motivated by the recent work of Almeida in [3] who established results for the existence and uniqueness of solutions of FBVP involving a general form of fractional derivative. We shall consider a general form of the derivative, fractional derivative of a function with respect to another function, that includes other definitions

of operators for particular choice of a function. The new derivative herein generalizes the classical definitions of derivatives in the sense that the Riemann–Liouville [30], the Erdelyi–Kober [18, 29], and the Hadamard [20, 24] fractional derivatives are all recovered by choosing particular forms of  $\varphi(t)$ . Moreover, the results of this paper generalize the work established in the papers [7, 8, 32].

In this paper, we discuss the existence and multiplicity of positive solutions of FBVPs defined within  $\varphi$ -Riemann–Liouville and  $\varphi$ -Caputo operators. For our purpose, we convert FBVPs into equivalent integral equations via constructing Green functions for the proposed problems. The technique of fixed point theorems is employed to prove the main results.

The FBVP under consideration has the form

$$D_{0+}^{\alpha,\varphi} z(x) + f(x, z(x)) = 0, \quad x \in (0, 1), \tag{1.1}$$

and is associated with two different boundary conditions

$$z(0) = 0, \quad z(1) = 0, \tag{1.2}$$

$$z(0) = 0, \quad z(1) = \beta z(\eta), \tag{1.3}$$

where  $1 < \alpha \leq 2$ ,  $0 < (\varphi(1) - \varphi(0))^{\alpha-1} - \beta(\varphi(\eta) - \varphi(0))^{\alpha-1} < 1$ , and  $D_{0+}^{\alpha,\varphi}$  is the  $\varphi$ -Riemann–Liouville fractional derivative.

Moreover, we study FBVP of the form

$$\begin{cases} {}^c D_{0+}^{\alpha,\varphi} z(x) - f(x, z(x)) = 0, & x \in (0, 1), \\ z(0) + z'(0) = 0, & z(1) + z'(1) = 0, \end{cases} \tag{1.4}$$

where  $1 < \alpha \leq 2$ ,  ${}^c D_{0+}^{\alpha,\varphi}$  is the  $\varphi$ -Caputo fractional derivative.

Furthermore, the nonlinear  $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is continuous and the function  $\varphi : [0, 1] \rightarrow [0, 1]$  is a strictly increasing function such that  $\varphi \in C^2[0, 1]$ ,  $\varphi'(x) \neq 0$  for all  $x \in [0, 1]$ .

The paper is divided into four sections. Section 1 presents a descriptive introduction. Section 2 states some essential definitions and lemmas that we utilize to prove the main results. Section 3 is devoted to proving the main existence results for FBVP (1.1)–(1.2), (1.1)–(1.3), and (1.4), respectively. Section 4 demonstrates illustrative examples that show consistency to the main theorems.

## 2 Preliminaries

In this part of the paper, we assemble some essential definitions and fixed point theorems that will be used throughout the remaining part of the paper. Besides, some auxiliary lemmas are proved prior to proceeding to the main results of this paper.

**Definition 2.1** ([4]) Let  $z : [a, b] \rightarrow \mathbb{R}$  be an integrable function and  $\varphi : [a, b] \rightarrow \mathbb{R}$  be an increasing function such that, for all  $x \in [a, b]$ ,  $\varphi'(x) \neq 0$ . The  $\varphi$ -Riemann–Liouville fractional integral of a function  $z$  is defined as follows:

$$I_{a+}^{\alpha,\varphi} z(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \varphi'(v) (\varphi(x) - \varphi(v))^{\alpha-1} z(v) dv. \tag{2.1}$$

**Definition 2.2** ([4]) The  $\varphi$ -Riemann–Liouville fractional derivative of order  $\alpha$ , with  $n = [\alpha] + 1$ , of a function  $z$  corresponding to  $\varphi$ -Riemann–Liouville fractional integral (2.1) is defined as follows:

$$D_{a^+}^{\alpha,\varphi} z(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n \int_a^x \varphi'(v) (\varphi(x) - \varphi(v))^{n-\alpha-1} z(v) dv.$$

**Definition 2.3** ([4]) The  $\varphi$ -Caputo fractional derivative of order  $\alpha$ , with  $n = [\alpha] + 1$ , is defined as

$${}^c D_{a^+}^{\alpha,\varphi} z(x) = D_{a^+}^{\alpha,\varphi} \left[ z(x) - \sum_{k=0}^{n-1} \frac{z_\varphi^{[k]}(a)}{k!} (\varphi(x) - \varphi(a))^k \right],$$

where  $z \in C^{n-1}[a, b]$ .

In what follows, we convert the FBVPs into integral equations via Green functions.

**Lemma 2.4** Let  $f \in C[0, 1]$  and  $1 < \alpha \leq 2$ . Then the FBVP

$$\begin{cases} D_{a^+}^{\alpha,\varphi} z(x) + f(x) = 0, & x \in (0, 1), \\ z(0) = 0, & z(1) = 0, \end{cases} \tag{2.2}$$

is equivalent to

$$z(x) = \int_0^1 G(x, v) \varphi'(v) f(v) dv,$$

where

$$G(x, v) = \frac{\Upsilon(x)}{\Gamma(\alpha)} \begin{cases} (\varphi(1) - \varphi(v))^{\alpha-1} - \frac{1}{\Upsilon(x)} (\varphi(x) - \varphi(v))^{\alpha-1}, & 0 \leq v \leq x \leq 1, \\ (\varphi(1) - \varphi(v))^{\alpha-1}, & 0 \leq x \leq v \leq 1, \end{cases} \tag{2.3}$$

with  $K(x) = \varphi(x) - \varphi(0)$  and  $\Upsilon(x) = \frac{(K(x))^{\alpha-1}}{(K(1))^{\alpha-1}}$ .

*Proof* The general solution of FBVP (2.2) is given by

$$z(x) = c_1 (\varphi(x) - \varphi(0))^{\alpha-1} + c_2 (\varphi(x) - \varphi(0))^{\alpha-2} - I_{0^+}^{\alpha,\varphi} f(x),$$

i.e.,

$$z(x) = c_1 (K(x))^{\alpha-1} + c_2 (K(x))^{\alpha-2} - I_{0^+}^{\alpha,\varphi} f(x).$$

By using the given conditions  $z(0) = z(1) = 0$ , we obtain

$$\begin{aligned} z(x) &= \frac{\Upsilon(x)}{\Gamma(\alpha)} \int_0^1 \varphi'(v)(\varphi(1) - \varphi(v))^{\alpha-1} f(v) \, dv \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^x \varphi'(v)(\varphi(x) - \varphi(v))^{\alpha-1} f(v) \, dv \\ &= \int_0^1 G(x, v) \varphi'(v) f(v) \, dv, \end{aligned}$$

where  $G$  is defined as in equation (2.3). □

**Lemma 2.5** *The following conditions are satisfied by the Green function  $G$  defined by equation (2.3):*

- (i)  $G(x, v) > 0$  for all  $x, v \in (0, 1)$ ,
- (ii) For  $v \in (0, 1)$ , there exists a positive function  $\gamma$  such that

$$\min_{x \in [1/4, 3/4]} G(x, v) \geq \gamma(v) \max_{x \in [0, 1]} G(x, v).$$

*Proof* We prove (i): Since  $\varphi$  is a strictly increasing function, we have  $\varphi(1) > \varphi(v)$  whenever  $v < 1$ . So one can easily conclude from equation (2.3) that, for  $0 \leq x \leq v \leq 1$ ,  $G(x, v) > 0$ . And for  $0 \leq v \leq x \leq 1$ , we consider

$$\varphi(1) - \varphi(x) > 0.$$

Multiplying both sides by  $(\varphi(v) - \varphi(0)) > 0$ , we have

$$(\varphi(1) - \varphi(x))(\varphi(v) - \varphi(0)) > 0,$$

which implies

$$\varphi(1)\varphi(v) + \varphi(0)\varphi(x) > \varphi(1)\varphi(0) + \varphi(v)\varphi(x).$$

Multiplying both sides of the above inequality by  $-1$ , we get

$$-\varphi(1)\varphi(0) - \varphi(v)\varphi(x) > -\varphi(1)\varphi(v) - \varphi(0)\varphi(x).$$

Adding  $\varphi(1)\varphi(x) + \varphi(0)\varphi(v)$  to both sides, we obtain

$$(\varphi(x) - \varphi(0))(\varphi(1) - \varphi(v)) > (\varphi(x) - \varphi(v))(\varphi(1) - \varphi(0)).$$

Raising both sides to the power  $(\alpha - 1)$  and then dividing by  $(\varphi(x) - \varphi(0))^{\alpha-1}$ , we get

$$(\varphi(1) - \varphi(v))^{\alpha-1} - \frac{(\varphi(1) - \varphi(0))^{\alpha-1}}{(\varphi(x) - \varphi(0))^{\alpha-1}} (\varphi(x) - \varphi(v))^{\alpha-1} > 0.$$

Hence  $G(x, v) > 0$  for  $x, v \in (0, 1)$ .

We prove (ii): Let us denote  $K(x) = \varphi(x) - \varphi(0)$ ,  $\Upsilon(x) = \left(\frac{K(x)}{K(1)}\right)^{\alpha-1}$ . Note that

$$\begin{aligned} g_1(x, \nu) &= \frac{\Upsilon(x)}{\Gamma(\alpha)} (\varphi(1) - \varphi(\nu))^{\alpha-1} - \frac{1}{\Gamma(\alpha)} (\varphi(x) - \varphi(\nu))^{\alpha-1} \\ &\leq \frac{(\varphi(1) - \varphi(\nu))^{\alpha-1}}{(K(1))^{\alpha-1} \Gamma(\alpha)}, \quad 0 \leq \nu \leq x \leq 1, \end{aligned}$$

and

$$\begin{aligned} g_2(x, \nu) &= \frac{\Upsilon(x)}{\Gamma(\alpha)} (\varphi(1) - \varphi(\nu))^{\alpha-1} \\ &\leq \frac{(\varphi(1) - \varphi(\nu))^{\alpha-1}}{(K(1))^{\alpha-1} \Gamma(\alpha)}, \quad 0 \leq x \leq \nu \leq 1. \end{aligned}$$

Thus, we have

$$\max_{x \in [0,1]} G(x, \nu) \leq \frac{(\varphi(1) - \varphi(\nu))^{\alpha-1}}{(K(1))^{\alpha-1} \Gamma(\alpha)}.$$

Observing that  $G(x, \nu)$  is a decreasing function for  $\nu \leq x$ , and it is an increasing function for  $x \leq \nu$ , we deduce that

$$\min_{x \in [1/4, 3/4]} G(x, \nu) = \begin{cases} g_1(3/4, \nu), & \nu \in (0, k), \\ g_2(1/4, \nu), & \nu \in [k, 1], \end{cases}$$

where  $1/4 < k < 3/4$  is a unique solution of the equation

$$\left[ \Upsilon\left(\frac{3}{4}\right) (\varphi(1) - \varphi(\nu)) \right]^{\alpha-1} - \left( \varphi\left(\frac{3}{4}\right) - \varphi(\nu) \right)^{\alpha-1} = \left[ \Upsilon\left(\frac{1}{4}\right) (\varphi(1) - \varphi(\nu)) \right]^{\alpha-1}.$$

Then the conclusion follows by setting

$$\gamma(\nu) = \begin{cases} \frac{(K(3/4))^{\alpha-1} (\varphi(1) - \varphi(\nu))^{\alpha-1} - (\varphi(3/4) - \varphi(\nu))^{\alpha-1} (K(1))^{\alpha-1}}{(\varphi(1) - \varphi(\nu))^{\alpha-1}}, & \nu \in (0, k), \\ (K(1/4))^{\alpha-1}, & \nu \in [k, 1]. \end{cases} \quad \square$$

For the sake of convenience, we denote

$$K(x) = \varphi(x) - \varphi(0), \quad \mu := [K(1)]^{\alpha-1} - \beta [K(\eta)]^{\alpha-1}.$$

**Lemma 2.6** *Let  $f \in C[0, 1]$  and  $1 < \alpha \leq 2, 0 < \mu < 1$ . Then the FBVP*

$$\begin{cases} D_{a^+}^{\alpha, \varphi} z(x) + f(x) = 0, & x \in (0, 1), \\ z(0) = 0, & z(1) = \beta z(\eta), \end{cases} \tag{2.4}$$

has a solution

$$z(x) = \int_0^1 H(x, \nu) \varphi'(\nu) f(\nu) d\nu,$$

where

$$H(x, v) = (\mu\Gamma(\alpha))^{-1}\Pi(x) \tag{2.5}$$

with

$$\Pi(x) = \begin{cases} [K(x)(\varphi(1) - \varphi(v))]^{\alpha-1} - \beta[K(x)(\varphi(\eta) - \varphi(v))]^{\alpha-1} - \mu(\varphi(x) - \varphi(v))^{\alpha-1}, & 0 \leq v \leq x \leq 1, v \leq \eta, \\ [K(x)(\varphi(1) - \varphi(v))]^{\alpha-1} - \mu(\varphi(x) - \varphi(v))^{\alpha-1}, & 0 < \eta \leq v \leq x \leq 1, \\ [K(x)(\varphi(1) - \varphi(v))]^{\alpha-1} - \beta[K(x)(\varphi(\eta) - \varphi(v))]^{\alpha-1}, & 0 \leq x \leq v \leq \eta < 1, \\ [K(x)(\varphi(1) - \varphi(v))]^{\alpha-1}, & 0 \leq x \leq v \leq 1, \eta \leq v. \end{cases}$$

**Lemma 2.7** *The function H defined by (2.5) satisfies  $H(x, v) > 0$  for  $x, v \in (0, 1)$ .*

*Proof* For  $0 \leq v \leq x \leq 1, v \leq \eta$ , we let  $K(x) = \varphi(x) - \varphi(0)$ ,  $\Pi(x) = \frac{K(v)}{K(x)}$ , and

$$h(x, v) = [K(x)(\varphi(1) - \varphi(v))]^{\alpha-1} - \beta[K(x)(\varphi(\eta) - \varphi(v))]^{\alpha-1} - \mu(\varphi(x) - \varphi(v))^{\alpha-1}.$$

Then we have

$$\begin{aligned} h(x, v) &= [K(x)(\varphi(1) - \varphi(v))]^{\alpha-1} - \beta[K(x)(\varphi(\eta) - \varphi(v))]^{\alpha-1} - \mu(\varphi(x) - \varphi(v))^{\alpha-1}, \\ h(x, v) &= (K(x))^{\alpha-1} [(\varphi(1) - \varphi(v))^{\alpha-1} - \beta(\varphi(\eta) - \varphi(v))^{\alpha-1} - \mu(1 - \Pi(x))^{\alpha-1}]. \end{aligned}$$

Consider

$$h(x) = [(\varphi(1) - \varphi(v))^{\alpha-1} - \beta(\varphi(\eta) - \varphi(v))^{\alpha-1} - \mu(1 - \Pi(x))^{\alpha-1}].$$

Thus, we obtain

$$h'(x) = -(\alpha - 1)\mu(1 - \Pi(x))^{\alpha-2} \frac{(K(v))\varphi'(x)}{(K(x))^2} < 0,$$

which implies that  $h(x)$  is a decreasing function for  $0 \leq v \leq x \leq 1, v \leq \eta$ . Moreover, we note that, for  $x \in (0, 1)$ , we have

$$\begin{aligned} h(1) &= [(\varphi(1) - \varphi(v))^{\alpha-1} - \beta(\varphi(\eta) - \varphi(v))^{\alpha-1} - \mu(1 - \Pi(1))^{\alpha-1}] \\ &= [-\beta(\varphi(\eta) - \varphi(v))^{\alpha-1} + \beta(K(\eta))^{\alpha-1}(1 - \Pi(1))^{\alpha-1}]. \end{aligned}$$

By adding and subtracting  $\varphi(0)$  in the first term of the above equation, we get

$$h(1) = \beta(K(\eta))^{\alpha-1} [(1 - \Pi(1))^{\alpha-1} - (1 - \Pi(\eta))^{\alpha-1}] > 0.$$

Hence,  $h(x, v) > 0$  for  $0 \leq v \leq x \leq 1, v \leq \eta$ .

For  $\eta \leq v \leq x$ , we let

$$h(x, v) = (K(x))^{\alpha-1} (\varphi(1) - \varphi(v))^{\alpha-1} - [(K(1))^{\alpha-1} - \beta(K(\eta))^{\alpha-1}] (\varphi(x) - \varphi(v))^{\alpha-1}.$$

It follows that

$$h(x, v) > (K(x))^{\alpha-1} (\varphi(1) - \varphi(v))^{\alpha-1} - (K(1))^{\alpha-1} (\varphi(x) - \varphi(v))^{\alpha-1}.$$

Adding and subtracting  $\varphi(0)$ , we have

$$\begin{aligned} h(x, v) &= (K(x))^{\alpha-1} (\varphi(1) - \varphi(0) + \varphi(0) - \varphi(v))^{\alpha-1} \\ &\quad - (K(1))^{\alpha-1} (\varphi(x) - \varphi(0) + \varphi(0) - \varphi(v))^{\alpha-1}. \end{aligned}$$

Therefore, we obtain

$$h(x, v) = [K(1)K(x)]^{\alpha-1} [(1 - \Pi(1))^{\alpha-1} - (1 - \Pi(x))^{\alpha-1}] > 0.$$

For  $x \leq v \leq \eta$ , we let

$$\begin{aligned} h(x, v) &= (K(x))^{\alpha-1} (\varphi(1) - \varphi(v))^{\alpha-1} - \beta(K(x))^{\alpha-1} (\varphi(\eta) - \varphi(v))^{\alpha-1} \\ &= (K(x))^{\alpha-1} [(\varphi(1) - \varphi(v))^{\alpha-1} - \beta(\varphi(\eta) - \varphi(v))^{\alpha-1}] \\ &= (K(x))^{\alpha-1} [(\varphi(1) - \varphi(v))^{\alpha-1} (1 - \Pi(1))^{\alpha-1} \\ &\quad - \beta(\varphi(\eta) - \varphi(v))^{\alpha-1} (1 - \Pi(\eta))^{\alpha-1}]. \end{aligned}$$

Since  $\mu > 0$ , we obtain

$$h(x, v) > (K(x))^{\alpha-1} \mu (1 - \Pi(\eta))^{\alpha-1} > 0.$$

Clearly, for  $0 \leq x \leq v \leq 1$ ,  $\eta \leq v$ ,  $h(x, v) > 0$ .

Hence  $H(x, v) > 0$  for  $x, v \in (0, 1)$ . □

**Lemma 2.8** *Let  $f \in C[a, b]$  and  $1 < \alpha \leq 2$ , then the FBVP*

$$\begin{cases} {}^c D_{0+}^{\alpha, \varphi} z(x) - f(x) = 0, & x \in (0, 1), \\ z(0) + z'(0) = 0, & z(1) + z'(1) = 0, \end{cases}$$

has a solution  $z(x) = \int_0^1 W(x, v) \varphi'(v) f(v) dv$ , where

$$W(x, v) = (\Gamma(\alpha - 1) [\varphi(1) - \varphi(0) + \varphi'(1) - \varphi'(0)])^{-1} P(x), \tag{2.6}$$

where

$$P(x) = \begin{cases} (\varphi'(0) + \varphi(0) - \varphi(x)) [(\varphi(1) - \varphi(v))^{\alpha-2} + \frac{1}{\alpha-1} (\varphi(1) - \varphi(v))^{\alpha-1}] \\ \quad + \frac{(\varphi(1) - \varphi(0)) + (\varphi'(1) - \varphi'(0))}{\alpha-1} (\varphi(x) - \varphi(v))^{\alpha-1}, & 0 \leq v \leq x \leq 1, \\ (\varphi'(0) + \varphi(0) - \varphi(x)) [(\varphi(1) - \varphi(v))^{\alpha-2} + \frac{1}{\alpha-1} (\varphi(1) - \varphi(v))^{\alpha-1}], \\ & 0 \leq x \leq v \leq 1. \end{cases}$$



**Lemma 2.9** *Let  $\varphi(x) \leq \varphi(0) + \varphi'(0)$ , then the function  $W$  defined by (2.6) satisfies  $W(x, \nu) > 0$  for all  $x, \nu \in (0, 1)$ . Besides, there exists a positive function  $\nu \in (0, 1)$  such that*

$$\min_{x \in [1/4, 3/4]} W(x, \nu) \geq \nu(\nu)M(\nu), \quad \nu \in (0, 1),$$

and

$$\max_{x \in [0, 1]} W(x, \nu) \leq M(\nu).$$

*Proof* Under the assumption  $\varphi(x) \leq \varphi(0) + \varphi'(0)$ , it is clear that  $W(x, \nu) > 0$  for all  $x, \nu \in (0, 1)$ .

Further assume that

$$W_1(x, \nu) = (\varphi'(0) + \varphi(0) - \varphi(x)) \left[ (\varphi(1) - \varphi(\nu))^{\alpha-2} + \frac{1}{\alpha-1} (\varphi(1) - \varphi(\nu))^{\alpha-1} \right] + \frac{(\varphi(1) - \varphi(0)) + (\varphi'(1) - \varphi'(0))}{\alpha-1} (\varphi(x) - \varphi(\nu))^{\alpha-1}, \quad 0 \leq \nu \leq x \leq 1,$$

and

$$W_2(x, \nu) = (\varphi'(0) + \varphi(0) - \varphi(x)) \left[ (\varphi(1) - \varphi(\nu))^{\alpha-2} + \frac{1}{\alpha-1} (\varphi(1) - \varphi(\nu))^{\alpha-1} \right], \quad 0 \leq x \leq \nu \leq 1.$$

Since  $W_1(x, \nu)$  is continuous for all  $x \in [1/4, 3/4]$ , we have

$$W_1(x, \nu) \geq (\varphi'(0) + \varphi(0) - \varphi(3/4)) \left[ (\varphi(1) - \varphi(\nu))^{\alpha-2} + \frac{1}{\alpha-1} (\varphi(1) - \varphi(\nu))^{\alpha-1} \right]$$

and

$$\max_{x \in [0, 1]} W_1(x, \nu) \leq (\varphi(1) + \varphi'(1)) \left[ (\varphi(1) - \varphi(\nu))^{\alpha-2} + \frac{2}{\alpha-1} ((\varphi(1) - \varphi(\nu))^{\alpha-1}) \right].$$

For some  $\nu \in (0, 1)$ ,  $W_2(x, \nu)$  is a decreasing function with respect to  $x$ . So, we obtain

$$\min_{x \in [1/4, 3/4]} W_2(x, \nu) = W_2(3/4, \nu) \quad \text{and} \quad \max_{x \in [0, 1]} W_2(x, \nu) = W_2(0, \nu),$$

and

$$W_2(0, \nu) < (\varphi(1) + \varphi'(1)) \left[ (\varphi(1) - \varphi(\nu))^{\alpha-2} + \frac{2}{\alpha-1} ((\varphi(1) - \varphi(\nu))^{\alpha-1}) \right].$$

Hence, for  $\nu \in (0, 1)$ , we have  $\min_{x \in [1/4, 3/4]} W(x, \nu) \geq m(\nu)$ , where

$$m(\nu) = (\varphi'(0) + \varphi(0) - \varphi(3/4)) \left[ (\varphi(1) - \varphi(\nu))^{\alpha-2} + \frac{1}{\alpha-1} (\varphi(1) - \varphi(\nu))^{\alpha-1} \right],$$

$\max_{x \in [0,1]} W(x, \nu) \leq M(\nu)$ , where

$$M(\nu) = (\varphi(1) + \varphi'(1)) \left[ (\varphi(1) - \varphi(\nu))^{\alpha-2} + \frac{2}{\alpha-1} (\varphi(1) - \varphi(\nu))^{\alpha-1} \right].$$

The conclusion follows by setting  $\nu(\nu) = \frac{m(\nu)}{M(\nu)}$

$$\nu(\nu) = \frac{\varphi'(0) + \varphi(0) - \varphi(3/4)}{\varphi'(1) + \varphi(1)} \frac{(\alpha-1)(\varphi(1) - \varphi(\nu))^{\alpha-2} + (\varphi(1) - \varphi(\nu))^{\alpha-1}}{(\alpha-1)(\varphi(1) - \varphi(\nu))^{\alpha-2} + 2(\varphi(1) - \varphi(\nu))^{\alpha-1}}, \quad \nu \in (0, 1).$$

□

**Definition 2.10** Let  $P$  be a cone of a real Banach space  $B$  and  $\theta : P \rightarrow [0, \infty)$  be a continuous map such that

$$\theta(\lambda z + (1 - \lambda)w) \geq \lambda\theta(z) + (1 - \lambda)\theta(w)$$

for all  $z, w \in P$  and  $0 \leq \lambda \leq 1$ . Then  $\theta$  is said to be a nonnegative continuous concave functional on  $P$ .

We shall rely on the following fixed point theorems to prove the main results.

**Lemma 2.11** ([13]) *Let  $B$  be a Banach space,  $P \subset B$  be a cone. Let  $\Omega_1, \Omega_2$  be open balls in  $B$  with  $0 \in \Omega_1$ , with  $\overline{\Omega}_1 \subset \Omega_2$ . Also, assume that  $\mathcal{F} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow P$  is a completely continuous operator such that, either*

- (i)  $\|\mathcal{F}z\| \leq \|z\|$  for  $z \in P \cap \partial\Omega_1$  and  $\|\mathcal{F}z\| \geq \|z\|$ ,  $z \in P \cap \partial\Omega_2$ , or
- (ii)  $\|\mathcal{F}z\| \geq \|z\|$  for  $z \in P \cap \partial\Omega_1$  and  $\|\mathcal{F}z\| \leq \|z\|$ ,  $z \in P \cap \partial\Omega_2$ .

Then  $\mathcal{F}$  has at least one fixed point in  $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ .

**Lemma 2.12** ([19]) *Suppose that  $B$  is a real Banach space,  $P \subset B$  is a cone,  $P_c = \{z \in P : \|z\| \leq c\}$ . Let  $\theta$  be a nonnegative continuous concave functional on  $P$  such that  $\theta(z) \leq \|z\|$  for all  $z \in \overline{P}_c$ , and  $P(\theta, b, d) = \{z \in P : b \leq \theta(z), \|z\| \leq d\}$ . Assume that  $\mathcal{F} : \overline{P}_c \rightarrow \overline{P}_c$  is completely continuous and there exist constants  $0 < a < b < d \leq c$  such that*

- (H<sub>1</sub>)  $\{z \in P(\theta, b, d) : \theta(\mathcal{F}z) > b\} \neq \emptyset$  and  $\theta(Pz) > b$  for  $z \in P(\theta, b, d)$ ;
- (H<sub>2</sub>)  $\|\mathcal{F}z\| < a$  for  $z \leq a$ ;
- (H<sub>3</sub>)  $\theta(\mathcal{F}z) > b$  for  $z \in P(\theta, b, c)$  with  $\|\mathcal{F}z\| > d$ .

Then  $\mathcal{F}$  has at least three fixed points  $z_1, z_2$ , and  $z_3$  with  $\|z_1\| < a$ ,  $b < \theta(z_2)$ ,  $a < \|z_3\|$  with  $\theta(z_3) < b$ .

### 3 Existence results

The following section is devoted to stating and proving the existence results for problems (1.1)–(1.2), (1.1)–(1.3), and (1.4).

Let  $E = C[0, 1]$  be a Banach space equipped with the norm  $\|z\| = \max_{x \in [0,1]} |z(x)|$ , and let  $P, R \subset E$  defined by

$$P = \{z \in E : z(x) \geq 0\} \quad \text{and} \quad R = \{z \in E : z(x) \geq \nu(\nu)\|z\|\}$$

be the cones, where  $\nu(\nu)$  is defined later.

**Lemma 3.1** *Let the operator  $T : P \rightarrow E$  be defined by*

$$Tz(x) = \int_0^1 G(x, v)\varphi'(v)f(v, z(v)) \, dv,$$

*then  $T : P \rightarrow P$  is a completely continuous operator.*

*Proof* Because of nonnegativity and continuity of  $G(x, v)$  and  $f(v, z(v))$ , the operator  $T : P \rightarrow P$  is continuous. Further assume that  $\Omega \subset P$  is bounded, and for all  $z \in \Omega$ , there exists a constant  $r_1 > 0$  such that  $\|z\| \leq r_1$ . Let  $L = \max_{x \in [0,1], z \in [0,r_1]} |f(x, z)| + 1$ , then for  $z \in \Omega$ , we have

$$\begin{aligned} |Tz(x)| &\leq \max_{x \in [0,1]} \int_0^1 G(x, v)\varphi'(v)f(v, z(v)) \, dv \\ &\leq L \int_0^1 \frac{(\varphi(1) - \varphi(v))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1}} \varphi'(v) \, dv \\ &= \frac{L(\varphi(1) - \varphi(0))}{\alpha}. \end{aligned}$$

For each  $z \in \Omega$ ,  $x_1, x_2 \in [0, 1]$ ,  $x_1 < x_2$ , we have

$$|Tz(x_2) - Tz(x_1)| = \int_0^1 |G(x_2, v) - G(x_1, v)|\varphi'(v)f(v, z(v)) \, dv.$$

Consider  $S = |G(x_2, v) - G(x_1, v)|$ ,

$$\begin{aligned} S = &\left| \frac{(\varphi(1) - \varphi(v))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1} \Gamma(\alpha)} [(\varphi(x_2) - \varphi(0))^{\alpha-1} - (\varphi(x_1) - \varphi(0))^{\alpha-1}] \right. \\ &\left. + \frac{1}{\Gamma(\alpha)} [(\varphi(x_1) - \varphi(v))^{\alpha-1} - (\varphi(x_2) - \varphi(v))^{\alpha-1}] \right|. \end{aligned}$$

By applying the mean value theorem, we obtain

$$|G(x_2, v) - G(x_1, v)| = |x_2 - x_1| \left[ \frac{(\varphi(1) - \varphi(v))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1} \Gamma(\alpha)} h'(\xi) + \frac{1}{\Gamma(\alpha)} g'(\eta) \right].$$

Therefore, as  $x_1 \rightarrow x_2$ ,  $|Tz(x_2) - Tz(x_1)| \rightarrow 0$ . Hence by the Arzela–Ascoli theorem,  $T : P \rightarrow P$  is completely continuous. □

**Lemma 3.2** *Let the operator  $\mathcal{F} : P \rightarrow P$  be defined by*

$$\mathcal{F}z(x) = \int_0^1 H(x, v)\varphi'(v)f(v, z(v)) \, dv.$$

*Then  $\mathcal{F} : P \rightarrow P$  is completely continuous.*

*Proof* Since  $f$ ,  $G$ , and  $\varphi'$  are nonnegative and continuous, the operator  $\mathcal{F} : P \rightarrow P$  is continuous. We also assume that  $\tilde{\Omega} \subset P$  is bounded, that is, for all  $z \in \tilde{\Omega}$ ,  $\|z\| \leq r_2$ , for some  $r_2 > 0$ , a positive constant.

Let  $\delta = \max_{x \in [0,1], z \in [0, \rho_2]} |f(x, z)| + 1$ , then for  $z \in \tilde{\Omega}$ ,

$$\begin{aligned} |\mathcal{F}u| &\leq \left| \int_0^1 H(x, v) \varphi'(v) f(v, z(v)) \, dv \right| \\ &\leq \varsigma \delta \int_0^1 (\varphi(1) - \varphi(v))^{\alpha-1} \varphi'(v) \, dv = \frac{\varsigma \delta (\varphi(1))^\alpha}{\alpha}, \end{aligned}$$

where  $\varsigma := \frac{(\varphi(1) - \varphi(0))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1} - b(\varphi(\eta) - \varphi(0))^{\alpha-1} \Gamma(\alpha)}$ . By applying the mean value theorem, we can show that, for  $x_1, x_2 \in [0, 1]$ ,  $x_1 < x_2$ ,  $|H(x_2, v) - H(x_1, v)| \rightarrow 0$  as  $x_1 \rightarrow x_2$ . Thus, by the Arzela–Ascoli theorem,  $\mathcal{F} : P \rightarrow P$  is completely continuous.  $\square$

**Lemma 3.3** *Let the operator  $Q : R \rightarrow R$  be defined by*

$$Qz(x) = \int_0^1 W(x, v) \varphi'(v) f(v, z(v)) \, dv.$$

*Then  $Q : R \rightarrow R$  is completely continuous.*

The proof of the above statement is straightforward and hence is omitted.

In the sequel, we make use of the following notations:

$$\begin{aligned} N &= \left( \int_{1/4}^{3/4} \gamma(v) \frac{(\varphi(1) - \varphi(v))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1} \Gamma(\alpha)} \varphi'(v) \, dv \right)^{-1}, \\ M &= \frac{\Gamma(\alpha + 1)}{(\varphi(1) - \varphi(0))}. \end{aligned}$$

**Theorem 3.4** *Assume that  $f(x, z) \geq 0$  is a continuous function on  $[0, 1] \times [0, \infty)$ . Let  $\rho_2 > \rho_1 > 0$  be two positive constants so that*

- (A<sub>1</sub>)  $f(x, z) \leq M\rho_2$  for  $(x, z) \in [0, 1] \times [0, \rho_2]$ ;
- (A<sub>2</sub>)  $f(x, z) \geq N\rho_1$  for  $(x, z) \in [1/4, 3/4] \times [0, \rho_1]$ .

*Then there exists at least one positive solution (say  $z_0$ ) of FBVP (1.1)–(1.2) such that  $\rho_1 \leq \|z_0\| \leq \rho_2$ .*

*Proof* By Lemma 3.1, we have  $T : P \rightarrow P$  is completely continuous. Let  $\Omega_1 = \{z \in P : \|u\| < \rho_1\}$ . For  $z \in \partial\Omega_1$ , we have  $0 \leq z(x) \leq \rho_1$  for all  $x \in [0, 1]$  such that assumption (A<sub>2</sub>) holds. For  $x \in [1/4, 3/4]$ , we find that

$$\begin{aligned} Tz(x) &= \int_0^1 G(x, v) \varphi'(v) f(v, z(v)) \, dv \\ &\geq N\rho_1 \int_{1/4}^{3/4} \gamma(v) \frac{(\varphi(1) - \varphi(v))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1} \Gamma(\alpha)} \varphi'(v) \, dv = \rho_1. \end{aligned}$$

Thus, for  $z \in \partial\Omega_1$ , we have

$$\|Tz\| \geq \|z\|.$$

On the other hand, let  $\Omega_2 = \{z \in P : \|z\| < \rho_2\}$ . For  $z \in \partial\Omega_2$ , we have that  $0 \leq z(x) \leq \rho_2$  for all  $x \in [0, 1]$  such that assumption  $(A_1)$  holds. For  $x \in [0, 1]$ , we find that

$$\begin{aligned} \|Tz(x)\| &= \max_{x \in [0,1]} \int_0^1 G(x, v)\varphi'(v)f(v, z(v)) \, dv \\ &\leq M\rho_2 \int_0^1 \frac{(\varphi(1) - \varphi(v))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1}\Gamma(\alpha)} \varphi'(v) \, dv \\ &= M\rho_2 \frac{(\varphi(1) - \varphi(0))}{\Gamma(\alpha + 1)} = \rho_2 = \|u\|. \end{aligned}$$

Hence by (ii) of Lemma 2.11, it follows that problem (1.1)–(1.2) has a positive solution with  $\rho_1 \leq \|z_0\| \leq \rho_2$ . □

**Theorem 3.5** *Let  $f(x, z)$  be continuous and  $a, b, c$  be positive constants with  $a < b < c$  such that the following assumptions hold:*

- (C<sub>1</sub>) for  $(x, z) \in [0, 1] \times [0, a], f(x, z) < Ma$ ;
- (C<sub>2</sub>) for  $(x, z) \in [1/4, 3/4] \times [b, c], f(x, z) \geq Nb$ ;
- (C<sub>3</sub>) for  $(x, z) \in [0, 1] \times [0, c], f(x, z) \leq Mc$ .

*Then there exist at least three positive solutions  $z_1, z_2, z_3$  of FBVP (1.1)–(1.2), with*

$$\begin{aligned} \max_{x \in [0,1]} |z_1(x)| < a, \quad b < \min_{x \in [1/4,3/4]} |z_2(x)| < \max_{x \in [0,1]} |z_2(x)| \leq c, \\ a < \max_{x \in [0,1]} |z_3(x)| \leq c, \quad \min_{x \in [1/4,3/4]} |z_3(x)| < b. \end{aligned}$$

*Proof* For  $z \in \bar{P}_c$  and  $\|z\| \leq c$ , let assumption  $(C_3)$  hold. Then we have

$$\begin{aligned} \|Tz\| &= \max_{x \in [0,1]} \left| \int_0^1 G(x, v)\varphi'(v)f(v, z(v)) \, dv \right| \\ &\leq Mc \int_0^1 \frac{(\varphi(1) - \varphi(v))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1}\Gamma(\alpha)} \varphi'(v) \, dv \\ &= Mc \frac{(\varphi(1) - \varphi(0))}{\Gamma(\alpha + 1)} = c. \end{aligned}$$

Therefore,  $T : \bar{P}_c \rightarrow \bar{P}_c$ . Similarly, we can show that, for  $z \in \bar{P}_a$ , condition  $(H_2)$  of Lemma 2.12 is fulfilled.

To check condition  $(H_1)$  of Lemma 2.12, let  $z(x) = \frac{b+c}{2}, x \in [0, 1]$ . Clearly  $\theta(z) = \theta(\frac{b+c}{2}) > b$ , as  $z(x) = \frac{b+c}{2} \in P(\theta, b, c)$ . Therefore,  $\{z \in P(\theta, b, c) : \theta(z) > b\} \neq \emptyset$ . Thus, if  $z \in P(\theta, b, c)$ , then for  $x \in [1/4, 3/4], b \leq z(x) \leq c$ . Moreover, from assumption  $(C_2)$ , we have  $f(x, z(x)) \geq Nb$  for  $x \in [1/4, 3/4]$ . Thus, for all  $z \in P(\theta, b, c)$ , we have

$$\theta(Tz) = \min_{x \in [1/4,3/4]} |(Tz)(x)| > Nb \int_{1/4}^{3/4} \gamma(v) \frac{(\varphi(1) - \varphi(v))^{\alpha-1}}{(\varphi(1) - \varphi(0))^{\alpha-1}\Gamma(\alpha)} \varphi'(v) \, dv = b.$$

Thus condition  $(H_1)$  of Lemma 2.12 is satisfied.

Hence there exist at least three positive solutions  $z_1, z_2, z_3$  of problem (1.1), (1.2) with

$$\begin{aligned} \max_{x \in [0,1]} |z_1(x)| < a, \quad b < \min_{x \in [1/4,3/4]} |z_2(x)| < \max_{x \in [0,1]} |z_2(x)| \leq c, \\ a < \max_{x \in [0,1]} |z_3(x)| \leq c, \quad \min_{x \in [1/4,3/4]} |z_3(x)| < b. \end{aligned} \quad \square$$

Now, we prove the existence result for problem (1.1)–(1.3).

**Theorem 3.6** *Let  $f(x, z)$  satisfy the condition*

$$|f(x, z) - f(x, w)| < \tilde{\gamma}(x)|z - w|, \quad \text{for } x \in [0, 1], z, w \in [0, \infty). \tag{3.1}$$

If

$$\int_0^1 (\varphi(v) - \varphi(0))^{\alpha-1} (\varphi(1) - \varphi(v))^{\alpha-1} \varphi'(v) \tilde{\gamma}(v) dv < \Gamma(\alpha)\mu, \tag{3.2}$$

then there exists a unique positive solution of FBVP (1.1)–(1.3).

*Proof* We want to prove that the operator  $\mathcal{F}^n$  is a contraction for sufficiently large  $n$ . For  $z, w \in P$ , we have

$$\begin{aligned} |(\mathcal{F}z - \mathcal{F}w)(x)| &= \int_0^1 H(x, v) \varphi'(v) |f(v, z(v)) - f(v, w(v))| dv \\ &< \frac{\|z - w\| (\varphi(x) - \varphi(0))^{\alpha-1}}{\Gamma(\alpha)\mu} \int_0^1 (\varphi(1) - \varphi(v))^{\alpha-1} \varphi'(v) \tilde{\gamma}(v) dv \\ &= \frac{L\|z - w\| (\varphi(x) - \varphi(0))^{\alpha-1}}{\Gamma(\alpha)\mu}, \end{aligned}$$

where  $L = \int_0^1 (\varphi(1) - \varphi(v))^{\alpha-1} \varphi'(v) \tilde{\gamma}(v) dv$ . Consequently, if we set  $K = \int_0^1 (\varphi(v) - \varphi(0))^{\alpha-1} \times (\varphi(1) - \varphi(v))^{\alpha-1} \varphi'(v) \tilde{\gamma}(v) dv$ , then

$$\begin{aligned} |(\mathcal{F}^2z - \mathcal{F}^2w)(x)| &= \int_0^1 H(x, v) \varphi'(v) |f(v, (\mathcal{F}z)(v)) - f(v, (\mathcal{F}w)(v))| dv \\ &< \frac{L\|z - w\| (\varphi(x) - \varphi(0))^{\alpha-1}}{(\Gamma(\alpha)\mu)^2} K \\ &= \frac{LK\|z - w\| (\varphi(x) - \varphi(0))^{\alpha-1}}{(\Gamma(\alpha)\mu)^2}, \end{aligned}$$

and by induction, we obtain

$$|(\mathcal{F}^nz - \mathcal{F}^nw)(x)| \leq \frac{LK^{n-1}\|z - w\| (\varphi(x) - \varphi(0))^{\alpha-1}}{(\Gamma(\alpha)\mu)^n}.$$

Using condition (3.2) and letting  $\frac{LK^{n-1}}{(\Gamma(\alpha)\mu)^n} < \frac{1}{2}$  for sufficiently large  $n$ , we have

$$\|\mathcal{F}^nz - \mathcal{F}^nw\| \leq \frac{1}{2}\|z - w\|.$$

This completes the proof. □

Next we prove the existence results for FBVP (1.4). To complete this, we assume a particular form for  $f$ . That is,

(P<sub>1</sub>) Let  $f(x, z) = \kappa f_1(x)f_2(z)$ , where  $\kappa$  is a positive constant. Besides, assume

$$f_0 = \lim_{z \rightarrow 0} \frac{f_2(z)}{z}, \quad f_\infty = \lim_{z \rightarrow \infty} \frac{f_2(z)}{z}.$$

**Theorem 3.7** *Suppose that (P<sub>1</sub>) holds.*

(P<sub>2</sub>) *If  $f_0 = 0, f_\infty = \infty$ , then FBVP (1.4) possesses a positive solution for all  $\kappa$  positive.*

(P<sub>3</sub>) *If  $f_0 = \infty, f_\infty = 0$ , then FBVP (1.4) possesses a positive solution for all  $\kappa$  positive.*

*Proof* From Lemma 3.3, we have that the operator  $Q : R \rightarrow R$  is completely continuous. Let  $\tilde{\Omega}_\rho = \{z \in R : \|z\| < \rho\}$ , then the operator  $Q : \tilde{\Omega}_\rho \rightarrow R$  defined as

$$Qz(x) = \kappa \int_0^1 W(x, v)\varphi'(v)f_1(v)f_2(z(v)) \, dv$$

is completely continuous. The hypothesis  $f_0 = 0$  implies, for  $\epsilon < (\kappa \int_0^1 M(v)\varphi'(v)f_1(v) \, dv)^{-1}$ , that there exists  $\rho_1 > 0$  such that  $f_2(z) < \epsilon|z|$ , whenever  $|z| < \rho_1$ . Let  $\tilde{\Omega}_{\rho_1} = \{z \in R : \|z\| < \rho_1\}$ , then for  $z \in R \cap \partial\tilde{\Omega}_{\rho_1}$ , we have

$$\begin{aligned} |Qz(x)| &\leq \epsilon \kappa \int_0^1 M(v)\varphi'(v)f_1(v)|z(v)| \, dv \\ &< \rho_1 \epsilon \kappa \int_0^1 M(v)\varphi'(v)f_1(v) \, dv < \rho_1. \end{aligned}$$

Thus, we have  $\|Qz\| \leq \|z\|$ . Furthermore, as  $f_\infty = \infty$ , there exists  $M^* > 0$  such that  $f_2(z) > N|z|$ , whenever  $|z| > M^*$ , where  $N > (\kappa \int_{1/4}^{3/4} v^2(v)M(v)f_1(v)\varphi'(v) \, dv)^{-1}$ . Choose  $\rho_2 > \{\rho_1, \frac{M^*}{v(v)}\}$ , and let  $\tilde{\Omega}_{\rho_2} = \{z \in R : \|z\| < \rho_2\}$ . Then, for  $z \in R \cap \partial\tilde{\Omega}_{\rho_2}$ , we have  $z(x) > M^*$  for  $x \in [1/4, 3/4]$ . We have

$$\begin{aligned} Qz(x) &\geq \kappa \int_{1/4}^{3/4} W(x, v)f_1(v)f_2(z(v))\varphi'(v) \, dv \\ &\geq \kappa N \int_{1/4}^{3/4} v^2(v)M(v)f_1(v)\|z(v)\|\varphi'(v) \, dv > \|z\|. \end{aligned}$$

Thus, for  $z \in R \cap \partial\tilde{\Omega}_{\rho_2}$ , we have  $\|Qz\| \geq \|z\|$ . Hence, by (i) of Lemma 2.11,  $Q$  has at least one fixed point  $u \in R \cap (\tilde{\Omega}_{\rho_2} / \tilde{\Omega}_{\rho_1})$ , and Lemma 2.8 implies  $u \in R \cap (\tilde{\Omega}_{\rho_2} / \tilde{\Omega}_{\rho_1})$  is the positive solution of FBVP (1.4).

The assumption  $f_0 = \infty$  implies that there exists  $\rho_1 > 0$  such that  $f_2(z) > N|z|$ , whenever  $|z| < \rho_1$  for  $N > (\kappa \int_{1/4}^{3/4} v^2(v)M(v)f_1(v)\varphi'(v) \, dv)^{-1}$ . Let  $\tilde{\Omega}_{\rho_1} = \{z \in R : \|z\| < \rho_1\}$ . Thus, for  $z \in R \cap \partial\tilde{\Omega}_{\rho_1}$ , we have

$$\begin{aligned} Qz(x) &\geq \kappa \int_{1/4}^{3/4} W(x, v)f_1(v)f_2(z(v))\varphi'(v) \, dv \\ &\geq \kappa N \int_{1/4}^{3/4} v^2(v)M(v)f_1(v)\|z(v)\|\varphi'(v) \, dv > \|z\|. \end{aligned}$$

Thus, for  $z \in R \cap \partial\tilde{\Omega}_{\rho_1}$ , we have  $\|Qz\| \geq \|z\|$ .

The hypothesis  $f_\infty = 0$  implies that there exists  $M^* > 0$  such that  $f_2(z) < \epsilon|u|$  for  $|u| > M^*$ ,  $\epsilon \in (2\kappa \int_0^1 M(v)f_1(v)\varphi'(v) dv)^{-1}$ . Therefore,  $f_2(z) \leq \epsilon|z| + L^*$  for  $z \in [0, \infty)$ , where  $L^* = \max_{0 \leq z \leq M^*} f_2(z) + 1$ . Let  $\tilde{\Omega}_{\rho_2} = \{z \in R : \|z\| < \rho_2\}$ , where  $\rho_2 > \{\rho_1, 2\kappa L^* \int_0^1 M(v)f_1(v)\varphi'(v) dv\}$ , then for  $z \in R \cap \partial\tilde{\Omega}_{\rho_2}$ , we have

$$\begin{aligned} |Qz(x)| &\leq \kappa \int_0^1 M(v)f_1(v)|z(v)|\varphi'(v) dv + \kappa L^* \int_0^1 M(v)f_1(v)\varphi'(v) dv \\ &\leq \rho_2 \kappa \int_0^1 M(v)f_1(v)\varphi'(v) dv + \kappa L^* \int_0^1 M(v)f_1(v)\varphi'(v) dv \\ &\leq \frac{\rho_2}{2} + \frac{\rho_2}{2} = \rho_2. \end{aligned}$$

Therefore, we have  $\|Qz\| \leq \|z\|$ , for  $z \in R \cap \partial\tilde{\Omega}_{\rho_2}$ . Thus, by (ii) of Lemma 2.11,  $Q$  has at least one fixed point  $z \in R \cap (\tilde{\Omega}_{\rho_2}/\tilde{\Omega}_{\rho_1})$ , and Lemma 2.8 implies that  $z \in R \cap (\tilde{\Omega}_{\rho_2}/\tilde{\Omega}_{\rho_1})$  is a positive solution of FBVP (1.4).  $\square$

**Theorem 3.8** *Let condition  $(P_1)$  hold. If  $f_0 = 0$  or  $f_\infty = 0$ , then there exists  $\kappa_0 > 0$  such that, for all  $\kappa > \kappa_0$ , FBVP (1.4) has a positive solution.*

*Proof* For  $\rho > 0$ , define  $R_\rho = \{z \in R : \|z\| < \rho\}$ . Then the operator  $Q : R_\rho \rightarrow R$  is completely continuous. If we fix  $\rho_1 > 0$ , define  $\kappa_0 = \rho_1(m_{\rho_1} \int_{1/4}^{3/4} v(v)M(v)f_1(v)\varphi'(v) dv)^{-1}$  and  $\tilde{\Omega}_{\rho_1} = \{z \in R : \|z\| < \rho_1\}$ , where  $m_{\rho_1} = \min_{v_{\rho_1} \leq z \leq \rho_1} f_2(z)$ . From condition  $(P_1)$ ,  $m_{\rho_1} > 0$ , then for  $z \in R \cap \partial\tilde{\Omega}_{\rho_1}$ , we have

$$\begin{aligned} \min_{x \in [1/4, 3/4]} Qz(x) &\geq \kappa \int_{1/4}^{3/4} v(v)M(v)f_1(v)f_2(z(v))\varphi'(v) dv \\ &> \kappa_0 m_{\rho_1} \int_{1/4}^{3/4} v(v)M(v)f_1(v)\varphi'(v) dv \\ &= \rho_1 = \|z\|, \end{aligned}$$

thus,

$$\|Qz\| > \|z\| \quad \text{for } z \in R \cap \partial\tilde{\Omega}_{\rho_1}, \kappa > \kappa_0.$$

Now if  $f_0 = 0$ , then for  $\epsilon \in (0, (\kappa \int_0^1 M(v)\varphi'(v)f_1(v) dv)^{-1})$ , there exists  $\tilde{\rho}_2 > 0$  such that  $f_2(z) < \epsilon|z|$ , whenever  $|z| < \tilde{\rho}_2$ . Let  $\tilde{\Omega}_{\rho_2} = \{z \in R : \|z\| < \rho_2\}$ ,  $\rho_2 < \{\rho_1, \tilde{\rho}_2\}$ , then for  $z \in R \cap \partial\tilde{\Omega}_{\rho_2}$ , we have

$$\begin{aligned} |Qz(x)| &\leq \epsilon \kappa \int_0^1 M(v)\varphi'(v)f_1(v)|z(v)| dv \\ &< \rho_2 \epsilon \kappa \int_0^1 M(v)\varphi'(v)f_1(v) dv < \rho_2, \end{aligned}$$

thus, we have  $\|Qz\| \leq \|z\|$ . The assumption  $f_\infty = 0$  implies that there exists  $M^* > 0$  such that  $f_2(z) < \epsilon|z|$  for  $|z| > M^*$ ,  $\epsilon \in (0, (2\kappa \int_0^1 M(v)f_1(v)\varphi'(v) dv)^{-1})$ . Therefore,  $f_2(z) \leq \epsilon|z| + L^*$  for  $u \in [0, \infty)$ , where  $L^* = \max_{0 \leq z \leq M^*} f_2(z) + 1$ . Let  $\tilde{\Omega}_{\rho_3} = \{z \in R : \|z\| < \rho_3\}$ , where  $\rho_3 >$



$\{\rho_1, 2\kappa L^* \int_0^1 M(v)f_1(v)\varphi'(v) dv\}$ , then for  $z \in R \cap \partial\tilde{\Omega}_{\rho_3}$ , we have

$$\begin{aligned} |Qz(x)| &\leq \kappa \in \int_0^1 M(v)f_1(v)|z(v)|\varphi'(v) dv + \kappa L^* \int_0^1 M(v)f_1(v)\varphi'(v) dv \\ &\leq \rho_3 \kappa \in \int_0^1 M(v)f_1(v)\varphi'(v) dv + \kappa L^* \int_0^1 M(v)f_1(v)\varphi'(v) dv \\ &\leq \frac{\rho_3}{2} + \frac{\rho_3}{2} = \rho_3, \end{aligned}$$

thus, we have  $\|Qz\| \leq \|z\|$  for  $z \in R \cap \partial\tilde{\Omega}_{\rho_3}$ . Hence, by Lemma 2.11,  $Q$  has a fixed point  $z \in R \cap (\tilde{\Omega}_{\rho_2}/\tilde{\Omega}_{\rho_1})$ , or  $z \in R \cap (\tilde{\Omega}_{\rho_3}/\tilde{\Omega}_{\rho_1})$ , according to  $f_0 = 0$  or  $f_\infty = 0$ , respectively. Therefore, for  $\kappa > \kappa_0$ , there exists a positive solution of FBVP (1.4).  $\square$

Similarly one can state and prove the following theorems.

**Theorem 3.9** *Let condition  $(P_1)$  hold. If  $f_0 = f_\infty = \infty$ , then there exists  $\kappa_0 > 0$  such that, for all  $0 < \kappa < \kappa_0$ , FBVP (1.4) has two positive solutions.*

**Theorem 3.10** *Let condition  $(P_1)$  hold. If  $f_0 = f_\infty = 0$ , then there exists  $\lambda_0 > 0$  such that, for all  $\lambda > \lambda_0$ , FBVP (1.4) has two positive solutions.*

#### 4 Illustrative examples

Corresponding to the proposed problems, we provide the following examples that demonstrate consistency to the main theorems.

*Example 4.1* Consider the FBVP

$$\begin{cases} D_{0^+}^{\frac{3}{2}, \frac{e^x}{3}} z(x) + (1+x^2)e^x \cos^2(z(x)) = 0, & x \in (0, 1), \\ z(0) = 0, & z(1) = 0, \end{cases} \tag{4.1}$$

where  $\alpha = \frac{3}{2}$ ,  $\varphi(x) = \frac{e^x}{3}$ ,  $\rho_1 = 0.1$ ,  $\rho_2 = 1.6$ , and  $f(x, z(x)) = (1+x^2)e^x \cos^2(z(x))$ . It follows that

$$f(x, z(x)) \leq (1+x^2)e^x \approx 5.436 \leq M\rho_2 \approx 5.584,$$

where  $M = \frac{\Gamma(\alpha+1)}{(\varphi(1)-\varphi(0))} \approx 3.49$ , and

$$f(x, z(x)) \geq (1+x^2)e^x(0.999) \approx 2.718 \geq N\rho_1 \approx 0.6884,$$

where

$$N = \left( \int_{\frac{1}{4}}^{\frac{3}{4}} \gamma(v) \frac{(\varphi(1)-\varphi(v))^{\alpha-1}}{(\varphi(1)-\varphi(0))^{\alpha-1} \Gamma(\alpha)} \varphi'(v) dv \right)^{-1} \approx 6.844 \quad \text{for } \gamma\left(\frac{3}{4}\right) \approx 0.61.$$

Hence all the conditions of Theorem 3.4 are satisfied. Therefore, FBVP (4.1) has at least one positive solution satisfying  $0.1 \leq \|z_0\| \leq 1.6$ .

*Example 4.2* Consider the FBVP

$$\begin{cases} D_{0+}^{\frac{3}{2}, \frac{x^2+1}{2}} z(x) + \frac{x}{2}(1+z^2) = 0, & x \in (0, 1), \\ z(0) = 0, & z(1) = \frac{1}{2}z(\frac{1}{2}), \end{cases} \tag{4.2}$$

where  $\alpha = \frac{3}{2}, \beta = \eta = \frac{1}{2}, f(x, z(x)) = \frac{x}{2}(1+z^2), \varphi(x) = \frac{(x^2+1)}{2}$ . Then we have

$$|f(x, z(x)) - f(x, w(x))| = \frac{x}{2}|z^2 - w^2| \leq \frac{x}{2}|z + w|.$$

Furthermore, we get

$$\int_0^1 (\varphi(v) - \varphi(0))^{\alpha-1} (\varphi(1) - \varphi(v))^{\alpha-1} \varphi'(v) \tilde{\gamma}(v) dv = \frac{1}{4} \int_0^1 (1-v^2)^{1/2} v^3 dv = 0.033$$

and

$$\mu = (\varphi(1) - \varphi(0))^{\alpha-1} - \beta(\varphi(\eta) - \varphi(0))^{\alpha-1} = \frac{3}{4} \sqrt{\frac{1}{2}} \approx 0.53.$$

Hence all the conditions of Theorem 3.6 are satisfied. Therefore there exists a unique positive solution of FBVP (4.2).

*Example 4.3* Consider the following FBVP:

$$\begin{cases} {}^c D_{0+}^{\frac{3}{2}, \sin(x)} z(x) - \kappa e^{-x} \frac{u^3(x)}{1+z^2(x)} = 0, & x \in (0, 1), \\ z(0) + z'(0) = 0, & z(1) + z'(1) = 0, \end{cases} \tag{4.3}$$

where  $\alpha = \frac{3}{2}, \rho_1 = 1, f(x, z(x)) = \kappa e^{-x} \frac{u^3(x)}{1+z^2(x)}$ , and  $\varphi(x) = \sin(x)$ . It is clear that

$$f(x, z(x)) = \kappa f_1(x) f_2(z(x)),$$

where  $f_1(x) = e^{-x}$  and  $f_2(z(x)) = \frac{z^3(x)}{1+z^2(x)}$ . By simple computations, we conclude that

$$\begin{aligned} \nu \left( \frac{3}{4} \right) &= \frac{\varphi'(0) + \varphi(0) - \varphi(\frac{3}{4})}{\varphi'(1) + \varphi(1)} \frac{(\alpha - 1)(\varphi(1) - \varphi(\frac{3}{4}))^{\alpha-2} + (\varphi(1) - \varphi(\frac{3}{4}))^{\alpha-1}}{(\alpha - 1)(\varphi(1) - \varphi(\frac{3}{4}))^{\alpha-2} + 2(\varphi(1) - \varphi(\frac{3}{4}))^{\alpha-1}} \\ &\approx 0.780955 < 1, \end{aligned}$$

$$m_{\rho_1} = \min_{0.781 \leq z \leq 1} \frac{z^3(x)}{1+z^2} \approx 0.25,$$

and

$$\begin{aligned} \kappa_0 &= \rho_1 \left( m_{\rho_1} \int_{\frac{1}{4}}^{\frac{3}{4}} \nu(v) M(v) f_1(v) \varphi'(v) dv \right)^{-1} \\ &\approx \left( 1.1715 \int_{\frac{1}{4}}^{\frac{3}{4}} e^{-v} \cos(v) dv \right)^{-1} \\ &\approx 3.1716. \end{aligned}$$

Moreover,  $f_0 = \lim_{z \rightarrow 0} \frac{f_2(z(x))}{z} = 0$ . Hence all the conditions of Theorem 3.8 are satisfied. Therefore, FBVP (4.3) has a positive solution.

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#### Authors' contributions

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