SEPARATION PROPERTIES IN THE PRIMITIVE IDEAL SPACE OF A MULTIPLIER ALGEBRA

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Abstract. We use recent work on spectral synthesis in multiplier algebras to give an intrinsic characterization of the separable C*-algebras A for which $\mathrm{Orc}(M(A)) = 1$, i.e. for which the relation of inseparability on the topological space of primitive ideals of the multiplier algebra $M(A)$ is an equivalence relation. This characterization has applications to the calculation of norms of inner derivations and other elementary operators on A and $M(A)$. For example, we give necessary and sufficient conditions on the ideal structure of a separable C*-algebra A for the norm of every inner derivation to be twice the distance of the implementing element to the centre of $M(A)$.

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1. Introduction

The origin of the paper lies in the following problem: suppose that A is a non-unital C^*-algebra and that $\mathrm{Prim}(A)$, the primitive ideal space of A with the hull-kernel topology, has some topological property. In what circumstances does $\mathrm{Prim}(M(A))$ (where $M(A)$ is the multiplier algebra of A) inherit the same or a similar property? For instance, if $\mathrm{Prim}(A)$ is Hausdorff, when is $\mathrm{Prim}(M(A))$ Hausdorff? This question is non-trivial, even in the simplest case when A is n-homogeneous, and in Theorem 2.1 we give an example of a 2-homogeneous C*-algebra A for which $\mathrm{Prim}(M(A))$ is non-Hausdorff.

Although $\mathrm{Prim}(M(A))$ in Theorem 2.1 is non-Hausdorff, it does have the property that the relation $\sim$ on $\mathrm{Prim}(M(A))$ is an equivalence relation (where for a C*-algebra B we say that $P \sim Q$ if the primitive ideals P and Q cannot be separated by disjoint open sets in $\mathrm{Prim}(B)$). In general the relation $\sim$ is reflexive and symmetric but not necessarily transitive. When $\sim$ is an equivalence relation on $\mathrm{Prim}(A)$ we say that $\mathrm{Orc}(A) = 1$, and otherwise the *connecting order* $\mathrm{Orc}(A)$ is greater than 1 (for an explanation of this terminology, see [40]). C*-algebras with $\mathrm{Orc}(A) = 1$ include von Neumann and AW*-algebras and their quotients, numerous group C*-algebras (see Section 3), the spoke-algebras of [13], and also *quasi-standard* C*-algebras (i.e. those for which the relation $\sim$ is an open equivalence relation on $\mathrm{Prim}(A)$ [9]).

The main purpose of this paper is to characterize the separable C*-algebras A for which $\mathrm{Orc}(M(A)) = 1$. Since $\mathrm{Prim}(A)$ is canonically homeomorphic to an open subset of $\mathrm{Prim}(M(A))$ [37, 4.1.10], it is easily seen that such algebras necessarily have $\mathrm{Orc}(A) = 1$. Regarding A as a $C_0(X)$-algebra in a natural way, we show in Theorem 4.2 that a necessary condition for $\mathrm{Orc}(M(A)) = 1$ is that $M(A)$ should have spectral synthesis as a $C(\beta X)$-algebra in the sense of [12] (where βX is the Stone-Čech compactification of X).

Combining this with the characterization of spectral synthesis in [12, Corollary 3.10], it follows that $\mathrm{Orc}(M(A)) = 1$ if and only if $\mathrm{Orc}(A) = 1$ and the Glimm ideals of A are locally modular (Corollary 4.7).

Part of our interest in the condition $\mathrm{Orc}(A) = 1$ arises from its connection with the derivation constant $K_s(A)$ (see Section 5). Indeed, if A is a unital non-commutative C^*-algebra, $K_s(A)$ takes the optimal value $1/2$ if and only if $\mathrm{Orc}(A) = 1$ [40, Theorem 4.4]. Thus Corollary 4.7 allows us to characterize the separable C*-algebras A for which $K_s(M(A)) = 1/2$ (Corollary 5.1).

We now describe the structure of the paper. In Section 2 we give the example of a 2-homogeneous C^*-algebra A such that $\mathrm{Prim}(M(A))$ is non-Hausdorff. In Section 3, we solve an old problem for σ-unital C^*-algebras with $\mathrm{Orc}(A) = 1$ by showing that if primitive ideals P and Q can be separated by disjoint open sets then they can be separated by a continuous function (Theorem 3.3). The solution of this problem is needed in Section 4 for the proofs of Theorem 4.6 and Corollary 4.7. On the other hand, we show in Example 3.4 that if $\mathrm{Prim}(A)$ is not σ-compact then it is possible to have $\mathrm{Orc}(A) = 1$ and primitive ideals P and Q which can be separated by disjoint open sets but not by continuous functions.

In Section 4 we show that if A is a separable $C_0(X)$-algebra then an ideal J_x of A which is not locally modular gives rise to an ideal H_x of $M(A)$ which is not 2-primal (Theorem 4.2). This leads to Theorem 4.6 which uses local modularity to link n-primality of the ideals J_x with n-primality of the ideals H_x. From this, the characterization of separable C*-algebras A for which $\mathrm{Orc}(M(A)) = 1$ follows easily (Corollary 4.7).

If A is a non-unital C^*-algebra then the multiplier algebra $M(A)$ provides the natural setting for the study of elementary operators. In particular, if $T : A \to A$ is an elementary operator of the form $T(x) = \sum_{i=1}^{n} a_i x b_i$ $(x \in A)$, where $a_i, b_i \in M(A)$ $(1 \leq i \leq n)$ then T extends to $M(A)$, without increase of norm, by using the same formula for $x \in M(A)$. In Section 5 we see how the results of the previous section can be used in estimating the norms of elementary operators on $M(A)$ and in particular the derivation constants $K(M(A))$ and $K_s(M(A))$ introduced in [3] (see Section 5 for the definitions). For non-commutative A, the optimal value of $K(M(A))$ is $K(M(A)) = 1/2$, which occurs precisely when the norm of every inner derivation of A (or of $M(A)$) is equal to twice the distance of the implementing element to the centre of $M(A)$. We show, for example, that if A is a separable non-commutative C^*-algebra then $K(M(A)) = 1/2$ if and only if every Glimm ideal of A is 3-primal and locally modular (Theorem 5.2). We also give applications to the canonical contraction $\Theta_Z : M(A) \otimes_{Z,h} M(A) \to \mathrm{CB}(A)$ from the central Haagerup tensor product of $M(A)$ to the space of completely bounded linear maps on a C^*-algebra A (Theorem 5.7 and Theorem 5.9).

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2. A 2-homogeneous C^*-algebra with $\mathrm{Prim}(M(A))$ non-Hausdorff

We begin with an example of a 2-homogeneous C*-algebra A for which $\mathrm{Prim}(M(A))$ is non-Hausdorff. Recall that a C*-algebra A is *n-homogeneous* if every irreducible representation A has the same finite dimension n. If A is n-homogeneous then A is quasi-central (i.e. no primitive ideal of A contains the centre of A) and $\mathrm{Prim}(A)$ is Hausdorff [28, Theorem

4.2]. Furthermore $M(A)$ is n-subhomogeneous (i.e. every irreducible representation is of dimension bounded by n).

It is known that if A is an n-homogeneous C*-algebra then $M(A)$ is n-homogeneous if and only if A is of 'finite type' [34, Remark 3.3] (see also [16, 3.16(ii)]). For examples where A is separable and not of finite type, see [39, Corollary 4.8] and [38, Example 3.5]. If A is not of finite type then it is known that $\mathrm{Orc}(M(A)) = 1$ and that $M(A)$ forms a continuous field of C*-algebras over the Stone-Čech compactification of $\mathrm{Prim}(A)$ [10, Corollary 4.10] (see also the proof of [34, Lemma 3.4]). Some of the fibre algebras corresponding to points in the Stone-Čech remainder are isomorphic to $M_n(\mathbb{C})$ [34, Lemma 3.4] but little is known about the others; in particular, it appears to be previously unknown whether $\mathrm{Prim}(M(A))$ can be non-Hausdorff (i.e. whether the fibre algebra corresponding to a point in the Stone-Čech remainder can be non-simple).

In this section we exhibit an example with $\mathrm{Prim}(M(A))$ non-Hausdorff. We are grateful to Chuck Akemann, Larry Brown, Bojan Magajna, and Chris Phillips for discussions on this subject.

Theorem 2.1. *There exists a 2-homogeneous C*-algebra A for which* $\mathrm{Prim}(M(A))$ *is non-Hausdorff.*

Proof. For a completely regular Hausdorff space W, set $W^* = \beta W \setminus W$. Let $\Lambda = \beta\mathbb{R} \setminus \mathbb{N}^*$. Then Λ is a locally compact Hausdorff space, since $\mathbb{N}^*$ is a closed subset of $\beta\mathbb{R}$, but Λ is not normal [23, 6P]. To see this, note that since $\Lambda \supseteq \mathbb{R}$, $\beta\Lambda = \beta\mathbb{R}$ [23, 6.7]. The closure of $\Lambda \setminus \mathbb{R}$ in $\beta\mathbb{R}$ is equal to $\mathbb{R}^*$, and hence $X := \Lambda \setminus \mathbb{R}$ and $Y := \mathbb{N}$ are disjoint closed subsets of Λ whose closures in $\beta\Lambda = \beta\mathbb{R}$ are not disjoint, since they both contain $\mathbb{N}^*$. The space Λ is pseudocompact (i.e. every continuous real-valued function on Λ is bounded) [23, 6P]. Since Λ is locally compact, it follows that $\Lambda \times \Lambda$ is also pseudocompact [43, 8.21]. This implies that $\beta(\Lambda \times \Lambda) = \beta\Lambda \times \beta\Lambda$ by Glicksberg's theorem [43, 8.12]. Note also that, since $\beta\Lambda = \beta\mathbb{R}$, $\Lambda^* = \mathbb{N}^*$.

Now set $S = X \times Y$ and $T = Y \times X$. Then S and T are homeomorphic disjoint closed subsets of $\Lambda \times \Lambda$, and their closures in $\beta\Lambda \times \beta\Lambda$ both contain $\mathbb{N}^* \times \mathbb{N}^*$. Let Θ be the homeomorphism of $\Lambda \times \Lambda$ that takes $(x, y) \in \Lambda \times \Lambda$ to (y, x). Then $\Theta(S) = T$ and $\Theta(T) = S$. Let $B = C_0(\Lambda \times \Lambda) \otimes M_2(\mathbb{C})$, and let

$$A = \{f \in B : f((x, y)) = zf(\Theta((x, y)))z^*, \ (x, y) \in S\},$$

where z is the self-adjoint unitary $\mathrm{diag}(1, -1)$. We claim that A is a 2-homogeneous C*-algebra but that $\mathrm{Prim}(M(A))$ is not Hausdorff.

First note that by standard representation theory, every irreducible representation of A is (unitarily equivalent to) a summand of the restriction to A of a point evaluation of B. By constructing suitable matrix-valued functions, it can be seen that each such restriction to A is irreducible, mapping A to $M_2(\mathbb{C})$. For example, for $(x, y) \in S$, let U and V be disjoint neighbourhoods of x and y respectively in Λ and let $g \in C_0(\Lambda \times \Lambda)$ such that g is supported in $U \times V$ and $g(x, y) = 1$. For $m \in M_2(\mathbb{C})$ define a cross-section f by $f(u, v) = g(u, v)m + g(v, u)z^*mz$ ($(u, v) \in \Lambda \times \Lambda$). Then $f \in A$ (since $z = z^*$) and $f(x, y) = m$. Hence A is 2-homogeneous.

For $(x, y) \in \Lambda \times \Lambda$, let $\pi_{(x,y)}$ denote the irreducible representation of A given by point evaluation at (x, y). Then $\pi_{(x,y)}$ extends to an irreducible representation $\tilde{\pi}_{(x,y)}$ of $M(A)$.

Hence for $b \in M(A)$, the map $\sigma(b)$ given by $\sigma(b)((x,y)) = \tilde{\pi}_{(x,y)}(b)$ defines a bounded $M_2(\mathbb{C})$-valued cross-section over $\Lambda \times \Lambda$. By constructing suitable scalar-valued functions $f \in A$ and using the fact that $bf \in A$, it can be checked that $\sigma(b)$ is a continuous cross-section, and satisfies $\sigma(b)((x,y)) = z\sigma(b)(\Theta((x,y)))z^*$ for $(x,y) \in S$. For example, for $(x,y) \in S$, let U, V, and g be as above. Define a cross-section f by

$$f(u,v) = g(u,v)1 + g(v,u)1 \qquad ((u,v) \in \Lambda \times \Lambda).$$

Then $f \in A$ and $f(u,v) = g(u,v)1$ for $(u,v) \in U \times V$. Let (u_α, v_α) be a net in $\Lambda \times \Lambda$ with $(u_\alpha, v_\alpha) \to (x,y)$. Then eventually $(u_\alpha, v_\alpha) \in U \times V$, so

$$\sigma(b)(u_\alpha, v_\alpha)g(u_\alpha, v_\alpha) = \sigma(b)(u_\alpha, v_\alpha)f(u_\alpha, v_\alpha) = (bf)(u_\alpha, v_\alpha) \to (bf)(x,y) = \sigma(b)(x,y).$$

Hence $\sigma(b)(u_\alpha, v_\alpha) \to \sigma(b)(x,y)$ because $g(u_\alpha, v_\alpha) \to 1$. Thus $\sigma(b)$ is continuous at (x,y), and

$$\sigma(b)((x,y)) = (bf)((x,y)) = z(bf)((y,x))z^* = z\sigma(b)((y,x))z^*.$$

Since $M(B)$ is the C*-algebra of all bounded continuous functions from $\Lambda \times \Lambda$ into $M_2(\mathbb{C})$ [1], the map $\sigma : b \mapsto \sigma(b)$ defines a *-homomorphism from $M(A)$ into $M(B)$ fixing elements of A; and since A is essential in $M(A)$, we have that $\ker \sigma = \{0\}$. It follows from standard theory that $\sigma(M(A))$ is the idealizer of A in $M(B)$. For $h \in M(B)$, each of the four entry functions has a unique extension to a continuous function on $\beta(\Lambda \times \Lambda) = \beta\Lambda \times \beta\Lambda$, and thus we may consider $M(B)$ as the C*-algebra of continuous $M_2(\mathbb{C})$-valued functions on $\beta\Lambda \times \beta\Lambda$.

For $w \in \mathbb{N}^* = \Lambda^*$, there exist nets (x_α) and (y_α) (both indexed by a neighbourhood base for w) in X and Y respectively such that $x_\alpha \to w$ and $y_\alpha \to w$. Then for $b \in M(A)$ we have $\sigma(b)((x_\alpha, y_\alpha)) \to \sigma(b)((w,w))$, and

$$z^*\sigma(b)((x_\alpha, y_\alpha))z = \sigma(b)((y_\alpha, x_\alpha)) \to \sigma(b)((w,w)).$$

Hence $\sigma(b)((w,w)) = z^*\sigma(b)((w,w))z$, so $\sigma(b)((w,w))$ is a diagonal matrix. Note that the constant cross-section $d_{1,1}$ given by $d_{1,1}((x,y)) = \mathrm{diag}(1,0)$ $((x,y) \in \beta(\Lambda \times \Lambda))$ belongs to the idealizer of A in $M(B)$, and likewise the constant cross-section $d_{2,2}$ given by $d_{2,2}((x,y)) = \mathrm{diag}(0,1)$. Hence the map $b \mapsto \sigma(b)((w,w))$ $(b \in M(A))$ is a *-homomorphism from $M(A)$ onto $\mathbb{C} \oplus \mathbb{C}$. Thus the net $(\tilde{\pi}_{x_\alpha, y_\alpha})$ converges to two distinct limits in $\widehat{M(A)}$. Finally, $\widehat{M(A)}$ is homeomorphic to $\mathrm{Prim}(M(A))$ since $M(A)$ is 2-subhomogeneous. □

The example just given makes heavy use of special properties which can only hold in a non-normal space such as Λ. This raises the question of whether $\mathrm{Prim}(M(A))$ can be non-Hausdorff if A is a σ-unital n-homogeneous C*-algebra (recall that a C^*-algebra A is *σ-unital* if it contains a strictly positive element or, equivalently, a countable approximate unit [37, 3.10.5]).

3. Separation by continuous functions

In this section we consider the following basic problem (which arises in the proofs of Theorem 4.6 and Corollary 4.7). Let A be a C*-algebra and suppose that the relation $\sim$ of Section 1 is an equivalence relation on $\mathrm{Prim}(A)$. Let $P, Q \in \mathrm{Prim}(A)$ with $P \not\sim Q$. Does there exist a continuous real function f on $\mathrm{Prim}(A)$ such that $f(P) \neq f(Q)$? The second author showed twenty years ago that this is the case if $\mathrm{Prim}(A)$ is compact [40, Corollary 2.7]. Here we use a method recently introduced by Lazar to exhibit such a function if $\mathrm{Prim}(A)$ is

σ-compact. On the other hand, we give an example to show that such a function may not exist if Prim(A) is not σ-compact.

Following Lazar [30], we work in a more general context. We say that a topological space X is *locally compact* if every point has a neighbourhood base consisting of compact sets. A subset F of X is a *limit set* if there is a net in X converging to all the points of F. Let $\mathcal{L}(X)$ be the set of closed limit sets of X and set $\mathcal{L}'(X) = \mathcal{L}(X) \setminus \{\emptyset\}$. Define a topology τ_s (the Fell topology [22]) on $\mathcal{L}(X)$ as follows. A base for τ_s consists of the family of all sets

$$U(C, \Phi) = \{F \in \mathcal{L}(X) : F \cap C = \emptyset;\ F \cap V \neq \emptyset, V \in \Phi\},$$

where C is a compact subset of X (possibly the empty set) and Φ is a finite family of open subsets of X. Then with this topology $\mathcal{L}(X)$ is a compact Hausdorff space, and $\mathcal{L}'(X)$ is a locally compact subspace. If X is σ-compact then $\mathcal{L}'(X)$ is σ-compact [30, Lemma 2.5].

The first lemma is [15, Vol.2: Exercise 15(b) on p.244]. For completeness, we give a proof in the Appendix.

Lemma 3.1. *Let Y be a locally compact, σ-compact Hausdorff space and let R be an equivalence relation on Y. If the graph of R is closed in $Y \times Y$ then Y/R is a normal Hausdorff space.*

The proof of the next result is modelled on the methods of Lazar [30].

Proposition 3.2. *Let X be a locally compact, σ-compact space and let $*$ be an equivalence relation on X. The following are equivalent:*
(i) *$X/*$ is a Hausdorff space;*
(ii) *$X/*$ is a normal Hausdorff space;*
(iii) *$*$ contains $\sim$ and the graph of $*$ is closed in $X \times X$.*

Proof. Let $q : X \to X/*$ denote the quotient map. Suppose that (i) holds. Let $x, y \in X$ with $x \sim y$ and let (x_α) be a net in X converging to both x and y. Then $(q(x_\alpha))$ converges to both $q(x)$ and $q(y)$, so $q(x) = q(y)$ since $X/*$ is Hausdorff. Hence $x * y$, so $*$ contains $\sim$. Now let (x_α, y_α) be a net in $X \times X$ with limit $(x, y) \in X \times X$ and with $x_\alpha * y_\alpha$ for each α. Then $q(x_\alpha) \to q(x)$ and $q(y_\alpha) \to q(y)$. Hence $q(x) = q(y)$, by the Hausdorffness of $X/*$, since $q(x_\alpha) = q(y_\alpha)$ for each α. Thus $x * y$, so the graph of $*$ is closed in $X \times X$. This shows that (iii) holds.

Now suppose that (iii) holds. Define a relation $\diamond$ on $\mathcal{L}'(X)$ by $F \diamond G$ if there exist $x \in F$ and $y \in G$ such that $x * y$. Then $\diamond$ is reflexive and symmetric. Since $*$ contains $\sim$ we see that $x * x'$ for all $x, x' \in F$ and hence if $F \diamond G$ then $x' * y'$ for all $x' \in F$ and $y' \in G$. Thus $\diamond$ is transitive, and hence we see that $\diamond$ is an equivalence relation on $\mathcal{L}'(X)$. We now show that the graph of $\diamond$ is closed. Let $(F, G) \in \mathcal{L}'(X) \times \mathcal{L}'(X)$ with $(F, G) \notin \diamond$. Let $x \in F$ and $y \in G$. Then $(x, y) \notin *$, and the graph of $*$ is closed, so there exist open neighbourhoods M of x and N of y such that the basic open neighbourhood $M \times N$ of (x, y) does not meet $*$. Set

$$M' = \{F' \in \mathcal{L}'(X) : F' \cap M \neq \emptyset\} \text{ and } N' = \{G' \in \mathcal{L}'(X) : G' \cap N \neq \emptyset\}.$$

Then $M' \times N'$ is an open neighbourhood of (F, G) in $\mathcal{L}'(X) \times \mathcal{L}'(X)$. If $(F', G') \in M' \times N'$ then there exist $x' \in F' \cap M$ and $y' \in G' \cap N$. Hence $(x', y') \in M \times N$ which does not meet $*$. Thus $(F', G') \notin \diamond$, so the graph of $\diamond$ is closed.

Since $\mathcal{L}'(X)$ is σ-compact [30, Lemma 2.5], it follows from Lemma 3.1 that $\mathcal{L}'(X)/\diamond$ is normal and Hausdorff. It remains to show, therefore, that $X/*$ is homeomorphic to $\mathcal{L}'(X)/\diamond$.

Let $Q : \mathcal{L}'(X) \to \mathcal{L}'(X)/\diamond$ denote the quotient map, and define a map $\chi : X/* \to \mathcal{L}'(X)/\diamond$ by $\chi(q(x)) := Q(F)$ where F is any closed limit set containing x. By definition of $\diamond$, the map χ is well-defined and injective, and it is trivially surjective. Let V be a closed subset of $\mathcal{L}'(X)/\diamond$ and set $W = q^{-1}(\chi^{-1}(V))$. Let (x_α) be a net in W with limit $x \in X$. For each α let $F_\alpha \in \mathcal{L}'(X)$ with $x_\alpha \in F_\alpha$ and note that $Q(F_\alpha) \in V$ for each α. By the τ_s-compactness of $\mathcal{L}(X)$, and by passing to a subnet if necessary, we may assume that there exists $F \in \mathcal{L}(X)$ with $F_\alpha \to F$. Then $x \in F$ by [30, Lemma 2.1], so $F \in \mathcal{L}'(X)$. It follows that $Q(F) \in V$, since Q is continuous and V is closed. Thus $\chi(q(x)) \in V$, so $x \in W$. This shows that W is a closed subset of X and hence that $\chi^{-1}(V)$ is closed in $X/*$. It follows that the map χ is continuous.

Now let V be a closed subset of $X/*$ and set $W = q^{-1}(V)$. Let $Y = \{F \in \mathcal{L}'(X) : F \subseteq W\}$. Then Y is $\diamond$-saturated since W is $*$-saturated. Let $F \in \mathcal{L}'(X) \setminus Y$. Then there exists $x \in F$ such that $x \in U := X \setminus W$. Hence the set $\{G \in \mathcal{L}'(X) : G \cap U \neq \emptyset\}$ is a τ_s-neighbourhood of F disjoint from Y. It follows that Y is closed and hence that $Q(Y) = \chi(V)$ is closed in $\mathcal{L}'(X)/\diamond$. Thus χ is a homeomorphism and (ii) holds. □

Before proceeding, we need to introduce some further terminology. A weaker relation than $\sim$ on Prim(A) is the relation $\approx$ which is defined by $P \approx Q$ if P and Q cannot be separated by a continuous bounded real-valued function. The relation $\approx$ is always an equivalence relation, and the $\approx$-classes are called Glimm classes. The kernel of a Glimm class (i.e. the intersection of all the primitive ideals in the class) is called a Glimm ideal of A, and the set of Glimm ideals is denoted Glimm(A). If A is unital then these are precisely the ideals of A generated by the maximal ideals of the centre of A [24]. The usual topology on Glimm(A) is the weakest topology such that the functions on Glimm(A) induced by the continuous bounded functions on Prim(A) are all continuous [17]. This topology is Hausdorff and completely regular. The map which takes each primitive ideal of A to the unique Glimm ideal which it contains is called the complete regularization map, and is continuous.

A closed two-sided ideal J of a C*-algebra A is said to be *primal* if whenever $n \geq 2$ and $J_1, J_2, \ldots, J_n$ are ideals of A with product $J_1 J_2 \ldots J_n = \{0\}$ then at least one of the J_i is contained in J. This concept arose in [4] where it was shown that a state of A is a weak*-limit of factorial states if and only if the kernel of its Gelfand-Naimark-Segal representation is primal. The ideal J is primal if and only if there is a net in Prim(A) which converges to every point in (a dense subset of) Prim(A/J) (see [4, Proposition 3.2]). If the variable integer n in the definition of a primal ideal is replaced by a fixed integer $n \geq 2$, we obtain the notion of an *n-primal* ideal J. By [8, Lemma 1.3], the ideal J is n-primal if and only if the ideal $\bigcap_{i=1}^n P_i$ is primal whenever $P_1, \ldots, P_n$ are primitive ideals of A containing J.

If an ideal J is 2-primal then $P \sim Q$ whenever P and Q are primitive ideals containing J. Hence P and Q lie in the same Glimm class. It follows that every 2-primal ideal contains a unique Glimm ideal. The condition that every Glimm ideal be 2-primal is equivalent to requiring that for $P, Q \in$ Prim(A), $P \sim Q$ if and only if $P \approx Q$. Evidently this condition implies that $\sim$ is an equivalence relation on Prim(A), i.e. that Orc$(A) = 1$, and the question naturally arises as to whether the reverse implication holds. It is already known that this is so when A is unital [40, Corollary 2.7], and we are now able to extend this to the case when A is σ-unital.

Theorem 3.3. *Let A be a σ-unital C*-algebra.*

(a) *The relation* $\approx$ *is the smallest equivalence relation on* $\mathrm{Prim}(A)$ *which contains* $\sim$ *and which has closed graph in* $\mathrm{Prim}(A) \times \mathrm{Prim}(A)$.

(b) *Define a relation* $*$ *on* $\mathrm{Prim}(A)$ *by* $P * Q$ *if* P *and* Q *belong to the same* $\sim$*-component. Then the graph of* $*$ *is closed in* $\mathrm{Prim}(A) \times \mathrm{Prim}(A)$ *if and only if every Glimm class is a* $\sim$*-component of* $\mathrm{Prim}(A)$.

(c) *If* $\mathrm{Orc}(A) = 1$ *then* $\sim$ *and* $\approx$ *coincide on* $\mathrm{Prim}(A)$ *and every Glimm ideal of* A *is* 2*-primal.*

Proof. (a) Let $\mathcal{R}$ be the set of all equivalence relations on $\mathrm{Prim}(A)$ which contain $\sim$ and which have closed graph. Clearly the equivalence relation $\approx$ contains $\sim$, and we now show that the graph of $\approx$ is closed. Let (P_α, Q_α) be a net in $\mathrm{Prim}(A) \times \mathrm{Prim}(A)$ with limit $(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(A)$. If $P \not\approx Q$ then there is a continuous bounded real-valued function f on $\mathrm{Prim}(A)$ such that $f(P) \neq f(Q)$. Hence eventually $f(P_\alpha) \neq f(Q_\alpha)$. It follows that the graph of $\approx$ is closed, and hence that $\approx$ belongs to $\mathcal{R}$. Now let $R = \bigcap\{S : S \in \mathcal{R}\}$. Then R is a equivalence relation on $\mathrm{Prim}(A)$ containing $\sim$ and with closed graph. Thus R is contained in $\approx$. On the other hand, since A is σ-unital, $\mathrm{Prim}(A)$ is σ-compact and so, by Proposition 3.2, $\mathrm{Prim}(A)/R$ is a normal Hausdorff space and hence is completely regular. This implies that R contains $\approx$, and thus R coincides with $\approx$.

(b) Clearly $*$ is the smallest equivalence relation on $\mathrm{Prim}(A)$ containing $\sim$. Hence by part (a), $*$ coincides with $\approx$ if and only if the graph of $*$ is closed. On the other hand $*$ coincides with $\approx$ if and only if every $\approx$-class is a $\sim$-component of $\mathrm{Prim}(A)$.

(c) Let (P_α, Q_α) be a net in $\mathrm{Prim}(A)\times\mathrm{Prim}(A)$ with limit $(P, Q) \in \mathrm{Prim}(A)\times\mathrm{Prim}(A)$. If $P \not\sim Q$ then eventually $P_\alpha \not\sim Q_\alpha$. Thus the graph of $\sim$ is always closed in $\mathrm{Prim}(A)\times\mathrm{Prim}(A)$. If $\mathrm{Orc}(A) = 1$ then $\sim$ itself is an equivalence relation with closed graph, so $\sim$ and $\approx$ coincide by part (a). Hence every Glimm ideal of A is 2-primal. □

Since the relations $\sim$ and $\approx$ can be defined on any topological space, the proof of Theorem 3.3 may be applied to any locally compact, σ-compact space X instead of $\mathrm{Prim}(A)$, with the analogous conclusions.

One application of Theorem 3.3 is to the following problem. For $N \geq 3$, let G_N be the 'threadlike' nilpotent Lie group of dimension N [8] (so that G_3 is the continuous Heisenberg group). It was shown in [8, Corollary 3.10] that $\sim$ is an equivalence relation on $\mathrm{Prim}(A)$ for the group C*-algebra $A = C^*(G_N)$ except when $N \equiv 0 \pmod 4$ with $N > 4$. If N is odd then $\sim$ and $\approx$ coincide on $\mathrm{Prim}(A)$ [8, Corollary 3.7 and remark after Corollary 3.10], but the relation $\approx$ was left undetermined in the case $N \equiv 2 \pmod 4$ (see [14, p.1426]). It follows from Theorem 3.3(c) that $\sim$ and $\approx$ must coincide in this case.

It is worth remarking that group C*-algebras with $\mathrm{Orc}(A) = 1$ are surprisingly common. For example, in addition to the 'threadlike' nilpotent Lie groups G_N just mentioned, $A = C^*(G)$ has this property if G is any of the following groups: $SL(2, \mathbb{C})$ (see [20] and [19, 18.9.13]); an amenable [SIN] group [27]; a universal simply connected, 2-step nilpotent Lie group W_n ($n \geq 2$) [8, Corollary 2.8]; any simply connected, nilpotent Lie group of dimension not exceeding 6. For this last claim, note that this class contains thirty-two indecomposable non-abelian groups, listed in [35]. All but $G_{5,2}$, $G_{5,4}$ and $G_{6,18}$ were observed to have $\mathrm{Orc}(C^*(G)) = 1$ in [5] or [8]. The orbit data in [35] is sufficient to determine the relation $\sim$ in the three remaining cases, from which it follows that $\mathrm{Orc}(C^*(G)) = 1$ for these too. Finally, if G is a Type I group and H is any other locally compact group then $C^*(G \times H)$ is isomorphic to the unique C^*-tensor product $C^*(G) \otimes C^*(H)$ and has primitive ideal space canonically homeomorphic to $\mathrm{Prim}(C^*(G)) \times \mathrm{Prim}(C^*(H))$. It follows easily from this that $\mathrm{Orc}(C^*(G \times H)) = 1$ if and only if $\mathrm{Orc}(C^*(G)) = 1 = \mathrm{Orc}(C^*(H))$ (cf. [26, Lemmas 3.1 and 3.2]).

We close this section with an example where $\mathrm{Orc}(A) = 1$ but $\sim$ and $\approx$ do not coincide.

Example 3.4. *A (non-σ-unital) C*-algebra A for which $\sim$ is an equivalence relation (i.e. for which* $\mathrm{Orc}(A) = 1$*) but for which $\sim$ is not equal to $\approx$.*

Let X be a non-normal locally compact Hausdorff space with disjoint closed subsets Y and Z which cannot be completely separated, i.e. such that there is no continuous function on X taking the value 1 on Y and 0 on Z. Set $B = C_0(X)$, and let C (respectively D) be the *-homomorphic image of B in $C^b(Y)$ (respectively $C^b(Z)$) obtained by restricting functions in B to Y (respectively Z). By [9, Theorem 3.6] there are quasi-standard C*-algebras E and F with Glimm ideals G and H respectively such that $C \cong E/G$ and $D \cong F/H$. Let A_1 be the direct sum of B, E and F, and let A be the C*-subalgebra of A_1 consisting of those elements $(b, e, f) \in A_1$ such that $b|_Y = e + G$ in C and $b|_Z = f + H$ in D.

Then $\sim$ is an equivalence relation on $\mathrm{Prim}(A)$ with the primitive ideals associated with Y forming one $\sim$-class, and the primitive ideals associated with Z forming another. But there is no continuous bounded function on $\mathrm{Prim}(A)$ separating these two $\sim$-classes, and therefore together they form a single Glimm class. Hence $\mathrm{Orc}(A) = 1$ but $\sim$ and $\approx$ do not coincide on $\mathrm{Prim}(A)$. Note that the graph of $\sim$ is automatically closed by the proof of Theorem 3.3(c), and hence this example shows that parts (a), (b), and (c) of Theorem 3.3 can all fail in the non-σ-unital case.

4. Characterizing $\mathrm{Orc}(M(A)) = 1$

In this section we prove the central result of the paper which is to give an intrinsic characterization of the separable C*-algebras A for which $\mathrm{Orc}(M(A)) = 1$, or equivalently, for which every Glimm ideal of $M(A)$ is 2-primal.

For a proper, closed, two-sided ideal J in a C*-algebra A, let $\tilde{J}$ denote the strict closure of J in $M(A)$ (see [10, Proposition 1.1]). Note that for, $P \in \mathrm{Prim}(A)$, $\tilde{P}$ is the unique primitive ideal of $M(A)$ such that $\tilde{P} \cap A = P$. We begin with a lemma relating the primality of J to that of $\tilde{J}$.

Lemma 4.1. *Let A be a C*-algebra and let J be a closed, two-sided ideal in A with strict closure $\tilde{J}$ in $M(A)$. Then for $n \geq 2$, J is n-primal (respectively primal) in A if and only if $\tilde{J}$ is n-primal (respectively primal) in $M(A)$.*

Proof. Since an ideal is primal if and only if it is n-primal for all $n \geq 2$, it is enough to prove the result for n-primality. If J is n-primal then $\tilde{J}$ is n-primal by [10, Lemma 4.5] and its proof.

Conversely, suppose that $\tilde{J}$ is n-primal. Let $P_1, \ldots, P_n \in \mathrm{Prim}(A/J)$. Then by the n-primality of $\tilde{J}$ there is a net in $\mathrm{Prim}(M(A))$ converging to $\tilde{P}_1, \ldots, \tilde{P}_n$, and by the density of $\mathrm{Prim}(A)^\sim$ in $\mathrm{Prim}(M(A))$ this net may be chosen to be of the form $(\tilde{P}_\alpha)$ where each $P_\alpha \in \mathrm{Prim}(A)$. Hence (P_α) converges to $P_1, \ldots, P_n$, so J is n-primal. □

Next we give the main technical result of this section. It is convenient to work in the context of $C_0(X)$-algebras. Recall that a C*-algebra A is a *$C_0(X)$-algebra* if there is a continuous map ϕ from $\mathrm{Prim}(A)$ to the locally compact Hausdorff space X. The map ϕ is called the *base map* and the image of ϕ is denoted X_ϕ. For $x \in X_\phi$, we define $H(x) := \phi^{-1}(x)$ and $J_x := \ker(H(x))$. The map ϕ has a unique extension to a continuous map $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$, such that $\overline{\phi}(\tilde{P}) = \phi(P)$ for all $P \in \mathrm{Prim}(A)$; and for $x \in X_\phi$ we define $H_x := \ker((\overline{\phi})^{-1}(x))$. Then $J_x \subseteq H_x \subseteq \tilde{J}_x$, so H_x is strictly closed if and only if $H_x = \tilde{J}_x$. When $H_x = \tilde{J}_x$ we say that spectral synthesis holds at x [12]. For further details, see [10] and [12].

Theorem 4.2. *Let A be a separable $C_0(X)$-algebra with base map ϕ and let $x \in X_\phi$ with H_x not strictly closed. Then H_x is not 2-primal.*

Proof. Let $b \in \tilde{J}_x$ such that $\|b + H_x\| = 1$, and let Q be a primitive ideal of $M(A)$ containing H_x such that $\|b + Q\| > 1/2$. Set $U = \{S \in \mathrm{Prim}(M(A)) : \|b + S\| > 1/2\}$, an open neighbourhood of Q in $\mathrm{Prim}(M(A))$, and set $V = \{P \in \mathrm{Prim}(A) : \tilde{P} \in U\}$. Since $\mathrm{Prim}(A)^{\sim}$ is a dense open subset of $\mathrm{Prim}(M(A))$, V is open in $\mathrm{Prim}(A)$ and $\tilde{V}$ is dense in U. Let $T \in H(x)$. If $\tilde{T} \not\sim Q$ then H_x is not 2-primal, so we may assume that $\tilde{T} \sim Q$. Then $\tilde{T}$ lies in the closure of U in $\mathrm{Prim}(M(A))$ and hence in the closure of $\tilde{V}$. Thus T lies in the closure of V in $\mathrm{Prim}(A)$. Since A is separable, T has a countable neighbourhood base in $\mathrm{Prim}(A)$ [19, 3.3.4] and so there exists a sequence (P_n) in V such that $P_n \to T$.

Let I_b be the closed ideal of A generated by b, that is, the norm-closure of AbA. Using an approximate identity for A, we obtain that $ab, ba \in I_b$ $(a \in A)$ and hence $b \in \tilde{I}_b$. Set $W = \phi(\mathrm{Prim}(I_b))$ (where $\mathrm{Prim}(I_b)$ is regarded as a subset of $\mathrm{Prim}(A)$ in the usual way). If $P \in \mathrm{Prim}(A)$ and $P \supseteq I_b$ then $b \in \tilde{I}_b \subseteq \tilde{P}$ and so $P \notin V$. It follows that $V \subseteq \mathrm{Prim}(I_b)$ and hence $\phi(V) \subseteq W$. On the other hand, if $P \in H(x)$ then $b \in \tilde{J}_x \subseteq \tilde{P}$ and so $I_b \subseteq \tilde{P} \cap A = P$. Thus $x \notin W$. Since A is separable, X_ϕ is perfectly normal [12, Lemma 3.9] and so there is a continuous function $\rho : X_\phi \to [0, \infty)$ with zero set equal to the closed set $\{x\}$. In particular, $\rho(y) > 0$ for all $y \in W$.

We shall use the following observation. Let f be any continuous bounded function on W. Then $f \circ \phi$ defines a continuous bounded function on $\mathrm{Prim}(I_b)$, and hence defines a unique central multiplier z_f of I_b such that

$$z_f c + (P \cap I_b) = f(\phi(P))(c + (P \cap I_b))$$

for all $c \in I_b$ and $P \in \mathrm{Prim}(A)$ such that $P \not\supseteq I_b$. We now define a double centralizer (R, L) on A by $R(a) = z_f(ab)$, $L(a) = z_f(ba)$ $(a \in A)$. Since $I_b = I_b^2$, $(z_f c)a = z_f(ca)$ and $a(cz_f) = (ac)z_f$ for all $a \in A$ and $c \in I_b$. Thus, for $a_1, a_2 \in A$,

$$R(a_1)a_2 = z_f(a_1 b a_2) = (a_1 b a_2)z_f = a_1 L(a_2).$$

Hence by [37, 3.12.3] there exists $b_f \in M(A)$ such that $b_f a = z_f(ba)$ and $ab_f = z_f(ab)$ for all $a \in A$. Let $y \in W$, $a \in A$ and $P \in \mathrm{Prim}(A)$ with $P \supseteq J_y$. If $I_b \not\subseteq P$ then

$$b_f a - f(y)ba = z_f(ba) - f(y)ba \in P \cap I_b \subseteq P.$$

On the other hand, if $I_b \subseteq P$ then $b_f a - f(y)ba = z_f(ba) - f(y)ba \in I_b \subseteq P$. Thus in either case, $b_f a - f(y)ba \in P$. Since this is true for all such P, $b_f a - f(y)ba \in J_y$ and similarly $ab_f - f(y)ab \in J_y$. Hence $b_f - f(y)b \in \tilde{J}_y$.

Recall that $P_n \in V$ and set $x_n = \phi(P_n)$ so that $x_n \in \phi(V) \subseteq W$ and

$$\|b + \tilde{J}_{x_n}\| \geq \|b + \tilde{P}_n\| > 1/2 \qquad (n \geq 1).$$

Since ϕ is continuous, $x_n \to x$ and hence $\rho(x_n) \to 0$. By passing to a subsequence, we may suppose that $\rho(x_n)$ is strictly decreasing to zero. Let $g : (0,\infty) \mapsto (0,\infty)$ be a continuous function such that $g(\rho(x_n)) = \frac{2}{\pi(2n-1)}$ $(n \geq 1)$. Let f be the continuous bounded function on W defined by

$$f(y) = \sin\left(\frac{1}{g(\rho(y))}\right)$$

$(y \in W)$. Then $f(x_{2m}) = -1$ and $f(x_{2m-1}) = 1$ $(m \geq 1)$. Let f^+ and f^- be the positive and negative parts of f. Then f^+ and f^- are non-zero continuous bounded functions on W and $f^+f^- = 0$. Set $b_1 = b_{f^+}$ and $b_2 = b_{f^-}$. Then $b_1Ab_2 = \{0\}$. For $c \in M(A)$ and $a \in A$, $b_1cb_2a = \lim b_1cu_\lambda b_2a = 0$ (where (u_λ) is an approximate identity for A) and similarly $ab_1cb_2 = 0$. Thus $b_1cb_2 = 0$ and so $b_1M(A)b_2 = \{0\}$. Hence $J_1J_2 = \{0\}$ where J_1 and J_2 are the smallest closed ideals in $M(A)$ containing b_1 and b_2 respectively. On the other hand, since $b_1 - f^+(x_{2n-1})b \in \tilde{J}_{2n-1}$, we have

$$\|b_1 + H_{x_{2n-1}}\| \geq \|b_1 + \tilde{J}_{x_{2n-1}}\| = |(f^+)(x_{2n-1})|\ \|b + \tilde{J}_{x_{2n-1}}\| > 1/2$$

for all n, and similarly $\|b_2 + H_{x_{2n}}\| > 1/2$ for all n. Hence neither b_1 nor b_2 belong to H_x, by the upper semi-continuity of norm functions on X. Thus H_x is not 2-primal. □

If ϕ is the complete regularization map for $\mathrm{Prim}(A)$ then H_x is a Glimm ideal of $M(A)$ (see the remarks after Proposition 4.4). Thus we see that if A is separable then necessary conditions for $\mathrm{Orc}(M(A)) = 1$ are that $\mathrm{Orc}(A) = 1$ and that every Glimm ideal of $M(A)$ which does not contain A should be strictly closed. This latter property has been called 'global spectral synthesis' in [12].

We give two examples which illustrate Theorem 4.2. The second of these shows that when A is non-separable it is possible for H_x to be 2-primal but not strictly closed.

Example 4.3. (i) Let A be the C*-algebra of sequences $x = (x_n)$ of 2×2 complex matrices such that $x_n \to \mathrm{diag}(\lambda(x), 0)$ as $n \to \infty$. Then $M(A)$ is the C*-algebra of sequences $y = (y_n)$ of matrices whose off-diagonal terms converge to zero, whose $(1,1)$-entries converge to a limit $\tilde{\lambda}(y)$, and whose $(2,2)$-entries form a bounded sequence. Set $J_\infty = \ker\lambda$, and for $n \geq 1$, set $J_n = \{x \in A : x_n = 0\}$. It is well-known that $\mathrm{Prim}(A) = \{J_n : n \geq 1\} \cup \{J_\infty\}$ and is homeomorphic to the Hausdorff space $\mathbb{N} \cup \{\infty\}$. Hence $\mathrm{Prim}(A) = \mathrm{Glimm}(A)$. It is not difficult to see that $\tilde{J}_\infty = \ker\tilde{\lambda}$ while

$$H_\infty = \{y \in M(A) : y_n \to 0\} = J_\infty.$$

Hence H_∞ is not strictly closed (see also [12, Proposition 2.6(iii)]) so it follows from Theorem 4.2 that H_∞ is not 2-primal. This can also be seen directly by considering, for instance, the ideals of elements in $M(A)$ whose even (respectively, odd) terms are zero.

(ii) Let $X = \beta\mathbb{N}$ and let $y \in \beta\mathbb{N} \setminus \mathbb{N}$. Let A be the C*-algebra of continuous functions f from X into the 2×2 complex matrices such that $f(y) = \mathrm{diag}(\lambda(f), 0)$. Routine arguments show that $M(A)$ is the algebra of all continuous functions f from X into the 2×2 complex matrices such that $f(y) = \mathrm{diag}(\lambda(f), \mu(f))$ (cf. [13, Example 12]). Then, as in example (i), $\mathrm{Prim}(A)$ is homeomorphic to the base-space $\beta\mathbb{N}$ and every primitive ideal of A is a Glimm ideal. Set $J_y = \{f \in A : \lambda(f) = 0\}$. Then $\tilde{J}_y = \{f \in M(A) : \lambda(f) = 0\}$ while $H_y = \{f \in M(A) : \lambda(f) = \mu(f) = 0\}$. Hence H_y is not strictly closed. Theorem 4.2 does not apply, since A is non-separable, and in fact H_y is primal.

We now need some further definitions [12]. Let A be a $C_0(X)$-algebra with base map $\phi : \mathrm{Prim}(A) \to X$. For $x \in X_\phi$ we say that ϕ is *locally closed at* x [29, §13.XIV] if whenever Y is a closed subset of $\mathrm{Prim}(A)$ such that x lies in the closure of $\phi(Y)$ then $x \in \phi(Y)$, that is, $Y \cap H(x)$ is non-empty.

For the next definition, it is helpful to have the following notation. For $x \in X_\phi$, let $\partial H(x)$ be the boundary and $U(x)$ the interior of $H(x)$ in $\mathrm{Prim}(A)$. We say that J_x is *locally modular* if for each $P \in \partial H(x)$ there exists a relatively open neighbourhood V of P in $\mathrm{Prim}(A) \setminus U(x)$ such that $A/\ker V$ is a unital C*-algebra. Examples of locally modular ideals are given in [12, Section 3 and Examples 6.4].

For a useful formulation, which is equivalent to local modularity when A is σ-unital, we need a slight variant of the definition of $\sim$. For $Q, R \in \mathrm{Prim}(M(A)) \setminus \tilde{U}(x)$ we say that $Q \sim_x R$ if there is a net (P_α) in $\mathrm{Prim}(A) \setminus U(x)$ such that $(\tilde{P}_\alpha)$ converges to both Q and R. If A is σ-unital then J_x is locally modular if and only if for all $P \in \partial H(x)$ and $R \in \mathrm{Prim}(M(A)/A)$, $\tilde{P} \not\sim_x R$ [12, Lemma 3.3].

Proposition 4.4. *Let A be a separable $C_0(X)$-algebra with base map ϕ. Let $x \in X_\phi$ and let $n \geq 2$. Then the following are equivalent:*
(a) *H_x is n-primal (respectively, primal);*
(b) *J_x is n-primal (respectively, primal) and H_x is strictly closed;*
(c) *J_x is n-primal (respectively, primal) and locally modular, and ϕ is locally closed at x.*

Proof. (a)⇒(b) If H_x is n-primal then so too is the containing ideal $\tilde{J}_x$, and then J_x is n-primal by Lemma 4.1. Furthermore H_x must be strictly closed by Theorem 4.2.
(b)⇒(a) This follows from Lemma 4.1 since $\tilde{J}_x$ is the strict closure of H_x.
(b)⇔(c) This follows from [12, Corollary 3.10]. □

Any C^*-algebra A may be viewed as a $C_0(X)$-algebra where the base map ϕ is the complete regularization map and where X is either $\mathrm{Glimm}(A)$ (if this is locally compact) or the Stone-Čech compactification of $\mathrm{Glimm}(A)$. In either case $X_\phi = \mathrm{Glimm}(A)$. From now on, for the sake of generality and definiteness, we will take $X = \beta\,\mathrm{Glimm}(A)$ as the target of the complete regularization map for A. As noted in [6, p. 2054] (with slightly different notation), if $G \in X_\phi = \mathrm{Glimm}(A)$ then $J_G = G$ and $H_G = \iota(G)$, where $\iota : \beta\,\mathrm{Glimm}(A) \to \mathrm{Glimm}(M(A))$ is the canonical homeomorphism described in [10, Proposition 4.7]. There is a continuous map $\overline{\phi} : \mathrm{Prim}(M(A)) \to X = \beta\,\mathrm{Glimm}(A)$ such that $\overline{\phi}(\tilde{P}) = \phi(P)$ for all $P \in \mathrm{Prim}(A)$ and $\iota \circ \overline{\phi} = \phi_{M(A)}$, where $\phi_{M(A)} : \mathrm{Prim}(M(A)) \to \mathrm{Glimm}(M(A))$ is the complete regularization map for $M(A)$ [6, p. 2054]. For $x \in X = \beta\,\mathrm{Glimm}(A)$, $H_x = \bigcap\{Q \in \mathrm{Prim}(A) : \overline{\phi}(Q) = x\}$ by definition. But $\overline{\phi}(Q) = x$ if and only if $\phi_{M(A)}(Q) = \iota(x)$ and so $H_x = \iota(x) \in \mathrm{Glimm}(M(A))$. Since ι is surjective,

$$\mathrm{Glimm}(M(A)) = \{H_x : x \in \beta\,\mathrm{Glimm}(A)\}.$$

The topology on $X_\phi = \mathrm{Glimm}(A)$ is the complete regularization topology, and this coincides with the quotient topology induced from $\mathrm{Prim}(A)$ by ϕ if A is σ-unital [30, Theorem 2.6].

Proposition 4.5. *Let A be a σ-unital $C_0(X)$-algebra where ϕ is the complete regularization map for* $\mathrm{Prim}(A)$ *and suppose that* $\mathrm{Orc}(A) = 1$. *Suppose that the Glimm ideal J_x is locally modular for all $x \in X_\phi$. Then, for all $x \in X_\phi$, ϕ is locally closed at x and H_x is strictly closed.*

Proof. The main thing is to show that ϕ is locally closed at x for all $x \in X_\phi$. As a matter of fact, this follows from [12, Theorem 5.3], but the proof of that theorem is somewhat involved and we can give a much simpler proof in the present case where $\mathrm{Orc}(A) = 1$ by exploiting the new result in Theorem 3.3(c).

Let Y be a closed subset of $\mathrm{Prim}(A)$. We are required to show that $\phi(Y)$ is closed in X_ϕ. But A is σ-unital and so it is enough to show that $\phi^{-1}(\phi(Y))$ is closed [30, Theorem 2.6]. Since $\sim$ and $\approx$ coincide on $\mathrm{Prim}(A)$ (Theorem 3.3(c)), this amounts to showing that $Y^1 := \{Q \in \mathrm{Prim}(A) : \exists P \in Y \text{ with } P \sim Q\}$ is closed.

Let (Q_α) be a net in Y^1 with $Q_\alpha \to Q \in \mathrm{Prim}(A)$. Set $x = \phi(Q)$. Suppose, for a contradiction, that $Q \notin Y^1$. For each α there exists $P_\alpha \in Y$ such that $P_\alpha \sim Q_\alpha$. By the compactness of $\mathrm{Prim}(M(A))$, and by passing to a subnet if necessary, we may assume that $\tilde{P}_\alpha \to R$ for some $R \in \mathrm{Prim}(M(A))$. Then $R \sim \tilde{Q}$. If $R \in \tilde{Y}$ then $Q \in Y^1$, contrary to our assumption. Hence $R \supseteq A$, since Y is closed. If $Q_\alpha \in H(x)$ for any α then $P_\alpha \in H(x)$ since $P_\alpha \sim Q_\alpha$. This implies that $H(x) \subseteq Y^1$, since $\sim$ and $\approx$ coincide, and hence that $Q \in H(x) \subseteq Y^1$, contrary to our assumption. Thus we see that no Q_α belongs to $H(x)$. It follows that Q lies in the boundary of $H(x)$.

Let U and V be open neighbourhoods of $\tilde{Q}$ and of R respectively in $\mathrm{Prim}(M(A))$. Set $U' = \{P \in \mathrm{Prim}(A) : \tilde{P} \in U\}$ and $V' = \{P \in \mathrm{Prim}(A) : \tilde{P} \in V\}$. Then there exists α such that $Q_\alpha \in U'$ and $P_\alpha \in V'$. Since $Q_\alpha, P_\alpha \notin H(x)$, $U'' = U' \setminus H(x)$ and $V'' = V' \setminus H(x)$ are open neighbourhoods of Q_α and P_α respectively. Since $Q_\alpha \sim P_\alpha$, there exists $P \in \mathrm{Prim}(A)$ such $P \in U'' \cap V''$. Hence $P \notin H(x)$ and $\tilde{P} \in U \cap V$. It follows that $\tilde{Q} \sim_x R$. By [12, Lemma 3.3], this contradicts the local modularity hypothesis on J_x. Thus $Q \in Y^1$ and Y^1 is closed. It follows, as above, that ϕ is locally closed at each $x \in X_\phi$. Hence, using the local modularity of J_x again, H_x is strictly closed for all $x \in X_\phi$ [12, Proposition 3.4]. □

Theorem 4.6. *Let A be a separable C^*-algebra and let ϕ be the complete regularization map for* $\mathrm{Prim}(A)$. *Then for $n \geq 2$, H_x is n-primal (respectively, primal) for all $x \in \beta X_\phi$ if and only if J_x is locally modular and n-primal (respectively, primal) for all $x \in X_\phi$.*

Proof. Suppose first that H_x is n-primal (respectively primal) for all $x \in \beta X_\phi$. Then it follows from Proposition 4.4 that J_x is locally modular and n-primal (respectively primal) for all $x \in X_\phi$.

For the converse, it suffices to assume that A is σ-unital rather than separable. Suppose that J_x is locally modular and n-primal (respectively primal) for all $x \in X_\phi$. In particular, this implies that $\mathrm{Orc}(A) = 1$. By Proposition 4.5, H_x is strictly closed for all $x \in X_\phi$. Hence $H_x = \tilde{J}_x$ is n-primal (respectively primal) for all $x \in X_\phi = \mathrm{Glimm}(A)$ by Lemma 4.1. Since A is σ-unital, X_ϕ is normal [11, p. 366]. Recalling that $\mathrm{Glimm}(M(A)) = \{H_x : x \in \beta\,\mathrm{Glimm}(A)\}$, we obtain from [6, Corollary 3.3] that H_x is n-primal (respectively primal) for all $x \in \beta X_\phi$. □

Translating Theorem 4.6 back into the language of Glimm ideals, we obtain our main result. We observe that condition (b) below is intrinsic to A.

Corollary 4.7. *Let A be a separable C^*-algebra. Then the following are equivalent:*

(a) $\mathrm{Orc}(M(A)) = 1$*;*
(b) $\mathrm{Orc}(A) = 1$ *and every Glimm ideal of A is locally modular;*
(c) $\mathrm{Orc}(A) = 1$ *and H_G is strictly closed in $M(A)$ for all $G \in \mathrm{Glimm}(A)$.*

Proof. Let ϕ be the complete regularization map for $\mathrm{Prim}(A)$ so that $X_\phi = \mathrm{Glimm}(A)$ and the ideals J_x $(x \in X_\phi)$ are the Glimm ideals of A.
(a) $\Rightarrow$ (b). Suppose that $\mathrm{Orc}(M(A)) = 1$. Then $\mathrm{Orc}(A) = 1$ since $\mathrm{Prim}(A)$ is homeomorphic to an open subset of $\mathrm{Prim}(M(A))$. Furthermore, since $M(A)$ is unital, H_x is 2-primal for all $x \in \beta X_\phi$ [40, Corollary 2.7] and so J_x is locally modular for all $x \in X_\phi$ by Theorem 4.6.
(b) $\Rightarrow$ (c). This follows from Proposition 4.5.
(c) $\Rightarrow$ (a). Suppose that (c) holds. Since $\mathrm{Orc}(A) = 1$, J_x is 2-primal for all $x \in X_\phi$ by Theorem 3.3(c). Hence, as in the proof of Theorem 4.6, $H_x = \tilde{J}_x$ is 2-primal for all $x \in X_\phi = \mathrm{Glimm}(A)$ by Lemma 4.1 and so every Glimm ideal of $M(A))$ is 2-primal by [6, Corollary 3.3]. □

As noted before Proposition 4.5, any C^*-algebra A may be regarded as a C(X)-algebra where X is the Stone-Čech compactification of $\mathrm{Glimm}(A)$ and the base map ϕ is the complete regularization map. The corresponding structure map $\mu : C(X) \to Z(M(A))$ satisfies $\mu(h) = \theta_A(h \circ \phi)$ where $\theta_A : C^b(\mathrm{Prim}(A)) \to Z(M(A))$ is the Dauns-Hofmann isomorphism (see [10, p. 74]). In this case, the algebra $Z'(A) := \mu(C(X)) \cap A$ (see [10, Section 2] and [12, Section 2]) coincides with $Z(A)$. To see this, let $z \in Z(A)$ and let $f = \theta_A^{-1}(z) \in C^b(\mathrm{Prim}(A))$. By the definition of the complete regularization topology on $\mathrm{Glimm}(A)$, there exists $g \in C^b(\mathrm{Glimm}(A))$ such that $f = g \circ \phi$. Let $h \in C(X)$ be the unique extension of g. Then

$$\mu(h) = \theta_A(h \circ \phi) = \theta_A(g \circ \phi) = \theta_A(f) = z.$$

We now introduce the following notation. Let A be a C*-algebra with centre $Z(A)$ and let ϕ be the complete regularization map for $\mathrm{Prim}(A)$. Set $U_\phi = \{x \in X_\phi : J_x \not\supseteq Z(A)\}$ and let $W_\phi = X_\phi \setminus U_\phi$. Then U_ϕ is an open subset of X_ϕ (see [12, Section 2] and [10, Section 2]). Recall that a C*-algebra A is said to be *quasi-central* if no primitive ideal of A contains $Z(A)$. If A is quasi-central then $U_\phi = X_\phi (= \mathrm{Glimm}(A))$ and so every Glimm ideal of A is locally modular (see [12, Section 3]).

Corollary 4.8. *Let A be a separable C^*-algebra with* $\mathrm{Orc}(A) = 1$ *and suppose that every Glimm ideal of A is locally modular.*

(i) *Suppose that $Z(A) = \{0\}$. Then A is a direct sum of primitive C^*-algebras.*

(ii) *Suppose that* $\mathrm{Prim}(A)$ *is a T_1-space. Then $A = B \oplus C$ where B is quasi-central and C is a direct sum of simple C^*-algebras.*

Proof. (i) Let ϕ denote the complete regularization map for $\mathrm{Prim}(A)$. By Proposition 4.5, H_x is strictly closed for all $x \in X_\phi$. Since $Z(A) = \{0\}$, X_ϕ is discrete by [12, Corollary 4.4(ii)]. Let $x \in X_\phi$. By Theorem 3.3(c), J_x is a 2-primal ideal of A. Suppose that K and L are (closed two-sided) ideals of A such that $KL \subseteq J_x$. Using the canonical correspondence between open subsets of $\mathrm{Prim}(A)$ and ideals of A, we have

$$\mathrm{Prim}(K) \cap \mathrm{Prim}(L) \subseteq \mathrm{Prim}(J_x) = \mathrm{Prim}(A) \setminus H(x).$$

Since $H(x)$ is a clopen subset of $\mathrm{Prim}(A)$, $\mathrm{Prim}(K) \cap H(x)$ and $\mathrm{Prim}(L) \cap H(x)$ are disjoint open subsets of $\mathrm{Prim}(A)$ and so one or other is contained in $\mathrm{Prim}(J_x)$ by the 2-primality of J_x. Hence either $\mathrm{Prim}(K)$ or $\mathrm{Prim}(L)$ is contained in $\mathrm{Prim}(J_x)$ and so J_x is a prime ideal of A. Since A is separable, J_x is a primitive ideal of A [37, 4.3.6]. Hence A is a direct sum of primitive C^*-algebras.

(ii) Let $x \in X_\phi$. Since $\mathrm{Orc}(A) = 1$ and J_x is locally modular, it follows from [12, Corollary 6.3] that either $x \in U_\phi$ or $H(x)$ has non-empty interior. In the latter case, there exists a separated point M of $\mathrm{Prim}(A)$ in the interior of $H(x)$ [19, 3.9.4]. But M is a maximal ideal since $\mathrm{Prim}(A)$ is a T_1-space, and thus the singleton $\{M\}$ is a $\sim$-class in $\mathrm{Prim}(A)$ (by definition of a separated point). But $\mathrm{Orc}(A) = 1$ and hence $H(x) = \{M\}$ by Theorem 3.3(c) and so $M = J_x$. Since $\{M\}$ is a clopen subset of $\mathrm{Prim}(A)$, its characteristic function is continuous and so x is an isolated point in X_ϕ. It follows that W_ϕ is a discrete open subset of X_ϕ. Set $B = \ker W_\phi$ and $C = \ker U_\phi$. Then B is quasi-central and C is a direct sum of simple C*-algebras and $A = B \oplus C$. □

If A is as in Corollary 4.8 but $\mathrm{Prim}(A)$ is not a T_1-space then A can have more complicated structure, see [12, Example 6.4(i)].

As Corollary 4.8 shows, the local modularity condition in Corollary 4.7(b) is a restrictive one. However, if $A = C^*(G)$ where G is an [SIN] group then A has a central approximate identity (see [36] and [33, Section 1]). Hence A is quasi-central and so every Glimm ideal of A is locally modular.

5. Applications to norms of elementary operators

In this section we give some applications of the results of the previous section to norms of elementary operators. We begin with the definitions of the constants K and K_s associated with inner derivations, see [25], [3]. Let A be a C*-algebra and let $a \in A$. Then a simple application of the triangle inequality shows that

$$\|D(a, A)\| \leq 2d(a, Z(A)) \qquad (1)$$

where $D(a, A)$ is the inner derivation generated by a and $d(a, Z(A))$ is the distance from a to $Z(A)$, the centre of A. Define $K(A)$ to be the smallest number in $[0, \infty]$ such that

$$K(A)\|D(a, A)\| \geq d(a, Z(A))$$

for all $a \in A$. If the elements a are restricted to being self-adjoint then the corresponding constant is denoted $K_s(A)$. Clearly $K_s(A) \leq K(A)$, and it follows from (1) that $1/2 \leq K_s(A)$ unless A is commutative.

It was shown in [40, Theorem 4.4] that if A is a unital non-commutative C*-algebra then $K_s(A) = 1/2$ if and only if $\mathrm{Orc}(A) = 1$. If A is a non-unital C^*-algebra then, as discussed in Section 1, the multiplier algebra $M(A)$ provides the natural setting in which to study inner derivations and their norms. By applying the unital result above to the algebra $M(A)$ and combining with Corollary 4.7, we obtain the following.

Theorem 5.1. *Let A be a separable, non-commutative C*-algebra. Then $K_s(M(A)) = 1/2$ if and only if* $\mathrm{Orc}(A) = 1$ *and every Glimm ideal of A is locally modular.*

In a similar direction, it was shown in [41, Theorems 3.2 and 3.3] that for a unital non-commutative C*-algebra A, $K(A) = 1/2$ if and only if every Glimm ideal of A is 3-primal. Thus from Theorem 4.6 we obtain the following.

Theorem 5.2. *Let A be a separable, non-commutative C*-algebra. Then $K(M(A)) = 1/2$ if and only if every Glimm ideal of A is locally modular and* 3*-primal.*

In [6, Section 4] there are several examples of second countable, locally compact groups G such that $K(M(C^*(G))) = 1/2$.

It was also shown in [41, Theorems 3.3 and 3.4] that if A is a unital C*-algebra then $K(A) = 1/\sqrt{3}$ if and only if every Glimm ideal of A is 2-primal (i.e. $\mathrm{Orc}(A) = 1$) but A has a Glimm ideal which is not 3-primal. Applying this to $M(A)$, we obtain the following from Theorem 4.6.

Theorem 5.3. *Let A be a separable C^*-algebra. Then $K(M(A)) = 1/\sqrt{3}$ if and only if every Glimm ideal of A is locally modular and 2-primal but at least one Glimm ideal of A is not 3-primal.*

Proof. Suppose first that $K(M(A)) = 1/\sqrt{3}$. Then every Glimm ideal of $M(A)$ is locally modular and 2-primal but at least one is not 3-primal. By Theorem 4.6 (first with $n = 2$ and then with $n = 3$), every Glimm ideal of A is locally modular and 2-primal but at least one is not 3-primal.

Conversely, suppose that every Glimm ideal of A is locally modular and 2-primal but at least one Glimm ideal of A is not 3-primal. Then by Theorem 4.6 (with $n = 2$ and $n = 3$), every Glimm ideal of $M(A)$ is 2-primal but $M(A)$ has at least one Glimm ideal which is not 3-primal. Hence $K(M(A)) = 1/\sqrt{3}$. □

On the other hand, it was shown in [40, Theorem 4.4] that if A is a unital non-commutative C^*-algebra then $K_s(A) = \mathrm{Orc}(A)/2$. Thus Corollary 4.7 implies the following.

Theorem 5.4. *Let A be a separable C^*-algebra. Then the following are equivalent:*
(i) *either* $\mathrm{Orc}(A) > 1$ *or A has a Glimm ideal which is not locally modular;*
(ii) $K(M(A)) \geq K_s(M(A)) \geq 1$.

For the computation of $K(M(C^*(G)))$ for a number of group C*-algebras, see [6] and [7]. It was shown in [6, Theorem 4.8] that if G is a second countable $[\mathrm{FD}]^-$-group of type I then either $K(M(C^*(G))) \leq 1/2$ or $K(M(C^*(G))) = 1$ and hence, in particular, $K(M(C^*(G))) \neq 1/\sqrt{3}$. This raised the question of finding a locally compact group G such that $K(M(C^*(G))) = 1/\sqrt{3}$ [6, p. 2052]. By using results of Losert [33], we show below that this is not possible for second countable groups which are amenable or CCR or almost connected. We begin with a lemma.

Lemma 5.5. *Let A be a C^*-algebra of the form $A = B \oplus C$ where B is a separable C^*-algebra with centre $Z(B) = \{0\}$ and C is a quasi-central, quasi-standard C^*-algebra. Then either $K(M(A)) \leq 1/2$ or $K(M(A)) \geq 1$. Equivalently, $K(M(A)) \neq 1/\sqrt{3}$.*

Proof. If $\mathrm{Orc}(M(A)) \geq 2$ then $K(M(A)) \geq 1$ [40, Theorem 4.4], so we suppose from now on that $\mathrm{Orc}(M(A)) = 1$. Since $M(A) = M(B) \oplus M(C)$, $\mathrm{Orc}(M(B)) = 1$ and hence B is a direct sum of primitive C^*-algebras by Corollary 4.8. Then $\mathrm{Glimm}(B)$ is discrete and B is quasi-standard by [9, Theorem 3.4]. We now have that $M(B)$ is quasi-standard [10, Corollary 4.9] and so also is $M(C)$ [10, Corollary 4.10]. It follows that $M(A)$ is quasi-standard and hence $K(M(A)) \leq 1/2$ [41, Theorem 3.2]. The final part of the statement follows from the trichotomy in [41, p. 569]. □

For the next result, we recall that if a locally compact group G has a compact normal subgroup N then there is a canonical *-isomorphism ϵ_N from $C^*(G/N)$ onto a direct summand of $C^*(G)$ such that ϵ_N is a right inverse for the canonical *-homomorphism $\tau_N : C^*(G) \to C^*(G/N)$ and hence $C^*(G) = \ker \tau_N \oplus \epsilon_N(C^*(G/N))$. More generally, the same holds true if N is a closed normal subgroup with property T (see [33, Section 2] and [42, Section 3]).

Theorem 5.6. *Let $A = C^*(G)$ where G is a second countable locally compact group which is either (i) amenable or (ii) CCR or (iii) almost connected. Then either $K(M(A)) \leq 1/2$ or $K(M(A)) \geq 1$. Equivalently, $K(M(A)) \neq 1/\sqrt{3}$.*

Proof. Since G is second countable, A is separable and hence so is any C^*-subalgebra. If $Z(A) = \{0\}$ then the result follows from Lemma 5.5 with $C = \{0\}$. So we assume from now on that $Z(A) \neq \{0\}$.
(i) Suppose that G is amenable. By [33, Corollary 1.2], G is an IN-group. As discussed in [33, proof of Corollary 1.3] (see also [32, proof of Proposition 1.2]), G has a compact normal subgroup K such that G/K is a SIN-group and

$$Z(L^1(G)) = \epsilon_K(Z(L^1(G/K))) \subseteq \epsilon_K(C^*(G/K)).$$

Since G is amenable, $Z(A)$ is the norm-closure of $Z(L^1(G))$ [33, Theorem 1.1] and hence is contained in $\epsilon_K(C^*(G/K))$. Writing $B = \ker \tau_K$, we have $Z(B) = \{0\}$. Since G/K is an amenable SIN-group, $C^*(G/K)$ is quasi-central [33, Corollary 1.3] and quasi-standard [27] and hence the same holds for $C := \epsilon_K(C^*(G/K))$. The result follows from Lemma 5.5.
(ii) Suppose instead that G is a CCR group. By [33, Theorem 3.1], G has a closed normal subgroup N with property T such that G/N is an amenable SIN-group and $\epsilon_N(Z(C^*(G/N))) = Z(C^*(G))$. With $B := \ker \tau_N$ and $C := \epsilon_K(C^*(G/N))$ we have $Z(B) = \{0\}$ and, as before, C is quasi-central and quasi-standard. The result follows from Lemma 5.5.
(iii) Finally, suppose instead that G is almost connected. By [33, Theorem 5.1 and its proof], G has a closed normal subgroup N with property T such that G/N is an amenable IN-group and $\epsilon_N(Z(C^*(G/N))) = Z(C^*(G)) \neq \{0\}$. Arguing as in (i) (with G replaced by G/N), we obtain $C^*(G/N) = B \oplus C$ where $Z(B) = \{0\}$ and C is quasi-central and quasi-standard. The result follows by applying Lemma 5.5 to the decomposition $C^*(G) = (\ker \tau_N \oplus \epsilon_N(B)) \oplus \epsilon_N(C)$. □

Now let A be a C*-algebra and let Z be the centre of $M(A)$. Let $A \otimes_h A$ denote the Haagerup tensor product. Then the central Haagerup tensor product $A \otimes_{Z,h} A$ is the quotient of $A \otimes_h A$ by the subspace J_A defined to be the closure of the span of elements of the form $az \otimes b - a \otimes zb$ $(a, b \in A,\ z \in Z)$. Let CB(A) be the space of completely bounded maps on A and let θ be the contractive linear map from $A \otimes_h A \to \mathrm{CB}(A)$ defined on elementary tensors by

$$\theta\left(\sum_{i=1}^{n} a_i \otimes b_i\right)(x) = \sum_{i=1}^{n} a_i x b_i \qquad (x \in A).$$

Thus $\theta(u)$ is an elementary operator whenever u is an elementary tensor. Furthermore $J_A \subseteq \ker \theta$, so θ induces a contractive map $\theta_Z : A \otimes_{Z,h} A \to \mathrm{CB}(A)$ where $\theta_Z(u + J_A) = \theta(u)$ $(u \in A \otimes_h A)$. It was shown in [14, Theorem 3.8] that θ_Z is injective if and only if every Glimm ideal of A is 2-primal and that θ_Z is isometric if and only if every Glimm ideal of A is primal.

If A is non-unital then we can also consider the extended map $\overline{\theta}_Z : M(A) \otimes_{Z,h} M(A) \to \mathrm{CB}(M(A))$ and its restriction $\Theta_Z : M(A) \otimes_{Z,h} M(A) \to \mathrm{CB}(A)$ defined by

$$\Theta_Z(u + J_{M(A)}) = \overline{\theta}_Z(u + J_{M(A)})|_A \qquad (u \in M(A) \otimes_h M(A)).$$

It was shown in [13, Corollary 9] that Θ_Z is isometric if and only if every Glimm ideal of $M(A)$ is primal. Thus Theorem 4.6 implies the following.

Theorem 5.7. *Let A be a separable C*-algebra. The canonical contraction*

$$\Theta_Z : M(A) \otimes_{Z,h} M(A) \to \mathrm{CB}(A)$$

is isometric if and only if every Glimm ideal of A is locally modular and primal.

Theorem 4.2 requires a separability hypothesis, or something similar, and so the theorems thus far in this section are valid only for separable C*-algebras. The problem of determining the σ-unital C*-algebras A for which $\mathrm{Orc}(M(A)) = 1$ seems to be formidable, even if one makes the further assumption that A is quasi-standard. If A has enough non-unital quotients, however, then this problem can be solved [6, Theorem 3.6]. Theorem 5.9 will supplement that result. We begin with a proposition which augments the results in [31] and [18] concerning the relationship between modular ideals and non-trivial centre. It shows that the density hypothesis in [6, Theorem 3.6] can be replaced by the assumption of trivial centre.

Proposition 5.8. *Let A be a σ-unital quasi-standard C*-algebra. The following conditions are equivalent:*

(i) *the centre $Z(A) = \{0\}$;*
(ii) *the set $S = \{P \in \mathrm{Prim}(A) : A/P$ is non-unital$\}$ is dense in* $\mathrm{Prim}(A)$.

Proof. (ii) $\Rightarrow$ (i) This is an elementary consequence of the lower semi-continuity of norm functions (see, for example, [31, Section 1] and [6, Proposition 2.1]).
(i) $\Rightarrow$ (ii). Note that if $P \in \mathrm{Prim}(A)$ and A/P is unital then the assignment $a + P \to a + \tilde{P}$ $(a \in A)$ defines a *-isomorphism of A/P onto $M(A)/\tilde{P}$ because the image is a unital essential ideal of $M(A)/\tilde{P}$. Let u be a strictly positive element in A with $\|u\| = 1$. Suppose that the set S is not dense in $\mathrm{Prim}(A)$ and that W is a non-empty open subset of $\mathrm{Prim}(A)$ with A/P unital for all $P \in W$. The norm function $P \mapsto \|(1-u) + \tilde{P}\|$ $(P \in \mathrm{Prim}(A))$ is lower semi-continuous on the Baire space $\mathrm{Prim}(A)$ and thus there is a dense subset of $\mathrm{Prim}(A)$ consisting of points of continuity of this norm function (see the proof of [19, Appendice B18]). Let $Q \in W$ be such a point of continuity. Since A/Q is unital, $\|(1-u) + \tilde{Q}\| < 1$ and hence there is an open neighbourhood W' of Q in $\mathrm{Prim}(A)$ such that $\|(1-u) + \tilde{P}\| < (1+\alpha)/2$ for all $P \in W'$, where $\alpha = \|(1-u) + \tilde{Q}\|$.

Now let $R \in W'$ and let $S \in \mathrm{Prim}(A)$ with $R \sim S$. Then $\|(1-u) + \tilde{S}\| \leq (1+\alpha)/2$ for otherwise the set $\{P \in \mathrm{Prim}(A) : \|(1-u) + \tilde{P}\| > (1+\alpha)/2\}$ is an open neighbourhood of S disjoint from W', contradicting the fact that $S \sim R$. Let W'' be the $\sim$-saturation of W' and let $Y = \phi(W'')$, where ϕ is the complete regularization map for $\mathrm{Prim}(A)$. Then Y is an open subset of $X_\phi = \mathrm{Glimm}(A)$ since A is quasi-standard (see [9, Proposition 3.2 and Theorem 3.3]).

Let $f : [0,1] \to [0,1]$ be a continuous function such that $f(0) = 0$ and $f([(1-\alpha)/2, 1]) = \{1\}$. Let $v = f(u) \in A$. For all $P \in W''$, the spectrum of $u + \tilde{P}$ is contained in $[(1-\alpha)/2, 1]$ and so $\|(1-v) + \tilde{P}\| = 0$ which implies that $v + P$ is the identity of A/P. Let $h : X_\phi \to [0,1]$ be a non-zero continuous function vanishing off Y and let $z = \theta_A(h \circ \phi) \in Z(M(A))$, where $\theta_A : C^b(\mathrm{Prim}(A)) \to Z(M(A))$ is the Dauns-Hofmann isomorphism. If $P \in W''$ then $zv + P = h(\phi(P))(v + P)$ and if $P \in \mathrm{Prim}(A) \setminus W''$ then $zv + P = h(\phi(P))(v + P) = 0$ because $\phi(P) \notin Y$. Hence zv is a non-zero central element in A. ☐

Recall that a completely regular Hausdorff space X is an F-space if disjoint cozero sets in X can be separated by disjoint zero sets.

Theorem 5.9. *Let A be a σ-unital quasi-standard C^*-algebra and suppose that the centre $Z(A) = \{0\}$. Let $\Theta_Z : M(A) \otimes_{Z,h} M(A) \to \mathrm{CB}(A)$ be the canonical contraction. Then the following are equivalent:*
(i) $\mathrm{Glimm}(A)$ *is an F-space;*
(ii) $\mathrm{Orc}(M(A)) = 1$*;*
(iii) $K(M(A)) = K_s(M(A)) = 1/2$*;*
(iv) *every Glimm ideal of $M(A)$ is primal;*
(v) *the map Θ_Z is isometric;*
(vi) *the map Θ_Z is injective.*

Proof. The equivalence of (i)– (iv) follows from [6, Theorem 3.6] and Proposition 5.8. The implication (iv)⇒(v) follows from [13, Corollary 9] while (v)⇒(vi) is trivial. Now suppose that (vi) holds. Let $u \in M(A) \otimes_h M(A)$ and suppose that $\overline{\theta}_Z(u + J_{M(A)}) = 0$. Then $\Theta_Z(u + J_{M(A)}) = \overline{\theta}_Z(u + J_{M(A)})|_A = 0$, so $u \in J_{M(A)}$ by assumption. Hence $\overline{\theta}_Z$ is injective, so every Glimm ideal of $M(A)$ is 2-primal by [14, Theorem 3.8]. Thus (ii) holds. □

Let A be a σ-unital quasi-standard C*-algebra with $Z(A) = \{0\}$. Then Theorem 5.9 shows that every Glimm ideal of $M(A)$ is primal if and only if $\mathrm{Glimm}(A)$ is an F-space. On the other hand, $M(A)$ satisfies the stronger condition of being quasi-standard if and only if $\mathrm{Glimm}(A)$ is a basically disconnected space [10, Corollary 4.9].

Notice, in conclusion, that the density of the subset S in Lemma 5.8 implies that if A is a σ-unital quasi-standard C^*-algebra with $Z(A) = \{0\}$ then any locally modular Glimm ideal of A must be an isolated point in $\mathrm{Glimm}(A)$. On the other hand, there exist F-spaces such as $\beta\mathbb{N} \setminus \mathbb{N}$ which have no isolated points. Thus the local modularity condition of Corollary 4.7 loses its significance for the problem of characterizing $\mathrm{Orc}(M(A)) = 1$ when the separability assumption on A is dropped.

6. Appendix: Proof of lemma 3.1

Lemma 3.1 [15, Vol.2: Exercise 15(b) on p.244] *Let X be a locally compact, Hausdorff, σ-compact space and let R be an equivalence relation on X whose graph is closed in $X \times X$. Then X/R is Hausdorff and normal.*

Proof. Let $x \in X$. Since the graph of R is closed, the equivalence class $[x] = \{y \in X : yRx\}$ is closed in X. It follows that singleton subsets of X/R are closed, i.e. X/R is T_1. So it suffices to prove that X/R is normal. Let $q : X \to X/R$ be the quotient map.

First note that if Y is a compact subset of X then $q(Y)$ is closed. To see this, let $Z = q^{-1}(q(Y))$ and let (x_α) be a net in Z converging to a limit $x \in X$. Then for each α there exists $y_\alpha \in Y$ such that $x_\alpha R y_\alpha$. Using the compactness of Y, we may obtain a subnet (y_β) of (y_α) such that (y_β) converges to a limit $y \in Y$. But then xRy since the graph of R is closed, so $x \in Z$. Hence $q(Y)$ is closed by definition of the quotient topology.

Since X is σ-compact and locally compact, standard arguments lead to the existence of a sequence (U_n) of open subsets of X such that $\overline{U}_n$ is a compact subset of U_{n+1} for all $n \geq 1$ and $X = \bigcup_{n=1}^{\infty} U_n$. For $n \geq 1$, let $X_n = \overline{U}_n$ and let R_n be the restriction of R to X_n, that is $R_n = R \cap (X_n \times X_n)$. Since the graph of R is closed, the graph of R_n is closed in $X_n \times X_n$.

By [15, Vol.1, Prop.8 on p.105], X_n/R_n is compact and Hausdorff and hence normal. Let $q_n : X_n \to X_n/R_n$ be the quotient map and let $i_n : X_n/R_n \to X/R$ be given by $i_n([x]) = q(x)$ $(x \in X_n)$. Then i_n is a bijection between X_n/R_n and $q(X_n)$.

We now show that i_n is a homeomorphism between X_n/R_n and $q(X_n)$. Note that $q(X_n)$ is closed by the work of the second paragraph above. Let W be a subset of $q(X_n)$. We claim that the following is true: W is closed if and only if $Z := q^{-1}(W)$ is closed if and only if $Z \cap X_n$ is closed if and only if $q_n(Z \cap X_n)$ is closed. The first and third equivalences follow from the definition of the quotient topology together with the fact that $Z \cap X_n$ is an R_n-saturated subset of X_n. For the second equivalence, one direction is trivial. For the other direction, note that if $Z \cap X_n$ is closed then it is compact and hence $Z = q^{-1}(q(Z \cap X_n))$ is closed by the second paragraph above. Since $i_n \circ q_n(Z \cap X_n) = W$ it follows that i_n is a homeomorphism. Hence $q(X_n)$ is normal for all $n \geq 1$.

Next note that a subset W of X/R is closed if (and only if) $W \cap q(X_n)$ is a closed subset of $q(X_n)$ for all $n \geq 1$. Indeed, suppose that $W \cap q(X_n)$ is closed for all $n \geq 1$ and let (x_α) be a net in $Z := q^{-1}(W)$ converging to a limit $x \in X$. Then $x \in U_m$ for some m and hence eventually $x_\alpha \in U_m \subseteq X_m$. Thus eventually $q(x_\alpha) \in W \cap q(X_m)$ and $q(x_\alpha) \to q(x)$. Since $W \cap q(X_m)$ is a closed subset of $q(X_m)$, it follows that $q(x) \in W \cap q(X_m)$ and hence that $x \in Z$. Thus Z is closed, so W is closed. By taking complements, it follows that a subset W of X/R is open if and only if $W \cap q(X_n)$ is an open subset of $q(X_n)$ for all $n \geq 1$.

Now let A and B be disjoint closed subsets of X/R. For each $n \geq 1$, set $A_n = A \cap q(X_n)$ and $B_n = B \cap q(X_n)$. Then A_1 and B_1 are closed and disjoint so, working in $q(X_1)$, by two applications of normality there exist disjoint closed neighbourhoods C_1 and D_1 of A_1 and B_1 respectively. Then $A_2 \cup C_1$ and $B_2 \cup D_1$ are disjoint closed subsets of $q(X_2)$ so, working in $q(X_2)$, there exist disjoint closed neighbourhoods C_2 and D_2 of $A_2 \cup C_1$ and $B_2 \cup D_1$ respectively. Proceeding inductively, for each $n \geq 2$ we may find in $q(X_n)$ disjoint closed neighbourhoods C_n and D_n of $A_n \cup C_{n-1}$ and $B_n \cup D_{n-1}$ respectively. For each $n \geq 1$, let V_n be the interior of C_n in $q(X_n)$ and W_n the interior of D_n in $q(X_n)$. Set $V = \bigcup_{n=1}^{\infty} V_n$ and $W = \bigcup_{n=1}^{\infty} W_n$. Since C_n and D_n both increase with n, so too do V_n and W_n. For $N \geq 1$, $V \cap q(X_N) = \bigcup_{n \geq N}(V_n \cap q(X_N))$ which is open in $q(X_N)$. Hence V, and similarly W, is open in X by the previous paragraph. Furthermore for $m, n \geq 1$, $V_m \cap W_n \subseteq C_m \cap D_n = \emptyset$, and hence V and W are disjoint. Since $A_n \subseteq V_n$ and $B_n \subseteq W_n$ for each $n \geq 1$, we see that $A \subseteq V$ and $B \subseteq W$. Hence X/R is normal as required. □

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