Finite Element Approximation of an Allen-Cahn/Cahn-Hilliard System†

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Abstract

We consider an Allen-Cahn/Cahn-Hilliard system with a non-degenerate mobility and (i) a logarithmic free energy and (ii) a non-smooth free energy (the deep quench limit). This system arises in the modelling of phase separation and ordering in binary alloys. In particular we prove in each case that there exists a unique solution for sufficiently smooth initial data. Further, we prove an error bound for a fully practical piecewise linear finite element approximation of (i) and (ii) in one and two space dimensions (and three space dimensions for constant mobility). The error bound being optimal in the deep quench limit. In addition an iterative scheme for solving the resulting nonlinear discrete system is analysed. Finally some numerical experiments are presented.

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^d, d \leq 3$, with a Lipschitz boundary $\partial\Omega$. We consider the Allen-Cahn/Cahn-Hilliard system with varying mobility and logarithmic free energy:
$(\mathbf{P}_\theta)$ Find $\{u_\theta(x,t), v_\theta(x,t), w_\theta(x,t), z_\theta(x,t)\}$ such that

$$\begin{aligned}
\tfrac{\partial u_\theta}{\partial t} &= \nabla\,.\,(b(u_\theta,v_\theta)\nabla w_\theta) && \text{in } \Omega_T := \Omega\times(0,T), && (1.1\text{a})\\
\rho\,\tfrac{\partial v_\theta}{\partial t} &= -\,b(u_\theta,v_\theta)\,z_\theta && \text{in } \Omega_T, && (1.1\text{b})\\
w_\theta &= -\gamma\,\Delta u_\theta + \theta\,[\,\phi(u_\theta+v_\theta)+\phi(u_\theta-v_\theta)\,] - \alpha\,u_\theta && \text{in } \Omega_T, && (1.1\text{c})\\
z_\theta &= -\gamma\,\Delta v_\theta + \theta\,[\,\phi(u_\theta+v_\theta)-\phi(u_\theta-v_\theta)\,] - \beta\,v_\theta && \text{in } \Omega_T, && (1.1\text{d})\\
u_\theta(x,0) &= u^0(x), \quad v_\theta(x,0) = v^0(x) && \forall\, x\in\Omega, && (1.1\text{e})\\
\tfrac{\partial u_\theta}{\partial\nu} &= \tfrac{\partial v_\theta}{\partial\nu} = b(u_\theta,v_\theta)\,\tfrac{\partial w_\theta}{\partial\nu} = 0 && \text{on } \partial\Omega\times(0,T); && (1.1\text{f})
\end{aligned}$$

where ν is normal to $\partial\Omega$ and ρ, γ, θ, α and β are given positive constants. On introducing $\Phi \in C[0,1]$ such that

$$\Phi(s) := \Phi^+(s) + \Phi^+(1-s), \qquad \text{where } \Phi^+(s) := s\,\ln\,s; \qquad (1.2\text{a})$$

then the monotone function $\phi \in C^\infty(0,1)$ in (1.1c,d) is defined to be

$$\phi(s) := \Phi'(s) := \ln s - \ln(1-s) \equiv \phi^+(s) - \phi^+(1-s), \tag{1.2b}$$

where $\phi^+ \equiv (\Phi^+)'$. The singularities in ϕ at 0 and 1 force $(u_\theta \pm v_\theta)(x,t) \in (0,1)$, see Theorem 2.2 below; that is, $\{u_\theta(x,t), v_\theta(x,t)\} \in \mathcal{Q}$ *a.e.* in Ω_T, where $\mathcal{Q} := \{\{s_1,s_2\} \in \mathbf{R}^2 : 0 < s_1 + s_2 < 1,\ 0 < s_1 - s_2 < 1\}$. The mobility $b \in C(\overline{\mathcal{Q}})$ in (1.1a,b) is non-negative such that

$$0 \leq b_{\min} \leq b(s_1,s_2) \leq b_{\max} \qquad \forall\ \{s_1,s_2\} \in \overline{\mathcal{Q}}. \tag{1.3}$$

On introducing the total free energy

$$\mathcal{J}_\theta(u_\theta, v_\theta) := \int_\Omega \{\ \tfrac{1}{2}\gamma\,[\,|\nabla u_\theta|^2 + |\nabla v_\theta|^2\,] + \Psi_\theta(u_\theta,v_\theta)\ \}\ \mathrm{d}x, \tag{1.4a}$$

where

$$\Psi_\theta(u_\theta,v_\theta) := \theta\,[\,\Phi(u_\theta+v_\theta) + \Phi(u_\theta - v_\theta)\,] + \tfrac{1}{2}\,[\,\alpha\,u_\theta\,(1-u_\theta) - \beta\,v_\theta^2\,]; \tag{1.4b}$$

it follows that $\frac{\delta \mathcal{J}_\theta}{\delta u_\theta} = w_\theta + \frac{1}{2}\alpha$ and $\frac{\delta \mathcal{J}_\theta}{\delta v_\theta} = z_\theta$. On noting this and that (1.1a,f) implies that $\int_\Omega \frac{\partial u_\theta}{\partial t}\,\mathrm{d}x = 0$, it follows that (1.1a–f) can be viewed as a (weighted) gradient flow in $H^{-1} \times L^2$ with $\mathcal{J}_\theta$ non-increasing in time, t. Furthermore, the choice $u^0 \equiv \frac{1}{2}$ yields that $u_\theta \equiv \frac{1}{2}$ and $w_\theta \equiv -\frac{1}{2}\alpha$. Hence (1.1d) collapses to $z_\theta = -\gamma\Delta v_\theta + 2\theta\phi(\frac{1}{2} + v_\theta) - \beta v_\theta$ and the system (P$_\theta$) to a logarithmic Allen-Cahn equation with a varying mobility. Whereas, the choice $v^0 \equiv 0$ yields that $v_\theta \equiv z_\theta \equiv 0$. Hence (1.1c) collapses to $w_\theta = -\gamma\Delta u_\theta + 2\theta\phi(u_\theta) - \alpha u_\theta$ and the system (P$_\theta$) to a logarithmic Cahn-Hilliard equation with a varying mobility. Therefore for general initial data $\{u^0(x), v^0(x)\} \in \overline{\mathcal{Q}}$, for all $x \in \Omega$, (P$_\theta$) can be considered as a system encompassing both the logarithmic Allen-Cahn and Cahn-Hilliard equations with varying mobility.

The system (1.1a–f) was derived in [11] to model the simultaneous order-disorder and phase separation in binary alloys on a BCC lattice, for example in Fe-Al alloys. Here u_θ denotes the average concentration of one of the components and, as noted above, is a conserved quantity; and v_θ is a non-conserved order parameter. The parameter θ denotes the absolute temperature. We note that if $8\theta < \max\{\alpha,\beta\}$, then Ψ_θ is non-convex.

Existence, uniqueness and regularity have been established for (1.1a–f) with constant mobility and with $\Phi(s)$ in (1.2a) replaced by the quartic s^4 in [10]. In [12] and [23] formal asymptotics are used to describe the long time behaviour of the system (1.1a–f) close to the deep quench limit ($\theta = 0$). We note that the local minima of $\Psi_\theta(u_\theta,v_\theta)$ are transcendentally close, $O(e^{-c/\theta})$, to $\{\frac{1}{2}, \pm\frac{1}{2}\}$, $\{0,0\}$ and $\{1,0\}$; the vertices of $\mathcal{Q}$, for $\theta \approx 0$. The first pair of local minimizers are known as ordered variants and the second pair as disordered phases. Which pair are global minimizers depends on the ordering of α and β, since $\Psi_0(\frac{1}{2},\pm\frac{1}{2}) = \frac{1}{8}\,(\alpha - \beta)$ and $\Psi_0(0,0) = \Psi_0(1,0) = 0$. Partitions between an ordered variant and a disordered phase or between the two disordered phases are known as interphase boundaries (IPBs), whereas partitions between the two ordered variants are known as antiphase boundaries (APBs); see [12] and [23] for details. Finally, existence of a weak solution in one space dimension was established in [15] for the degenerate mobility $b(s_1, s_2) := b_1(s_1)\, b_2(s_2)$, where $b_2(s) := \frac{1}{4} - s^2$ and $b_1(s) := b_2(s - \frac{1}{2})$, which vanishes in the pure regions: the two ordered variants, $\{\frac{1}{2}, \pm\frac{1}{2}\}$, and the two disordered phases comprising of just one of the two components of the alloy, $\{0, 0\}$ and $\{1, 0\}$. This specific choice for b leads to a number of mathematical difficulties since it is degenerate, i.e. $b_{\min} = 0$ in (1.3). The key difficulty being that there is no uniqueness proof, as is common for fourth order degenerate parabolic problems; see e.g. [16] for the logarithmic Cahn-Hilliard equation with a degenerate mobility ((P_θ) with the above degenerate b and $v^0 \equiv 0$).

A simpler model is to consider for example $b(s_1, s_2) := b_1^\sigma(s_1)\, b_2^\sigma(s_2)$, where for $\sigma \geq 0$

$$b_2^\sigma(s) := \begin{cases} (\frac{1}{2} + \sigma)^2 - s^2 & \text{if } s \in [0, \frac{1}{2}(1+\sigma)], \\ \frac{1}{4}\sigma^2 \,[\,1 + (1+\sigma)\,(s - \frac{1}{2})^{-1}\,] & \text{if } s \geq \frac{1}{2}(1+\sigma), \\ b_2^\sigma(-s) & \text{if } s \leq 0 \end{cases} \tag{1.5}$$

and $b_1^\sigma(s) := b_2^\sigma(s - \frac{1}{2})$. The choice $\sigma = 0$ yields the degenerate mobility mentioned above, whereas for $\sigma > 0$ the resulting mobility $b \in C^1(\mathbf{R}^2)$ and is non-degenerate over $\mathbf{R}^2$. For the purposes of the analysis in this paper, a general mobility b is extended to $\mathbf{R}^2$ so that $b \in C^1(\mathbf{R}^2)$ and is non-degenerate over $\mathbf{R}^2$ if b is non-degenerate over $\overline{\mathcal{Q}}$. Such a non-degenerate mobility is also considered in [15], where existence and regularity of a solution to (P_θ) is then proved for u^0, $v^0 \in H^1(\Omega)$, $|\Omega|^{-1} \int_\Omega u^0(x)\,\mathrm{d}x \in (0,1)$ and $\{u^0(x), v^0(x)\} \in \overline{\mathcal{Q}}$ for all $x \in \Omega$. However, the question of uniqueness is not addressed. Throughout the paper, we will assume that

$$b \in C(\mathbf{R}^2), \qquad 0 \leq b_{\min} \leq b(s_1, s_2) \leq b_{\max} \qquad \forall\ \{s_1, s_2\} \in \mathbf{R}^2. \tag{1.6a}$$

For the majority of our results we require the further restrictions

$$b_{\min} > 0, \quad b \in C^1(\mathbf{R}^2) \quad \text{with} \quad |\tfrac{\partial}{\partial s_i} b(s_1, s_2)| \leq C \quad \forall\ \{s_1, s_2\} \in \mathbf{R}^2,\ i = 1,\, 2. \tag{1.6b}$$

There are two major difficulties in studying problem (P_θ) under the above assumptions on b. One is that ϕ is singular on the edges of $\mathcal{Q}$ and therefore equations (1.1c,d) have no meaning if $u_\theta \pm v_\theta = 0$ or 1 in an open set of non-zero measure. Secondly, establishing uniqueness of a solution is considerably more difficult for varying mobility. In the next section we prove a uniqueness result for (P_θ) assuming a smoother class of initial data than that quoted above for the existence result in [15], see Theorem 2.2 below.

In the above we have adopted the standard notation for Sobolev spaces, denoting the norm of $W^{m,p}(\Omega)$ ($m \in \mathbf{N}$, $p \in [1, \infty]$) by $\|\cdot\|_{m,p}$ and semi-norm by $|\cdot|_{m,p}$. For $p = 2$, $W^{m,2}(\Omega)$ will be denoted by $H^m(\Omega)$ with the associated norm and semi-norm written, as respectively, $\|\cdot\|_m$ and $|\cdot|_m$. Throughout $(\cdot,\cdot)$ denotes the standard L^2 inner product over Ω and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. In addition we define

$$\fint \eta := \tfrac{1}{|\Omega|}(\eta, 1) \qquad \forall\, \eta \in L^2(\Omega).$$

For a number of reasons it is more convenient to rewrite (1.1c,d), by adding and subtracting, as

$$w_\theta + z_\theta = -\gamma\,\Delta(u_\theta + v_\theta) + 2\,\theta\,\phi(u_\theta + v_\theta) - \alpha\,u_\theta - \beta\,v_\theta \qquad \text{in } \Omega_T, \tag{1.7a}$$

$$w_\theta - z_\theta = -\gamma\,\Delta(u_\theta - v_\theta) + 2\,\theta\,\phi(u_\theta - v_\theta) - \alpha\,u_\theta + \beta\,v_\theta \qquad \text{in } \Omega_T. \tag{1.7b}$$

Clearly (1.7a,b) can be written more succinctly as

$$w_\theta \pm z_\theta = -\gamma\,\Delta(u_\theta \pm v_\theta) + 2\,\theta\,\phi(u_\theta \pm v_\theta) - (\alpha\,u_\theta \pm \beta\,v_\theta) \qquad \text{in } \Omega_T \tag{1.8}$$

and we adopt this convention throughout this paper. It then follows, see Section 2, that the deep quench limit of (P$_\theta$); that is, the limit as $\theta \to 0$; is the free boundary problem:

(P) Find $\{u(x,t), v(x,t), w(x,t), z(x,t)\}$ such that

$$\tfrac{\partial u}{\partial t} = \nabla\,.\,(b(u,v)\nabla w) \qquad \text{in } \Omega_T, \tag{1.9a}$$

$$\rho\,\tfrac{\partial v}{\partial t} = -\,b(u,v)\,z \qquad \text{in } \Omega_T, \tag{1.9b}$$

$$w \pm z \in -\gamma\,\Delta(u \pm v) + \partial\mathcal{I}_{[0,1]}(u \pm v) - (\alpha\,u \pm \beta\,v) \qquad \text{in } \Omega_T, \tag{1.9c}$$

$$u(x,0) = u^0(x), \quad v(x,0) = v^0(x) \qquad \forall\, x \in \Omega, \tag{1.9d}$$

$$\tfrac{\partial u}{\partial \nu} = \tfrac{\partial v}{\partial \nu} = b(u,v)\,\tfrac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T); \tag{1.9e}$$

where $\partial\mathcal{I}_{[0,1]}$ is the subdifferential of the indicator function $\mathcal{I}_{[0,1]}$ for the set $[0,1]$; that is $\mathcal{I}_{[0,1]}(s) = 0$ if $s \in [0,1]$ and $+\infty$ otherwise. Furthermore, we prove a uniqueness result for (P) and error bounds between u and u_θ, and v and v_θ; see Theorem 2.3 below.

In Sections 3 and 4 we consider continuous piecewise linear finite element approximations of (P) and (P$_\theta$) under the following respective assumptions on the mesh:

(A) Let Ω be a convex polyhedron. Let $\mathcal{T}^h$ be a quasi-uniform partitioning of Ω into disjoint open simplices κ with $h_\kappa := \text{diam}(\kappa)$ and $h := \max_{\kappa\in\mathcal{T}^h} h_\kappa$, so that $\overline{\Omega} = \cup_{\kappa\in\mathcal{T}^h}\overline{\kappa}$.

(A$_\theta$) In addition to the assumptions (A), it is assumed that $\mathcal{T}^h$ is a (weakly) acute partitioning; that is for (a) $d = 2$ for any pair of adjacent triangles the sum of opposite angles relative to the common side does not exceed π. (b) $d = 3$ the angle between any two faces of the same tetrahedron does not exceed $\pi/2$.

Associated with $\mathcal{T}^h$ is the finite element space

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi\,|_\kappa \ \text{ is linear } \forall\, \kappa \in \mathcal{T}^h\} \subset H^1(\Omega).$$

Let $\pi^h : C(\overline{\Omega}) \to S^h$ be the interpolation operator such that $\pi^h \eta(x_j) = \eta(x_j)$ $(j = 1 \to J)$, where $\{x_j\}_{j=1}^J$ is the set of nodes of $\mathcal{T}^h$. A discrete semi-inner product on $C(\overline{\Omega})$, is then defined by

$$(\eta_1, \eta_2)^h := \int_\Omega \pi^h(\eta_1(x)\,\eta_2(x))\,\mathrm{d}x \equiv \sum_{j=1}^J m_j\,\eta_1(x_j)\,\eta_2(x_j), \tag{1.10}$$

where $0 < m_j \leq Ch^d$. We introduce the weighted H^1 projection $Q_\gamma^h : H^1(\Omega) \to S^h$ defined by

$$\gamma\,(\nabla(I - Q_\gamma^h)\eta, \nabla\chi) + (\,(I - Q_\gamma^h)\eta, \chi) = 0 \quad \forall\,\chi \in S^h. \tag{1.11}$$

For the approximation of (P), we require in addition

$$K^h := \{\chi \in S^h : \chi(x_j) \in [0,1],\ j = 1 \to J\}.$$

Let $0 \equiv t_0 < t_1 < \cdots t_{N-1} < t_N \equiv T$ be a partitioning of $[0,T]$ into possibly variable time steps $\tau_n := t_n - t_{n-1}$, $n = 1 \to N$. Let $\tau := \max_{n=1\to N} \tau_n$. In this paper we consider the following fully practical finite element approximations of (P_θ) and (P):

$(\mathbf{P}_\theta^{h,\tau})$ Let $U_\theta^0 \equiv Q_\gamma^h u^0$ and $V_\theta^0 \equiv Q_\gamma^h v^0$. For $n = 1 \to N$ find $\{U_\theta^n, V_\theta^n, W_\theta^n, Z_\theta^n\}$ $\in [S^h]^4$ such that

$$\left(\tfrac{U_\theta^n - U_\theta^{n-1}}{\tau_n}, \chi\right)^h + \left(b(U_\theta^{n-1}, V_\theta^{n-1})\nabla W_\theta^n, \nabla\chi\right) = 0 \quad \forall\,\chi \in S^h, \tag{1.12a}$$

$$\rho\,\left(\tfrac{V_\theta^n - V_\theta^{n-1}}{\tau_n}, \chi\right)^h + \left(b(U_\theta^{n-1}, V_\theta^{n-1})\,Z_\theta^n, \chi\right) = 0 \quad \forall\,\chi \in S^h, \tag{1.12b}$$

$$\begin{aligned}&\gamma(\nabla(U_\theta^n \pm V_\theta^n), \nabla\chi) + 2\,\theta\,(\phi(U_\theta^n \pm V_\theta^n), \chi)^h \\ &\qquad = (W_\theta^n \pm Z_\theta^n + \alpha\,U_\theta^{n-1} \pm \beta\,V_\theta^{n-1}, \chi)^h \quad \forall\,\chi \in S^h.\end{aligned} \tag{1.12c}$$

$(\mathbf{P}^{h,\tau})$ Let $U^0 \equiv Q_\gamma^h u^0$ and $V^0 \equiv Q_\gamma^h v^0$. For $n = 1 \to N$ find $\{U^n, V^n, W^n, Z^n\}$ $\in [S^h]^4$ such that $U^n \pm V^n \in K^h$ and

$$\left(\tfrac{U^n - U^{n-1}}{\tau_n}, \chi\right)^h + \left(b(U^{n-1}, V^{n-1})\nabla W^n, \nabla\chi\right) = 0 \quad \forall\,\chi \in S^h, \tag{1.13a}$$

$$\rho\,\left(\tfrac{V^n - V^{n-1}}{\tau_n}, \chi\right)^h + \left(b(U^{n-1}, V^{n-1})\,Z^n, \chi\right) = 0 \quad \forall\,\chi \in S^h, \tag{1.13b}$$

$$\begin{aligned}&\gamma(\nabla(U^n \pm V^n), \nabla(\chi - (U^n \pm V^n))) \\ &\qquad \geq (W^n \pm Z^n + \alpha\,U^{n-1} \pm \beta\,V^{n-1}, \chi - (U^n \pm V^n))^h \quad \forall\,\chi \in K^h.\end{aligned} \tag{1.13c}$$

It is notationally convenient to introduce for $n \geq 1$

$$U_{(\theta)}(\cdot,t) := \tfrac{t - t_{n-1}}{\tau_n}\,U_{(\theta)}^n(\cdot) + \tfrac{t_n - t}{\tau_n}\,U_{(\theta)}^{n-1}(\cdot) \qquad t \in [t_{n-1}, t_n]. \tag{1.14}$$

The notation "$\cdot_{(\theta)}$" adopted in (1.14) and throughout is abbreviation for either "with" or "without" the subscript "θ". In addition, we adopt similar notation for the other variables, $V_{(\theta)}$, $W_{(\theta)}$ and $Z_{(\theta)}$. It is the main purpose of this paper to prove the following error bounds for the approximations $(\mathrm{P}_\theta^{h,\tau})$ and $(\mathrm{P}^{h,\tau})$:

THEOREM. 1.1. *Let $u^0 \in H^3(\Omega)$, $v^0 \in H^2(\Omega)$, $\frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} = 0$ on $\partial\Omega$ and $\delta \in (0, \frac{1}{2})$ be such that $\|u^0 \pm v^0 - \frac{1}{2}\|_{0,\infty} \leq \frac{1}{2} - \delta$. Let the assumptions $(A_{(\theta)})$ hold for the approximation $(P_{(\theta)}^{h,\tau})$. Let either $d \leq 2$ with b satisfying (1.6a,b) or $d = 3$ with $b > 0$ constant. Then for all $h \leq h_0$ such that $\|Q_\gamma^h(u^0 \pm v^0) - \frac{1}{2}\|_{0,\infty} \leq \frac{1}{2}(1-\delta)$ if $\theta > 0$, ($\frac{1}{2}$ if $\theta = 0$), and for all partitions $\{\tau_n\}_{n=1}^N$ of $[0,T]$ such that $\tau_{n-1}/\tau_n \leq C$, $n = 2 \to N$, the unique solutions $\{U_\theta^n, V_\theta^n\}_{n=0}^N$, $\{U^n, V^n\}_{n=0}^N$ to $(P_\theta^{h,\tau})$, $(P^{h,\tau})$ satisfy the error bounds*

$$
\begin{aligned}
&\|u_\theta - U_\theta\|^2_{L^2(0,T;H^1(\Omega))} + \|u_\theta - U_\theta\|^2_{L^\infty(0,T;(H^1(\Omega))')} \\
&\qquad + \|v_\theta - V_\theta\|^2_{L^2(0,T;H^1(\Omega))} + \|v_\theta - V_\theta\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\leq C_b(T)\,\tau^2 + \begin{cases} C_b(T)\,h^{\frac{4}{3}}(\ln\frac{1}{h})^{\frac{2(d-1)}{3}} & \text{if } d \leq 2, \\ C_b(T)\,h & \text{if } d = 3 \text{ and } b \text{ is constant;} \end{cases}
\end{aligned} \tag{1.15}
$$

$$
\begin{aligned}
&\|u - U\|^2_{L^2(0,T;H^1(\Omega))} + \|u - U\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|v - V\|^2_{L^2(0,T;H^1(\Omega))} \\
&\qquad + \|v - V\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_b(T)\left[\tau^2 + h^2\right].
\end{aligned} \tag{1.16}
$$

We remark that the error bound (1.16) is optimal, whereas the error bound (1.15) is optimal in time, but probably not in space. It should be noted that the singular nature of the nonlinearity $\phi(\cdot)$ and the use of numerical integration on those terms in (1.12c), which leads to a fully practical scheme, make the analysis of the spatial error in the approximation of (P_θ) by $(\mathrm{P}_\theta^{h,\tau})$, particularly delicate. We remark also that a "standard" error analysis in time would require bounds on $\|\frac{\partial^2 u_{(\theta)}}{\partial t^2}\|_{L^1(0,T;(H^1(\Omega))')}$ and $\|\frac{\partial^2 v_{(\theta)}}{\partial t^2}\|_{L^1(0,T;L^2(\Omega))}$. Unfortunately, these are not available for (P_θ) and (P) due to the singular nature of $\phi(\cdot)$ and the variational inequality structure, respectively. However, by adapting the approach developed in the papers [21] and [22] for analysing the time discretization error of the backward Euler method applied to "subgradient flows", it is possible to prove an optimal error bound in time for the discretizations $(\mathrm{P}_{(\theta)}^{h,\tau})$ without having bounds on these second time derivatives.

The layout of this paper is as follows. In the next section we extend the results of [6, §2] for the scalar Cahn-Hilliard equation with concentration dependent mobility to (P_θ). We introduce a regularized version, $(\mathrm{P}_{\theta,\varepsilon})$, of (P_θ) by regularizing the singular ϕ. Firstly we prove some ε independent stability bounds for the solution $\{u_{\theta,\varepsilon}, v_{\theta,\varepsilon}, w_{\theta,\varepsilon}, z_{\theta,\varepsilon}\}$. Passing to the limit, $\varepsilon = 0$, we prove existence of a solution $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ to (P_θ). We prove uniqueness of these solutions to $(\mathrm{P}_{\theta,\varepsilon})$ and (P_θ), and an error bound for this regularization procedure under a number of regularity assumptions; which are shown to hold for sufficiently smooth initial data and either $d \leq 2$ and $b(\cdot)$ satisfying both

(1.6a,b) or $d = 3$ and $b > 0$ constant. Finally in Section 2 we show that (P) is the $\theta \to 0$ limit of (P_θ), extending the uniqueness and regularity results for (P_θ) to (P). In section 3 we introduce "semidiscrete finite element approximations" $(\mathrm{P}^h_{\theta,\varepsilon})$, (P^h_θ) and (P^h) of $(\mathrm{P}_{\theta,\varepsilon})$, (P_θ) and (P); respectively. These are not true semidiscrete finite element approximations for non-constant mobility, since for technical reasons the mobility b is "frozen"; that is, $b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})$, $b(u_\theta, v_\theta)$ and $b(u,v)$ appear in $(\mathrm{P}^h_{\theta,\varepsilon})$, (P^h_θ) and (P^h), respectively. Hence for non-constant mobility the problems $(\mathrm{P}^h_{\theta,\varepsilon})$, (P^h_θ) and (P^h) are not computable. We prove error bounds between the unique solutions of $(\mathrm{P}_{(\theta)})$ and $(\mathrm{P}^h_{(\theta)})$. In the case $\theta > 0$, this is via an error bound between the unique solutions of $(\mathrm{P}_{\theta,\varepsilon})$ and $(\mathrm{P}^h_{\theta,\varepsilon})$, and an error bound for the regularization procedures on (P_θ) and (P^h_θ). In Section 4 well-posedness and a number of stability bounds are proved for $(\mathrm{P}^{h,\tau}_{(\theta)})$. Then adapting the approach in [22], we prove an optimal a priori error bound in time between $(\mathrm{P}^h_{(\theta)})$ and $(\mathrm{P}^{h,\tau}_{(\theta)})$. Moreover, for the case of constant mobility this is an optimal a posteriori error bound. Combining the regularization, spatial and temporal error bounds above we obtain the desired error bounds (1.15) and (1.16). In Section 5 the iterative algorithm in [8, §3 & 4] for the scalar Cahn-Hilliard equation with concentration dependent mobility is extended to the nonlinear algebraic system arising from the discretizations $(\mathrm{P}^{h,\tau}_{(\theta)})$ at each time level. Moreover, global convergence is proved. Finally in section 6 we report on some numerical experiments in one space dimension illustrating the error bounds (1.15) and (1.16).

Throughout C denotes a generic constant independent of the four key parameters, θ, ε, h and τ. In addition $C(a_1, \cdots, a_I)$ denotes a constant depending on the non-negative parameters $\{a_i\}_{i=1}^I$, such that $C(a_1, \cdots, a_I) \leq C$ if $a_i \leq C$ for $i = 1 \to I$. For notational convenience we write $C_b \equiv C(b_{\min}^{-1})$ and $C_b(a_1, \cdots, a_I) \equiv C(b_{\min}^{-1}, a_1, \cdots, a_I)$.

2 The Continuous Problems and Regularization

2.1 Logarithmic Free Energy

In order to analyse (P_θ), we employ a regularization procedure. The logarithmic convex function Φ is replaced for $\varepsilon \in (0, \frac{1}{2})$ by the twice continuously differentiable convex function

$$\Phi_\varepsilon(s) := \Phi^+_\varepsilon(s) + \Phi^+_\varepsilon(1-s), \tag{2.1a}$$

$$\text{where} \qquad \Phi^+_\varepsilon(s) := \begin{cases} \frac{1}{2\varepsilon}(s^2 - \varepsilon^2) + s \ln \varepsilon & \text{if } s \leq \varepsilon, \\ \Phi^+(s) & \text{if } \varepsilon \leq s. \end{cases} \tag{2.1b}$$

We note for future reference that

$$\Phi_\varepsilon(s) \geq \begin{cases} \frac{1}{2\varepsilon}\left([s]_-^2 + [s-1]_+^2 - \varepsilon^2 \right) & \text{if } s \leq 0 \text{ or } s \geq 1, \\ \Phi(\frac{1}{2}) = -\ln 2 & \text{if } s \in [0,1]; \end{cases} \tag{2.2}$$

where $[\cdot]_- := \min\{\cdot, 0\}$ and $[\cdot]_+ := \max\{\cdot, 0\}$. The monotone function

$$\phi_\varepsilon(s) := \Phi'_\varepsilon(s) \equiv \phi^+_\varepsilon(s) - \phi^+_\varepsilon(1-s), \qquad \text{where } \phi^+_\varepsilon(s) := (\Phi^+_\varepsilon)'(s), \tag{2.3}$$

has the following properties: For all $\varepsilon \in (0, \frac{1}{2})$

$$\phi_\varepsilon(s) \geq \phi(s) \quad \text{if } s \in (0, \varepsilon), \qquad \phi(s) \geq \phi_\varepsilon(s) \quad \text{if } s \in (1-\varepsilon, 1). \tag{2.4}$$

For $\varepsilon \in (0, \frac{1}{4})$ and for all $r,\, s \in \mathbf{R}$

$$4\,(r-s)^2 \leq (\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s) \tag{2.5a}$$

and

$$\begin{aligned}(\phi_\varepsilon(r) - \phi_\varepsilon(s))^2 &\leq \phi'_\varepsilon(\tfrac{1}{2} + \max\{|r - \tfrac{1}{2}|, |s - \tfrac{1}{2}|\})(\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s) \\ &\leq \tfrac{4}{3\varepsilon}(\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s).\end{aligned} \tag{2.5b}$$

In addition, if $r, s \leq \varepsilon$ or $r, s \geq 1-\varepsilon$ then

$$\tfrac{1}{\varepsilon}(r-s)^2 \leq (\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s). \tag{2.6}$$

For later use we need to bound below $\Psi_{\theta,\varepsilon}$, the corresponding regularized version of Ψ_θ (see 1.4b). To achieve this we first note from using a Young's inequality that for all $\{s_1, s_2\} \in \mathbf{R}^2$

$$\begin{aligned}\alpha\, s_1\,(1-s_1) - \beta\, s_2^2 &= \tfrac{1}{4}\,[2\,\alpha\,(r_1 + r_2) - (\alpha+\beta)\,(r_1^2 + r_2^2) + 2\,(\beta - \alpha)\,r_1\,r_2\,] \\ &\geq \tfrac{1}{2}\sum_{i=1}^{2}[\,\alpha\, r_i - \max\{\alpha,\beta\}\, r_i^2\,],\end{aligned} \tag{2.7}$$

where $r_1 := s_1 + s_2$ and $r_2 := s_1 - s_2$. Next we note, again from using a Young's inequality, that for all $r \in \mathbf{R}$

$$\begin{aligned}2\,[\,\alpha\, r - \max\{\alpha,\beta\}\, r^2\,] \geq &-\max\{\alpha, 4\beta - 3\alpha\} \\ &- \max\{3\alpha, 4\beta - \alpha\}\,(\,[r]_-^2 + [r-1]_+^2\,).\end{aligned} \tag{2.8}$$

Combining (2.2), (2.7) and (2.8) we have for all $\varepsilon < \varepsilon_0 := \min\{\frac{1}{4}, \frac{2\theta}{\max\{3\alpha, 4\beta-\alpha\}}\}$ and for all $\{s_1, s_2\} \in \mathbf{R}^2$ that

$$\begin{aligned}\Psi_{\theta,\varepsilon}(s_1, s_2) &:= \theta\,[\,\Phi_\varepsilon(s_1 + s_2) + \Phi_\varepsilon(s_1 - s_2)\,] + \tfrac{1}{2}\,[\,\alpha\, s_1\,(1-s_1) - \beta\, s_2^2\,] \\ &\geq -\tfrac{1}{4}\,\max\{\alpha, 4\beta - 3\alpha\} - \theta\,(\,\tfrac{1}{4} + 2\,\ln\,2\,) + \tfrac{\theta}{4\varepsilon}\,(\,[s_1 + s_2]_-^2 \\ &\quad + \,[s_1 - s_2]_-^2 + [s_1 + s_2 - 1]_+^2 + [s_1 - s_2 - 1]_+^2\,).\end{aligned} \tag{2.9}$$

For later purposes, we recall also the following well-known Sobolev interpolation results, e.g. see [1]: Let $p \in [1, \infty]$, $m \geq 1$,

$$r \in \begin{cases} [p, \infty] & \text{if } m - \frac{d}{p} > 0, \\ [p, \infty) & \text{if } m - \frac{d}{p} = 0, \\ [p, -\frac{d}{m-(d/p)}] & \text{if } m - \frac{d}{p} < 0; \end{cases}$$

and $\mu := \frac{d}{m}\left(\frac{1}{p} - \frac{1}{r}\right)$. Then there is a constant C depending only on Ω, p, r and m such that

$$|v|_{0,r} \leq C\, |v|_{0,p}^{1-\mu}\, \|v\|_{m,p}^{\mu} \quad \forall\ v \in W^{m,p}(\Omega). \tag{2.10}$$

Replacing ϕ by ϕ_ε in (1.1c,d), we obtain ($\mathrm{P}_{\theta,\varepsilon}$), the regularized version of (P_θ). Adopting the notation throughout that "$\cdot_{\theta(,\varepsilon)}$" is an abbreviation for either "with" or "without" the subscript "ε", and noting (1.8); the weak formulations of (P_θ) and ($\mathrm{P}_{\theta,\varepsilon}$) are then:

($\mathbf{P}_{\theta(,\varepsilon)}$) Find $\{u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)}, w_{\theta(,\varepsilon)}, z_{\theta(,\varepsilon)}\}$ such that $u_{\theta(,\varepsilon)}(\cdot,0) = u^0(\cdot)$, $v_{\theta(,\varepsilon)}(\cdot,0) = v^0(\cdot)$ and for *a.e.* $t \in (0,T)$

$$\langle \tfrac{\partial u_{\theta(,\varepsilon)}}{\partial t}, \eta\rangle + (\, b(u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)})\nabla w_{\theta(,\varepsilon)}, \nabla\eta) = 0 \quad \forall\ \eta \in H^1(\Omega), \tag{2.11a}$$

$$\rho\, (\tfrac{\partial v_{\theta(,\varepsilon)}}{\partial t}, \eta) + (\, b(u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)})\, z_{\theta(,\varepsilon)}, \eta) = 0 \quad \forall\ \eta \in L^2(\Omega), \tag{2.11b}$$

$$\begin{aligned}\gamma\, (\nabla(u_{\theta(,\varepsilon)} \pm v_{\theta(,\varepsilon)}), \nabla\eta) + (2\,\theta\,\phi_{(\varepsilon)}(u_{\theta(,\varepsilon)} \pm v_{\theta(,\varepsilon)}) - (\alpha\, u_{\theta(,\varepsilon)} \pm \beta\, v_{\theta(,\varepsilon)}), \eta)&\\ = (w_{\theta(,\varepsilon)} \pm z_{\theta(,\varepsilon)}, \eta) \quad \forall\ \eta \in H^1(\Omega).&\end{aligned} \tag{2.11c}$$

It is convenient to introduce the "inverse Laplacian" operator $\mathcal{G} : \mathcal{F} \to \Xi$ such that

$$(\nabla \mathcal{G} f, \nabla\eta) = \langle f, \eta\rangle \quad \forall\ \eta \in H^1(\Omega), \tag{2.12}$$

where

$$\mathcal{F} := \left\{ f \in (H^1(\Omega))' : \langle f, 1\rangle = 0 \right\}, \qquad \Xi := \{\xi \in H^1(\Omega) : (\xi, 1) = 0\}. \tag{2.13}$$

The well-posedness of $\mathcal{G}$ follows from the Lax-Milgram theorem and the Poincaré inequality

$$|\eta|_{0,p} \leq C(\, |\eta|_{1,p} + |(\eta,1)|\,) \quad \forall\ \eta \in W^{1,p}(\Omega) \quad \text{and} \quad p \in [1,\infty]. \tag{2.14}$$

One can define a norm on $\mathcal{F}$ by

$$\|f\|_{-1} := |\mathcal{G} f|_1 \equiv \langle f, \mathcal{G} f\rangle^{\frac{1}{2}} \quad \forall\ f \in \mathcal{F}. \tag{2.15}$$

We note also for future reference that using a Young's inequality yields for all $f \in \mathcal{F}$, $\eta \in H^1(\Omega)$ and for all $\mu > 0$ that

$$\langle f, \eta\rangle \equiv (\nabla \mathcal{G} f, \nabla\eta) \leq \|f\|_{-1}\, |\eta|_1 \leq \tfrac{1}{2\mu}\|f\|_{-1}^2 + \tfrac{\mu}{2}|\eta|_1^2. \tag{2.16}$$

In addition it follows from (2.12), (2.10) and (2.14) that

$$|\nabla \mathcal{G} f|_0 \leq C|f|_{0,r} \quad \forall\ f \in L^r(\Omega) \cap \mathcal{F}; \tag{2.17}$$

where $r = 1, 1+\zeta, \frac{6}{5}$, for any $\zeta \in (0, \frac{1}{5})$, for $d = 1,2,3$ respectively.

Assuming that $b_{\min} > 0$ and given q_i measurable in Ω, it is also convenient to introduce the operator $\mathcal{G}_{q_1,q_2} : \mathcal{F} \to \Xi$ such that

$$(b(q_1,q_2)\nabla \mathcal{G}_{q_1,q_2} f, \nabla\eta) = \langle f, \eta\rangle \quad \forall\ \eta \in H^1(\Omega). \tag{2.18}$$

It follows for all q_i measurable in Ω and $f \in \mathcal{F}$ that

$$\begin{aligned}|\nabla \mathcal{G} f|_0^2 &\equiv \langle f, \mathcal{G} f\rangle \equiv (b(q_1,q_2)\nabla \mathcal{G}_{q_1,q_2} f, \nabla \mathcal{G} f)\\ &\leq b_{\max}^{\frac{1}{2}} |[b(q_1,q_2)]^{\frac{1}{2}} \nabla \mathcal{G}_{q_1,q_2} f|_0 \, |\nabla \mathcal{G} f|_0.\end{aligned}$$

Similarly we have that

$$\begin{aligned}|[b(q_1,q_2)]^{\frac{1}{2}} \nabla \mathcal{G}_{q_1,q_2} f|_0^2 &\equiv \langle f, \mathcal{G}_{q_1,q_2} f\rangle \equiv (\nabla \mathcal{G} f, \nabla \mathcal{G}_{q_1,q_2} f)\\ &\leq b_{\min}^{-\frac{1}{2}} |\nabla \mathcal{G} f|_0 \, |[b(q_1,q_2)]^{\frac{1}{2}} \nabla \mathcal{G}_{q_1,q_2} f|_0.\end{aligned}$$

Combining the above, it follows for all q_i measurable in Ω and $f \in \mathcal{F}$ that

$$b_{\min} |[b(q_1,q_2)]^{\frac{1}{2}} \nabla \mathcal{G}_{q_1,q_2} f|_0^2 \leq |\nabla \mathcal{G} f|_0^2 \leq b_{\max} |[b(q_1,q_2)]^{\frac{1}{2}} \nabla \mathcal{G}_{q_1,q_2} f|_0^2. \qquad (2.19)$$

Let q_i be measurable in Ω, $f \in \mathcal{F}$ and $\eta \in H^1(\Omega)$, then

$$\begin{aligned}\langle f, \eta\rangle \equiv (b(q_1,q_2)\nabla \mathcal{G}_{q_1,q_2} f, \nabla \eta) &\leq b_{\max}^{\frac{1}{2}} |[b(q_1,q_2)]^{\frac{1}{2}} \nabla \mathcal{G}_{q_1,q_2} f|_0 |\eta|_1\\ &= b_{\max}^{\frac{1}{2}} \langle f, \mathcal{G}_{q_1,q_2} f\rangle^{\frac{1}{2}} |\eta|_1 \qquad (2.20)\end{aligned}$$

so that an analogue of (2.16) holds. Similarly to (2.17), we have from (2.18), (2.10) and (2.14) that

$$|\nabla \mathcal{G}_{q_1,q_2} f|_0 \leq C\, b_{\min}^{-1} |f|_{0,r} \quad \forall\ f \in L^r(\Omega) \cap \mathcal{F}; \qquad (2.21)$$

where $r = 1, 1+\zeta, \frac{6}{5}$, for any $\zeta \in (0, \frac{1}{5})$, for $d = 1,2,3$ respectively.

For a.e. $t \in (0,T)$, let $q_i(\cdot,t)$ be measurable in Ω and $f(\cdot,t) \in L^2(\Omega) \cap \mathcal{F}$ be such that $\frac{\partial q_i}{\partial t}(\cdot,t)$, $\frac{\partial f}{\partial t}(\cdot,t) \in L^2(\Omega)$. If b satisfies (1.6a,b), then by differentiating (2.18) with respect to t and setting $\eta \equiv \mathcal{G}_{q_1,q_2} f$ we obtain that

$$\begin{aligned}(\tfrac{\partial f}{\partial t}, \mathcal{G}_{q_1,q_2} f) &= (\tfrac{\partial}{\partial t}[b(q_1,q_2)\nabla \mathcal{G}_{q_1,q_2} f], \nabla \mathcal{G}_{q_1,q_2} f)\\ &= (\tfrac{\partial}{\partial t} b(q_1,q_2), |\nabla \mathcal{G}_{q_1,q_2} f|^2) + (b(q_1,q_2)\nabla \tfrac{\partial}{\partial t}[\mathcal{G}_{q_1,q_2} f], \nabla \mathcal{G}_{q_1,q_2} f)\\ &= (\tfrac{\partial}{\partial t} b(q_1,q_2), |\nabla \mathcal{G}_{q_1,q_2} f|^2) + (\tfrac{\partial}{\partial t}[\mathcal{G}_{q_1,q_2} f], f). \qquad (2.22)\end{aligned}$$

Hence applying (2.22) and noting (2.18) yields that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{G}_{q_1,q_2} f, f) &= (\tfrac{\partial}{\partial t}[\mathcal{G}_{q_1,q_2} f], f) + (\mathcal{G}_{q_1,q_2} f, \tfrac{\partial f}{\partial t})\\ &= 2\,(\mathcal{G}_{q_1,q_2} \tfrac{\partial f}{\partial t}, f) - (\tfrac{\partial}{\partial t} b(q_1,q_2), |\nabla \mathcal{G}_{q_1,q_2} f|^2). \qquad (2.23)\end{aligned}$$

We note for future reference that if $\frac{\partial q_i}{\partial t}(\cdot,t) \in L^2(\Omega)$ for a.e. $t \in (0,T)$ and if b satisfies (1.6a,b) then (2.10) yields for all $\eta \in H^1(\Omega)$ that

$$|(\tfrac{\partial}{\partial t} b(q_1,q_2), \eta^2)| \leq C\, |\tfrac{\partial}{\partial t} b(q_1,q_2)|_0 \, |\eta|_{0,4}^2 \leq C\, |\tfrac{\partial}{\partial t} b(q_1,q_2)|_0 \, |\eta|_0^{2-\frac{d}{2}} \, \|\eta\|_1^{\frac{d}{2}}\,. \quad (2.24)$$

In addition if q_i, $\widehat{q}_i \in H^1(\Omega)$ and b satisfies (1.6a,b), then (2.10) yields that

$$|[b(\widehat{q}_1,\widehat{q}_2)]^{\frac{1}{2}} \nabla (\mathcal{G}_{q_1,q_2} - \mathcal{G}_{\widehat{q}_1,\widehat{q}_2}) f|_0$$

$$
\begin{aligned}
&\equiv ([b(\widehat{q}_1,\widehat{q}_2)-b(q_1,q_2)]\nabla\mathcal{G}_{q_1,q_2}f,\nabla(\mathcal{G}_{q_1,q_2}-\mathcal{G}_{\widehat{q}_1,\widehat{q}_2})f)^{\frac{1}{2}} \\
&\leq b_{\min}^{-\frac{1}{2}}\,|[b(\widehat{q}_1,\widehat{q}_2)-b(q_1,q_2)]\nabla\mathcal{G}_{q_1,q_2}f|_0 \\
&\leq C\,b_{\min}^{-\frac{1}{2}}\,|\nabla\mathcal{G}_{q_1,q_2}f|_{0,4}\sum_{i=1}^{2}|q_i-\widehat{q}_i|_{0,4} \\
&\leq C\,b_{\min}^{-\frac{1}{2}}\,|\mathcal{G}_{q_1,q_2}f|_1^{1-\frac{d}{4}}\,\|\mathcal{G}_{q_1,q_2}f\|_2^{\frac{d}{4}}\sum_{i=1}^{2}[\,|q_i-\widehat{q}_i|_0^{1-\frac{d}{4}}\,\|q_i-\widehat{q}_i\|_1^{\frac{d}{4}}\,]. \qquad (2.25)
\end{aligned}
$$

Adapting an argument in [17], we now find a bound on $\|\mathcal{G}_{q_1,q_2}f\|_2$ when b satisfies (1.6a,b), $q_i \in H^2(\Omega)$, $f \in L^2(\Omega)\cap\mathcal{F}$ and Ω is a convex polyhedron or $\partial\Omega \in C^{1,1}$. It follows from the standard regularity estimate $|\cdot|_2 \leq C|\Delta\cdot|_0$, $\mathcal{G}_{q_1,q_2}f \in \Xi$, (2.21), (2.18) and (2.10) with $r_1 = \infty, \frac{2}{\zeta}, 6$, $\mu_1 = \frac{1}{2}, 1-\zeta, 1$ and $r_2 = 2, \frac{2}{1-\zeta}, 3$, $\mu_2 = 0, \zeta, \frac{1}{2}$ for all $\zeta \in (0,1)$, when $d = 1,2,3$ respectively, that

$$
\begin{aligned}
b_{\min}\|\mathcal{G}_{q_1,q_2}f\|_2 &\leq C\,[\,b_{\min}|\Delta\mathcal{G}_{q_1,q_2}f|_0+|f|_0\,] \leq C\,[\,|b(q_1,q_2)\Delta\mathcal{G}_{q_1,q_2}f|_0+|f|_0\,] \\
&= C\,[\,|\,(\sum_{i=1}^{2}\tfrac{\partial b}{\partial q_i}\,\nabla q_i\,)\,.\,\nabla\mathcal{G}_{q_1,q_2}f+f|_0+|f|_0\,] \\
&\leq C\,[\,(\,\sum_{i=1}^{2}|\nabla q_i|_{0,r_1}\,)\,|\nabla\mathcal{G}_{q_1,q_2}f|_{0,r_2}+|f|_0\,] \\
&\leq C\,[\,(\,\sum_{i=1}^{2}|q_i|_1^{1-\mu_1}\,\|q_i\|_2^{\mu_1}\,)\,|\mathcal{G}_{q_1,q_2}f|_1^{1-\mu_2}\,\|\mathcal{G}_{q_1,q_2}f\|_2^{\mu_2}+|f|_0\,]. \qquad (2.26)
\end{aligned}
$$

Hence applying a Young's inequality to (2.26) and noting (2.21) yields that

$$
\begin{aligned}
\|\mathcal{G}_{q_1,q_2}f\|_2 &\leq C_b\,[\,(\,\sum_{i=1}^{2}|q_i|_1^{\mu_1}\,\|q_i\|_2^{\mu_2}\,)\,|\mathcal{G}_{q_1,q_2}f|_1+|f|_0\,] \qquad (2.27\text{a}) \\
&\leq C_b(\|q_i\|_2)\,|f|_0 \quad \forall\ f \in L^2(\Omega)\cap\mathcal{F}, \qquad (2.27\text{b})
\end{aligned}
$$

where $\mu_1 = \frac{1}{2}, \frac{\zeta}{1-\zeta}, 0$, for all $\zeta \in (0,1)$, and $\mu_2 = 2^{d-2}$ when $d = 1,2,3$ respectively. Finally, it follows from (2.24), (2.10), (2.19), (2.15), (2.27b), (2.16) and a Young's inequality that for all $f \in \Xi$

$$
\begin{aligned}
|(\tfrac{\partial}{\partial t}b(q_1,q_2),|\nabla\mathcal{G}_{q_1,q_2}f|^2)| &\leq C_b\,|\tfrac{\partial}{\partial t}b(q_1,q_2)|_0\,\|f\|_{-1}^{2-\frac{d}{4}}\,|f|_1^{\frac{d}{4}} \\
&\leq \tfrac{\gamma}{8}\,|f|_1^2+C_b(\|q_i\|_2)\,|\tfrac{\partial}{\partial t}b(q_1,q_2)|_0^{\frac{8}{8-d}}\,\|f\|_{-1}^2. \qquad (2.28)
\end{aligned}
$$

Assuming that $b_{\min} > 0$ and given q_i measurable in Ω, it is also convenient for later purposes to introduce the operator $\mathcal{M}_{q_1,q_2} : L^2(\Omega) \to L^2(\Omega)$ such that

$$
(b(q_1,q_2)\mathcal{M}_{q_1,q_2}f,\eta) = (f,\eta) \quad \forall\ \eta \in L^2(\Omega). \qquad (2.29)
$$

As $\mathcal{M}_{q_1,q_2} f \equiv f/b$ we have for all $f \in L^p(\Omega)$, $p \in [2,\infty]$, that

$$b_{\max}^{-1}|f|_{0,p} \leq |\mathcal{M}_{q_1,q_2} f|_{0,p} \leq b_{\min}^{-1}|f|_{0,p}. \tag{2.30}$$

For a.e. $t \in (0,T)$, let $q_i(\cdot,t)$ be measurable in Ω and $f(\cdot,t) \in L^2(\Omega)$ be such that $\frac{\partial q_i}{\partial t}(\cdot,t)$, $\frac{\partial f}{\partial t}(\cdot,t) \in L^2(\Omega)$. If b satisfies (1.6a,b), then we have the analogue of (2.23)

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{M}_{q_1,q_2} f, f) = 2\,(\mathcal{M}_{q_1,q_2} \tfrac{\partial f}{\partial t}, f) - (\tfrac{\partial}{\partial t} b(q_1,q_2), |\mathcal{M}_{q_1,q_2} f|^2). \tag{2.31}$$

Similarly if q_i, $\widehat{q}_i \in H^1(\Omega)$ and b satisfies (1.6a,b), we have the analogue of (2.25)

$$\begin{aligned}
&|[b(\widehat{q}_1,\widehat{q}_2)]^{\frac{1}{2}}(\mathcal{M}_{q_1,q_2} - \mathcal{M}_{\widehat{q}_1,\widehat{q}_2}) f|_0 \\
&\quad \leq C\, b_{\min}^{-\frac{1}{2}}\, |\mathcal{M}_{q_1,q_2} f|_{0,4} \sum_{i=1}^{2} |q_i - \widehat{q}_i|_{0,4} \\
&\quad \leq C\, b_{\min}^{-\frac{1}{2}}\, |\mathcal{M}_{q_1,q_2} f|_0^{1-\frac{d}{4}}\, \|\mathcal{M}_{q_1,q_2} f\|_1^{\frac{d}{4}} \sum_{i=1}^{2} [\, |q_i - \widehat{q}_i|_0^{1-\frac{d}{4}}\, \|q_i - \widehat{q}_i\|_1^{\frac{d}{4}}\,].
\end{aligned} \tag{2.32}$$

We now find a bound on $\|\mathcal{M}_{q_1,q_2} f\|_2$ when b satisfies (1.6a,b), $q_i \in H^2(\Omega)$ and $f \in H^1(\Omega)$. Similarly to (2.26) and (2.27a), it follows from (2.29), (2.10), a Young's inequality and (2.30) that

$$\begin{aligned}
|\mathcal{M}_{q_1,q_2} f|_1 = |\nabla([b(q_1,q_2)]^{-1} f)|_0 &\leq b_{\min}^{-1}\,[\, |(\nabla b(q_1,q_2))\, \mathcal{M}_{q_1,q_2} f|_0 + |f|_1\,] \\
&\leq C_b\,[\, (\sum_{i=1}^{2} |q_i|_1^{\mu_1}\, \|q_i\|_2^{\mu_2}\,)\, |\mathcal{M}_{q_1,q_2} f|_0 + |f|_1\,], &(2.33\text{a}) \\
&\leq C_b(\|q_i\|_2)\, \|f\|_1 \quad \forall\, f \in H^1(\Omega), &(2.33\text{b})
\end{aligned}$$

where μ_i are defined as in (2.27a). Similarly to (2.28), it follows from (2.24), (2.10), (2.33b), (2.30) and a Young's inequality that for all $f \in H^1(\Omega)$

$$|(\tfrac{\partial}{\partial t} b(q_1,q_2), |\mathcal{M}_{q_1,q_2} f|^2)| \leq \tfrac{\gamma}{8}\, \|f\|_1^2 + C_b(\|q_i\|_2)\, |\tfrac{\partial}{\partial t} b(q_1,q_2)|_0^{\frac{4}{4-d}}\, |f|_0^2. \tag{2.34}$$

Choosing $\eta \equiv 1$ in (2.11a) yields that $\langle \frac{\partial u_{\theta(,\varepsilon)}}{\partial t}, 1\rangle = 0$, i.e. $(u_{\theta(,\varepsilon)}(\cdot,t),1) = (u^0(\cdot),1)$ for all t. Hence it follows from (2.11a,b), (2.18), (1.6b), (2.14), (2.29) and (2.11c) with $\eta \equiv 1$ that

$$w_{\theta(,\varepsilon)} \equiv -\mathcal{G}_{u_{\theta(,\varepsilon)},v_{\theta(,\varepsilon)}} \frac{\partial u_{\theta(,\varepsilon)}}{\partial t} + \lambda_{\theta(,\varepsilon)}, \qquad z_{\theta(,\varepsilon)} \equiv -\rho\, \mathcal{M}_{u_{\theta(,\varepsilon)},v_{\theta(,\varepsilon)}} \frac{\partial v_{\theta(,\varepsilon)}}{\partial t} \tag{2.35a}$$

$$\text{and} \quad \lambda_{\theta(,\varepsilon)} := -\!\!\!\!\!\!\int [\, \theta\, \phi_{(\varepsilon)}(u_{\theta(,\varepsilon)} + v_{\theta(,\varepsilon)}) + \theta\, \phi_{(\varepsilon)}(u_{\theta(,\varepsilon)} - v_{\theta(,\varepsilon)}) - \alpha\, u_{\theta(,\varepsilon)}\,]. \tag{2.35b}$$

Therefore for $b_{\min} > 0$, $(\mathrm{P}_{\theta(,\varepsilon)})$ can be rewritten as:
Find $\{u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)}\}$ such that $u_{\theta(,\varepsilon)}(\cdot,0) = u^0(\cdot)$, $v_{\theta(,\varepsilon)}(\cdot,0) = v^0(\cdot)$ and for *a.e.* $t \in (0,T)$, $(u_{\theta(,\varepsilon)}(\cdot,t),1) = (u^0(\cdot),1)$ and

$$\gamma\,(\nabla(u_{\theta(,\varepsilon)} \pm v_{\theta(,\varepsilon)}), \nabla\eta) + (\mathcal{G}_{u_{\theta(,\varepsilon)},v_{\theta(,\varepsilon)}} \frac{\partial u_{\theta(,\varepsilon)}}{\partial t} - \lambda_{\theta(,\varepsilon)} \pm \rho\, \mathcal{M}_{u_{\theta(,\varepsilon)},v_{\theta(,\varepsilon)}} \frac{\partial v_{\theta(,\varepsilon)}}{\partial t}, \eta)$$

$$+ \,(2\,\theta\,\phi_{(\varepsilon)}(u_{\theta(,\varepsilon)} \pm v_{\theta(,\varepsilon)}) - (\alpha\, u_{\theta(,\varepsilon)} \pm \beta\, v_{\theta(,\varepsilon)}), \eta) = 0 \quad \forall\, \eta \in H^1(\Omega), \tag{2.36}$$

where $\lambda_{\theta(,\varepsilon)}$ is defined by (2.35b) and $\{w_{\theta(,\varepsilon)}, z_{\theta(,\varepsilon)}\}$ can be obtained from (2.35a). Theorems 2.1 and 2.2 below are extensions to (P_θ) of Theorems 2.1 and 2.2 in [6] for the scalar Cahn-Hilliard equation with concentration dependent mobility.

THEOREM. 2.1. *Let $d \le 3$ and u^0, $v^0 \in H^1(\Omega)$ be such that $\|u^0 \pm v^0 - \frac{1}{2}\|_{0,\infty} \le \frac{1}{2}$ and $|\,⨍\,(u^0 - \frac{1}{2})| < \frac{1}{2} - \delta$ for some $\delta \in (0, \frac{1}{2})$. If b satisfies (1.6a) with $b_{\min} > 0$, then for all $\theta \le \theta_{\max}$, where $\theta_{\max} > 0$ is arbitrary, and for all $\varepsilon < \varepsilon_0$ there exists $\{u_{\theta,\varepsilon}, v_{\theta,\varepsilon}, w_{\theta,\varepsilon}, z_{\theta,\varepsilon}\}$ solving $(P_{\theta,\varepsilon})$ such that the following stability bounds hold independently of ε and θ*

$$\|u_{\theta,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} + \|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} + \|v_{\theta,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} + \|\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)} \le C, \tag{2.37a}$$

$$\theta\,[\,\|[u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon}]_-\|^2_{L^\infty(0,T;L^2(\Omega))} + \|[u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon} - 1]_+\|^2_{L^\infty(0,T;L^2(\Omega))}\,] \le C\,\varepsilon. \tag{2.37b}$$

In addition the following stability bounds hold independently of ε and θ

$$\|\lambda_{\theta,\varepsilon}\|_{L^2(0,T)} + \|w_{\theta,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \|z_{\theta,\varepsilon}\|_{L^2(\Omega_T)} + \theta\,\|\phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})\|_{L^2(\Omega_T)} \le C_b(T) \tag{2.38}$$

and if Ω is a convex polyhedron or $\partial\Omega \in C^{1,1}$

$$\|u_{\theta,\varepsilon}\|_{L^2(0,T;H^2(\Omega))} + \|v_{\theta,\varepsilon}\|_{L^2(0,T;H^2(\Omega))} \le C_b(T). \tag{2.39}$$

Furthermore, if either $b > 0$ is constant or b satisfies (1.6b) and

$$\|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}\|_{L^{\frac{4}{4-d}}(0,T;L^2(\Omega))} + \|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}\|_{L^{\frac{16}{12-3d}}(0,T;(H^1(\Omega))')} + \|u_{\theta,\varepsilon}\|_{L^{2(d-1)}(0,T;H^2(\Omega))} + \|\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}\|_{L^{\frac{16}{12-3d}}(0,T;L^2(\Omega))} + \|v_{\theta,\varepsilon}\|_{L^{2(d-1)}(0,T;H^2(\Omega))} \le C_b(\varepsilon^{-1}, \theta^{-1}, T); \tag{2.40}$$

then the solution $\{u_{\theta,\varepsilon}, v_{\theta,\varepsilon}, w_{\theta,\varepsilon}, z_{\theta,\varepsilon}\}$ of $(P_{\theta,\varepsilon})$ is unique.

PROOF. Existence follows from standard arguments using Galerkin approximations and then passing to the limit, e.g. extend the $d = 1$ argument in [15, Theorem 2.1]. The choices of η below can be justified in a similar way.

Adding together the $\pm$ regularized versions of (2.36) with $\eta \equiv \frac{\partial}{\partial t}(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})$, respectively, noting (2.18), (2.29), (2.3), (2.9) and $\langle \frac{\partial u_{\theta,\varepsilon}}{\partial t}, 1\rangle = 0$, and integrating over $(0,t)$ yields for all $t \in (0,T)$ that

$$\tfrac{\gamma}{2}\,[\,|u_{\theta,\varepsilon}(\cdot,t)|_1^2 + |v_{\theta,\varepsilon}(\cdot,t)|_1^2\,] + (\Psi_{\theta,\varepsilon}(u_{\theta,\varepsilon}(\cdot,t), v_{\theta,\varepsilon}(\cdot,t)), 1) + \int_0^t |\,[b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})]^{\frac{1}{2}} \nabla \mathcal{G}_{u_{\theta,\varepsilon}, v_{\theta,\varepsilon}} \tfrac{\partial u_{\theta,\varepsilon}}{\partial s}(\cdot,s)|_0^2 \,\mathrm{d}s$$

$$+ \ \rho \int_0^t |\,[\,b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})\,]^{\frac{1}{2}} \mathcal{M}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial v_{\theta,\varepsilon}}{\partial s}(\cdot, s)|_0^2 \,\mathrm{d}s$$
$$= \tfrac{\gamma}{2}\,[\,|u^0|_1^2 + |v^0|_1^2\,] + (\Psi_{\theta,\varepsilon}(u^0, v^0), 1) \leq C, \qquad (2.41)$$

where we have noted (2.1a,b) and the assumptions on u^0 and v^0. The bounds (2.37b) follow immediately from (2.41) and (2.9). The bounds (2.37a) follow from noting (2.41), (2.9), (2.37b), (2.14), (2.19), (2.15) and (2.30).

Noting (2.35a), (2.14) and (2.41) yields that

$$\begin{aligned}\|w_{\theta,\varepsilon} - \lambda_{\theta,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} &\leq C\,\|\nabla \mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial u_{\theta,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)} \\ &\leq C\, b_{\min}^{-\frac{1}{2}}\, \|[b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})]^{\frac{1}{2}} \nabla \mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial u_{\theta,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)} \\ &\leq C\, b_{\min}^{-\frac{1}{2}}. \end{aligned} \qquad (2.42)$$

Choosing $\eta \equiv 2\theta\,\phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon}) - \lambda_{\theta,\varepsilon}$ in the regularized versions of (2.36), respectively, noting that $\phi_\varepsilon'(\cdot) \in [4, \frac{4}{3}\varepsilon^{-1}]$, see (2.5a,b), and applying a Young's inequality yields for *a.e.* $t \in (0,T)$ that

$$\begin{aligned}&\tfrac{3}{2}\gamma\theta\,\varepsilon\,|\phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})|_1^2 + \tfrac{1}{2}\,|2\,\theta\,\phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon}) - \lambda_{\theta,\varepsilon}|_0^2 \\ &\quad\leq \tfrac{1}{2}\,|\mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial u_{\theta,\varepsilon}}{\partial t} \pm \rho\,\mathcal{M}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial v_{\theta,\varepsilon}}{\partial t} - (\alpha u_{\theta,\varepsilon} \pm \beta v_{\theta,\varepsilon})|_0^2. \end{aligned} \qquad (2.43)$$

Integrating the above over $t \in (0,T)$, noting (2.14), (2.37a) and (2.41) yields that

$$\|2\,\theta\,\phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon}) - \lambda_{\theta,\varepsilon}\|_{L^2(\Omega_T)} \leq C\, b_{\min}^{-\frac{1}{2}}. \qquad (2.44)$$

Adding together the $\pm$ regularized versions of (2.36) with $\eta \equiv (u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon}) - \mu$, respectively, for any $\mu \in \mathbf{R}$ yields for *a.e.* $t \in (0,T)$ that

$$\begin{aligned}(\lambda_{\theta,\varepsilon}, \mu - u_{\theta,\varepsilon}) &= -\gamma\,[\,|u_{\theta,\varepsilon}|_1^2 + |v_{\theta,\varepsilon}|_1^2\,] + \alpha\,(u_{\theta,\varepsilon}, u_{\theta,\varepsilon} - \mu) + \beta\,|v_{\theta,\varepsilon}|_0^2 \\ &\quad - \ (\mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial u_{\theta,\varepsilon}}{\partial t}, u_{\theta,\varepsilon}) - \rho\,(\mathcal{M}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial v_{\theta,\varepsilon}}{\partial t}, v_{\theta,\varepsilon}) \\ &\quad + \ \theta\,(\phi_\varepsilon(u_{\theta,\varepsilon} + v_{\theta,\varepsilon}), \mu - (u_{\theta,\varepsilon} + v_{\theta,\varepsilon})) \\ &\quad + \ \theta\,(\phi_\varepsilon(u_{\theta,\varepsilon} - v_{\theta,\varepsilon}), \mu - (u_{\theta,\varepsilon} - v_{\theta,\varepsilon})) \\ &\leq C\,(1 + |\mu|) + C\,b_{\min}^{-\frac{1}{2}}\,|\,[b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})]^{\frac{1}{2}} \nabla \mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial u_{\theta,\varepsilon}}{\partial t}|_0\,|u_{\theta,\varepsilon}|_0 \\ &\quad + \ C\,b_{\min}^{-\frac{1}{2}}\,|\,[b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})]^{\frac{1}{2}} \mathcal{M}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial v_{\theta,\varepsilon}}{\partial t}|_0\,|v_{\theta,\varepsilon}|_0 \\ &\quad + \ \theta\,(2\,\Phi_\varepsilon(\mu) - \Phi_\varepsilon(u_{\theta,\varepsilon} + v_{\theta,\varepsilon}) - \Phi_\varepsilon(u_{\theta,\varepsilon} - v_{\theta,\varepsilon}), 1), \end{aligned}$$

where we have noted (2.14), (2.37a) and the convexity of Φ_ε. Hence it follows on choosing $\mu = 0$ and 1, and noting (2.1a,b), (2.2), (2.37a) and the assumptions on u^0 that

$$\begin{aligned}\delta\,|\Omega|\,|\lambda_{\theta,\varepsilon}| &\leq C + C\,b_{\min}^{-\frac{1}{2}}\,|\,[b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})]^{\frac{1}{2}} \nabla \mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial u_{\theta,\varepsilon}}{\partial t}|_0 \\ &\quad + \ C\,b_{\min}^{-\frac{1}{2}}\,|\,[b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})]^{\frac{1}{2}} \mathcal{M}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \tfrac{\partial v_{\theta,\varepsilon}}{\partial t}|_0. \end{aligned} \qquad (2.45)$$

Squaring the above, integrating over $t \in (0,T)$ and noting (2.41) yields that

$$\|\lambda_{\theta,\varepsilon}\|_{L^2(0,T)} \leq C(T)\, b_{\min}^{-\frac{1}{2}}. \tag{2.46}$$

Combining (2.46), (2.42) and (2.44), and recalling (2.35a) and (2.41) yields the desired result (2.38). Finally (2.39) follows from (2.11c), (2.35a,b), (2.37a), (2.38) and standard elliptic regularity theory.

We now consider the question of uniqueness of the solution to $(\mathrm{P}_{\theta,\varepsilon})$. Assuming that the regularized version of (2.36) has two solutions $\{u^i_{\theta,\varepsilon}, v^i_{\theta,\varepsilon}\}$, $i = 1, 2$, with corresponding $\{w^i_{\theta,\varepsilon}, z^i_{\theta,\varepsilon}, \lambda^i_{\theta,\varepsilon}\}$ defined by (2.35a,b), it follows that for *a.e.* $t \in (0,T)$ $\overline{u}_{\theta,\varepsilon}(\cdot,t) := (u^1_{\theta,\varepsilon} - u^2_{\theta,\varepsilon})(\cdot,t) \in \Xi$, $\overline{v}_{\theta,\varepsilon} := v^1_{\theta,\varepsilon} - v^2_{\theta,\varepsilon}$ and $\overline{\lambda}_{\theta,\varepsilon} := \lambda^1_{\theta,\varepsilon} - \lambda^2_{\theta,\varepsilon}$ satisfy

$$\begin{aligned}
&\gamma\,|\overline{u}_{\theta,\varepsilon} \pm \overline{v}_{\theta,\varepsilon}|_1^2 + 2\,\theta\,(\phi_\varepsilon(u^1_{\theta,\varepsilon} \pm v^1_{\theta,\varepsilon}) - \phi_\varepsilon(u^2_{\theta,\varepsilon} \pm v^2_{\theta,\varepsilon}), \overline{u}_{\theta,\varepsilon} \pm \overline{v}_{\theta,\varepsilon}) \\
&\quad + ((\mathcal{G}_1 \tfrac{\partial u^1_{\theta,\varepsilon}}{\partial t} - \mathcal{G}_2 \tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}) \pm \rho\,(\mathcal{M}_1 \tfrac{\partial v^1_{\theta,\varepsilon}}{\partial t} - \mathcal{M}_2 \tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}) - \overline{\lambda}_{\theta,\varepsilon}, \overline{u}_{\theta,\varepsilon} \pm \overline{v}_{\theta,\varepsilon}) \\
&\qquad = (\alpha\,\overline{u}_{\theta,\varepsilon} \pm \beta\,\overline{v}_{\theta,\varepsilon}, \overline{u}_{\theta,\varepsilon} \pm \overline{v}_{\theta,\varepsilon}),
\end{aligned} \tag{2.47}$$

where for notational convenience $b_i \equiv b(u^i_{\theta,\varepsilon}, v^i_{\theta,\varepsilon})$, $\mathcal{G}_i \equiv \mathcal{G}_{u^i_{\theta,\varepsilon}, v^i_{\theta,\varepsilon}}$ and $\mathcal{M}_i \equiv \mathcal{M}_{u^i_{\theta,\varepsilon}, v^i_{\theta,\varepsilon}}$, $i = 1, 2$. Adding together the $\pm$ versions of (2.47) and noting the monotonicity of ϕ_ε yields for *a.e.* $t \in (0,T)$

$$\begin{aligned}
&\gamma\,[\,|\overline{u}_{\theta,\varepsilon}|_1^2 + |\overline{v}_{\theta,\varepsilon}|_1^2\,] + ((\mathcal{G}_1 \tfrac{\partial u^1_{\theta,\varepsilon}}{\partial t} - \mathcal{G}_2 \tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}), \overline{u}_{\theta,\varepsilon}) \\
&\quad + \rho\,((\mathcal{M}_1 \tfrac{\partial v^1_{\theta,\varepsilon}}{\partial t} - \mathcal{M}_2 \tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}), \overline{v}_{\theta,\varepsilon}) \leq \alpha\,|\overline{u}_{\theta,\varepsilon}|_0^2 + \beta\,|\overline{v}_{\theta,\varepsilon}|_0^2.
\end{aligned} \tag{2.48}$$

If b satisfies (1.6a,b), on noting (2.23), (2.31), (2.24), (2.20), (2.25), (2.29), (2.32), (2.27a), (2.33a), (2.30), (2.19), (2.16), (2.15) and applying Hölder's and Young's inequalities it follows that for any $\omega \in [0,2]$ and for *a.e.* $t \in (0,T)$

$$\begin{aligned}
&\gamma\,[\,|\overline{u}_{\theta,\varepsilon}|_1^2 + |\overline{v}_{\theta,\varepsilon}|_1^2\,] + \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{G}_1\overline{u}_{\theta,\varepsilon}, \overline{u}_{\theta,\varepsilon}) + \tfrac{\rho}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{M}_1\overline{v}_{\theta,\varepsilon}, \overline{v}_{\theta,\varepsilon}) \\
&\quad \leq \alpha\,|\overline{u}_{\theta,\varepsilon}|_0^2 + \beta\,|\overline{v}_{\theta,\varepsilon}|_0^2 - \tfrac{1}{2}(\tfrac{\partial}{\partial t}b_1, |\nabla\mathcal{G}_1\overline{u}_{\theta,\varepsilon}|^2) - \tfrac{\rho}{2}(\tfrac{\partial}{\partial t}b_1, |\mathcal{M}_1\overline{v}_{\theta,\varepsilon}|^2) \\
&\qquad - \;((\mathcal{G}_1 - \mathcal{G}_2)\tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}, \overline{u}_{\theta,\varepsilon}) - \rho\,((\mathcal{M}_1 - \mathcal{M}_2)\tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}, \overline{v}_{\theta,\varepsilon}) \\
&\quad \leq \alpha\,|\overline{u}_{\theta,\varepsilon}|_0^2 + \beta\,|\overline{v}_{\theta,\varepsilon}|_0^2 + C_b\left(\,|\overline{u}_{\theta,\varepsilon}|_0^{1-\frac{d}{4}}\,\|\overline{u}_{\theta,\varepsilon}\|_1^{\frac{d}{4}} + |\overline{v}_{\theta,\varepsilon}|_0^{1-\frac{d}{4}}\,\|\overline{v}_{\theta,\varepsilon}\|_1^{\frac{d}{4}}\,\right) \\
&\qquad \times \left[\,|\mathcal{G}_1\overline{u}_{\theta,\varepsilon}|_1^{1-\frac{d}{4}}\,\|\mathcal{G}_1\overline{u}_{\theta,\varepsilon}\|_2^{\frac{d}{4}}\,\|\tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}\|_{-1} + |\mathcal{M}_1\overline{v}_{\theta,\varepsilon}|_0^{1-\frac{d}{4}}\,\|\mathcal{M}_1\overline{v}_{\theta,\varepsilon}\|_1^{\frac{d}{4}}\,|\tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}|_0\right] \\
&\qquad + C\left(\,|\tfrac{\partial u^1_{\theta,\varepsilon}}{\partial t}|_0 + |\tfrac{\partial v^1_{\theta,\varepsilon}}{\partial t}|_0\right)\left[\,|\mathcal{G}_1\overline{u}_{\theta,\varepsilon}|_1^{2-\frac{d}{2}}\,\|\mathcal{G}_1\overline{u}_{\theta,\varepsilon}\|_2^{\frac{d}{2}}\right. \\
&\qquad \left. + |\mathcal{M}_1\overline{v}_{\theta,\varepsilon}|_0^{2-\frac{d}{2}}\,\|\mathcal{M}_1\overline{v}_{\theta,\varepsilon}\|_1^{\frac{d}{2}}\,\right] \\
&\quad \leq C_b\left(1 + \|\tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}\|_{-1}^{\omega} + |\tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}|_0^{\omega}\right)\left[\,|\overline{u}_{\theta,\varepsilon}|_0^{2-\frac{d}{2}}\,\|\overline{u}_{\theta,\varepsilon}\|_1^{\frac{d}{2}} + |\overline{v}_{\theta,\varepsilon}|_0^{2-\frac{d}{2}}\,\|\overline{v}_{\theta,\varepsilon}\|_1^{\frac{d}{2}}\,\right] \\
&\qquad + C_b\left(\,|\tfrac{\partial u^1_{\theta,\varepsilon}}{\partial t}|_0 + |\tfrac{\partial v^1_{\theta,\varepsilon}}{\partial t}|_0 + \|\tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}\|_{-1}^{2-\omega} + |\tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}|_0^{2-\omega}\,\right) \\
&\qquad \times \left[\,|\mathcal{G}_1\overline{u}_{\theta,\varepsilon}|_1^{2-\frac{d}{2}}\,\|\mathcal{G}_1\overline{u}_{\theta,\varepsilon}\|_2^{\frac{d}{2}} + |\mathcal{M}_1\overline{v}_{\theta,\varepsilon}|_0^{2-\frac{d}{2}}\,\|\mathcal{M}_1\overline{v}_{\theta,\varepsilon}\|_1^{\frac{d}{2}}\,\right]
\end{aligned}$$

$$\le C_b \left(1 + \|\tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}\|^{\omega}_{-1} + |\tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}|^{\omega}_{0}\right) \left[|\overline{u}_{\theta,\varepsilon}|_0^{2-\frac{d}{2}} \|\overline{u}_{\theta,\varepsilon}\|_1^{\frac{d}{2}} + |\overline{v}_{\theta,\varepsilon}|_0^{2-\frac{d}{2}} \|\overline{v}_{\theta,\varepsilon}\|_1^{\frac{d}{2}} \right]$$
$$+ C_b \left(|\tfrac{\partial u^1_{\theta,\varepsilon}}{\partial t}|_0 + |\tfrac{\partial v^1_{\theta,\varepsilon}}{\partial t}|_0 + \|\tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}\|^{2-\omega}_{-1} + |\tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}|^{2-\omega}_{0} \right)$$
$$\times \left\{ \left[|u^1_{\theta,\varepsilon}|_1^{\frac{\mu_1 d}{2}} \|u^1_{\theta,\varepsilon}\|_2^{\frac{\mu_2 d}{2}} + |v^1_{\theta,\varepsilon}|_1^{\frac{\mu_1 d}{2}} \|v^1_{\theta,\varepsilon}\|_2^{\frac{\mu_2 d}{2}} \right] \left(|\mathcal{G}_1 \overline{u}_{\theta,\varepsilon}|_1^2 + |\mathcal{M}_1 \overline{v}_{\theta,\varepsilon}|_0^2 \right) \right.$$
$$\left. + \left[|\mathcal{G}_1 \overline{u}_{\theta,\varepsilon}|_1^{2-\frac{d}{2}} |\overline{u}_{\theta,\varepsilon}|_0^{\frac{d}{2}} + |\mathcal{M}_1 \overline{v}_{\theta,\varepsilon}|_0^{2-\frac{d}{2}} |\overline{v}_{\theta,\varepsilon}|_1^{\frac{d}{2}} \right] \right\}$$
$$\le \tfrac{\gamma}{2} \left[|\overline{u}_{\theta,\varepsilon}|_1^2 + |\overline{v}_{\theta,\varepsilon}|_1^2 \right] + C_b \Big(1 + \|\tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}\|_{-1}^{\frac{8\omega}{4-d}} + |\tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}|_0^{\frac{8\omega}{4-d}} + |\tfrac{\partial u^1_{\theta,\varepsilon}}{\partial t}|_0^{\frac{4}{4-d}}$$
$$+ |\tfrac{\partial v^1_{\theta,\varepsilon}}{\partial t}|_0^{\frac{4}{4-d}} + \|\tfrac{\partial u^2_{\theta,\varepsilon}}{\partial t}\|_{-1}^{\frac{4(2-\omega)}{4-d}} + |\tfrac{\partial v^2_{\theta,\varepsilon}}{\partial t}|_0^{\frac{4(2-\omega)}{4-d}} + |u^1_{\theta,\varepsilon}|_1^{2\mu_1} \|u^1_{\theta,\varepsilon}\|_2^{2\mu_2}$$
$$+ |v^1_{\theta,\varepsilon}|_1^{2\mu_1} \|v^1_{\theta,\varepsilon}\|_2^{2\mu_2} \Big) \left[(\mathcal{G}_1 \overline{u}_{\theta,\varepsilon}, \overline{u}_{\theta,\varepsilon}) + (\mathcal{M}_1 \overline{v}_{\theta,\varepsilon}, \overline{v}_{\theta,\varepsilon}) \right], \tag{2.49}$$

where μ_i are defined as in (2.27a). Uniqueness then follows from choosing $\omega = \frac{2}{3}$, noting (2.37a), the assumptions (2.40), a Gronwall inequality, (2.14) and (2.35a,b). Clearly for constant $b > 0$ the uniqueness argument in (2.49) above is trivial and the assumptions (2.40) are not required. □

We note that the integral assumption on the initial data, in Theorem 2.1 above, only excludes the trivial (physically uninteresting) case of $u^0 \equiv 0$ or 1, when only one component of the alloy is present.

If $\frac{\partial q_1}{\partial t}(\cdot,t) \in \Xi$ and $\frac{\partial q_2}{\partial t}(\cdot,t) \in H^1(\Omega)$ for a.e. $t \in (0,T)$ and b satisfies (1.6a,b), then alternatively to (2.24), we have from (2.16) and (2.17) that for all $\eta \in L^{2(1+\zeta)}(\Omega)$

$$|(\tfrac{\partial}{\partial t} b(q_1,q_2), \eta^2)| \le |\Omega| \, |⨍ \tfrac{\partial q_2}{\partial t}| \, |⨍ [\tfrac{\partial b}{\partial q_2} \eta^2]| + \sum_{i=1}^{2} |(\tfrac{\partial q_i}{\partial t}, (I - ⨍)[\tfrac{\partial b}{\partial q_i} \eta^2])|$$
$$\le C \left[|\tfrac{\partial q_1}{\partial t}|_1 + \|\tfrac{\partial q_2}{\partial t}\|_1 \right] |\eta|^2_{0,2(1+\zeta)}$$
$$\le \tfrac{\gamma}{8} \left[|\tfrac{\partial q_1}{\partial t}|_1^2 + \|\tfrac{\partial q_2}{\partial t}\|_1^2 \right] + C |\eta|^4_{0,2(1+\zeta)}, \tag{2.50}$$

where $\zeta = 0$, any $\zeta \in (0, \frac{1}{5})$, $\zeta = \frac{1}{5}$, for $d = 1, 2, 3$ respectively.

Corollary. 2.1. *Let $u^0 \in H^3(\Omega)$, $v^0 \in H^2(\Omega)$, $\frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} = 0$ on $\partial\Omega$ and $\delta \in (0, \frac{1}{2})$ be such that $\|u^0 \pm v^0 - \frac{1}{2}\|_{0,\infty} \le \frac{1}{2} - \delta$. Let $d \le 3$ with either Ω being a convex polyhedron or $\partial\Omega \in C^{1,1}$. Let b satisfy (1.6a,b). Then for all $\theta \le \theta_{\max}$ and for all $\varepsilon \le \varepsilon_1$, where $\varepsilon_1 := \min\{\varepsilon_0, \delta\}$, solutions $\{u_{\theta,\varepsilon}, v_{\theta,\varepsilon}, w_{\theta,\varepsilon}, z_{\theta,\varepsilon}\}$ of $(P_{\theta,\varepsilon})$ are such that the following stability bounds hold independently of ε and θ*

$$\|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}\|_{L^\infty(0,T;(H^1(\Omega))')} + \|u_{\theta,\varepsilon}\|_{L^\infty(0,T;H^2(\Omega))}$$
$$+ \|\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}\|_{L^\infty(0,T;L^2(\Omega))} + \|v_{\theta,\varepsilon}\|_{L^\infty(0,T;H^2(\Omega))}$$
$$+ \theta \|\phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})\|_{L^\infty(0,T;L^2(\Omega))} + \|w_{\theta,\varepsilon}\|_{L^\infty(0,T;H^1(\Omega))} \le C_b(T) \tag{2.51}$$

for any $T > 0$ if $d \le 2$, and some $T > 0$ if $d = 3$. Hence the solution $\{u_{\theta,\varepsilon}, v_{\theta,\varepsilon}, w_{\theta,\varepsilon}, z_{\theta,\varepsilon}\}$ of $(P_{\theta,\varepsilon})$ is unique over Ω_T.

PROOF. Differentiating the regularized version of (2.11c) with respect to t and setting $\eta \equiv \frac{\partial}{\partial t}(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})$ and noting that $\phi'_\varepsilon \geq 0$, yields for *a.e.* $t \in (0,T)$

$$\gamma\,|\tfrac{\partial}{\partial t}(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})|_1^2 - (\alpha\,\tfrac{\partial u_{\theta,\varepsilon}}{\partial t} \pm \beta\,\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}, \tfrac{\partial}{\partial t}(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})) \\ \leq \langle \tfrac{\partial w_{\theta,\varepsilon}}{\partial t} \pm \tfrac{\partial z_{\theta,\varepsilon}}{\partial t}, \tfrac{\partial}{\partial t}(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})\rangle. \qquad (2.52)$$

Once again this differentiation and these choices of test function can be justified in the standard way by using a Galerkin approximation and then passing to the limit. Adding together the $\pm$ versions of (2.52), noting (2.11a,b) and (2.50) yields for *a.e.* $t \in (0,T)$ that

$$\begin{aligned}\gamma\,[\,|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}|_1^2 + |\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}|_1^2\,] &+ \tfrac{1}{2}\,\tfrac{\mathrm{d}}{\mathrm{d}t}(b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon}), |\nabla w_{\theta,\varepsilon}|^2 + \rho^{-1} z_{\theta,\varepsilon}^2) \\ &\leq \alpha\,|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}|_0^2 + \beta\,|\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}|_0^2 - \tfrac{1}{2}\,(\tfrac{\partial}{\partial t} b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon}), |\nabla w_{\theta,\varepsilon}|^2 + \rho^{-1} z_{\theta,\varepsilon}^2) \\ &\leq \tfrac{\gamma}{8}\,[\,|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}|_1^2 + |\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}|_1^2\,] + C\,[\,|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}|_0^2 + |\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}|_0^2\,] \\ &\quad + C\,[\,|w_{\theta,\varepsilon}|_{1,2(1+\zeta)}^4 + |z_{\theta,\varepsilon}|_{0,2(1+\zeta)}^4\,], \qquad (2.53)\end{aligned}$$

where $\zeta = 0$, any $\zeta \in (0, \frac{1}{5})$, $\zeta = \frac{1}{5}$, for $d = 1, 2, 3$ respectively. Next we note from (2.10), (2.35a), (2.27a), (2.37a) and a Young's inequality that

$$\begin{aligned}|w_{\theta,\varepsilon}|_{1,2(1+\zeta)}^4 &\leq C\,|w_{\theta,\varepsilon}|_1^{4(1-\omega)}\,\|\mathcal{G}_{u_{\theta,\varepsilon}, v_{\theta,\varepsilon}}\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}\|_2^{4\omega} \\ &\leq C_b\,|w_{\theta,\varepsilon}|_1^4\,\left(\|u_{\theta,\varepsilon}\|_2^{4\omega\mu_2} + \|v_{\theta,\varepsilon}\|_2^{4\omega\mu_2}\right) \\ &\quad + C_b\,\left[\,|w_{\theta,\varepsilon}|_1^{\frac{4(1-\omega)}{1-2\omega}} + |\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}|_0^2\right], \qquad (2.54)\end{aligned}$$

where μ_i are defined as in (2.27a) and $\omega = \frac{d\,\zeta}{2(1+\zeta)}$. Similarly to (2.54), we have from (2.10), (2.35a), (2.33a), (2.37a) and a Young's inequality that for any $\kappa > 0$

$$\begin{aligned}|z_{\theta,\varepsilon}|_{0,2(1+\zeta)}^4 &\leq C_b(\kappa^{-1})\,\left[\,|z_{\theta,\varepsilon}|_0^4\,\left(\|u_{\theta,\varepsilon}\|_2^{4\omega\mu_2} + \|v_{\theta,\varepsilon}\|_2^{4\omega\mu_2}\right) + |z_{\theta,\varepsilon}|_0^{\frac{4(1-\omega)}{1-2\omega}}\right] \\ &\quad + \kappa\,|\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}|_1^2. \qquad (2.55)\end{aligned}$$

It follows from standard elliptic regularity, (2.36), (2.37a), (2.35a,b), (2.43), (2.45) and (2.14) that

$$\begin{aligned}\|u_{\theta,\varepsilon}\|_2 + \|v_{\theta,\varepsilon}\|_2 &\leq C\,[\,1 + \theta\,|\phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})|_0 + |w_{\theta,\varepsilon}|_0 + |z_{\theta,\varepsilon}|_0\,] \\ &\leq C_b\,[\,1 + |w_{\theta,\varepsilon}|_1 + |z_{\theta,\varepsilon}|_0\,]\,. \qquad (2.56)\end{aligned}$$

Combining (2.53), (2.54), (2.55), (2.56) and noting (2.20) and (2.35a,b) yields that for *a.e.* $t \in (0,T)$

$$\begin{aligned}\gamma\,[\,|\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}|_1^2 + |\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}|_1^2\,] &+ \tfrac{\mathrm{d}}{\mathrm{d}t}(b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon}), |\nabla w_{\theta,\varepsilon}|^2 + \rho^{-1} z_{\theta,\varepsilon}^2) \\ &\leq C_b\,[\,1 + |w_{\theta,\varepsilon}|_1^{4(1+\mu)} + |z_{\theta,\varepsilon}|_0^{4(1+\mu)}\,]\end{aligned}$$

$$\leq C_b \left[1 + (b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon}), |\nabla w_{\theta,\varepsilon}|^2 + \rho^{-1} z_{\theta,\varepsilon}^2)^{2(1+\mu)} \right], \tag{2.57}$$

where $\mu = \frac{5}{2}\zeta$.

We set $B(t) := \max\{(b(u_{\theta,\varepsilon}(\cdot,t), v_{\theta,\varepsilon}(\cdot,t)), |\nabla w_{\theta,\varepsilon}(\cdot,t)|^2 + \rho^{-1}|z_{\theta,\varepsilon}(\cdot,t)|^2), 1\}$ for *a.e.* $t \in (0,T)$. It follows from (2.38) that

$$\int_0^T B(t)\,\mathrm{d}t \leq C_b(T). \tag{2.58}$$

From (2.57) and the above notation we have for *a.e.* $t \in (0,T)$ that

$$\tfrac{\mathrm{d}}{\mathrm{d}t} B \leq C_b\, B^{2(1+\mu)}. \tag{2.59}$$

Via a Galerkin approximation in the standard way, one can show from (2.36) and (2.35a) that

$$\begin{aligned} B(0) &\leq C\,\left[\, 1 + |w_{\theta,\varepsilon}(\cdot,0)|_1^2 + |z_{\theta,\varepsilon}(\cdot,0)|_0^2 \,\right] \\ &\leq C\,\left[\, 1 + \|u^0\|_3^2 + \|v^0\|_2^2 + \|\phi_\varepsilon(u^0 \pm v^0)\|_1^2 \,\right] \\ &\leq C\,\left[\, 1 + \|u^0\|_3^2 + \|v^0\|_2^2 \,\right] \;\leq\; C, \end{aligned} \tag{2.60}$$

provided $u^0 \in H^3(\Omega)$, $v^0 \in H^2(\Omega)$, $\frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} = 0$ on $\partial\Omega$ and $\varepsilon \leq \varepsilon_1$. For $d = 1$, i.e. $\mu = 0$, it follows from (2.59) and (2.58) that for $t \in (0,T)$

$$B(t) \leq e^{C_b \int_0^t B(s)\,\mathrm{d}s}\, B(0) \leq C_b(T)\, B(0). \tag{2.61}$$

For $d = 2$, i.e. for any $\mu \in (0, \frac{1}{2})$, it follows from (2.59) that for *a.e.* $t \in (0,T)$

$$-\tfrac{1}{2\mu}\tfrac{\mathrm{d}}{\mathrm{d}t} B^{-2\mu} \leq C_b\, B. \tag{2.62}$$

Hence we have from (2.62) and (2.58) that

$$\begin{aligned} B(t) &\leq [\, 1 - 2\mu\, C_b\, [B(0)]^{2\mu} \int_0^t B(s)\,\mathrm{d}s\,]^{-\frac{1}{2\mu}}\, B(0) \\ &\leq [1 + 4\mu\, C_b\, [B(0)]^{2\mu} \int_0^t B(s)\,\mathrm{d}s\,]^{\frac{1}{2\mu}}\, B(0) \\ &\leq e^{C_b\,[B(0)]^{2\mu} \int_0^t B(s)\,\mathrm{d}s}\, B(0) \;\leq\; C_b(T, B(0)) \qquad t \in (0,T); \end{aligned} \tag{2.63}$$

provided $\mu \in (0, \frac{1}{2})$ is chosen sufficiently small so that

$$4\mu\, C_b\, [B(0)]^{2\mu} \int_0^T B(s)\,\mathrm{d}s \leq 1. \tag{2.64}$$

For $d = 3$, i.e. $\mu = \frac{1}{2}$, it follows from (2.59) that for *a.e.* $t \in (0,T)$

$$-\tfrac{1}{1+2\mu}\tfrac{\mathrm{d}}{\mathrm{d}t} B^{-(1+2\mu)} \leq C_b. \tag{2.65}$$

Hence we have from (2.65) that

$$\begin{aligned} B(t) &\leq [\,1-(1+2\mu)\,C_b\,[B(0)]^{1+2\mu}\,t\,]^{-\frac{1}{(1+2\mu)}}\,B(0) \\ &\leq [\,1+2(1+2\mu)\,C_b\,[B(0)]^{1+2\mu}\,t\,]^{\frac{1}{1+2\mu}}\,B(0) \\ &\leq e^{C_b\,[B(0)]^{1+2\mu}\,t}\,B(0) \leq C_b(T,B(0)) \qquad\qquad t\in(0,T); \qquad (2.66) \end{aligned}$$

provided T is such that $2(1+2\mu)\,C_b\,[B(0)]^{1+2\mu}\,T \leq 1$.

The second, fifth and eighth bounds in (2.51) then follow from (2.61), (2.63), (2.66), (2.60), (2.35a,b), (2.45), (2.19), (2.15), (2.30) and (2.14). The first and fourth bounds in (2.51) then follow from the fifth and eighth, (2.35a), (2.30), (2.57) and (2.60). Similarly, the third, sixth and seventh bounds in (2.51) follow from the fifth and eighth, (2.35a) and (2.30). Finally, uniqueness of a solution to $(\mathrm{P}_{\theta,\varepsilon})$ over Ω_T follows from the bounds (2.51) and the assumptions (2.40) on noting (2.16). □

THEOREM. 2.2. *If $d \leq 3$, b satisfies (1.6a) with $b_{\min} > 0$ and the assumptions on u^0 and v^0 of Theorem 2.1 hold, then for all $\theta \leq \theta_{\max}$ there exists $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ solving (P_θ) such that the following stability bounds hold independently of θ*

$$\begin{aligned} \|u_\theta\|_{L^\infty(0,T;H^1(\Omega))} + \|\tfrac{\partial u_\theta}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} \\ + \|v_\theta\|_{L^\infty(0,T;H^1(\Omega))} + \|\tfrac{\partial v_\theta}{\partial t}\|_{L^2(\Omega_T)} \leq C, \qquad (2.67\mathrm{a}) \end{aligned}$$

$$\begin{aligned} \|\lambda_\theta\|_{L^2(0,t)} + \|w_\theta\|_{L^2(0,T;H^1(\Omega))} + \|z_\theta\|_{L^2(\Omega_T)} \\ + \theta\,\|\phi(u_\theta \pm v_\theta)\|_{L^2(\Omega_T)} \leq C_b(T); \qquad (2.67\mathrm{b}) \end{aligned}$$

where the latter implies that $(u_\theta \pm v_\theta)(x,t) \in (0,1)$ for a.e. $(x,t) \in \Omega_T$. In addition if Ω is convex polyhedron or $\partial\Omega \in C^{1,1}$, then

$$\|u_\theta\|_{L^2(0,T;H^2(\Omega))} + \|v_\theta\|_{L^2(0,T;H^2(\Omega))} \leq C_b(T). \qquad (2.68)$$

Furthermore, if u^0, v^0 and b satisfy the assumptions of Corollary 2.1 then solutions $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ of (P_θ) are such that the following stability bounds hold independently of θ

$$\begin{aligned} &\|\tfrac{\partial u_\theta}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial u_\theta}{\partial t}\|_{L^\infty(0,T;(H^1(\Omega))')} + \|u_\theta\|_{L^\infty(0,T;H^2(\Omega))} \\ &\quad + \|\tfrac{\partial v_\theta}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial v_\theta}{\partial t}\|_{L^\infty(0,T;L^2(\Omega))} + \|v_\theta\|_{L^\infty(0,T;H^2(\Omega))} \\ &\quad + \theta\,\|\phi(u_\theta \pm v_\theta)\|_{L^\infty(0,T;L^2(\Omega))} + \|w_\theta\|_{L^\infty(0,T;H^1(\Omega))} \leq C_b(T) \qquad (2.69) \end{aligned}$$

for any $T > 0$ if $d \leq 2$, and some $T > 0$ if $d = 3$. Moreover, the solution $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ of (P_θ) is unique over Ω_T and we have for all $\varepsilon \leq \varepsilon_1$ that

$$\begin{aligned} &\|u_\theta - u_{\theta,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} + \|u_\theta - u_{\theta,\varepsilon}\|^2_{L^\infty(0,T;(H^1(\Omega))')} \\ &\quad + \|v_\theta - v_{\theta,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} + \|v_\theta - v_{\theta,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_b(T)\,\theta^{-1}\,\varepsilon. \qquad (2.70) \end{aligned}$$

Note that if $b > 0$ is constant the solution $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ of (P_θ) is unique over Ω_T for any $T > 0$ and the bounds (2.70) hold under the minimal assumptions on u^0 and v^0 of Theorem 2.1.

Proof. As the bounds (2.37a) and (2.38) are independent of ε, it follows that there exist $u_\theta \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$, $v_\theta \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, $w_\theta \in L^2(0,T;H^1(\Omega))$, $z_\theta \in L^2(\Omega_T)$, $\phi^\star_\pm \in L^2(\Omega_T)$ and a subsequence $\{u_{\theta,\varepsilon'}, v_{\theta,\varepsilon'}, w_{\theta,\varepsilon'}, z_{\theta,\varepsilon'}\}$ such that as $\varepsilon' \to 0$

$$\begin{aligned}
&u_{\theta,\varepsilon'} \to u_\theta \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ weak–star and in } H^1(0,T;(H^1(\Omega))') \text{ weakly},\\
&v_{\theta,\varepsilon'} \to v_\theta \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ weak–star and in } H^1(0,T;L^2(\Omega)) \text{ weakly},\\
&w_{\theta,\varepsilon'} \to w_\theta \text{ in } L^2(0,T;H^1(\Omega)) \text{ weakly} \quad\text{and}\quad z_{\theta,\varepsilon'} \to z_\theta \text{ in } L^2(\Omega_T) \text{ weakly},\\
&\phi_{\varepsilon'}(u_{\theta,\varepsilon'} \pm v_{\theta,\varepsilon'}) \to \phi^\star_\pm \text{ in } L^2(\Omega_T) \text{ weakly}. \qquad (2.71)
\end{aligned}$$

The first two lines of (2.71) imply that $u_{\theta,\varepsilon'} \to u_\theta$ and $v_{\theta,\varepsilon'} \to v_\theta$ in $L^2(\Omega_T)$ strongly and *a.e.* as $\varepsilon' \to 0$, see [18]. Hence adapting the argument in the proof of Theorem 2.1 in [2], it follows that $\phi^\star_\pm \equiv \phi(u_\theta \pm v_\theta)$. Therefore taking the limit $\varepsilon' \to 0$ in the regularized version of (2.11c) yields that $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ solves the corresponding non-regularized version of (2.11c). It follows from (1.6a), the *a.e.* convergence of $u_{\theta,\varepsilon'} \to u_\theta$ and $v_{\theta,\varepsilon'} \to v_\theta$ that $b(u_{\theta,\varepsilon'}, v_{\theta,\varepsilon'}) \to b(u_\theta, v_\theta)$ in $L^p(\Omega_T)$, any $p \in [1,\infty)$, as $\varepsilon' \to 0$. Noting this, (2.71) and a density argument we have for *a.e.* $t \in (0,T)$ that as $\varepsilon' \to 0$

$$(b(u_{\theta,\varepsilon'}, v_{\theta,\varepsilon'})\nabla w_{\theta,\varepsilon'}, \nabla\eta) \to (b(u_\theta, v_\theta)\nabla w_\theta, \nabla\eta) \qquad \forall\ \eta \in H^1(\Omega) \quad (2.72a)$$

$$\text{and} \qquad (b(u_{\theta,\varepsilon'}, v_{\theta,\varepsilon'})\, z_{\theta,\varepsilon'}, \eta) \to (b(u_\theta, v_\theta)\, z_\theta, \eta) \qquad \forall\ \eta \in L^2(\Omega). \quad (2.72b)$$

Therefore taking the limit $\varepsilon' \to 0$ in the regularized versions of (2.11a,b) yields, on noting (2.72a,b) and (2.71), that $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ solves (P_θ). Hence we have existence of a solution $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ of (P_θ), satisfying (2.67a,b) on noting (2.35a,b). The regularity results (2.68) and (2.69) follow from the ε independent bounds (2.39) and (2.51). Uniqueness of a solution to (P_θ) over Ω_T then follows as for ($\mathrm{P}_{\theta,\varepsilon}$), see (2.47)–(2.49).

We now prove an error bound between the unique solutions $\{u_\theta, v_\theta\}$ and $\{u_{\theta,\varepsilon}, v_{\theta,\varepsilon}\}$ of problems (P_θ) and ($\mathrm{P}_{\theta,\varepsilon}$). Let $e_u := u_\theta - u_{\theta,\varepsilon}$ and $e_v := v_\theta - v_{\theta,\varepsilon}$. Subtraction of the regularized form of (2.36) from their non-regularized form with $\eta \equiv e_u \pm e_v$, then adding together these $\pm$ versions and noting (2.23) and (2.31) yields for *a.e.* $t \in (0,T)$ that

$$\begin{aligned}
&\gamma\,[\,|e_u|_1^2 + |e_v|_1^2\,] + \theta\,(\phi(u_\theta + v_\theta) - \phi_\varepsilon(u_{\theta,\varepsilon} + v_{\theta,\varepsilon}), e_u + e_v)\\
&\quad + \theta\,(\phi(u_\theta - v_\theta) - \phi_\varepsilon(u_{\theta,\varepsilon} - v_{\theta,\varepsilon}), e_u - e_v) + \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{G}_\theta\, e_u, e_u) + \tfrac{\rho}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{M}_\theta\, e_v, e_v)\\
&\qquad \le \alpha\,|e_u|_0^2 + \beta\,|e_v|_0^2 - \tfrac{1}{2}(\tfrac{\partial}{\partial t}b_\theta, |\nabla\mathcal{G}_\theta\, e_u|^2) - \tfrac{\rho}{2}(\tfrac{\partial}{\partial t}b_\theta, |\mathcal{M}_\theta\, e_v|^2)\\
&\qquad\quad - ((\mathcal{G}_\theta - \mathcal{G}_{\theta,\varepsilon})\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}, e_u) - \rho\,((\mathcal{M}_\theta - \mathcal{M}_{\theta,\varepsilon})\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}, e_v), \qquad (2.73)
\end{aligned}$$

where for notational convenience $b_{\theta(,\varepsilon)} \equiv b(u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)})$, $\mathcal{G}_{\theta(,\varepsilon)} \equiv \mathcal{G}_{u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)}}$, $\mathcal{M}_{\theta(,\varepsilon)} \equiv \mathcal{M}_{u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)}}$. From (2.5a) and (2.6) it follows that for *a.e.* $t \in (0,T)$

$$(\phi_\varepsilon(u_\theta \pm v_\theta) - \phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon}), e_u \pm e_v) \ge \varepsilon^{-1}\int_{\Omega^\pm_\varepsilon(t)} (e_u \pm e_v)^2\,\mathrm{d}x, \qquad (2.74)$$

where

$$\Omega_\varepsilon^\pm(t) := \{x \in \Omega : (u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})(x,t) \le (u_\theta \pm v_\theta)(x,t) \le \varepsilon \quad \text{or} \quad 1-\varepsilon \le (u_\theta \pm v_\theta)(x,t) \le (u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon})(x,t)\}.$$

Next we note from (2.4) that

$$0 < r \le \varepsilon \text{ and } r \le s \Longrightarrow (\phi_\varepsilon(r) - \phi(r))(r-s) \le 0, \tag{2.75a}$$

$$1-\varepsilon \le r < 1 \text{ and } s \le r \Longrightarrow (\phi_\varepsilon(r) - \phi(r))(r-s) \le 0. \tag{2.75b}$$

Hence it follows from (2.3), (2.75a,b) and a Young's inequality that for *a.e* $t \in (0,T)$

$$\begin{aligned}(\phi_\varepsilon(u_\theta \pm v_\theta) &- \phi(u_\theta \pm v_\theta), e_u \pm e_v) \\ &\le \int_{\Omega_\varepsilon^\pm(t)} (\phi_\varepsilon(u_\theta \pm v_\theta) - \phi(u_\theta \pm v_\theta))(e_u \pm e_v)\,\mathrm{d}x \\ &\le -\int_{\Omega_\varepsilon^\pm(t)} \phi(u_\theta \pm v_\theta)(e_u \pm e_v)\,\mathrm{d}x \\ &\le \tfrac{1}{2}\varepsilon^{-1}\int_{\Omega_\varepsilon^\pm(t)} (e_u \pm e_v)^2\,\mathrm{d}x + \tfrac{1}{2}\varepsilon\int_{\Omega_\varepsilon^\pm(t)} [\,\phi(u_\theta \pm v_\theta)\,]^2\,\mathrm{d}x. \end{aligned} \tag{2.76}$$

Using the bounds (2.74) and (2.76) in (2.73), it follows for *a.e.* $t \in (0,T)$ that

$$\begin{aligned}\gamma\,[\,|e_u|_1^2 + |e_v|_1^2\,] &+ \tfrac{\theta}{2\varepsilon}\int_{\Omega_\varepsilon^+(t)} (e_u + e_v)^2\,\mathrm{d}x + \tfrac{\theta}{2\varepsilon}\int_{\Omega_\varepsilon^-(t)} (e_u - e_v)^2\,\mathrm{d}x \\ &+ \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{G}_\theta\, e_u, e_u) + \tfrac{\rho}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{M}_\theta\, e_v, e_v) \\ &\le \alpha\,|e_u|_0^2 + \beta\,|e_v|_0^2 - \tfrac{1}{2}(\tfrac{\partial}{\partial t}b_\theta, |\nabla \mathcal{G}_\theta\, e_u|^2) - \tfrac{\rho}{2}(\tfrac{\partial}{\partial t}b_\theta, |\mathcal{M}_\theta\, e_v|^2) \\ &\quad - ((\mathcal{G}_\theta - \mathcal{G}_{\theta,\varepsilon})\tfrac{\partial u_{\theta,\varepsilon}}{\partial t}, e_u) - \rho\,((\mathcal{M}_\theta - \mathcal{M}_{\theta,\varepsilon})\tfrac{\partial v_{\theta,\varepsilon}}{\partial t}, e_v) \\ &\quad + \tfrac{\theta}{2}\,\varepsilon\int_{\Omega_\varepsilon^+(t)} [\,\phi(u_\theta + v_\theta)\,]^2\,\mathrm{d}x + \tfrac{\theta}{2}\,\varepsilon\int_{\Omega_\varepsilon^-(t)} [\,\phi(u_\theta - v_\theta)\,]^2\,\mathrm{d}x. \end{aligned} \tag{2.77}$$

The desired result (2.70) then follows from (2.77) on treating the first six terms on the right hand side as in the uniqueness proof, (2.49), by using (2.24), (2.20), (2.25), (2.29), (2.32), (2.27a), (2.33a), (2.30), (2.19), (2.16), (2.15) and applying Hölder's, Young's and Gronwall inequalities; noting (2.14) and the regularity results (2.69) and (2.51), which also deal with the remaining two terms on the left hand side.

For constant $b > 0$ the uniqueness argument for (P_θ), and hence the proof of the bound (2.70), simplifies considerably in that the stronger regularity assumptions (2.69) and (2.51) are not required. □

2.2 The Deep Quench Limit

Introducing

$$K := \{\,\eta \in H^1(\Omega) : \eta(x) \in [0,1] \text{ for } a.e.\ x \in \Omega\,\},$$

the weak formulation of (P), (1.9a–e), is then:
(P) Find $\{u, v, w, z\}$ such that $u(\cdot, 0) = u^0(\cdot)$, $v(\cdot, 0) = v^0(\cdot)$ and for *a.e.* $t \in (0, T)$, $u(\cdot, t) \pm v(\cdot, t) \in K$ and

$$\langle \tfrac{\partial u}{\partial t}, \eta \rangle + (\, b(u, v) \nabla w, \nabla \eta) = 0 \quad \forall\, \eta \in H^1(\Omega), \tag{2.78a}$$

$$\rho\, (\tfrac{\partial v}{\partial t}, \eta) + (\, b(u, v)\, z, \eta) = 0 \quad \forall\, \eta \in L^2(\Omega), \tag{2.78b}$$

$$\gamma\, (\nabla (u \pm v), \nabla (\eta - (u \pm v))) \geq (w \pm z + \alpha\, u \pm \beta\, v, \eta - (u \pm v)) \quad \forall\, \eta \in K. \tag{2.78c}$$

Then similarly to (2.36) and (2.35a,b), for $b_{\min} > 0$ the weak formulation of (P) can be rewritten as:
Find $\{u, v, \lambda\}$ such that $u(\cdot, 0) = u^0(\cdot)$, $v(\cdot, 0) = v^0(\cdot)$ and for *a.e.* $t \in (0, T)$, $u(\cdot, t) \pm v(\cdot, t) \in K$, $\lambda(t) \in \mathbf{R}$, $(u(\cdot, t), 1) = (u^0(\cdot), 1)$ and

$$\begin{aligned} \gamma\, (\nabla (u \pm v), \nabla (\eta - (u \pm v))) + (\mathcal{G}_{u,v} \tfrac{\partial u}{\partial t} - \lambda \pm \rho\, \mathcal{M}_{u,v} \tfrac{\partial v}{\partial t}, \eta - (u \pm v)) \\ \geq (\alpha\, u \pm \beta\, v, \eta - (u \pm v)) \quad \forall\, \eta \in K \end{aligned} \tag{2.79}$$

$$\text{with} \qquad w \equiv -\,\mathcal{G}_{u,v} \tfrac{\partial u}{\partial t} + \lambda, \qquad z \equiv -\rho\, \mathcal{M}_{u,v} \tfrac{\partial v}{\partial t}. \tag{2.80}$$

Choosing $\eta \equiv (u \pm v) + \frac{1}{2}\,(u \pm v)\,(1 - (u \pm v))$ and $(u \pm v) - \frac{1}{2}\,(u \pm v)\,(1 - (u \pm v))$ in (2.79) yields that for *a.e.* $t \in (0, T)$ that

$$\begin{aligned} &\gamma\, (\nabla (u \pm v), \nabla ((u \pm v)(1 - (u \pm v)))) \\ &\quad + (\mathcal{G}_{u,v} \tfrac{\partial u}{\partial t} - \lambda \pm \rho\, \mathcal{M}_{u,v} \tfrac{\partial v}{\partial t} - (\alpha\, u \pm \beta\, v), (u \pm v)(1 - (u \pm v))) = 0. \end{aligned} \tag{2.81}$$

Theorem. 2.3. *If $d \leq 3$, b satisfies (1.6a) with $b_{\min} > 0$ and the assumptions on u^0 and v^0 of Theorem 2.1 hold, then there exists $\{u, v, \lambda, w, z\}$ solving (P) such that*

$$u \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))'), \quad \lambda \in L^2(0, T), \tag{2.82a}$$

$$w \in L^2(0, T; H^1(\Omega)) \quad \textit{and} \quad v \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)). \tag{2.82b}$$

In addition if either Ω is a convex polyhedron or $\partial\Omega \in C^{1,1}$, then

$$u,\ v \in L^2(0, T; H^2(\Omega)). \tag{2.83}$$

Furthermore, if u^0, v^0 and b satisfy the assumptions of Corollary 2.1 then solutions $\{u, v, \lambda, w, z\}$ of (P) are such that

$$u \in L^\infty(0, T; H^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap W^{1,\infty}(0, T; (H^1(\Omega))'), \tag{2.84a}$$

$$v \in L^\infty(0, T; H^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap W^{1,\infty}(0, T; L^2(\Omega)), \tag{2.84b}$$

$$\lambda \in L^\infty(0, T) \quad \textit{and} \quad w \in L^\infty(0, T; H^1(\Omega)) \tag{2.84c}$$

for any $T > 0$ if $d \leq 2$, and some $T > 0$ if $d = 3$. Moreover, the solution $\{u, v\}$ of (P) is unique over Ω_T and we have for all $\theta < \frac{1}{2}$ that

$$\|u - u_\theta\|^2_{L^2(0,T;H^1(\Omega))} + \|u - u_\theta\|^2_{L^\infty(0,T;(H^1(\Omega))')}$$

$$+ \|v - v_\theta\|^2_{L^2(0,T;H^1(\Omega))} + \|v - v_\theta\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_b(T) \, [\theta \ln(\tfrac{1}{\theta})]^2. \quad (2.85)$$

Note that if $b > 0$ is constant the solution $\{u, v\}$ of (P) is unique over Ω_T for any $T > 0$ and the bounds (2.85) hold under the minimal assumptions on u^0 and v^0 of Theorem 2.1.

Proof. As the bounds (2.67a) and the first two bound in (2.67b) are independent of θ, it follows that there exist $u \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))')$, $v \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$, $w \in L^2(0, T; H^1(\Omega))$, $z \in L^2(\Omega_T)$ and a subsequence $\{u_{\theta'}, v_{\theta'}, w_{\theta'}, z_{\theta'}\}$ such that as $\theta' \to 0$

$$\begin{aligned}
&u_{\theta'} \to u \text{ in } L^\infty(0, T; H^1(\Omega)) \text{ weak–star and in } H^1(0, T; (H^1(\Omega))') \text{ weakly,} \\
&v_{\theta'} \to v \text{ in } L^\infty(0, T; H^1(\Omega)) \text{ weak–star and in } H^1(0, T; L^2(\Omega)) \text{ weakly,} \\
&w_{\theta'} \to w \text{ in } L^2(0, T; H^1(\Omega)) \text{ weakly} \quad \text{and} \quad z_{\theta'} \to z \text{ in } L^2(\Omega_T) \text{ weakly.} \quad (2.86)
\end{aligned}$$

The first two lines of (2.86) imply that $u_{\theta'} \to u$ and $v_{\theta'} \to v$ in $L^2(\Omega_T)$ strongly and *a.e.* as $\theta' \to 0$. Then similarly to (2.72a,b), noting this, (1.6a) and (2.86) it follows for *a.e.* $t \in (0, T)$ that as $\theta' \to 0$

$$(b(u_{\theta'}, v_{\theta'}) \nabla w_{\theta'}, \nabla \eta) \to (b(u, v) \nabla w, \nabla \eta) \qquad \forall\, \eta \in H^1(\Omega) \qquad (2.87a)$$

$$\text{and} \qquad (b(u_{\theta'}, v_{\theta'}) \, z_{\theta'}, \eta) \to (b(u, v) \, z, \eta) \qquad \forall\, \eta \in L^2(\Omega). \qquad (2.87b)$$

Taking the limit $\theta' \to 0$ in the non-regularized versions of (2.11a,b) yields, on noting (2.87a,b) and (2.86), that $\{u_\theta, v_\theta, w_\theta, z_\theta\}$ satisfies (2.78a,b). Next we note that

$$\liminf_{\theta' \to 0} |u_{\theta'} \pm v_{\theta'}|_1^2 \geq |u \pm v|_1^2. \qquad (2.88)$$

In addition the monotonicity of ϕ and the boundedness of Φ on K yields that

$$\begin{aligned}
&\limsup_{\theta' \to 0} \theta' \, (\phi(u_{\theta'} \pm v_{\theta'}), \eta - (u_{\theta'} \pm v_{\theta'})) \\
&\qquad \leq \lim_{\theta' \to 0} \theta' \, (\Phi(\eta) - \Phi(u_{\theta'} \pm v_{\theta'}), 1) = 0 \qquad \forall\, \eta \in K. \quad (2.89)
\end{aligned}$$

Taking the limit $\theta' \to 0$ in the non-regularized version of (2.11c) with $\eta \equiv \eta_1 - (u_{\theta'} \pm v_{\theta'})$ for any $\eta_1 \in K$ and noting (2.86), the strong convergence in $L^2(\Omega_T)$ of $u_{\theta'}$ and $v_{\theta'}$, (2.88) and (2.89) yields (2.78c) with $\eta \equiv \eta_1$. Hence we have existence of a solution $\{u, v, w, z\}$, satisfying (2.82a,b), of (P). The bound on λ following from (2.80), (2.19), (2.15) and the bounds on u and w. The regularity results (2.83) and (2.84a–c) follow from the θ independent bounds (2.68) and (2.69).

We now consider the uniqueness of this solution to (P). Assuming that (2.79) has two solutions $\{u^i, v^i, \lambda^i\}$, $i = 1, 2$, with corresponding $\{w^i, z^i\}$ defined by (2.80); then choosing $\eta \equiv u^j \pm v^j$ in the i^{th} version of (2.79), $j \neq i$, and adding together the resulting four inequalities yields for *a.e.* $t \in (0, T)$ that $\overline{u}(\cdot, t) := (u^1 - u^2)(\cdot, t) \in \Xi$, $\overline{v} := v^1 - v^2$ and $\overline{\lambda} := \lambda^1 - \lambda^2$ satisfy the analogue of (2.48) with all $\cdot_{\theta,\varepsilon}$ subscripts removed. Uniqueness of $\{u,v,z\}$ over Ω_T, then follows from the analogue of (2.49) on noting the regularity results (2.84a,b) and (2.80). From (2.81) and the uniqueness of $\{u,v\}$ it follows for *a.e.* $t \in (0,T)$ that $\overline{\lambda}\,(1,(u\pm v)(1-(u\pm v)))=0$. Hence it follows that $\{\lambda(t), w(\cdot,t)\}$ are unique, provided $\{u(\cdot,t),v(\cdot,t)\} \not\equiv \{\frac{1}{2},-\frac{1}{2}\}$ or $\{\frac{1}{2},\frac{1}{2}\}$ and this can be guaranteed if $\fint u^0 \neq \frac{1}{2}$.

We now prove an error bound between the unique solutions $\{u,v\}$ and $\{u_\theta,v_\theta\}$ of problems (P) and (P$_\theta$). Let $e_u := u - u_\theta$ and $e_v := v - v_\theta$. Choosing $\eta \equiv u_\theta \pm v_\theta \in K$, see (2.67b), in (2.78c) and $\eta \equiv e_u \pm e_v$ in the non-regularized version of (2.36); then adding together the resulting four (in)equalities and noting (2.23) and (2.31) yields for *a.e.* $t\in(0,T)$ that

$$
\begin{aligned}
\gamma\,[\,|e_u|_1^2 + |e_v|_1^2\,] &+ \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathcal{G}}\,e_u, e_u) + \tfrac{\rho}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathcal{M}}\,e_v, e_v) \\
&\le \alpha\,|e_u|_0^2 + \beta\,|e_v|_0^2 - \tfrac{1}{2}(\tfrac{\partial}{\partial t}b, |\nabla\overline{\mathcal{G}}\,e_u|^2) - \tfrac{\rho}{2}(\tfrac{\partial}{\partial t}b, |\overline{\mathcal{M}}\,e_v|^2) \\
&\quad - \;((\overline{\mathcal{G}} - \overline{\mathcal{G}}_\theta)\tfrac{\partial u_\theta}{\partial t}, e_u) - \rho\,((\overline{\mathcal{M}} - \overline{\mathcal{M}}_\theta)\tfrac{\partial v_\theta}{\partial t}, e_v) \\
&\quad + \theta\,(\phi(u_\theta + v_\theta), e_u + e_v) + \theta\,(\phi(u_\theta - v_\theta), e_u - e_v),
\end{aligned} \tag{2.90}
$$

where for notational convenience $b_{(\theta)} \equiv b(u_{(\theta)}, v_{(\theta)})$, $\overline{\mathcal{G}}_{(\theta)} \equiv \mathcal{G}_{u_{(\theta)},v_{(\theta)}}$ and $\overline{\mathcal{M}}_{(\theta)} \equiv \mathcal{M}_{u_{(\theta)},v_{(\theta)}}$. We note for all $r,\, s \in [0,1]$ that

$$
(r - s + \theta^2)\,(\ln(r+\theta^2) - \ln(1 - r + \theta^2) - \phi(s)) \ge 0 \tag{2.91a}
$$

$$
\text{and} \qquad |\ln(r+\theta^2)| \le -2\,\ln\,\theta \quad \text{if } \theta \in (0,\tfrac{1}{2}). \tag{2.91b}
$$

It follows from (2.91a,b) and a Hölder inequality that for *a.e.* $t \in (0,T)$

$$
\begin{aligned}
\theta\,(\phi(u_\theta \pm v_\theta), e_u \pm e_v)& \\
&\le C\,[\,\theta^3\ln(\tfrac{1}{\theta}) + \theta^3\,|\phi(u_\theta \pm v_\theta)|_0 + \theta\ln(\tfrac{1}{\theta})\,|e_u \pm e_v|_0\,].
\end{aligned} \tag{2.92}
$$

The desired result (2.85) then follows from (2.90) on firstly treating the first six terms on the right hand side as in the uniqueness proof, (2.49), by using (2.24), (2.20), (2.25), (2.29), (2.32), (2.27a), (2.33a), (2.30), (2.19), (2.16) and (2.15), secondly bounding the remaining two terms on the right hand side of (2.90) via (2.92), then applying Hölder's, Young's and Gronwall inequalities and noting (2.14) and the regularity results (2.84a–b) and (2.69).

Once again for constant $b > 0$ the uniqueness argument for (P), and hence the proof of the bound (2.85), simplifies considerably in that the stronger regularity assumptions (2.84a–b) and (2.69) are not required. □

3 Finite Element Approximations

3.1 Logarithmic Free Energy

Throughout this subsection we assume that the assumptions (A$_\theta$) hold. We introduce the following "semidiscrete finite element approximation" of (P$_{\theta(,\varepsilon)}$):

$(\mathbf{P}^h_{\theta(,\varepsilon)})$ Find $\{u^h_{\theta(,\varepsilon)}(\cdot,t), v^h_{\theta(,\varepsilon)}(\cdot,t), w^h_{\theta(,\varepsilon)}(\cdot,t), z^h_{\theta(,\varepsilon)}(\cdot,t)\} \in [S^h]^4$ such that $u^h_{\theta(,\varepsilon)}(\cdot,0) = Q^h_\gamma u^0(\cdot)$, $v^h_{\theta(,\varepsilon)}(\cdot,0) = Q^h_\gamma v^0(\cdot)$ and for $a.e.\ t \in (0,T)$

$$(\tfrac{\partial u^h_{\theta(,\varepsilon)}}{\partial t}, \chi)^h + (\, b(u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)}) \nabla w^h_{\theta(,\varepsilon)}, \nabla \chi) = 0 \quad \forall\, \chi \in S^h, \tag{3.1a}$$

$$\rho\,(\tfrac{\partial v^h_{\theta(,\varepsilon)}}{\partial t}, \chi)^h + (\, b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})\, z^h_{\theta(,\varepsilon)}, \chi) = 0 \quad \forall\, \chi \in S^h, \tag{3.1b}$$

$$\begin{aligned}\gamma\,(\nabla(u^h_{\theta(,\varepsilon)} \pm v^h_{\theta(,\varepsilon)}), \nabla\chi) + (2\,\theta\,\phi_{(\varepsilon)}(u^h_{\theta(,\varepsilon)} \pm v^h_{\theta(,\varepsilon)}) - (\alpha\, u^h_{\theta(,\varepsilon)} \pm \beta\, v^h_{\theta(,\varepsilon)}), \chi)^h \\ = (w^h_{\theta(,\varepsilon)} \pm z^h_{\theta(,\varepsilon)}, \chi)^h \quad \forall\, \chi \in S^h.\end{aligned} \tag{3.1c}$$

This is not a true semidiscrete finite element approximation for non-constant mobility since for technical reasons, see Remark 3.1 below, the mobility is "frozen"; that is, we have $b(u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)})$ in place of $b(u^h_{\theta(,\varepsilon)}, v^h_{\theta(,\varepsilon)})$. Hence for non-constant mobility the problems $(\mathrm{P}^h_{\theta(,\varepsilon)})$ are not computable.

In addition to the interpolation operator π^h and the weighted H^1 projection Q^h_γ, we introduce the "lumped" L^2 projection $\widehat{Q}^h_0 : L^2(\Omega) \to S^h$ such that

$$(\widehat{Q}^h_0 \eta, \chi)^h = (\eta, \chi) \quad \forall\, \chi \in S^h. \tag{3.2}$$

Below we recall some well-known, or easily derived, results concerning S^h and the above operators. For $m = 0$ or 1, we have that

$$|\chi|_{m,p_2} \leq C h^{\frac{d(p_1-p_2)}{p_1 p_2}} |\chi|_{m,p_1} \quad \forall\, \chi \in S^h, \quad 1 \leq p_1 \leq p_2 \leq \infty. \tag{3.3}$$

$$\|\chi\|_{0,\infty} \leq C\,(\ln \tfrac{1}{h})^{d-1}\, \|\chi\|_1 \quad \forall\, \chi \in S^h, \quad h \leq h_0, \quad d \leq 2. \tag{3.4}$$

$$\begin{aligned}&|(I - \pi^h)\eta|_{m,p_2} \leq C\, h^{\mu_1}\, |\eta|_{2,p_1} \quad \forall\, \eta \in W^{2,p_1}(\Omega);\\ &\qquad p_1 \leq p_2 \leq \infty, \text{ where } p_1 \geq 1 \text{ if } d \leq 2 \text{ or } p_1 > \tfrac{3}{2} \text{ if } d = 3,\\ &\qquad \text{and either } \mu_1 := 2 - m - d(\tfrac{1}{p_1} - \tfrac{1}{p_2}) > 0 \text{ or } \mu_1 \geq 0 \text{ if } p_2 < \infty.\end{aligned} \tag{3.5}$$

$$\begin{aligned}&|(I - Q^h_\gamma)\eta|_{m,p} \leq C\, h^{\mu_2}\, |\eta|_2 \quad \forall\, \eta \in H^2(\Omega), \quad p \in [2,\infty]\\ &\qquad \text{and either } \mu_2 := 2 - m - d(\tfrac{1}{2} - \tfrac{1}{p}) > 0 \text{ or } \mu_2 \geq 0 \text{ if } p < \infty.\end{aligned} \tag{3.6}$$

$$|(I - \widehat{Q}^h_0)\eta|_0 + h\, |(I - \widehat{Q}^h_0)\eta|_1 \leq C\, h\, |\eta|_1 \quad \forall\, \eta \in H^1(\Omega). \tag{3.7}$$

$$|\chi|_0 \leq |\chi|_h := [(\chi,\chi)^h]^{\frac{1}{2}} \leq (d+2)^{\frac{1}{2}}\, |\chi|_0 \quad \forall\, \chi \in S^h. \tag{3.8}$$

$$|(\chi_1, \chi_2) - (\chi_1, \chi_2)^h| \leq C\, h^{1+m}\, \|\chi_1\|_m\, \|\chi_2\|_1 \quad \forall\, \chi_1, \chi_2 \in S^h. \tag{3.9}$$

Since ϕ_ε is monotone, (2.5b), and the partitioning $\mathcal{T}^h$ is (weakly) acute it follows for all $\varepsilon \in (0, \frac{1}{4})$ that

$$\begin{aligned}|\nabla \pi^h[\phi_\varepsilon(\chi)]\,|_0^2 &\leq \phi'_\varepsilon(\tfrac{1}{2} + \|\chi - \tfrac{1}{2}\|_{0,\infty})\, (\nabla\chi, \nabla\pi^h[\phi_\varepsilon(\chi)])\\ &\leq \tfrac{4}{3}\,\varepsilon^{-1}\, (\nabla\chi, \nabla\pi^h[\phi_\varepsilon(\chi)]) \quad \forall\, \chi \in S^h,\end{aligned} \tag{3.10}$$

see [13] and [20, §2.4.2]. It is easily deduced from (3.5), (3.10) and (2.5b) for all $\eta \in H^2(\Omega)$ with $\frac{\partial\eta}{\partial\nu} = 0$ on $\partial\Omega$ that

$$|(I - \pi^h)\phi_\varepsilon(\eta)|_0^2 + h^2\, \|\,\pi^h[\phi_\varepsilon(\eta)]\,\|_1^2$$

$$\leq C\,\varepsilon^{-1}\,h^2\,[\,|\phi_\varepsilon(\eta)|_0^2 + (1+\varepsilon^{-1}\,h^2)\,|\eta|_2^2\,], \tag{3.11}$$

see [3, (3.25–26)]. If $d \leq 2$, then one can exploit the concavity of ϕ_ε^+, see (2.3), to show, on noting (3.5), that for all $\eta \in W^{2,1}(\Omega)$ with $\frac{\partial \eta}{\partial \nu} = 0$ on $\partial\Omega$ the improved bound

$$\begin{aligned} |(I-\pi^h)\phi_\varepsilon(\eta)\,|_{0,1} &\leq |(I-\pi^h)\phi_\varepsilon^+(\eta)\,|_{0,1} + |(I-\pi^h)\phi_\varepsilon^+(1-\eta)\,|_{0,1} \\ &\leq C\,h^2\,[\,|\phi_\varepsilon^+(\eta)|_{2,1} + |\phi_\varepsilon^+(1-\eta)|_{2,1}\,] \leq C\,\varepsilon^{-1}\,h^2\,|\eta|_{2,1} \end{aligned} \tag{3.12}$$

holds, see [5, (3.69–70)].

Similarly to (2.12), it is convenient to introduce the operator $\widehat{\mathcal{G}}^h : \mathcal{F}^c \to \Xi^h$ defined by

$$(\nabla \widehat{\mathcal{G}}^h f, \nabla \chi) = (f,\chi)^h \quad \forall\, \chi \in S^h, \tag{3.13}$$

where $\Xi^h := \{\xi^h \in S^h : (\xi^h, 1) = 0\} \subset \mathcal{F}^c := \{f \in C(\overline{\Omega}) : (f,1)^h = 0\}$. Note that the analogue of (2.16) holds: for all $f \in \mathcal{F}^c$, $\chi \in S^h$ and for all $\mu > 0$

$$(f,\chi)^h \equiv (\nabla \widehat{\mathcal{G}}^h f, \nabla \chi) \leq |\widehat{\mathcal{G}}^h f|_1 |\chi|_1 \leq \tfrac{1}{2\mu} |\widehat{\mathcal{G}}^h f|_1^2 + \tfrac{\mu}{2} |\chi|_1^2. \tag{3.14}$$

In addition we have for all $\xi^h \in \Xi^h$ that

$$h^2\,|\xi^h|_1 \leq C_1\,h\,|\xi^h|_h \leq C_2\,|\widehat{\mathcal{G}}^h \xi^h|_1 \leq C_3\,\|\xi^h\|_{-1} \leq C_4\,|\widehat{\mathcal{G}}^h \xi^h|_1. \tag{3.15}$$

The first inequality on the left is just an inverse inequality on noting (3.8) and holds for all $\xi^h \in S^h$. The second follows from the first and (3.14). The third and fourth follow from (2.12), (3.13) and (3.9); see (3.10) and (3.21) in [6]. For later use, we introduce also $\widehat{M}^h : C(\overline{\Omega}) \to S^h$ defined by

$$(\widehat{\mathcal{M}}^h f, \chi) = (f,\chi)^h \quad \forall\, \chi \in S^h. \tag{3.16}$$

Assuming that $b_{\min} > 0$ and given q_i measurable in Ω, we introduce the analogue of (2.18): $\mathcal{G}^h_{q_1,q_2} : \mathcal{F} \to \Xi^h$ such that

$$(b(q_1,q_2)\nabla \mathcal{G}^h_{q_1,q_2} f, \nabla \chi) = \langle f, \chi \rangle \quad \forall\, \chi \in S^h. \tag{3.17}$$

It follows immediately from (2.18), (3.17) and (3.5) that for all measurable q_i and $f \in \mathcal{F}$ that

$$\begin{aligned} |[b(q_1,q_2)]^{\frac{1}{2}} \nabla(\mathcal{G}_{q_1,q_2} - \mathcal{G}^h_{q_1,q_2}) f|_0 &\leq |[b(q_1,q_2)]^{\frac{1}{2}} \nabla (I-\pi^h)\mathcal{G}_{q_1,q_2} f|_0 \\ &\leq C\,h\,|\mathcal{G}_{q_1,q_2} f|_2. \end{aligned} \tag{3.18}$$

Similarly to (3.17), we introduce $\widehat{\mathcal{G}}^h_{q_1,q_2} : \mathcal{F}^c \to \Xi^h$ such that

$$(b(q_1,q_2)\nabla \widehat{\mathcal{G}}^h_{q_1,q_2} f, \nabla \chi) = (f,\chi)^h \quad \forall\, \chi \in S^h. \tag{3.19}$$

The analogues of (2.19) and (2.20) hold: for all $f \in \mathcal{F}^c$

$$b_{\min} |[b(q_1,q_2)]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^h_{q_1,q_2} f|_0^2 \leq |\nabla \widehat{\mathcal{G}}^h f|_0^2 \leq b_{\max} |[b(q_1,q_2)]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^h_{q_1,q_2} f|_0^2, \tag{3.20}$$

$$(f,\chi)^h \equiv (b(q_1,q_2)\nabla\widehat{\mathcal{G}}^h_{q_1,q_2} f, \nabla\chi) \le b_{\max}^{\frac{1}{2}} \left[(f, \widehat{\mathcal{G}}^h_{q_1,q_2} f)^h\right]^{\frac{1}{2}} |\chi|_1 \quad \forall\, \chi \in S^h. \quad (3.21)$$

It is easily deduced from (3.17), (3.19), (3.9) and (2.14) that for q_i measurable

$$\|(\mathcal{G}^h_{q_1,q_2} - \widehat{\mathcal{G}}^h_{q_1,q_2})\xi^h\|_1 \le C\, b_{\min}^{-1}\, h^2\, \|\xi^h\|_1 \quad \forall\, \xi^h \in \Xi^h. \quad (3.22)$$

It follows from (3.3) that for all $p \ge 2$

$$\begin{aligned}|\widehat{\mathcal{G}}^h_{q_1,q_2}\xi^h|_{1,p} &\le |\mathcal{G}_{q_1,q_2}\xi^h|_{1,p} + |(\mathcal{G}_{q_1,q_2} - \mathcal{G}^h_{q_1,q_2})\xi^h|_{1,p} \\ &\quad + C\, h^{\frac{(2-p)d}{2p}}\, |(\mathcal{G}^h_{q_1,q_2} - \widehat{\mathcal{G}}^h_{q_1,q_2})\xi^h|_1 \quad \forall\, \xi^h \in \Xi^h. \quad (3.23)\end{aligned}$$

It follows from (3.3), (3.5), (2.27b) and (3.18) that for all $p \ge 2$

$$\begin{aligned}&|(\mathcal{G}_{q_1,q_2} - \mathcal{G}^h_{q_1,q_2})\xi^h|_{1,p} \\ &\quad \le |(I-\pi^h)\mathcal{G}_{q_1,q_2}\xi^h|_{1,p} + C\, h^{\frac{(2-p)d}{2p}}\, |(\pi^h\,\mathcal{G}_{q_1,q_2} - \mathcal{G}^h_{q_1,q_2})\xi^h|_1 \\ &\quad \le C_b(\|q_i\|_2)\, h^{\frac{(2-p)d}{2p}}\, [\, h\,|\xi^h|_0 + |(I-\pi^h)\mathcal{G}_{q_1,q_2}\xi^h|_1 + |(\mathcal{G}_{q_1,q_2} - \mathcal{G}^h_{q_1,q_2})\xi^h|_1\,] \\ &\quad \le C_b(\|q_i\|_2)\, h^{1+\frac{(2-p)d}{2p}}\, |\xi^h|_0 \quad \forall\, \xi^h \in \Xi^h. \quad (3.24)\end{aligned}$$

Combining (3.23) and (3.24), and noting (2.14), (3.22), (3.15), (3.8), (2.10), (2.27b), (2.19), (2.15) and (2.16) yields for all $p \in [2,6]$ that

$$\begin{aligned}\|\widehat{\mathcal{G}}^h_{q_1,q_2}\xi^h\|_{1,p} &\le C_b(\|q_i\|_2)\, [\, |\mathcal{G}_{q_1,q_2}\xi^h|_{1,p} + h^{1+\frac{(2-p)d}{2p}}\, |\xi^h|_0\,] \\ &\le C_b(\|q_i\|_2)\, \|\xi^h\|_{-1}^{1-\varsigma}\, |\xi^h|_1^{\varsigma} \quad \forall\, \xi^h \in \Xi^h, \quad (3.25)\end{aligned}$$

where $\varsigma = \frac{d(p-2)}{4p}$. Finally the $\widehat{\mathcal{G}}^h_{q_1,q_2}$ analogue of (2.28) follows from (2.24), (3.25) and a Young's inequality:

$$\begin{aligned}|(\tfrac{\partial}{\partial t}b(q_1,q_2), |\nabla\widehat{\mathcal{G}}^h_{q_1,q_2}\xi^h|^2)| &\le \tfrac{\gamma}{8}\, |\xi^h|_1^2 + C_b(\|q_i\|_2)\, |\tfrac{\partial}{\partial t}b(q_1,q_2)|_0^{\frac{8}{8-d}}\, \|\xi^h\|_{-1}^2 \\ &\qquad \forall\, \xi^h \in \Xi^h. \quad (3.26)\end{aligned}$$

Assuming that $b_{\min} > 0$ and given q_i measurable in Ω, we introduce the analogue of (2.29): $\mathcal{M}^h_{q_1,q_2} : L^2(\Omega) \to S^h$ such that

$$(b(q_1,q_2)\mathcal{M}^h_{q_1,q_2} f, \chi) = (f,\chi) \quad \forall\, \chi \in S^h. \quad (3.27)$$

It follows immediately from (2.29), (3.27) and (3.7) that for all measurable q_i and $f \in H^1(\Omega)$ that

$$\begin{aligned}|[b(q_1,q_2)]^{\frac{1}{2}}(\mathcal{M}_{q_1,q_2} - \mathcal{M}^h_{q_1,q_2})f|_0 &\le |[b(q_1,q_2)]^{\frac{1}{2}}(I - \widehat{Q}^h_0)\mathcal{M}_{q_1,q_2} f|_0 \\ &\le C\, h\, |\mathcal{M}_{q_1,q_2} f|_1. \quad (3.28)\end{aligned}$$

Similarly to (3.27), we introduce $\widehat{\mathcal{M}}^h_{q_1,q_2} : C(\overline{\Omega}) \to S^h$ such that

$$(b(q_1,q_2)\widehat{\mathcal{M}}^h_{q_1,q_2} f, \chi) = (f,\chi)^h \quad \forall\, \chi \in S^h. \quad (3.29)$$

It follows from (3.16), (3.29) and (3.8) that

$$|\xi^h|_0^2 \le |\widehat{\mathcal{M}}^h \xi^h|_0^2 \le b_{\max}\, |[b(q_1,q_2)]^{\frac{1}{2}} \widehat{\mathcal{M}}^h_{q_1,q_2} \xi^h|_0^2 \\ \le b_{\max}\, b_{\min}^{-1}\, (d+2)^2\, |\xi^h|_0^2 \quad \forall\, \xi^h \in S^h. \tag{3.30}$$

It is easily deduced from (3.27), (3.29), and (3.9) that for q_i measurable

$$|(\mathcal{M}^h_{q_1,q_2} - \widehat{\mathcal{M}}^h_{q_1,q_2})\xi^h|_0 \le C\, b_{\min}^{-1}\, h\, \|\xi^h\|_1 \quad \forall\, \xi^h \in S^h. \tag{3.31}$$

It follows from (3.3) that for all $p \ge 2$

$$|\widehat{\mathcal{M}}^h_{q_1,q_2} \xi^h|_{0,p} \le |\mathcal{M}_{q_1,q_2}\xi^h|_{0,p} + |(\mathcal{M}_{q_1,q_2} - \mathcal{M}^h_{q_1,q_2})\xi^h|_{0,p} \\ + C\, h^{\frac{(2-p)d}{2p}}\, |(\mathcal{M}^h_{q_1,q_2} - \widehat{\mathcal{M}}^h_{q_1,q_2})\xi^h|_0 \quad \forall\, \xi^h \in S^h. \tag{3.32}$$

It follows from (3.3), (2.10), (3.7), (2.30), (2.33b) and (3.28) that for all $p \in [2,6]$

$$|(\mathcal{M}_{q_1,q_2} - \mathcal{M}^h_{q_1,q_2})\xi^h|_{0,p} \\ \le |(I - \widehat{Q}^h_0)\mathcal{M}_{q_1,q_2}\xi^h|_{0,p} + C\, h^{\frac{(2-p)d}{2p}}\, |(\widehat{Q}^h_0\, \mathcal{M}_{q_1,q_2} - \mathcal{M}^h_{q_1,q_2})\xi^h|_0 \\ \le C_b(\|q_i\|)\, h^{1+\frac{(2-p)d}{2p}}\, \|\xi^h\|_1 \quad \forall\, \xi^h \in S^h. \tag{3.33}$$

Combining (3.32) and (3.33), and noting (3.31), (3.15), (3.8), (2.30) and (2.10) yields for all $p \in [2,6]$ that

$$|\widehat{\mathcal{M}}^h_{q_1,q_2}\xi^h|_{0,p} \le C_b(\|q_i\|_2)\, [\, |\mathcal{M}_{q_1,q_2}\xi^h|_{0,p} + h^{1+\frac{(2-p)d}{2p}}\, \|\xi^h\|_1\,] \\ \le C_b(\|q_i\|_2)\, |\xi^h|_0^{1-2\varsigma}\, \|\xi^h\|_1^{2\varsigma} \quad \forall\, \xi^h \in S^h, \tag{3.34}$$

where $\varsigma = \frac{d(p-2)}{4p}$. Finally the $\widehat{\mathcal{M}}^h_{q_1,q_2}$ analogue of (2.34) follows from (2.24), (3.34) and a Young's inequality:

$$|(\tfrac{\partial}{\partial t} b(q_1,q_2), |\widehat{\mathcal{M}}^h_{q_1,q_2}\xi^h|^2)| \le \tfrac{\gamma}{8}\, \|\xi^h\|_1^2 + C_b(\|q_i\|_2)\, |\tfrac{\partial}{\partial t} b(q_1,q_2)|_0^{\frac{4}{4-d}}\, |\xi^h|_0^2 \\ \forall\, \xi^h \in S^h. \tag{3.35}$$

Choosing $\chi \equiv 1$ in (3.1a) yields that $(\frac{\partial u^h_{\theta(,\varepsilon)}}{\partial t}, 1)^h = 0$, i.e. $(u^h_{\theta(,\varepsilon)}(\cdot,t),1) = (Q^h_\gamma u^0(\cdot),1) = (u^0,1)$ for all t. Hence, similarly to (2.35a,b), it follows from (3.1a,b), (3.19), (1.6b), (2.14), (3.29) and (3.1c) with $\chi \equiv 1$ that

$$w^h_{\theta(,\varepsilon)} \equiv -\widehat{\mathcal{G}}^h_{u_{\theta(,\varepsilon)},v_{\theta(,\varepsilon)}} \frac{\partial u^h_{\theta(,\varepsilon)}}{\partial t} + \lambda^h_{\theta(,\varepsilon)}, \qquad z^h_{\theta(,\varepsilon)} \equiv -\rho\, \widehat{\mathcal{M}}^h_{u_{\theta(,\varepsilon)},v_{\theta(,\varepsilon)}} \frac{\partial v^h_{\theta(,\varepsilon)}}{\partial t} \tag{3.36a}$$

and $$\lambda^h_{\theta(,\varepsilon)} := \fint \pi^h[\,\theta\,\phi_{(\varepsilon)}(u^h_{\theta(,\varepsilon)} + v^h_{\theta(,\varepsilon)}) + \theta\,\phi_{(\varepsilon)}(u^h_{\theta(,\varepsilon)} - v^h_{\theta(,\varepsilon)}) - \alpha\, u^h_{\theta(,\varepsilon)}\,]. \tag{3.36b}$$

It is convenient to introduce

$$S^h_m := \{\, \chi \in S^h : \fint \chi = m := \fint u^0 \,\}. \tag{3.37}$$

Therefore for $b_{\min} > 0$, $(\mathrm{P}^h_{\theta(,\varepsilon)})$ can be rewritten as:
Find $\{u^h_{\theta(,\varepsilon)}, v^h_{\theta(,\varepsilon)}\} \in S^h_m \times S^h$ such that $u^h_{\theta(,\varepsilon)}(\cdot,0) = Q^h_\gamma u^0(\cdot)$, $v^h_{\theta(,\varepsilon)}(\cdot,0) = Q^h_\gamma v^0(\cdot)$ and for *a.e.* $t \in (0,T)$

$$\begin{aligned}
&\gamma\,(\nabla(u^h_{\theta(,\varepsilon)} \pm v^h_{\theta(,\varepsilon)}), \nabla\chi) \\
&\quad + (\widehat{\mathcal{G}}^h_{u_{\theta(,\varepsilon)},v_{\theta(,\varepsilon)}} \frac{\partial u^h_{\theta(,\varepsilon)}}{\partial t} - \lambda^h_{\theta(,\varepsilon)} \pm \rho\,\widehat{\mathcal{M}}^h_{u_{\theta(,\varepsilon)},v_{\theta(,\varepsilon)}} \frac{\partial v^h_{\theta(,\varepsilon)}}{\partial t}, \chi)^h \\
&\quad + (2\,\theta\,\phi_{(\varepsilon)}(u^h_{\theta(,\varepsilon)} \pm v^h_{\theta(,\varepsilon)}) - (\alpha\, u^h_{\theta(,\varepsilon)} \pm \beta\, v^h_{\theta(,\varepsilon)}), \chi)^h = 0 \quad \forall\, \chi \in S^h, \qquad (3.38)
\end{aligned}$$

where $\lambda^h_{\theta(,\varepsilon)}$ is defined by (3.36b) and $\{w^h_{\theta(,\varepsilon)}, z^h_{\theta(,\varepsilon)}\}$ can be obtained from (3.36a). Theorems 3.1 and 3.2 below are adaptations to (P^h_θ) of Theorems 3.1 and 3.2 in [7] for a multi-component Cahn-Hilliard system with a concentration dependent mobility matrix.

Theorem. 3.1. *Let $u^0 \in H^3(\Omega)$, $v^0 \in H^2(\Omega)$, $\frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} = 0$ on $\partial\Omega$ and $\delta \in (0,\frac{1}{2})$ be such that $\|u^0 \pm v^0 - \frac{1}{2}\|_{0,\infty} \le \frac{1}{2} - \delta$. Let b satisfy (1.6a,b) and let the assumptions (A_θ) hold. Then for all $\theta \le \theta_{\max}$, $\varepsilon \le \varepsilon_2$, where $\varepsilon_2(\theta,\delta) := \min\{\varepsilon_0, \frac{1}{2}\delta\}$, and for all $h \le h_0$ such that $\|Q^h_\gamma(u^0 \pm v^0) - \frac{1}{2}\|_{0,\infty} \le \frac{1}{2}(1-\delta)$ there exists a unique solution $\{u^h_{\theta(,\varepsilon)}, v^h_{\theta(,\varepsilon)}, w^h_{\theta(,\varepsilon)}, z^h_{\theta(,\varepsilon)}\}$ to $(P^h_{\theta(,\varepsilon)})$ on Ω_T for any $T > 0$ if $d \le 2$ or if $b > 0$ is constant, and for some $T > 0$ if $d = 3$ and b is non-constant, such that the following stability bounds hold independently of ε, θ and h*

$$\begin{aligned}
&\|u^h_{\theta(,\varepsilon)}\|_{L^\infty(0,T;H^1(\Omega))} + \|\tfrac{\partial u^h_{\theta(,\varepsilon)}}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} \\
&\qquad\qquad + \|v^h_{\theta(,\varepsilon)}\|_{L^\infty(0,T;H^1(\Omega))} + \|\tfrac{\partial v^h_{\theta(,\varepsilon)}}{\partial t}\|_{L^2(\Omega_T)} \le C, \qquad (3.39\mathrm{a}) \\
&\theta\,\|\pi^h[u^h_{\theta(,\varepsilon)} \pm v^h_{\theta(,\varepsilon)}]_-\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad\qquad + \theta\,\|\pi^h[u^h_{\theta(,\varepsilon)} \pm v^h_{\theta(,\varepsilon)} - 1]_+\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\,\varepsilon, \quad (3.39\mathrm{b}) \\
&\|\lambda^h_{\theta(,\varepsilon)}\|_{L^2(0,T)} + \|w^h_{\theta(,\varepsilon)}\|_{L^2(0,T;H^1(\Omega))} + \theta\,\|\pi^h[\phi_\varepsilon(u^h_{\theta(,\varepsilon)} \pm v^h_{\theta(,\varepsilon)})]\|_{L^2(\Omega_T)} \\
&\quad + \|z^h_{\theta(,\varepsilon)}\|_{L^2(\Omega_T)} + (\theta\,\varepsilon)^{\frac{1}{2}}\,\|\pi^h[\phi_\varepsilon(u^h_{\theta(,\varepsilon)} \pm v^h_{\theta(,\varepsilon)})]\|_{L^2(0,T;H^1(\Omega))} \le C_b(T). \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.39\mathrm{c})
\end{aligned}$$

Furthermore, the unique solutions of (P^h_θ) and $(P^h_{\theta,\varepsilon})$ satisfy $u^h_\theta \pm v^h_\theta(\cdot,t) \in K^h$ for a.e. $t \in (0,T)$ and

$$\begin{aligned}
&\|\tfrac{\partial u^h_{\theta(,\varepsilon)}}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial u^h_{\theta(,\varepsilon)}}{\partial t}\|_{L^\infty(0,T;(H^1(\Omega))')} \\
&\quad + \|\tfrac{\partial v^h_{\theta(,\varepsilon)}}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial v^h_{\theta(,\varepsilon)}}{\partial t}\|_{L^\infty(0,T;L^2(\Omega))} \le C_b(T), \qquad (3.40) \\
&\|u^h_\theta - u^h_{\theta,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} + \|u^h_\theta - u^h_{\theta,\varepsilon}\|^2_{L^\infty(0,T;(H^1(\Omega))')} \\
&\quad + \|v^h_\theta - v^h_{\theta,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} + \|v^h_\theta - v^h_{\theta,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega))} \le C_b(T)\,\theta^{-1}\,\varepsilon. \quad (3.41)
\end{aligned}$$

Proof. The existence of a solution to $(\mathrm{P}^h_{\theta,\varepsilon})$ with the resulting bounds (3.39a–c) for $\{u^h_{\theta,\varepsilon}, v^h_{\theta,\varepsilon}, w^h_{\theta,\varepsilon}, z^h_{\theta,\varepsilon}\}$ is a simple analogue of that for $(\mathrm{P}_{\theta,\varepsilon})$, see Theorem

2.1 above. We just highlight the three main differences. Firstly, it follows from (1.11) and the assumptions on u^0 and v^0 that for all $h > 0$

$$\|Q^h_\gamma u^0\|_1 \leq C_1\,\|u^0\|_1 \leq C_2 \quad \text{and} \quad \|Q^h_\gamma v^0\|_1 \leq C_3\,\|v^0\|_1 \leq C_4. \tag{3.42}$$

Secondly (3.6), the assumption $\|u^0 \pm v^0 - \frac{1}{2}\|_{0,\infty} \leq \frac{1}{2} - \delta$ for some $\delta \in (0, \frac{1}{2})$, (2.9) and (2.1a,b) yield that there exists a $h_0 > 0$ such that for all $h \leq h_0$

$$\|Q^h_\gamma(u^0 \pm v^0) - \tfrac{1}{2}\|_{0,\infty} \leq \tfrac{1}{2}(1-\delta) \Longrightarrow (\Psi_{\theta,\varepsilon}(Q^h_\gamma u^0, Q^h_\gamma v^0), 1)^h \leq C. \tag{3.43}$$

Finally, the last bound in (3.39c) follows by choosing $\chi \equiv 2\theta\,\pi^h[\phi_\varepsilon(u^h_{\theta,\varepsilon} \pm v^h_{\theta,\varepsilon})] - \lambda^h_{\theta,\varepsilon}$ in the regularized versions of (3.38) and noting (3.10), which yields the analogue of (2.43).

As the mobility is "frozen", the proof of uniqueness of a solution to $(\mathrm{P}^h_{\theta,\varepsilon})$ is simpler than that of $(\mathrm{P}_{\theta,\varepsilon})$. Once again, we just stress the main differences. Let $\overline{u}^h_{\theta,\varepsilon} := (u^h_{\theta,\varepsilon})_1 - (u^h_{\theta,\varepsilon})_2 \in \Xi^h$ and $\overline{v}^h_{\theta,\varepsilon} := (v^h_{\theta,\varepsilon})_1 - (v^h_{\theta,\varepsilon})_2 \in S^h$, where $\{(u^h_{\theta,\varepsilon})_i, (v^h_{\theta,\varepsilon})_i\}$ are solutions to $(\mathrm{P}^h_{\theta,\varepsilon})$. It follows from (3.38) and the $\widehat{\mathcal{G}}^h_{q_1,q_2}$, $\widehat{\mathcal{M}}^h_{q_1,q_2}$ analogues of (2.23), (2.31) that for *a.e.* $t \in (0,T)$

$$\begin{aligned}
\gamma\,[\,|\overline{u}^h_{\theta,\varepsilon}|_1^2 + |\overline{v}^h_{\theta,\varepsilon}|_1^2\,] &+ \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\overline{u}^h_{\theta,\varepsilon}, \overline{u}^h_{\theta,\varepsilon})^h + \tfrac{\rho}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\widehat{\mathcal{M}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\overline{v}^h_{\theta,\varepsilon}, \overline{v}^h_{\theta,\varepsilon})^h \\
&\leq \alpha\,|\overline{u}^h_{\theta,\varepsilon}|_h^2 + \beta\,|\overline{v}^h_{\theta,\varepsilon}|_h^2 \\
&\qquad - \tfrac{1}{2}(\tfrac{\partial}{\partial t}b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon}), |\nabla\widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\overline{u}^h_{\theta,\varepsilon}|^2 + \rho\,|\widehat{\mathcal{M}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\overline{v}^h_{\theta,\varepsilon}|^2).
\end{aligned} \tag{3.44}$$

Applying (3.21), (3.8), (3.30), (3.29), (3.26), (3.15), (3.20), (3.19), (3.35), on noting (1.6b) and (2.51), and Young and Gronwall inequalities to (3.44) yields the desired uniqueness result for $(\mathrm{P}^h_{\theta,\varepsilon})$.

The bounds (3.40) for $u^h_{\theta,\varepsilon}$, $v^h_{\theta,\varepsilon}$ follow in a similar fashion to their analogues for $u_{\theta,\varepsilon}$, $v_{\theta,\varepsilon}$; see (2.51). However, once again due to the "frozen" mobility their proof is simpler. Differentiating the regularized version of (3.38) with respect to t, choosing $\chi \equiv \frac{\partial}{\partial t}(u^h_{\theta,\varepsilon} \pm v^h_{\theta,\varepsilon})$, adding together these $\pm$ versions, noting $\phi'_\varepsilon(\cdot) \geq 0$ and the $\widehat{\mathcal{G}}^h_{q_1,q_2}$ and $\widehat{\mathcal{M}}^h_{q_1,q_2}$ analogues of (2.23) and (2.31), respectively, yields for *a.e.* $t \in (0,T)$ that

$$\begin{aligned}
\gamma\,[\,|\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t}|_1^2 + |\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t}|_1^2\,] &+ \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t}, \tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t})^h \\
&+ \tfrac{\rho}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\widehat{\mathcal{M}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t}, \tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t})^h \leq \alpha\,|\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t}|_h^2 + \beta\,|\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t}|_h^2 \\
&\qquad - \tfrac{1}{2}(\tfrac{\partial}{\partial t}b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon}), |\nabla\widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t}|^2 + \rho\,|\widehat{\mathcal{M}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t}|^2).
\end{aligned} \tag{3.45}$$

It follows from (3.1c), (1.11), $\frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} = 0$ on $\partial\Omega$ and (3.2) that

$$\begin{aligned}
(w^h_{\theta,\varepsilon} \pm z^h_{\theta,\varepsilon})(\cdot,0) &= \widehat{Q}^h_0[-\gamma\,\Delta + (I - Q^h_\gamma)](u^0 \pm v^0)(\cdot) \\
&\quad + 2\theta\,\pi^h[\phi_\varepsilon(Q^h_\gamma(u^0 \pm v^0))](\cdot) - Q^h_\gamma(\alpha\,u^0 \pm \beta\,v^0)(\cdot).
\end{aligned} \tag{3.46}$$

Hence it follows from (3.46), (3.36a), (3.7), (3.10), (3.43), (3.42), (3.2) and (3.8) as $\varepsilon \leq \frac{1}{2}\delta$ that

$$|\widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \frac{\partial u^h_{\theta,\varepsilon}}{\partial t}(\cdot,0)|_1 \leq C\,[\,\|u^0\|_3 + \|v^0\|_1 + \phi'_\varepsilon(1-\tfrac{1}{2}\delta)\,(\|u^0\|_1 + \|v^0\|_1)\,] \\ \leq C\,[\,\|u^0\|_3 + \|v^0\|_1\,], \tag{3.47a}$$

$$|\widehat{\mathcal{M}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \frac{\partial v^h_{\theta,\varepsilon}}{\partial t}(\cdot,0)|_0 \leq C\,[\,\|v^0\|_2 + \|u^0\|_1 + \phi_\varepsilon(1-\tfrac{1}{2}\delta)\,] \\ \leq C\,[\,\|v^0\|_2 + \|u^0\|_1\,]. \tag{3.47b}$$

Applying (3.21), (3.8), (3.30), (3.29), (3.26), (3.15), (3.20), (3.19), (3.35), (1.6b), (2.51) and Young and Gronwall inequalities to (3.45) yields, on noting (3.47a,b) and (2.14), the desired results (3.40) for $u^h_{\theta,\varepsilon}$ and $v^h_{\theta,\varepsilon}$.

Existence of a solution $\{u^h_\theta, v^h_\theta, w^h_\theta, z^h_\theta\}$ to (P^h_θ) with the corresponding bounds (3.39a–c) and (3.40) follow by letting $\varepsilon \to 0$ in $(\mathrm{P}^h_{\theta,\varepsilon})$; this is a simple adaption of the argument for (P_θ) from $(\mathrm{P}_{\theta,\varepsilon})$, see Theorem 2.2 above. Uniqueness of this solution follows as for $(\mathrm{P}^h_{\theta,\varepsilon})$. Finally, we need to prove the error bound (3.41). This is a semidiscrete analogue of the result (2.70), which is also proved in Theorem 2.2 above. This proof is easily adapted to prove (3.41) on noting (3.26), (3.35), (3.40) and the two results below. Firstly, on setting $e_u^{(h)} := u_\theta^{(h)} - u_{\theta,\varepsilon}^{(h)}$ and $e_v^{(h)} := v_\theta^{(h)} - v_{\theta,\varepsilon}^{(h)}$, we have from (3.21), the $\widehat{\mathcal{G}}^h_{q_1,q_2}$ version of (2.25), (3.19), (2.10), (3.25), (2.15) and a Young's inequality that

$$|((\widehat{\mathcal{G}}^h_{u_\theta,v_\theta} - \widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}})\frac{\partial u^h_{\theta,\varepsilon}}{\partial t}, e^h_u)^h| \\ \leq C\,|(\widehat{\mathcal{G}}^h_{u_\theta,v_\theta} - \widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}})\frac{\partial u^h_{\theta,\varepsilon}}{\partial t}|_1 \left[(\widehat{\mathcal{G}}^h_{u_\theta,v_\theta} e^h_u, e^h_u)^h\right]^{\frac{1}{2}} \\ \leq C_b\,(\,|e_u|_{0,4} + |e_v|_{0,4}\,)\,|\widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}} \frac{\partial u^h_{\theta,\varepsilon}}{\partial t}|_{1,4} \left[(\widehat{\mathcal{G}}^h_{u_\theta,v_\theta} e^h_u, e^h_u)^h\right]^{\frac{1}{2}} \\ \leq \|e_u\|_1^2 + \|e_v\|_1^2 + C_b(\|u_{\theta,\varepsilon}\|_2, \|v_{\theta,\varepsilon}\|_2)\,\|\frac{\partial u^h_{\theta,\varepsilon}}{\partial t}\|_1^2\,(\widehat{\mathcal{G}}^h_{u_\theta,v_\theta} e^h_u, e^h_u)^h. \tag{3.48}$$

Similarly, we have from (3.8), the $\widehat{\mathcal{M}}^h_{q_1,q_2}$ version of (2.32), (3.30), (3.29), (2.10), (3.34) and a Young's inequality that

$$|((\widehat{\mathcal{M}}^h_{u_\theta,v_\theta} - \widehat{\mathcal{M}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}})\frac{\partial v^h_{\theta,\varepsilon}}{\partial t}, e^h_v)^h| \\ \leq \|e_u\|_1^2 + \|e_v\|_1^2 + C_b(\|u_{\theta,\varepsilon}\|_2, \|v_{\theta,\varepsilon}\|_2)\,\|\frac{\partial v^h_{\theta,\varepsilon}}{\partial t}\|_1^2\,(\widehat{\mathcal{M}}^h_{u_\theta,v_\theta} e^h_v, e^h_v)^h. \tag{3.49}$$

□

THEOREM. 3.2. *Let the assumptions of Theorem 3.1 hold. Then we have for all* $\theta \leq \theta_{\max}$, $\varepsilon \leq \varepsilon_2$ *and for all* $h \leq h_0$, *such that* $\|Q^h_\gamma(u^0 \pm v^0) - \frac{1}{2}\|_{0,\infty} \leq \frac{1}{2}(1-\delta)$, *that the unique solutions of* $(P_{\theta,\varepsilon})$ *and* $(P^h_{\theta,\varepsilon})$ *satisfy*

$$\|u_{\theta,\varepsilon} - u^h_{\theta,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} + \|u_{\theta,\varepsilon} - u^h_{\theta,\varepsilon}\|^2_{L^\infty(0,T;(H^1(\Omega))')} \\ + \|v_{\theta,\varepsilon} - v^h_{\theta,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} + \|v_{\theta,\varepsilon} - v^h_{\theta,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$\leq \begin{cases} C_b(T)\left[\,h^2+\varepsilon^{-2}h^4(\ln\frac{1}{h})^{2(d-1)}\,\right] & \textit{if } d\leq 2,\\ C_b(T)\left[\,\varepsilon^{-1}h^2+\varepsilon^{-2}h^4\,\right] & \textit{if } d=3 \textit{ and } b \textit{ is constant.}\end{cases} \tag{3.50}$$

Moreover, the unique solutions of (P_θ) *and* (P_θ^h) *satisfy*

$$\begin{aligned}&\|u_\theta-u_\theta^h\|^2_{L^2(0,T;H^1(\Omega))}+\|u_\theta-u_\theta^h\|^2_{L^\infty(0,T;(H^1(\Omega))')}\\ &\qquad+\|v_\theta-v_\theta^h\|^2_{L^2(0,T;H^1(\Omega))}+\|v_\theta-v_\theta^h\|^2_{L^\infty(0,T;L^2(\Omega))}\\ &\quad\leq\begin{cases} C_b(T)\,h^{\frac{4}{3}}(\ln\frac{1}{h})^{\frac{2(d-1)}{3}} & \textit{if } d\leq 2,\\ C_b(T)\,h & \textit{if } d=3 \textit{ and } b \textit{ is constant.}\end{cases}\end{aligned} \tag{3.51}$$

Proof. For *a.e.* $t\in(0,T)$, we set $e_u:=u_{\theta,\varepsilon}-u^h_{\theta,\varepsilon}\in\Xi$, $e_v:=v_{\theta,\varepsilon}-v^h_{\theta,\varepsilon}$, $e_u^A:=(I-\pi^h)u_{\theta,\varepsilon}$, $e_v^A:=(I-\pi^h)v_{\theta,\varepsilon}$ and $e_u^h:=\pi^h u_{\theta,\varepsilon}-u^h_{\theta,\varepsilon}$, $e_v^h:=\pi^h v_{\theta,\varepsilon}-v^h_{\theta,\varepsilon}\in S^h$. We note for future reference that $⨍\, e_u^h=-⨍\, e_u^A$. On subtracting the regularized version of (3.38) from the regularized version of (2.36), it follows for *a.e.* $t\in(0,T)$ that

$$\begin{aligned}&\gamma(\nabla(e_u\pm e_v),\nabla\chi)+(\mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial e_u}{\partial t}\pm\rho\,\mathcal{M}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial e_v}{\partial t},\chi)\\ &\quad+2\theta\,(\phi_\varepsilon(u_{\theta,\varepsilon}\pm v_{\theta,\varepsilon})-\phi_\varepsilon(u^h_{\theta,\varepsilon}\pm v^h_{\theta,\varepsilon}),\chi)^h=(\alpha\,e_u\pm\beta\,e_v+(\lambda_{\theta,\varepsilon}-\lambda^h_{\theta,\varepsilon}),\chi)\\ &\quad+[\,(\widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t},\chi)^h-(\mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t},\chi)\,]\pm\rho\,[\,(\widehat{\mathcal{M}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t},\chi)^h\\ &\quad-(\mathcal{M}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t},\chi)\,]+[\,(2\theta\,\phi_\varepsilon(u_{\theta,\varepsilon}\pm v_{\theta,\varepsilon})-(\alpha\,u^h_{\theta,\varepsilon}\pm\beta\,v^h_{\theta,\varepsilon}),\chi)^h\\ &\quad-(2\theta\,\phi_\varepsilon(u_{\theta,\varepsilon}\pm v_{\theta,\varepsilon})-(\alpha\,u^h_{\theta,\varepsilon}\pm\beta\,v^h_{\theta,\varepsilon}),\chi)\,]\quad\forall\,\chi\in S^h.\end{aligned} \tag{3.52}$$

Adding together the $\pm$ versions of (3.52) with $\chi\equiv e_u^h\pm e_v^h$, respectively, and noting (2.5b), (2.16), (2.15), (2.14), (2.19), (2.30), (3.9), (3.20), (3.15), (3.8), (3.22), (3.24), (2.51), (3.31), (3.33) and a Young's inequality yields for *a.e.* $t\in(0,T)$ that

$$\begin{aligned}&\gamma\,[\,|e_u|_1^2+|e_v|_1^2\,]+(\mathcal{G}_\varepsilon\tfrac{\partial e_u}{\partial t},e_u)+\rho\,(\mathcal{M}_\varepsilon\tfrac{\partial e_v}{\partial t},e_v)\\ &\quad\leq\gamma\,[\,|e_u|_1^2+|e_v|_1^2\,]+(\mathcal{G}_\varepsilon\tfrac{\partial e_u}{\partial t},e_u)+\rho\,(\mathcal{M}_\varepsilon\tfrac{\partial e_v}{\partial t},e_v)\\ &\qquad+\theta\,(\phi_\varepsilon(u_{\theta,\varepsilon}+v_{\theta,\varepsilon})-\phi_\varepsilon(u^h_{\theta,\varepsilon}+v^h_{\theta,\varepsilon}),e_u^h+e_v^h)^h\\ &\qquad+\theta\,(\phi_\varepsilon(u_{\theta,\varepsilon}-v_{\theta,\varepsilon})-\phi_\varepsilon(u^h_{\theta,\varepsilon}-v^h_{\theta,\varepsilon}),e_u^h-e_v^h)^h\\ &\quad=(\alpha\,e_u,\overline{e}_u^h)+(\beta\,e_v,e_v^h)+\gamma\,[\,(\nabla e_u,\nabla e_u^A)+(\nabla e_v,\nabla e_v^A)\,]\\ &\qquad+(\mathcal{G}_\varepsilon\tfrac{\partial e_u}{\partial t},e_u^A)+\rho\,(\mathcal{M}_\varepsilon\tfrac{\partial e_v}{\partial t},e_v^A)-(\lambda_{\theta,\varepsilon}-\lambda^h_{\theta,\varepsilon},e_u^A)\\ &\qquad+[\,(\widehat{\mathcal{G}}^h_\varepsilon\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t},\overline{e}_u^h)^h-(\widehat{\mathcal{G}}^h_\varepsilon\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t},\overline{e}_u^h)\,]+((\widehat{\mathcal{G}}^h_\varepsilon-\mathcal{G}_\varepsilon)\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t},\overline{e}_u^h)\\ &\qquad+\rho\,[\,(\widehat{\mathcal{M}}^h_\varepsilon\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t},e_v^h)^h-(\widehat{\mathcal{M}}^h_\varepsilon\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t},e_v^h)\,]+\rho\,((\widehat{\mathcal{M}}^h_\varepsilon-\mathcal{M}_\varepsilon)\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t},e_v^h)\\ &\qquad+\theta\,[\,(\phi_\varepsilon(u_{\theta,\varepsilon}+v_{\theta,\varepsilon}),e_u^h+e_v^h)^h-(\phi_\varepsilon(u_{\theta,\varepsilon}+v_{\theta,\varepsilon}),e_u^h+e_v^h)\,]\\ &\qquad+\theta\,[\,(\phi_\varepsilon(u_{\theta,\varepsilon}-v_{\theta,\varepsilon}),e_u^h-e_v^h)^h-(\phi_\varepsilon(u_{\theta,\varepsilon}-v_{\theta,\varepsilon}),e_u^h-e_v^h)\,]\\ &\qquad+\alpha\,[\,(u^h_{\theta,\varepsilon},e_u^h)-(u^h_{\theta,\varepsilon},e_u^h)^h\,]+\beta\,[\,(v^h_{\theta,\varepsilon},e_v^h)-(v^h_{\theta,\varepsilon},e_v^h)^h\,]\end{aligned}$$

$$
\begin{aligned}
&\leq \tfrac{\gamma}{2}\,[\,|e_u|_1^2 + |e_v|_1^2\,] + C\,[\,\|e_u\|_{-1}^2 + |e_v|_0^2 + \|e_u^A\|_1^2 + \|e_v^A\|_1^2\,] \\
&\quad + C_b\,[\,\|\tfrac{\partial e_u}{\partial t}\|_{-1} + |\lambda_{\theta,\varepsilon}| + |\lambda^h_{\theta,\varepsilon}|\,]\,|e_u^A|_0 + C_b\,|\tfrac{\partial e_v}{\partial t}|_0\,|e_v^A|_0 \\
&\quad + C_b\,h^2\,[\,|\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t}|_0^2 + \|\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t}\|_1^2\,] + C\,h^4\,[\,\|u^h_{\theta,\varepsilon}\|_1^2 + \|v^h_{\theta,\varepsilon}\|_1^2\,] \\
&\quad + C\,h^4\,[\,\|\pi^h[\phi_\varepsilon(u_{\theta,\varepsilon} + v_{\theta,\varepsilon})]\|_1^2 + \|\pi^h[\phi_\varepsilon(u_{\theta,\varepsilon} - v_{\theta,\varepsilon})]\|_1^2\,] \\
&\quad + \theta\,|((I-\pi^h)\phi_\varepsilon(u_{\theta,\varepsilon} + v_{\theta,\varepsilon}), e_u^h + e_v^h)| \\
&\quad + \theta\,|((I-\pi^h)\phi_\varepsilon(u_{\theta,\varepsilon} - v_{\theta,\varepsilon}), e_u^h - e_v^h)|,
\end{aligned}
\tag{3.53}
$$

where $\overline{e}_u^h := (I - ⨍\,)e_u^h \in \Xi^h$ and for notational convenience $\mathcal{G}_\varepsilon \equiv \mathcal{G}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}$, $\widehat{\mathcal{G}}^h_\varepsilon \equiv \widehat{\mathcal{G}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}$, $\mathcal{M}_\varepsilon \equiv \mathcal{M}_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}$ and $\widehat{\mathcal{M}}^h_\varepsilon \equiv \widehat{\mathcal{M}}^h_{u_{\theta,\varepsilon},v_{\theta,\varepsilon}}$.

We now bound the terms on the right hand side of (3.53). In the case $d \leq 2$ we have from (3.4) and (3.12) for *a.e.* $t \in (0,T)$ that

$$
\begin{aligned}
&|((I-\pi^h)\phi_\varepsilon(u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon}), e_u^h \pm e_v^h)| \\
&\qquad\qquad \leq C\,\varepsilon^{-1}\,h^2\,(\ln \tfrac{1}{h})^{d-1}\,|u_{\theta,\varepsilon} \pm v_{\theta,\varepsilon}|_{2,1}\,\|e_u^h \pm e_v^h\|_1.
\end{aligned}
\tag{3.54}
$$

Combining (3.53), (3.11) and (3.54), and noting (2.23), (2.28), (2.31), (2.34) (2.51), (3.5) and a Young's inequality yields for *a.e.* $t \in (0,T)$ that

$$
\begin{aligned}
&\gamma\,[\,|e_u|_1^2 + |e_v|_1^2\,] + \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{G}_\varepsilon e_u, e_u) + \rho\,\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{M}_\varepsilon e_v, e_v) \\
&\quad \leq \tfrac{3\gamma}{4}\,[\,|e_u|_1^2 + |e_v|_1^2\,] + C_b(T)\,[\,1 + |\tfrac{\partial}{\partial t}b(u_{\theta,\varepsilon}, v_{\theta,\varepsilon})|_0^{\frac{4}{4-d}}\,]\,[\,\|e_u\|_{-1}^2 + |e_v|_0^2\,] \\
&\qquad + C\,(h^2 + \varepsilon^{-2}\,h^4\,(\ln \tfrac{1}{h})^{2(d-1)})\,[\,\|u_{\theta,\varepsilon}\|_2^2 + \|v_{\theta,\varepsilon}\|_2^2\,] \\
&\qquad + C_b\,h^2\,[\,|\tfrac{\partial u^h_{\theta,\varepsilon}}{\partial t}|_0^2 + \|\tfrac{\partial v^h_{\theta,\varepsilon}}{\partial t}\|_1^2 + \|\tfrac{\partial e_u}{\partial t}\|_{-1}^2 + |\tfrac{\partial e_v}{\partial t}|_0^2 + |\lambda_{\theta,\varepsilon}|^2 + |\lambda^h_{\theta,\varepsilon}|^2\,] \\
&\qquad + C\,\varepsilon^{-1}\,h^4\,[\,|\phi_\varepsilon(u_{\theta,\varepsilon} + v_{\theta,\varepsilon})|_0^2 + |\phi_\varepsilon(u_{\theta,\varepsilon} - v_{\theta,\varepsilon})|_0^2\,] \\
&\qquad + C\,h^4\,[\,\|u^h_{\theta,\varepsilon}\|_1^2 + \|v^h_{\theta,\varepsilon}\|_1^2\,].
\end{aligned}
\tag{3.55}
$$

The desired result (3.50) for $d \leq 2$ then follows from applying a Gronwall inequality to (3.55) and noting (1.6b), (2.51), (2.38), (3.39a,c), (3.40), (2.14), (2.19), (2.15), (2.30) and (3.6).

For the case $d = 3$ we do not have the bound (3.54), so we have to use (3.11) instead and this leads to the inferior bound in (3.50).

Finally the bounds (3.51) follow immediately from (3.50), (2.70) and (3.41) on choosing $\varepsilon = Ch^{\frac{4}{3}}(\ln \frac{1}{h})^{\frac{2(d-1)}{3}} \leq \varepsilon_2$ if $d \leq 2$ and $\varepsilon = Ch \leq \varepsilon_2$ if $d = 3$ and b is constant. □

Remark. 3.1. *If we replaced $b(u_{\theta(,\varepsilon)}, v_{\theta(,\varepsilon)})$ by $b(u^h_{\theta(,\varepsilon)}, v^h_{\theta(,\varepsilon)})$ in (3.1a,b); that is, considered the natural semidiscrete finite element approximation, then this would lead to a number of difficulties. Uniqueness of a solution to $(P^h_{\theta(,\varepsilon)})$ on Ω_T and the regularization bound (3.41) would not follow immediately; since the present proofs use for example the bounds (3.26) and (3.35) which exploit the $H^2(\Omega)$ bounds on $u_{\theta(,\varepsilon)}$ and $v_{\theta(,\varepsilon)}$, see (2.51) and (2.69).*

3.2 The Deep Quench Limit

Similarly to (3.38) and (3.36a) the corresponding "semidiscrete finite element approximation" of (P) can be rewritten as:
($\mathbf{P}^h$) Find $\{u^h, v^h, \lambda^h\} \in S^h_m \times S^h \times \mathbf{R}$ such that $u^h(\cdot,0) = Q^h_\gamma u^0(\cdot)$, $v^h(\cdot,0) = Q^h_\gamma v^0(\cdot)$ and for *a.e.* $t \in (0,T)$, $u^h(\cdot,t) \pm v^h(\cdot,t) \in K^h$ and

$$\gamma\,(\nabla(u^h \pm v^h), \nabla(\chi - (u^h \pm v^h))) + (\widehat{\mathcal{G}}^h_{u,v}\,\tfrac{\partial u^h}{\partial t} \pm \rho\,\widehat{\mathcal{M}}^h_{u,v}\,\tfrac{\partial v^h}{\partial t}, \chi - (u^h \pm v^h))^h \geq (\lambda^h + \alpha\,u^h \pm \beta\,v^h, \chi - (u^h \pm v^h))^h \quad \forall\ \chi \in K^h \tag{3.56}$$

$$\text{with} \qquad w^h \equiv -\widehat{\mathcal{G}}^h_{u,v}\,\tfrac{\partial u^h}{\partial t} + \lambda^h, \qquad z^h \equiv -\rho\,\widehat{\mathcal{M}}^h_{u,v}\,\tfrac{\partial v^h}{\partial t}. \tag{3.57}$$

The following theorem is an adaption to (P^h) of Theorem 3.3 in [7] for a deep quench multi-component Cahn-Hilliard system with a concentration dependent mobility matrix.

THEOREM. 3.3. *Let u^0, v^0 and b satisfy the assumptions of Theorem 3.1. Let the assumptions (A) hold. Then for all $h \leq h_0$ such that $\|Q^h_\gamma(u^0 \pm v^0) - \frac{1}{2}\|_{0,\infty} \leq \frac{1}{2}$ there exists a solution $\{u^h, v^h, \lambda^h, w^h, z^h\}$ to (P^h) on Ω_T for any $T > 0$ if $d \leq 2$ or if $b > 0$ is constant, and for some $T > 0$ if $d = 3$ and b is non-constant, such that the following stability bounds hold independently of h*

$$\|u^h\|_{L^\infty(0,T;H^1(\Omega))} + \|\tfrac{\partial u^h}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} + \|v^h\|_{L^\infty(0,T;H^1(\Omega))} + \|\tfrac{\partial v^h}{\partial t}\|_{L^2(\Omega_T)} \leq C, \tag{3.58a}$$

$$\|\tfrac{\partial u^h}{\partial t}\|_{L^\infty(0,T;(H^1(\Omega))')} + \|\tfrac{\partial u^h}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\tfrac{\partial v^h}{\partial t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\tfrac{\partial v^h}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\lambda^h\|_{L^\infty(0,T)} \leq C_b(T). \tag{3.58b}$$

In addition the solution $\{u^h, v^h\}$ is unique over Ω_T. Furthermore, these unique solutions $\{u, v\}$ and $\{u^h, v^h\}$ of (P) and (P^h) satisfy

$$\|u - u^h\|^2_{L^2(0,T;H^1(\Omega))} + \|u - u^h\|^2_{L^\infty(0,T;(H^1(\Omega))')} + \|v - v^h\|^2_{L^2(0,T;H^1(\Omega))} + \|v - v^h\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_b(T)\,h^2. \tag{3.59}$$

PROOF. One could prove existence of a solution to (P^h) and the corresponding bounds (3.58a,b) by passing to the limit $\theta \to 0$ in (P^h_θ). This would be an analogue of the existence proof for (P), see Theorem 2.3 above. However, this approach would require the more restrictive assumptions (A_θ) on the mesh. An alternative approach is to discretize (P^h) in time yielding the analogue of ($\mathrm{P}^{h,\tau}$) with a "frozen" b. Then prove existence and a priori bounds for this fully discrete scheme, which is a simple adaption of Theorem 4.3 below, and then pass to the limit $\tau \to 0$ to prove the existence of a solution to (P^h) and the corresponding bounds (3.58a,b), which are the analogues of (3.39a,c) and (3.40).

Uniqueness of the solution $\{u^h, v^h, z^h\}$ to (P^h) follows as for ($\mathrm{P}^h_{\theta(,\varepsilon)}$), see (3.44). Similarly to (P), see the proof of Theorem 2.3 above, we can not guarantee the uniqueness of λ^h and hence w^h if $\fint u^0 = \frac{1}{2}$.

Finally we prove the error bound (3.59). For *a.e.* $t \in (0,T)$, we set $e_u := u - u^h \in \Xi$, $e_v := v - v^h$, $e_u^A := (I - \pi^h)u$, $e_v^A := (I - \pi^h)v$ and $e_u^h := \pi^h u - u^h$, $e_v^h := \pi^h v - v^h \in S^h$. Choosing $\eta \equiv u^h \pm v^h \in K^h \subset K$ in (2.79), $\chi \equiv \pi^h[u \pm v] \in K^h$ in (3.56), adding together the four resulting inequalities and rearranging yields, similarly to (3.53), for *a.e.* $t \in (0,T)$ that

$$\begin{aligned}
&\gamma\,[\,|e_u|_1^2 + |e_v|_1^2\,] + (\mathcal{G}_{u,v}\tfrac{\partial e_u}{\partial t}, e_u) + \rho\,(\mathcal{M}_{u,v}\tfrac{\partial e_v}{\partial t}, e_v) \\
&\quad\le (\alpha\, e_u, \overline{e}_u^h) + (\beta\, e_v, e_v^h) + \gamma\,[\,(\nabla e_u, \nabla e_u^A) + (\nabla e_v, \nabla e_v^A)\,] \\
&\qquad + (\mathcal{G}_{u,v}\tfrac{\partial e_u}{\partial t}, e_u^A) + \rho\,(\mathcal{M}_{u,v}\tfrac{\partial e_v}{\partial t}, e_v^A) \\
&\qquad + (\gamma\Delta u - \mathcal{G}_{u,v}\tfrac{\partial u}{\partial t} + \alpha\, u + \lambda^h, e_u^A) + (\gamma\Delta v - \rho\,\mathcal{M}_{u,v}\tfrac{\partial v}{\partial t} + \beta\, v, e_v^A) \\
&\qquad + [\,(\widehat{\mathcal{G}}_{u,v}^h\tfrac{\partial u^h}{\partial t}, \overline{e}_u^h)^h - (\widehat{\mathcal{G}}_{u,v}^h\tfrac{\partial u^h}{\partial t}, \overline{e}_u^h)\,] + ((\widehat{\mathcal{G}}_{u,v}^h - \mathcal{G}_{u,v})\tfrac{\partial u^h}{\partial t}, \overline{e}_u^h) \\
&\qquad + \rho\,[\,(\widehat{\mathcal{M}}_{u,v}^h\tfrac{\partial v^h}{\partial t}, e_v^h)^h - (\widehat{\mathcal{M}}_{u,v}^h\tfrac{\partial v^h}{\partial t}, e_v^h)\,] + \rho\,((\widehat{\mathcal{M}}_{u,v}^h - \mathcal{M}_{u,v})\tfrac{\partial v^h}{\partial t}, e_v^h) \\
&\qquad + \alpha\,[\,(u^h, e_u^h) - (u^h, e_u^h)^h\,] + \beta\,[\,(v^h, e_v^h) - (v^h, e_v^h)^h\,], \qquad (3.60)
\end{aligned}$$

where $\overline{e}_u^h := (I - \fint\,)e_u^h \in \Xi^h$. On noting the bounds (2.84a,b) and (3.58a,b), the remainder of the proof of (3.59) follows the techniques used in (3.53) and (3.55) above. □

4 Fully Discrete Approximations

4.1 Logarithmic Free Energy

We now consider the fully discrete approximation $(\mathrm{P}_\theta^{h,\tau})$, see (1.12a–c), to (P_θ). Choosing $\chi \equiv 1$ in (1.12a) yields that $\fint U_\theta^n = \fint U_\theta^0 = \fint u^0$, $n = 1 \to N$. Hence, similarly to (3.36a,b), it follows from (1.12a,b), (3.19), (1.6b), (2.14), (3.29) and (1.12c) with $\chi \equiv 1$ that for $n = 1 \to N$

$$W_\theta^n \equiv -\widehat{\mathcal{G}}_{n-1}^{\theta,h}\left(\frac{U_\theta^n - U_\theta^{n-1}}{\tau_n}\right) + \Lambda_\theta^n, \quad Z_\theta^n \equiv -\rho\,\widehat{\mathcal{M}}_{n-1}^{\theta,h}\left(\frac{V_\theta^n - V_\theta^{n-1}}{\tau_n}\right) \qquad (4.1a)$$

$$\text{and} \quad \Lambda_\theta^n := \fint \pi^h[\,\theta\,\phi(U_\theta^n + V_\theta^n) + \theta\,\phi(U_\theta^n - V_\theta^n) - \alpha\, U_\theta^{n-1}\,]; \qquad (4.1b)$$

where for notational convenience we set $\widehat{\mathcal{G}}_{n-1}^{\theta,h} \equiv \widehat{\mathcal{G}}_{U_\theta^{n-1},V_\theta^{n-1}}^h$ and $\widehat{\mathcal{M}}_{n-1}^{\theta,h} \equiv \widehat{\mathcal{M}}_{U_\theta^{n-1},V_\theta^{n-1}}^h$ throughout this section. Therefore for $b_{\min} > 0$, $(\mathrm{P}_\theta^{h,\tau})$ can be rewritten as:

Let $U_\theta^0 \equiv Q_\gamma^h u^0$ and $V_\theta^0 \equiv Q_\gamma^h v^0$. For $n = 1 \to N$, find $\{U_\theta^n, V_\theta^n\} \in S_m^h \times S^h$ such that

$$\begin{aligned}
&\gamma\,(\nabla(U_\theta^n \pm V_\theta^n), \nabla\chi) + (\widehat{\mathcal{G}}_{n-1}^{\theta,h}\left(\frac{U_\theta^n - U_\theta^{n-1}}{\tau_n}\right) - \Lambda_\theta^n \pm \rho\,\widehat{\mathcal{M}}_{n-1}^{\theta,h}\left(\frac{V_\theta^n - V_\theta^{n-1}}{\tau_n}\right), \chi)^h \\
&\quad + (2\,\theta\,\phi(U_\theta^n \pm V_\theta^n) - (\alpha\, U_\theta^{n-1} \pm \beta\, V_\theta^{n-1}), \chi)^h = 0 \quad \forall\, \chi \in S^h, \qquad (4.2)
\end{aligned}$$

where Λ_θ^n is defined by (4.1b) and $\{W_\theta^n, Z_\theta^n\}$ can be obtained from (4.1a). For later use, we introduce the discrete Lyapunov functional $\mathcal{J}_{\theta(,\varepsilon)}^h : [S^h]^2 \to \mathbf{R}$ defined by

$$\mathcal{J}_{\theta(,\varepsilon)}^h(\chi_1, \chi_2) := \tfrac{\gamma}{2}\,[\,|\chi_1|_1^2 + |\chi_2|_1^2\,] + (\Psi_{\theta(,\varepsilon)}(\chi_1, \chi_2), 1)^h \quad \forall\, \chi_1,\, \chi_2 \in S^h. \quad (4.3)$$

Theorems 4.1 and 4.2 below are adaptations to $(\mathrm{P}_\theta^{h,\tau})$ of Theorems 4.1 and 4.2 in [7] for a multi-component Cahn-Hilliard system with a concentration dependent mobility matrix.

THEOREM. 4.1. *Let $\theta \leq \theta_{\max}$ and b satisfy (1.6a) with $b_{\min} > 0$. Let the assumptions on u^0, v^0 of Theorem 2.1 and (A_θ) hold. Then for all $h \leq h_1$ such that $\|Q_\gamma^h(u^0 \pm v^0) - \frac{1}{2}\|_{0,\infty} \leq \frac{1}{2}$ and for all time partitions $\{\tau_n\}_{n=1}^N$ of $[0,T]$ there exists a unique solution $\{U_\theta^n, V_\theta^n, W_\theta^n, Z_\theta^n\}_{n=1}^N$ to $(P_\theta^{h,\tau})$ such that*

$$\max_{n=0\to N} [\,\|U_\theta^n\|_1^2 + \|V_\theta^n\|_1^2\,] + \sum_{n=1}^N [\,\|U_\theta^n - U_\theta^{n-1}\|_1^2 + |V_\theta^n - V_\theta^{n-1}|_1^2\,] + \sum_{n=1}^N \tau_n\,[\,\|\tfrac{U_\theta^n - U_\theta^{n-1}}{\tau_n}\|_{-1}^2 + |\tfrac{V_\theta^n - V_\theta^{n-1}}{\tau_n}|_0^2 \leq C, \tag{4.4a}$$

$$\sum_{n=1}^N \tau_n\,[\,\theta^2\,|\,\pi^h[\phi(U_\theta^n \pm V_\theta^n)]\,|_0^2 + |\Lambda_\theta^n|^2 + \|W_\theta^n\|_1^2 + |Z_\theta^n|_0^2\,] \leq C_b. \tag{4.4b}$$

Furthermore, $U_\theta^n \in S_m^h$ and $U_\theta^n \pm V_\theta^n \in K^h$, $n = 0 \to N$; in fact

$$0 < U_\theta^n \pm V_\theta^n < 1 \qquad n = 1 \to N. \tag{4.5}$$

PROOF. From our assumptions on u^0, v^0, $Q_\gamma^h(u^0 \pm v^0)$, and (1.11), (3.42) and (1.4b) we have for all $h \leq h_1$ that

$$U^0 \in S_m^h, \quad U_\theta^0 \pm V_\theta^0 \in K^h, \quad \|U_\theta^0\|_1^2 + \|V_\theta^0\|_1^2 + (\Psi_\theta(U_\theta^0, V_\theta^0), 1)^h \leq C. \tag{4.6}$$

For $n = 1 \to N$, given $U_\theta^{n-1} \in S_m^h$ with $U_\theta^{n-1} \pm V_\theta^{n-1} \in K^h$ and $\|U_\theta^{n-1}\|_1 + \|V_\theta^{n-1}\|_1 \leq C$, we prove existence of $\{U_\theta^n, V_\theta^n\}$ satisfying (4.2) by considering the regularized version for $\varepsilon < \varepsilon_0$:
Find $\{U_{\theta,\varepsilon}^n, V_{\theta,\varepsilon}^n\} \in S_m^h \times S^h$ such that

$$\gamma\,(\nabla(U_{\theta,\varepsilon}^n \pm V_{\theta,\varepsilon}^n), \nabla\chi) + (\widehat{\mathcal{G}}_{n-1}^{\theta,h}\left(\frac{U_{\theta,\varepsilon}^n - U_\theta^{n-1}}{\tau_n}\right) \pm \rho\,\widehat{\mathcal{M}}_{n-1}^{\theta,h}\left(\frac{V_{\theta,\varepsilon}^n - V_\theta^{n-1}}{\tau_n}\right), \chi)^h + (2\,\theta\,\phi_\varepsilon(U_{\theta,\varepsilon}^n \pm V_{\theta,\varepsilon}^n) - \Lambda_{\theta,\varepsilon}^n - (\alpha\,U_\theta^{n-1} \pm \beta\,V_\theta^{n-1}), \chi)^h = 0 \quad \forall\,\chi \in S^h, \tag{4.7}$$

$$\text{where} \quad \Lambda_{\theta,\varepsilon}^n := \fint \pi^h[\,\theta\,\phi_\varepsilon(U_{\theta,\varepsilon}^n + V_{\theta,\varepsilon}^n) + \theta\,\phi_\varepsilon(U_{\theta,\varepsilon}^n - V_{\theta,\varepsilon}^n) - \alpha\,U_\theta^{n-1}\,]. \tag{4.8}$$

Existence and uniqueness of $\{U_{\theta,\varepsilon}^n, V_{\theta,\varepsilon}^n\}$ follows by noting that (4.7) is the Euler-Lagrange equation of the strictly convex minimization problem

$$\min_{\chi_1 \in S_m^h,\,\chi_2 \in S^h} \Big\{ \tfrac{\gamma}{2}\,[\,|\chi_1|_1^2 + |\chi_2|_1^2\,] + \theta\,(\,\Phi_\varepsilon(\chi_1 + \chi_2) + \Phi_\varepsilon(\chi_1 - \chi_2), 1)^h + \tfrac{1}{2\tau_n}\,|\,[\,b_{n-1}^\theta\,]^{\frac{1}{2}}\nabla\widehat{\mathcal{G}}_{n-1}^{\theta,h}(\chi_1 - U_\theta^{n-1})\,|_0^2 - \alpha\,(U_\theta^{n-1}, \chi_1)^h + \tfrac{\rho}{2\tau_n}\,|\,[\,b_{n-1}^\theta\,]^{\frac{1}{2}}\widehat{\mathcal{M}}_{n-1}^{\theta,h}(\chi_2 - V_\theta^{n-1})\,|_0^2 - \beta\,(V_\theta^{n-1}, \chi_2)^h \Big\}, \tag{4.9}$$

where for notational convenience we set $b^{\theta}_{n-1} \equiv b(U^{n-1}_{\theta}, V^{n-1}_{\theta})$ throughout this section. Choosing $\chi \equiv (U^n_{\theta,\varepsilon} - U^n_{\theta}) \pm (V^n_{\theta,\varepsilon} - V^n_{\theta}) \in S^h$ in (4.7), then adding together these $\pm$ versions yields that

$$\begin{aligned}
&\gamma\,[\,(\nabla U^n_{\theta,\varepsilon}, \nabla(U^n_{\theta,\varepsilon} - U^{n-1}_{\theta})) + (\nabla V^n_{\theta,\varepsilon}, \nabla(V^n_{\theta,\varepsilon} - V^{n-1}_{\theta}))\,] \\
&\quad + \theta\,(\phi_\varepsilon(U^n_{\theta,\varepsilon} + V^n_{\theta,\varepsilon}), (U^n_{\theta,\varepsilon} - U^{n-1}_{\theta}) + (V^n_{\theta,\varepsilon} - V^{n-1}_{\theta}))^h \\
&\quad + \theta\,(\phi_\varepsilon(U^n_{\theta,\varepsilon} - V^n_{\theta,\varepsilon}), (U^n_{\theta,\varepsilon} - U^{n-1}_{\theta}) - (V^n_{\theta,\varepsilon} - V^{n-1}_{\theta}))^h \\
&\quad + \tau_n\,|\,[\,b^{\theta}_{n-1}]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^{\theta,h}_{n-1}\left(\tfrac{U^n_{\theta,\varepsilon} - U^{n-1}_{\theta}}{\tau_n}\right)|^2_0 + \tau_n\,\rho\,|\,[\,b^{\theta}_{n-1}]^{\frac{1}{2}} \widehat{\mathcal{M}}^{\theta,h}_{n-1}\left(\tfrac{V^n_{\theta,\varepsilon} - V^{n-1}_{\theta}}{\tau_n}\right)|^2_0 \\
&\qquad = \alpha\,(U^{n-1}_{\theta}, U^n_{\theta,\varepsilon} - U^{n-1}_{\theta})^h + \beta\,(V^{n-1}_{\theta}, V^n_{\theta,\varepsilon} - V^{n-1}_{\theta})^h. \qquad (4.10)
\end{aligned}$$

Rearranging (4.10) on noting the convexity of Φ_ε and the identity

$$2(s-r)s = s^2 - r^2 + (s-r)^2 \quad \forall\; r,\, s \in \mathbf{R}, \qquad (4.11)$$

yields that

$$\begin{aligned}
&\tfrac{\gamma}{2}\,[\,|U^n_{\theta,\varepsilon}|^2_1 + |V^n_{\theta,\varepsilon}|^2_1 - |U^{n-1}_{\theta}|^2_1 - |V^{n-1}_{\theta}|^2_1 + |U^n_{\theta,\varepsilon} - U^{n-1}_{\theta}|^2_1 + |V^n_{\theta,\varepsilon} - V^{n-1}_{\theta}|^2_1\,] \\
&\quad + \theta\,(\Phi_\varepsilon(U^n_{\theta,\varepsilon} + V^n_{\theta,\varepsilon}) + \Phi_\varepsilon(U^n_{\theta,\varepsilon} - V^n_{\theta,\varepsilon}), 1)^h \\
&\quad - \theta\,(\Phi_\varepsilon(U^{n-1}_{\theta} + V^{n-1}_{\theta}) + \Phi_\varepsilon(U^{n-1}_{\theta} - V^{n-1}_{\theta}), 1)^h \\
&\quad + \tau_n\,|\,[\,b^{\theta}_{n-1}]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^{\theta,h}_{n-1}\left(\tfrac{U^n_{\theta,\varepsilon} - U^{n-1}_{\theta}}{\tau_n}\right)|^2_0 + \tau_n\,\rho\,|\,[\,b^{\theta}_{n-1}]^{\frac{1}{2}} \widehat{\mathcal{M}}^{\theta,h}_{n-1}\left(\tfrac{V^n_{\theta,\varepsilon} - V^{n-1}_{\theta}}{\tau_n}\right)|^2_0 \\
&\qquad \leq \alpha\,(U^{n-1}_{\theta}, U^n_{\theta,\varepsilon} - U^{n-1}_{\theta})^h + \beta\,(V^{n-1}_{\theta}, V^n_{\theta,\varepsilon} - V^{n-1}_{\theta})^h \\
&\qquad \leq \tfrac{\alpha}{2}\,[\,|U^n_{\theta,\varepsilon}|^2_h - |U^{n-1}_{\theta}|^2_h\,] + \tfrac{\beta}{2}\,[\,|V^n_{\theta,\varepsilon}|^2_h - |V^{n-1}_{\theta}|^2_h\,]. \qquad (4.12)
\end{aligned}$$

On noting (4.3), (2.9), (2.1a,b), $U^n_{\theta,\varepsilon} - U^{n-1}_{\theta} \in \Xi^h$ and our assumptions on U^{n-1}_{θ}, V^{n-1}_{θ}; it follows from (4.12) that

$$\begin{aligned}
&\mathcal{J}^h_{\theta,\varepsilon}(U^n_{\theta,\varepsilon}, V^n_{\theta,\varepsilon}) + \tfrac{\gamma}{2}\,[\,|U^n_{\theta,\varepsilon} - U^{n-1}_{\theta}|^2_1 + |V^n_{\theta,\varepsilon} - V^{n-1}_{\theta}|^2_1\,] \\
&\quad + \tau_n\,|\,[\,b^{\theta}_{n-1}]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^{\theta,h}_{n-1}\left(\tfrac{U^n_{\theta,\varepsilon} - U^{n-1}_{\theta}}{\tau_n}\right)|^2_0 + \tau_n\,\rho\,|\,[\,b^{\theta}_{n-1}]^{\frac{1}{2}} \widehat{\mathcal{M}}^{\theta,h}_{n-1}\left(\tfrac{V^n_{\theta,\varepsilon} - V^{n-1}_{\theta}}{\tau_n}\right)|^2_0 \\
&\qquad \leq \mathcal{J}^h_{\theta,\varepsilon}(U^{n-1}_{\theta}, V^{n-1}_{\theta}) \leq C. \qquad (4.13)
\end{aligned}$$

Hence on noting (4.13), (4.3) and (2.9) there exist positive constants C_i, independent of θ, ε, h and τ_n, such that

$$C_1\,[\,\|U^n_{\theta,\varepsilon}\|^2_1 + \|V^n_{\theta,\varepsilon}\|^2_1\,] - C_2 \leq \mathcal{J}^h_{\theta,\varepsilon}(U^n_{\theta,\varepsilon}, V^n_{\theta,\varepsilon}) \leq C. \qquad (4.14)$$

On choosing $\chi \equiv \tau_n\,(2\theta\,\pi^h[\phi_\varepsilon(U^n_{\theta,\varepsilon} \pm V^n_{\theta,\varepsilon})] - \Lambda^n_{\theta,\varepsilon})$ in (4.7), we have the analogue of (2.43) on noting (3.10), a Young's inequality, (3.8), (2.14), (1.6a), our assumptions on U^{n-1}_{θ} and V^{n-1}_{θ}, and (4.13)

$$\tau_n\,[\,\tfrac{3}{2}\gamma\theta\,\varepsilon\,|\,\pi^h[\phi_\varepsilon(U^n_{\theta,\varepsilon} \pm V^n_{\theta,\varepsilon})]\,|^2_1 + \tfrac{1}{2}\,|2\,\theta\,\phi_\varepsilon(U^n_{\theta,\varepsilon} \pm V^n_{\theta,\varepsilon}) - \Lambda^n_{\theta,\varepsilon}|^2_h\,]$$

$$\leq \tfrac{1}{2}\tau_n \, |\widehat{\mathcal{G}}^{\theta,h}_{n-1}\left(\tfrac{U^n_{\theta,\varepsilon}-U^{n-1}_\theta}{\tau_n}\right) \pm \rho\, \widehat{\mathcal{M}}^{\theta,h}_{n-1}\left(\tfrac{V^n_{\theta,\varepsilon}-V^{n-1}_\theta}{\tau_n}\right) - (\alpha\, U^{n-1}_\theta \pm \beta\, V^{n-1}_\theta)|^2_h$$
$$\leq \tau_n\, C_b\,[\,1 + |\,[\,b^\theta_{n-1}]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^{\theta,h}_{n-1}\left(\tfrac{U^n_{\theta,\varepsilon}-U^{n-1}_\theta}{\tau_n}\right)|^2_0$$
$$+ |\,[\,b^\theta_{n-1}]^{\frac{1}{2}} \widehat{\mathcal{M}}^{\theta,h}_{n-1}\left(\tfrac{V^n_{\theta,\varepsilon}-V^{n-1}_\theta}{\tau_n}\right)|^2_0\,] \leq C_b. \qquad (4.15)$$

Similarly to (2.45), on choosing $\chi \equiv (U^n_{\theta,\varepsilon} \pm V^n_{\theta,\varepsilon}) - \mu$ in (4.7), adding together the resulting $\pm$ versions, choosing $\mu = 0$ and 1, and noting (3.19), (3.29), (4.14), the convexity of Φ_ε, (2.1a,b), (2.2), the assumptions on u^0 and (4.13) we have that

$$\tau_n\; |\Lambda^n_{\theta,\varepsilon}|^2 \leq \tau_n\, C_b\,[\,1 + |\,[\,b^\theta_{n-1}]^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^{\theta,h}_{n-1}\left(\tfrac{U^n_{\theta,\varepsilon}-U^{n-1}_\theta}{\tau_n}\right)|^2_0$$
$$+ |\,[\,b^\theta_{n-1}]^{\frac{1}{2}} \widehat{\mathcal{M}}^{\theta,h}_{n-1}\left(\tfrac{V^n_{\theta,\varepsilon}-V^{n-1}_\theta}{\tau_n}\right)|^2_0\,] \leq C_b. \qquad (4.16)$$

It follows from (4.14) that there exist $U^n_\theta \in S^h_m$ and $V^n_\theta \in S^h$ and subsequences $\{U^n_{\theta,\varepsilon'}, V^n_{\theta,\varepsilon'}\}$ such that $U^n_{\theta,\varepsilon'} \to U^n_\theta$ and $V^n_{\theta,\varepsilon'} \to V^n_\theta$ as $\varepsilon' \to 0$. It follows from (4.15) and (4.16) that there exist $\phi^n_\pm \in S^h$ such that $\pi^h[\phi_{\varepsilon'}(U^n_{\theta,\varepsilon'} \pm V^n_{\theta,\varepsilon'})] \to \phi^n_\pm$ as $\varepsilon' \to 0$. Noting that $\phi^{-1}_{(\varepsilon)} \in C^1(\mathbf{R})$ and $[\phi_\varepsilon]^{-1}(s) \to \phi^{-1}(s)$ as $\varepsilon \to 0$, for all $s \in \mathbf{R}$, we have that $\phi^n_\pm \equiv \pi^h[\phi(U^n_\theta \pm V^n_\theta)]$. Therefore we may pass to the limit $\varepsilon' \to 0$ in (4.7) to prove existence of a solution to (4.2) at time level t_n. Uniqueness of $\{U^n_\theta, V^n_\theta\}$ follows, as for $\{U^n_{\theta,\varepsilon}, V^n_{\theta,\varepsilon}\}$, from the monotonicity of ϕ. Hence noting (4.1a,b), we have existence and uniqueness of a solution $\{U^n_\theta, V^n_\theta, W^n_\theta, Z^n_\theta\}$ to (1.12a–c) at time level t_n. In addition non-regularized versions of the bounds (4.13)–(4.16) hold; that is, with the ε subscript removed and $\varepsilon = 0$ on the left hand side of (4.15). The non-regularized versions of (4.15), (4.16) and (4.14) immediately yield (4.5) and the first two bounds in (4.4a) for U^n_θ and V^n_θ. Hence we have these bounds for $n = 1 \to N$ by the above induction process. Finally, summing the non-regularized versions of (4.13), (4.15) and (4.16), and noting (3.8), (2.14), (4.1a,b), (3.20), (3.15) and (3.30) yield the summation bounds in (4.4a,b). □

Given u^0, v^0 satisfying the assumptions of Theorem 3.1 above, we introduce for the purposes of the analysis below W^0_θ, $Z^0_\theta \in S^h$ defined by

$$(W^0_\theta \pm Z^0_\theta, \chi)^h = -\gamma(\Delta(u^0 \pm v^0), \chi) + (2\theta\, \phi(U^0_\theta \pm V^0_\theta) - (\alpha\, U^0_\theta \pm \beta\, V^0_\theta), \chi)^h$$
$$\forall\, \chi \in S^h. \qquad (4.17)$$

Hence we have that

$$W^0_\theta \pm Z^0_\theta \equiv -\gamma\, \widehat{Q}^h_0(\Delta(u^0 \pm v^0)) + 2\,\theta\, \pi^h[\phi(U^0_\theta \pm V^0_\theta)] - (\alpha\, U^0_\theta \pm \beta\, V^0_\theta). \quad (4.18)$$

Similarly to (3.47a,b), it follows from (4.18), (3.7), (1.2b), (3.10), (4.6) and the assumptions on u^0, v^0 that for all $h \leq h_0$, defined as in Theorem 3.1 above,

$$\|W^0_\theta\|_1 + |Z^0_\theta|_0 \leq C\,[\,\|u^0\|_3 + \|v^0\|_2\,] \leq C. \qquad (4.19)$$

THEOREM. 4.2. *Let* $u^0 \in H^3(\Omega)$, $v^0 \in H^2(\Omega)$, $\frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} = 0$ *on* $\partial\Omega$ *and* $\delta \in (0, \frac{1}{2})$ *be such that* $\|u^0 \pm v^0 - \frac{1}{2}\|_{0,\infty} \le \frac{1}{2} - \delta$. *Let either* $d \le 2$ *with* b *satisfying (1.6a,b) or* $d = 3$ *with* $b > 0$ *constant. Let* $\theta \le \theta_{\max}$ *and the assumptions* (A_θ) *hold. Then for all* $h \le h_0$ *such that* $\|Q^h_\gamma(u^0 \pm v^0) - \frac{1}{2}\|_{0,\infty} \le \frac{1}{2}(1-\delta)$ *and for all partitions* $\{\tau_n\}_{n=1}^N$ *of* $[0,T]$ *such that* $\tau_{n-1}/\tau_n \le C$, $n = 2 \to N$, *the unique solution* $\{U^n_\theta, V^n_\theta, W^n_\theta, Z^n_\theta\}_{n=0}^N$ *to* $(P^{h,\Delta t}_\theta)$ *satisfies*

$$\begin{aligned}
&\sum_{n=1}^N \tau_n \, [\, \|\tfrac{U^n_\theta - U^{n-1}_\theta}{\tau_n}\|_1^2 + \|\tfrac{V^n_\theta - V^{n-1}_\theta}{\tau_n}\|_1^2 \,] + \max_{n=1\to N} [\, |[b^\theta_{n-1}]^{\frac{1}{2}} \nabla W^n_\theta|_0^2 + |[b^\theta_{n-1}]^{\frac{1}{2}} Z^n_\theta|_0^2 \,] \\
&\quad + \theta \sum_{n=1}^N \tau_n \, (\tfrac{\phi(U^n_\theta \pm V^n_\theta) - \phi(U^{n-1}_\theta \pm V^{n-1}_\theta)}{\tau_n}, \tfrac{(U^n_\theta \pm V^n_\theta) - (U^{n-1}_\theta \pm V^{n-1}_\theta)}{\tau_n})^h \\
&\quad + \sum_{n=1}^N [\, |[b^\theta_{n-1}]^{\frac{1}{2}} \nabla (W^n_\theta - W^{n-1}_\theta)|_0^2 + |[b^\theta_{n-1}]^{\frac{1}{2}} (Z^n_\theta - Z^{n-1}_\theta)|_0^2 \,] \\
&\quad + \sum_{n=1}^{N-1} |(b^\theta_n - b^\theta_{n-1}, |\nabla W^n_\theta|^2 + |Z^n_\theta|^2)| \le C_b.
\end{aligned} \tag{4.20}$$

PROOF. Let $D^n_\theta := (U^n_\theta - U^{n-1}_\theta)/\tau_n \in \Xi^h$ and $F^n_\theta := (V^n_\theta - V^{n-1}_\theta)/\tau_n \in S^h$, $n = 1 \to N$. It follows from (1.12c) and (4.17) for $n \ge 1$ that

$$\begin{aligned}
&\gamma \, \tau_n \, |D^n_\theta \pm F^n_\theta|_1^2 + 2\,\theta\, (\phi(U^n_\theta \pm V^n_\theta) - \phi(U^{n-1}_\theta \pm V^{n-1}_\theta), D^n_\theta \pm F^n_\theta)^h \\
&\quad = ((W^n_\theta - W^{n-1}_\theta) \pm (Z^n_\theta - Z^{n-1}_\theta), D^n_\theta \pm F^n_\theta)^h \\
&\quad\quad + \begin{cases} \gamma\, (\nabla[\,(u^0 - U^0_\theta) \pm (v^0 - V^0_\theta)\,], \nabla(D^1_\theta \pm F^1_\theta)) & \text{if } n = 1, \\ \tau_{n-1}\, (\alpha\, D^{n-1}_\theta \pm \beta\, F^{n-1}_\theta, D^n_\theta \pm F^n_\theta)^h & \text{if } n \ge 2. \end{cases}
\end{aligned} \tag{4.21}$$

Adding together the $\pm$ versions of (4.21) and noting (1.12a,b) with $\chi \equiv W^n_\theta - W^{n-1}_\theta$ and $\chi \equiv Z^n_\theta - Z^{n-1}_\theta$, respectively, and noting (4.11) and (1.11) yields for $n \ge 1$ that

$$\begin{aligned}
&\gamma\, \tau_n \, [\, |D^n_\theta|_1^2 + |F^n_\theta|_1^2 \,] + \theta\, (\phi(U^n_\theta + V^n_\theta) - \phi(U^{n-1}_\theta + V^{n-1}_\theta), D^n_\theta + F^n_\theta)^h \\
&\quad + \theta\, (\phi(U^n_\theta - V^n_\theta) - \phi(U^{n-1}_\theta - V^{n-1}_\theta), D^n_\theta - F^n_\theta)^h \\
&\quad + \tfrac{1}{2}\, (b^\theta_{n-1}, [\, |\nabla W^n_\theta|^2 + |\nabla (W^n_\theta - W^{n-1}_\theta)|^2 \,] + \rho^{-1}\, [\, |Z^n_\theta|^2 + |Z^n_\theta - Z^{n-1}_\theta|^2 \,]) \\
&\quad\quad = d^\theta_n := \tfrac{1}{2}\, (b^\theta_{n-1}, |\nabla W^{n-1}_\theta|^2 + \rho^{-1}\, |Z^{n-1}_\theta|^2) \\
&\quad\quad\quad + \begin{cases} [\,(U^0_\theta - u^0, D^1_\theta) + (V^0_\theta - v^0, F^1_\theta)\,] & \text{if } n = 1, \\ \tau_{n-1}\, [\, \alpha (D^{n-1}_\theta, D^n_\theta)^h + \beta\, (F^{n-1}_\theta, F^n_\theta)^h \,] & \text{if } n \ge 2. \end{cases}
\end{aligned} \tag{4.22}$$

It follows from (2.16), (3.15), (3.20), (2.29), (2.30), (3.30), (4.1a), (3.14), (3.8) and a Young's inequality that

$$\begin{aligned}
d^\theta_1 &\le \tfrac{1}{2}\, (b^\theta_0, |\nabla W^0_\theta|^2 + \rho^{-1} |Z^0_\theta|^2) + \tfrac{1}{4}\, (b^\theta_0, |\nabla W^1_\theta|^2 + \rho^{-1} |Z^1_\theta|^2) \\
&\quad\quad + C_b\, [\, |u^0 - U^0_\theta|_1^2 + |v^0 - V^0_\theta|_0^2], \qquad (4.23a) \\
d^\theta_n &\le \tfrac{1}{2}\, (b^\theta_{n-2}, |\nabla W^{n-1}_\theta|^2 + \rho^{-1} |Z^{n-1}_\theta|^2) + \tfrac{\gamma}{2}\, \tau_n\, [\, |D^n_\theta|_1^2 + |F^n_\theta|_0^2]
\end{aligned}$$

$$+ \tfrac{1}{2} \left([b^\theta_{n-1} - b^\theta_{n-2}], |\nabla W^{n-1}_\theta|^2 + \rho^{-1} |Z^{n-1}_\theta|^2 \right) \\ + C_b \, \tfrac{(\tau_{n-1})^2}{\tau_n} \, [\, \|D^{n-1}_\theta\|^2_{-1} + |F^{n-1}_\theta|^2_0 \,] \qquad \text{if } n \geq 2. \tag{4.23b}$$

Similarly to (2.50), we have from (1.6a,b), (2.16), (2.17), a Young's inequality and (3.3) that for $n \geq 2$

$$|([\tfrac{b^\theta_{n-1} - b^\theta_{n-2}}{\tau_{n-1}}], |\nabla W^{n-1}_\theta|^2 + \rho^{-1} |Z^{n-1}_\theta|^2)| \\ \leq \tfrac{\gamma}{4} [\, |D^{n-1}_\theta|^2_1 + \|F^{n-1}_\theta\|^2_1 \,] + C \, [\, |W^{n-1}_\theta|^4_{1,2(1+\zeta)} + |Z^{n-1}_\theta|^4_{0,2(1+\zeta)} \,] \\ \leq \tfrac{\gamma}{4} [\, |D^{n-1}_\theta|^2_1 + \|F^{n-1}_\theta\|^2_1 \,] + C \, [\, |W^{n-1}_\theta|^{4(1+\zeta)}_1 + |Z^{n-1}_\theta|^{4(1+\zeta)}_0 \,] + C \, \zeta \, h^{-2d}, \tag{4.24}$$

where $\zeta = 0$ for $d = 1$, and for any $\zeta > 0$ for $d = 2$. For b constant the above term is zero and the argument below simplifies considerably. We set $r_0 := 1$ and for $n \geq 1$

$$p_n := \tfrac{\gamma}{2} \, \tau_n \, [\, |D^n_\theta|^2_1 + |F^n_\theta|^2_1 \,], \qquad q_n := \tfrac{1}{2} \, (b^\theta_{n-1}, |\nabla W^n_\theta|^2 + \rho^{-1} |Z^n_\theta|^2), \\ r_n := \max\{p_n + q_n, \, r_{n-1}\}, \quad s_n := C \, [\, \|D^n_\theta\|^2_{-1} + |F^n_\theta|^2_0 + |F^{n+1}_\theta|^2_0 \,] + C \, \zeta h^{-2d} \\ \text{and} \qquad y_n := C \, q_n + s_n \, . \tag{4.25}$$

It follows from (4.4b) and if $\zeta \leq C h^{2d}$ that

$$\sum_{n=1}^{N-1} \tau_n \, y_n = C \, \sum_{n=1}^{N-1} \tau_n \, q_n + \sum_{n=1}^{N-1} \tau_n \, s_n \leq C_b \, (1 + \zeta \, h^{-2d}) \leq C_b. \tag{4.26}$$

It follows from (4.25), (4.22), (4.23a), (4.19) and (4.6) that $r_1 \leq C$. It follows from (4.25), (4.22), (4.23b), (4.24) and the assumption $\tau_{n-1}/\tau_n \leq C$ that for $n \geq 2$

$$\max\{1, \, q_{n-1}\} \leq r_{n-1} \leq r_n \leq r_{n-1} + C \, \tau_{n-1} \, q_{n-1}^{2(1+\zeta)} + \tau_{n-1} \, s_{n-1}. \tag{4.27}$$

For $d = 1$, i.e. $\zeta = 0$, it follows from (4.27) and (4.26) that for $n = 2 \to N$

$$r_n \leq (1 + C \tau_{n-1} \, q_{n-1}) \, r_{n-1} + \tau_{n-1} \, s_{n-1} \leq r_{n-1} \, \exp(C \tau_{n-1} \, q_{n-1}) + \tau_{n-1} \, s_{n-1} \\ \leq (r_1 + \sum_{i=1}^{n-1} \tau_i \, s_i) \, \exp(C \sum_{i=1}^{n-1} \tau_i \, q_i) \leq C. \tag{4.28}$$

For $d = 2$, i.e. $\zeta > 0$, it follows from the mean value theorem, (4.27) and (4.25) that

$$- \tfrac{1}{2\zeta} [\, r_n^{-2\zeta} - r_{n-1}^{-2\zeta} \,] \leq r_{n-1}^{-(1+2\zeta)} \, [\, r_n - r_{n-1} \,] \leq \tau_{n-1} \, y_{n-1} \qquad n \geq 2. \tag{4.29}$$

From summing (4.29) and noting (4.26) and that $r_1 \leq C$, we have that

$$r_n \leq \, [1 - 2 \, \zeta \, r_1^{2\zeta} \sum_{i=1}^{n-1} \tau_i \, y_i \,]^{-\frac{1}{2\zeta}} \, r_1 \leq [\, 1 + 4 \, \zeta \, r_1^{2\zeta} \sum_{i=1}^{n-1} \tau_i \, y_i \,]^{\frac{1}{2\zeta}} \, r_1$$

$$\leq r_1 \exp(2\, r_1^{2\zeta} \sum_{i=1}^{n-1} \tau_i\, y_i\,) \leq C \qquad n = 2 \to N; \tag{4.30}$$

provided $\zeta > 0$ is chosen sufficiently small so that $\zeta \leq C\, h^4$, as $d = 2$, and $4\,\zeta\, r_1^{2\zeta} \sum_{n=1}^{N-1} \tau_n\, y_n \leq 1$. Hence the second bound in (4.20) follows from (4.25), (4.28) and (4.30). The remaining bounds in (4.20) follow from summing (4.22) and noting (4.23a,b), (4.24), the second bound in (4.20), (4.4b) and (2.14). □

REMARK. 4.1. *The timestep constraint* $\tau_{n-1}/\tau_n \leq C$ *in Theorem 4.2 above arises solely from using the split time level approximation in (1.12c). If we replaced* $\alpha\, U_\theta^{n-1} \pm \beta\, V_\theta^{n-1}$ *in (1.12c) by* $\alpha\, U_\theta^n \pm \beta\, V_\theta^n$*, then this constraint could be removed. However, instead we would require* $\tau_n \max_{j \in J} b(U_\theta^{n-1}, V_\theta^{n-1})(x_j) \leq \min\{\frac{4\gamma}{\alpha^2}, \frac{\rho}{\beta}\}$, $n = 1 \to N$, *in order to guarantee the uniqueness of* $\{U_\theta^n, V_\theta^n\}_{n=1}^N$. *A similar comment applies in the deep quench limit.*

4.2 The Deep Quench Limit

Similarly to (4.2) and (4.1a) the fully discrete approximation of (P), $(\mathrm{P}^{h,\tau})$ see (1.13a–c), can be rewritten as:
Let $U^0 \equiv Q_\gamma^h u^0$ and $V^0 \equiv Q_\gamma^h v^0$. For $n = 1 \to N$, find $\{U^n, V^n, \Lambda^n\} \in S_m^h \times S^h \times \mathbf{R}$ such that $U^n \pm V^n \in K^h$ and

$$\begin{aligned} &\gamma(\nabla(U^n \pm V^n), \nabla(\chi - (U^n \pm V^n))) \\ &\quad + (\widehat{\mathcal{G}}_{n-1}^h \left(\tfrac{U^n - U^{n-1}}{\tau_n}\right) - \Lambda^n \pm \rho\, \widehat{\mathcal{M}}_{n-1}^h \left(\tfrac{V^n - V^{n-1}}{\tau_n}\right), \chi - (U^n \pm V^n))^h \\ &\qquad\qquad \geq (\alpha\, U^{n-1} \pm \beta\, V^{n-1}, \chi - (U^n \pm V^n))^h \quad \forall\, \chi \in K^h, \end{aligned} \tag{4.31}$$

where for notational convenience we set $\widehat{\mathcal{G}}_{n-1}^h \equiv \widehat{\mathcal{G}}_{U^{n-1}, V^{n-1}}^h$, $\widehat{\mathcal{M}}_{n-1}^h \equiv \widehat{\mathcal{M}}_{U^{n-1}, V^{n-1}}^h$ and $b_{n-1} \equiv b(U^{n-1}, V^{n-1})$ throughout this section. It follows for $n = 1 \to N$ that

$$W^n \equiv -\widehat{\mathcal{G}}_{n-1}^h \left(\tfrac{U^n - U^{n-1}}{\tau_n}\right) + \Lambda^n, \quad Z^n \equiv -\rho\, \widehat{\mathcal{M}}_{n-1}^h \left(\tfrac{V^n - V^{n-1}}{\tau_n}\right). \tag{4.32}$$

We introduce the analogue of (4.3), the discrete Lyapunov functional $\mathcal{J}^h : [S^h]^2 \to \mathbf{R}$ defined by

$$\mathcal{J}^h(\chi_1, \chi_2) := \tfrac{\gamma}{2}\,[\,|\chi_1|_1^2 + |\chi_2|_1^2\,] + (\Psi(\chi_1, \chi_2), 1)^h \qquad \forall\, \chi_1, \chi_2 \in S^h, \tag{4.33}$$

where on recalling (1.9c) and (1.4b)

$$\Psi(\chi_1, \chi_2) := \mathcal{I}_{[0,1]}(\chi_1 + \chi_2) + \mathcal{I}_{[0,1]}(\chi_1 - \chi_2) + \tfrac{1}{2}\,[\,\alpha\, \chi_1\,(1 - \chi_1) - \beta\, \chi_2^2\,]. \tag{4.34}$$

Theorem 4.3 below is an adaption to $(\mathrm{P}^{h,\tau})$ of Theorem 4.3 in [7] for a deep quench multi-component Cahn-Hilliard system with a concentration dependent mobility matrix.

THEOREM. 4.3. *Let b satisfy (1.6a) with $b_{\min} > 0$. Let the assumptions on u^0, v^0 of Theorem 2.1 and (A) hold. Then for all $h \leq h_1$ such that $\|Q^h_\gamma(u^0 \pm v^0) - \frac{1}{2}\|_{0,\infty} \leq \frac{1}{2}$ and for all time partitions $\{\tau_n\}_{n=1}^n$ of $[0,T]$ there exists a solution $\{U^n, V^n, \Lambda^n, W^n, Z^n\}_{n=1}^N$ to $(P^{h,\tau})$. Moreover $\{U^n, V^n, Z^n\}_{n=1}^N$ are unique and the following stability bounds hold such that*

$$\max_{n=0\to N} [\,\|U^n\|_1^2 + \|V^n\|_1^2\,] + \sum_{n=1}^N [\,\|U^n - U^{n-1}\|_1^2 + |V^n - V^{n-1}|_1^2\,]$$
$$+ \sum_{n=1}^N \tau_n\,[\,\|\tfrac{U^n - U^{n-1}}{\tau_n}\|_{-1}^2 + |\tfrac{V^n - V^{n-1}}{\tau_n}|_0^2\,] \leq C, \tag{4.35a}$$
$$\sum_{n=1}^N \tau_n\,[\,\|W^n\|_1^2 + |Z^n|_0^2 + |\Lambda^n|^2\,] \leq C_b. \tag{4.35b}$$

Furthermore if $u^0 \in H^3(\Omega)$, $v^0 \in H^2(\Omega)$, $\frac{\partial u^0}{\partial \nu} = \frac{\partial v^0}{\partial \nu} = 0$ on $\partial\Omega$ and if either $d \leq 2$ with b satisfying (1.6a,b) or $d = 3$ with $b > 0$ constant; then for all $h \leq h_1$ and for all partitions $\{\tau_n\}_{n=1}^N$ of $[0,T]$ such that $\tau_{n-1}/\tau_n \leq C$, $n = 2 \to N$, the solution $\{U^n, V^n, \Lambda^n, W^n, Z^n\}_{n=0}^N$ to $(P^{h,\Delta t})$ satisfies

$$\sum_{n=1}^N \tau_n\,[\,\|\tfrac{U^n - U^{n-1}}{\tau_n}\|_1^2 + \|\tfrac{V^n - V^{n-1}}{\tau_n}\|_1^2\,] + \max_{n=1\to N} [\,|[b_{n-1}]^{\frac{1}{2}}\nabla W^n|_0^2 + |[b_{n-1}]^{\frac{1}{2}} Z^n|_0^2\,]$$
$$+ \sum_{n=1}^N [\,|[b_{n-1}]^{\frac{1}{2}}\nabla(W^n - W^{n-1})|_0^2 + |[b_{n-1}]^{\frac{1}{2}}(Z^n - Z^{n-1})|_0^2\,]$$
$$+ \sum_{n=1}^{N-1} |(b_n - b_{n-1}, |\nabla W^n|^2 + |Z^n|^2)| + \max_{n=1\to N} |\Lambda^n|^2 \leq C_b. \tag{4.36}$$

PROOF. Existence and uniqueness of $\{U^n, V^n\}$ follows by noting that (4.31) is the Euler-Lagrange inequality of the strictly convex minimization problem

$$\min_{\substack{\chi_1 \in S^h_m,\, \chi_2 \in S^h \\ \chi_1 \pm \chi_2 \in K^h}} \Big\{ \tfrac{\gamma}{2}\,[\,|\chi_1|_1^2 + |\chi_2|_1^2\,] + \tfrac{1}{2\tau_n}\,|\,[\,b_{n-1}\,]^{\frac{1}{2}}\nabla\widehat{\mathcal{G}}^h_{n-1}(\chi_1 - U^{n-1})\,|_0^2$$
$$+ \tfrac{\rho}{2\tau_n}\,|\,[\,b_{n-1}\,]^{\frac{1}{2}}\widehat{\mathcal{M}}^h_{n-1}(\chi_2 - V^{n-1})\,|_0^2 - \alpha\,(U^{n-1}, \chi_1)^h - \beta\,(V^{n-1}, \chi_2)^h \Big\}. \tag{4.37}$$

Existence of the Lagrange multiplier Λ^n in (4.31) then follows from standard optimization theory. On noting (4.32) we then have the existence of W^n and the existence and uniqueness of Z^n. Once again, as for (P) and (P^h), we can not guarantee the uniqueness of Λ^n if $⨍\, u^0 = \frac{1}{2}$.

Choosing $\chi \equiv U^{n-1} \pm V^{n-1}$ in (4.31) yields the analogues (subscripts and superscripts θ, ε removed) of (4.13) and (4.14) on noting that $U^n \pm V^n \in K^h$;

that is, $\{U_j^n, V_j^n\} \in \overline{\mathcal{Q}}$ for all $j \in J$. The latter yields the first bound in (4.35a). Summing the former and noting (3.20), (3.15) and (3.30) yields the remaining bounds in (4.35a). Choosing $\chi \equiv \mu$ in (4.31), adding together the resulting $\pm$ versions and then choosing $\mu = 0$ and 1 yields the analogue (subscripts and superscripts θ, ε removed) of (4.16). Summing this analogue of (4.16) and noting (4.32), (3.20), (3.15), (3.30) and (4.35a) yields the bounds in (4.35b).

Given the further stated assumptions on u^0, v^0, then similarly to (4.17) we introduce for the purposes of the analysis below W^0, $Z^0 \in S^h$ defined by

$$(W^0 \pm Z^0, \chi)^h = -\gamma(\Delta(u^0 \pm v^0), \chi) - (\alpha\, U^0 \pm \beta\, V^0), \chi)^h \quad \forall\, \chi \in S^h. \quad (4.38)$$

Similarly to (4.19), we have that $\|W^0\|_1 + |Z^0|_0 \leq C$. Let $D^n := (U^n - U^{n-1})/\tau_n$ and $F^n := (V^n - V^{n-1})/\tau_n$ for $n \geq 1$. It follows from (1.13c) and (4.38) that the analogue of (4.21) holds; that is, all θ subscripts and superscripts and the $\theta\,\phi(\cdot)$ terms removed and "=" replaced by "$\leq$". Adding together these $\pm$ versions and noting (1.13a,b) with $\chi \equiv W^n - W^{n-1}$ and $\chi \equiv Z^n - Z^{n-1}$, respectively, and (4.11) and (1.11) yields for $n \geq 1$ that

$$\begin{aligned}
&\gamma\, \tau_n\, [\, |D^n|_1^2 + |F^n|_1^2\,] \\
&\quad + \tfrac{1}{2}\, (b_{n-1}, [\, |\nabla W^n|^2 + |\nabla(W^n - W^{n-1})|^2\,] + \rho^{-1}\, [\, |Z^n|^2 + |Z^n - Z^{n-1}|^2\,]\,) \\
&\qquad = d_n := \tfrac{1}{2}\, (b_{n-1}, |\nabla W^{n-1}|^2 + \rho^{-1}\, |Z^{n-1}|^2) \\
&\qquad\quad + \begin{cases} [\, (U^0 - u^0, D^1) + (V^0 - v^0, F^1)\,] & \text{if } n = 1, \\ \tau_{n-1}\, [\, \alpha(D^{n-1}, D^n)^h + \beta\, (F^{n-1}, F^n)^h\,] & \text{if } n \geq 2. \end{cases}
\end{aligned} \quad (4.39)$$

Following the remainder of the proof of Theorem 4.2 yields the bounds involving $\{U^n, V^n, W^n, Z^n\}_{n=0}^N$ in (4.36). The bound on the Lagrange multipliers, $\{\Lambda^n\}_{n=1}^N$ in (4.36) follows from the analogue of (4.16), discussed above, (4.32) and the maximum bounds on $\{|W^n|_1, |Z^n|_0\}_{n=1}^N$ in the first line of (4.36). $\square$

4.3 Error Analysis

We now adapt the framework in [22] for analysing the discretization error in the backward Euler method. In addition to (1.14), we introduce

$$U_{(\theta)}^{+}(\cdot, t) := U_{(\theta)}^n(\cdot) \qquad U_{(\theta)}^{-}(\cdot, t) := U_{(\theta)}^{n-1}(\cdot) \qquad t \in (t_{n-1}, t_n]. \quad (4.40)$$

On setting

$$\ell(t) := \tfrac{t_n - t}{\tau_n}, \qquad \overline{\tau}(t) := \tau_n \qquad t \in (t_{n-1}, t_n], \qquad n = 1 \to N; \quad (4.41)$$

it follows from (1.14) and (4.40) that for $a.e\ t \in (0, T)$

$$U_{(\theta)} - U_{(\theta)}^{+} = -\overline{\tau}\, \ell\, \tfrac{\partial U_{(\theta)}}{\partial t}, \qquad U_{(\theta)} - U_{(\theta)}^{-} = \overline{\tau}\, (1 - \ell)\, \tfrac{\partial U_{(\theta)}}{\partial t}. \quad (4.42)$$

Obviously, similar identities hold for the other variables, $V_{(\theta)}$, $W_{(\theta)}$, $Z_{(\theta)}$ and $\Lambda_{(\theta)}$. On recalling (4.3), (1.4b), (4.33) and (4.34); let $\mathcal{J}_{(\theta)}^{h,+} : [S^h]^2 \to \mathbf{R}$ be defined for all χ_1, $\chi_2 \in S^h$ by

$$\mathcal{J}_{\theta}^{h,+}(\chi_1, \chi_2) := \tfrac{\gamma}{2}\, [\, |\chi_1|_1^2 + |\chi_2|_1^2\,] + (\Psi_{(\theta)}^{+}(\chi_1, \chi_2), 1)^h, \quad (4.43a)$$

$$\text{where} \quad \Psi^+_{(\theta)}(\chi_1,\chi_2) := \Psi_{(\theta)}(\chi_1,\chi_2) - \tfrac{1}{2}\,[\,\alpha\chi_1\,(1-\chi_1) - \beta\,\chi_2^2\,]. \tag{4.43b}$$

We then introduce for *a.e.* $t \in (0,T)$ the residual

$$\begin{aligned}\mathcal{R}_{(\theta)} := &-(W^+_{(\theta)} + \alpha\,U^-_{(\theta)}, U_{(\theta)} - U^+_{(\theta)})^h - (Z^+_{(\theta)} + \beta\,V^-_{(\theta)}, V_{(\theta)} - V^+_{(\theta)})^h \\ &+ \left[\,J^{h,+}_{(\theta)}(U_{(\theta)}, V_{(\theta)}) - J^{h,+}_{(\theta)}(U^+_{(\theta)}, V^+_{(\theta)})\,\right].\end{aligned} \tag{4.44}$$

We introduce also

$$\begin{aligned}\mathcal{E}^n_{(\theta)} := &(\alpha\,U^{n-1}_{(\theta)} + W^n_{(\theta)}, \tfrac{U^n_{(\theta)} - U^{n-1}_{(\theta)}}{\tau_n})^h + (\beta\,V^{n-1}_{(\theta)} + Z^n_{(\theta)}, \tfrac{V^n_{(\theta)} - V^{n-1}_{(\theta)}}{\tau_n})^h \\ &- \tfrac{1}{\tau_n}\left[\,J^{h,+}_{(\theta)}(U^n_{(\theta)}, V^n_{(\theta)}) - J^{h,+}_{(\theta)}(U^{n-1}_{(\theta)}, V^{n-1}_{(\theta)})\,\right], \quad n = 1 \to N;\end{aligned} \tag{4.45}$$

$$\begin{aligned}\mathcal{D}^n_{(\theta)} := &(\alpha\,(U^{n-1}_{(\theta)} - U^{n-2}_{(\theta)}) + (W^n_{(\theta)} - W^{n-1}_{(\theta)}), \tfrac{U^n_{(\theta)} - U^{n-1}_{(\theta)}}{\tau_n})^h \\ &+ (\beta\,(V^{n-1}_{(\theta)} - V^{n-2}_{(\theta)}) + (Z^n_{(\theta)} - Z^{n-1}_{(\theta)}), \tfrac{V^n_{(\theta)} - V^{n-1}_{(\theta)}}{\tau_n})^h, \quad n = 2 \to N;\end{aligned} \tag{4.46a}$$

$$\begin{aligned}\mathcal{D}^1_{(\theta)} := &(\alpha\,U^0_{(\theta)} + W^1_{(\theta)}, \tfrac{U^1_{(\theta)} - U^0_{(\theta)}}{\tau_1})^h + (\beta\,V^0_{(\theta)} + Z^1_{(\theta)}, \tfrac{V^1_{(\theta)} - V^0_{(\theta)}}{\tau_1})^h \\ &- \gamma\,(\nabla U^0_{(\theta)}, \nabla(\tfrac{U^1_{(\theta)} - U^0_{(\theta)}}{\tau_1}))^h - \gamma\,(\nabla V^0_{(\theta)}, \nabla(\tfrac{V^1_{(\theta)} - V^0_{(\theta)}}{\tau_1}))^h \\ &- \begin{cases} \theta\,[\,(\phi(U^0_\theta + V^0_\theta) + \phi(U^0_\theta - V^0_\theta), \tfrac{U^1_\theta - U^0_\theta}{\tau_1})^h & \\ \quad + (\phi(U^0_\theta + V^0_\theta) - \phi(U^0_\theta - V^0_\theta), \tfrac{V^1_\theta - V^0_\theta}{\tau_1})^h\,] & \text{if } \theta > 0, \\ 0 & \text{if } \theta = 0. \end{cases}\end{aligned} \tag{4.46b}$$

It follows from the uniqueness of $U^n_{(\theta)},\ V^n_{(\theta)} \in S^h_m \times S^h$, $n = 0 \to N$, (4.1a) and (4.32) that $\mathcal{R}_{(\theta)}$ and $\{\mathcal{D}^n_{(\theta)}, \mathcal{E}^n_{(\theta)}\}_{n=1}^N$ are uniquely defined.

LEMMA. 4.1. *For $n = 1 \to N$ we have that*

$$\begin{aligned}&\tfrac{\gamma}{2}\,[\,|U^n_{(\theta)} - U^{n-1}_{(\theta)}|_1^2 + |V^n_{(\theta)} - V^{n-1}_{(\theta)}|_1^2\,] \leq \tau_n\,\mathcal{E}^n_{(\theta)} \\ &\quad \leq \tau_n\,\mathcal{D}^n_{(\theta)} - \tfrac{\gamma}{2}\,[\,|U^n_{(\theta)} - U^{n-1}_{(\theta)}|_1^2 + |V^n_{(\theta)} - V^{n-1}_{(\theta)}|_1^2\,] \leq \tau_n\,\mathcal{D}^n_{(\theta)}.\end{aligned} \tag{4.47}$$

For a.e. $t \in (0,T)$ we have that

$$\mathcal{R}_{(\theta)} \leq \ell\,\overline{\tau}\,\mathcal{E}_{(\theta)} \leq \ell\,\overline{\tau}\,\mathcal{D}_{(\theta)}, \tag{4.48a}$$

where

$$\mathcal{E}_{(\theta)}(t) := \mathcal{E}^n_{(\theta)}, \quad \mathcal{D}_{(\theta)}(t) := \mathcal{D}^n_{(\theta)}, \qquad t \in (t_{n-1}, t_n], \quad n = 1 \to N. \tag{4.48b}$$

Moreover under the assumptions of Theorem 1.1, we have that

$$\sum_{n=1}^N (\tau_n)^2\,\mathcal{E}^n_{(\theta)} \leq \tau^2 \sum_{n=1}^N \mathcal{D}^n_{(\theta)} \leq C_b\,\tau^2. \tag{4.49}$$

PROOF. The first inequality in (4.47) for $\theta > 0$ follows from the $\varepsilon = 0$ limit of (4.12) on noting (3.19), (3.29), (4.1a), (4.43a,b) and (1.4b). The deep quench limit ($\theta = 0$) of (4.47) follows from the corresponding analogue of (4.12); that is, all θ and ε subscripts and superscripts removed. Choosing $\chi \equiv (U_\theta^{n+1} - U_\theta^n) \pm (V_\theta^{n+1} - V_\theta^n)$ in (4.2), and $\chi \equiv U^{n+1} \pm V^{n+1}$ in (4.31), $n = 1 \to N-1$, adding together the resulting $\pm$ versions in each case, noting (4.1a), (4.32), (4.11) and the convexity of $\Psi^+_{(\theta)}$ yields the second inequality in (4.47) for $n = 2 \to N$. This inequality follows for $n = 1$ by comparing $\mathcal{D}^1_{(\theta)}$ and $\mathcal{E}^1_{(\theta)}$ and noting (4.11) and the convexity of $\Psi^+_{(\theta)}$. The bounds (4.48a) follow immediately from (4.48b), (4.44), (4.42), (4.41), the convexity of $J^{h,+}_{(\theta)}$ and (4.47).

The first inequality in (4.49) follows from (4.47). We now prove the second inequality. From (4.46a), (4.11), (3.19), (3.29), (4.1a), (4.32), (3.8), (4.20), (4.36) and the assumption $\tau_{n-1} \leq C\tau_n$ we have that

$$\begin{aligned}
\sum_{n=2}^N \mathcal{D}^n_{(\theta)} \leq{}& \tfrac{1}{2} \sum_{n=2}^N \tau_{n-1} \left[\alpha \left| \frac{U^{n-1}_{(\theta)} - U^{n-2}_{(\theta)}}{\tau_{n-1}} \right|_h^2 + \beta \left| \frac{V^{n-1}_{(\theta)} - V^{n-2}_{(\theta)}}{\tau_{n-1}} \right|_h^2 \right] \\
&+ \tfrac{1}{2} \sum_{n=2}^N \tau_{n-1} \left[\alpha \left| \frac{U^{n}_{(\theta)} - U^{n-1}_{(\theta)}}{\tau_{n}} \right|_h^2 + \beta \left| \frac{V^{n}_{(\theta)} - V^{n-1}_{(\theta)}}{\tau_{n}} \right|_h^2 \right] \\
&+ \tfrac{1}{2} \sum_{n=2}^N (b^{(\theta)}_{n-1} - b^{(\theta)}_{n-2}, |\nabla W^{n-1}_{(\theta)}|^2 + \rho^{-1} |Z^{n-1}_{(\theta)}|^2) \\
&+ \tfrac{1}{2} (b^{(\theta)}_0, |\nabla W^1_{(\theta)}|^2 + \rho^{-1} |Z^1_{(\theta)}|^2) \leq C_b. \qquad (4.50)
\end{aligned}$$

Finally it follows from (4.46b), (4.17), (4.38), (1.11), (3.19), (3.29) (4.1a), (4.32), (2.16), (3.42), (3.15), (3.20), (3.30), (4.20), (4.36), (4.19) and its $\theta = 0$ analogue that

$$\begin{aligned}
\mathcal{D}^1_{(\theta)} ={}& \gamma \left(\nabla(u^0 - U^0_{(\theta)}), \nabla\left(\frac{U^1_{(\theta)} - U^0_{(\theta)}}{\tau_1}\right)\right) + \left(W^1_{(\theta)} - W^0_{(\theta)}, \frac{U^1_{(\theta)} - U^0_{(\theta)}}{\tau_1}\right)^h \\
&+ \gamma \left(\nabla(v^0 - V^0_{(\theta)}), \nabla\left(\frac{V^1_{(\theta)} - V^0_{(\theta)}}{\tau_1}\right)\right) + \left(Z^1_{(\theta)} - Z^0_{(\theta)}, \frac{V^1_{(\theta)} - V^0_{(\theta)}}{\tau_1}\right)^h \\
\leq{}& \left(U^0_{(\theta)} - u^0, \frac{U^1_{(\theta)} - U^0_{(\theta)}}{\tau_1}\right) - (b^0_{(\theta)} \nabla W^0_{(\theta)}, \nabla W^1_{(\theta)}) \\
&+ \left(V^0_{(\theta)} - v^0, \frac{V^1_{(\theta)} - V^0_{(\theta)}}{\tau_1}\right) - \rho^{-1} (Z^0_{(\theta)}, Z^1_{(\theta)}) \leq C_b. \qquad (4.51)
\end{aligned}$$

Combining (4.50) and (4.51) yields the desired result (4.49). □

LEMMA. 4.2. *Let the assumptions of Theorem 1.1 hold. Then for a.e.* $t \in (0,T)$

$$\begin{aligned}
&\tfrac{\gamma}{2} [\, |E^u_{(\theta)}|_1^2 + |E^{u,+}_{(\theta)}|_1^2 + |E^v_{(\theta)}|_1^2 + |E^{v,+}_{(\theta)}|_1^2 \,] \\
&\quad + \tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} [\, (\widehat{\mathcal{G}}^h_{u_{(\theta)}, v_{(\theta)}} E^u_{(\theta)}, E^u_{(\theta)})^h + \rho\, (\widehat{\mathcal{M}}^h_{u_{(\theta)}, v_{(\theta)}} E^v_{(\theta)}, E^v_{(\theta)})^h \,] \\
&\quad \leq \alpha\, (E^{u,-}_{(\theta)}, E^u_{(\theta)})^h + \beta\, (E^{v,-}_{(\theta)}, E^v_{(\theta)})^h + \mathcal{R}_{(\theta)} \\
&\qquad - \tfrac{1}{2} (\tfrac{\partial}{\partial t} b(u_{(\theta)}, v_{(\theta)}), |\nabla \widehat{\mathcal{G}}^h_{u_{(\theta)}, v_{(\theta)}} E^u_{(\theta)}|^2 + \rho\, |\widehat{\mathcal{M}}^h_{u_{(\theta)}, v_{(\theta)}} E^v_{(\theta)}|^2)
\end{aligned}$$

$$- \, ((\widehat{\mathcal{G}}^h_{u_{(\theta)},v_{(\theta)}} - \widehat{\mathcal{G}}^h_{U^-_{(\theta)},V^-_{(\theta)}}) \tfrac{\partial U_{(\theta)}}{\partial t}, E^u_{(\theta)})^h$$
$$- \, \rho \, ((\widehat{\mathcal{M}}^h_{u_{(\theta)},v_{(\theta)}} - \widehat{\mathcal{M}}^h_{U^-_{(\theta)},V^-_{(\theta)}}) \tfrac{\partial V_{(\theta)}}{\partial t}, E^v_{(\theta)})^h, \tag{4.52}$$

where $E^{(\pm)}_{u(,\theta)} := u^h_{(\theta)} - U^{(\pm)}_{(\theta)} \in \Xi^h$ *and* $E^{(\pm)}_{v(,\theta)} := v^h_{(\theta)} - V^{(\pm)}_{(\theta)} \in S^h$.

Proof. Adopting the notation (1.14) and (4.40), then (4.2) can be restated as: Find $\{U_\theta, V_\theta\} \in H^1(0,T;S^h_m) \times H^1(0,T;S^h)$ such that $\{U_\theta(\cdot,0), V_\theta(\cdot,0)\} \equiv \{Q^h_\gamma u^0(\cdot), Q^h_\gamma v^0(\cdot)\}$ and for $a.e.$ $t \in (0,T)$

$$\gamma \, (\nabla(U^+_\theta \pm V^+_\theta), \nabla\chi) + (\widehat{\mathcal{G}}^h_{U^-_\theta,V^-_\theta} \tfrac{\partial U_\theta}{\partial t} - \Lambda^+_\theta \pm \rho \, \widehat{\mathcal{M}}^h_{U^-_\theta,V^-_\theta} \tfrac{\partial V_\theta}{\partial t}, \chi)^h$$
$$+ \, (2\,\theta\,\phi(U^+_\theta \pm V^+_\theta) - (\alpha \, U^-_\theta \pm \beta \, V^-_\theta), \chi)^h = 0 \quad \forall \, \chi \in S^h, \tag{4.53}$$

where $\Lambda^+_\theta := \fint \pi^h[\,\theta\,\phi(U^+_\theta + V^+_\theta) + \theta\,\phi(U^+_\theta - V^+_\theta) - \alpha\, U^-_\theta\,]$. Similarly (4.31) can be restated as:
Find $\{U, V\} \in H^1(0,T;S^h_m) \times H^1(0,T;S^h)$ such that $\{U_\theta(\cdot,0), V_\theta(\cdot,0)\} \equiv \{Q^h_\gamma u^0(\cdot), Q^h_\gamma v^0(\cdot)\}$ and for $a.e.$ $t \in (0,T)$ $U \pm V \in K^h$ and

$$\gamma \, (\nabla(U^+ \pm V^+), \nabla(\chi - (U^+ \pm V^+)))$$
$$+ \, (\widehat{\mathcal{G}}^h_{U^-,V^-} \tfrac{\partial U}{\partial t} - \Lambda^+ \pm \rho \, \widehat{\mathcal{M}}^h_{U^-,V^-} \tfrac{\partial V}{\partial t}, \chi - (U^+ \pm V^+))^h$$
$$\geq (\alpha \, U^- \pm \beta \, V^-, \chi - (U^+ \pm V^+))^h \quad \forall \, \chi \in K^h. \tag{4.54}$$

Choosing $\chi \equiv E^u_\theta \pm E^v_\theta$ in the non-regularized version of (3.38) and $\chi \equiv U \pm V$ in (3.56), adding together the two resulting (in)equalities in each case, and noting (4.11) and the convexity of $\Psi^+_{(\theta)}$ yields that

$$\tfrac{\gamma}{2} \, [\, |E^u_{(\theta)}|^2_1 + |E^v_{(\theta)}|^2_1 + |u^h_{(\theta)}|^2_1 + |v^h_{(\theta)}|^2_1 - |U_{(\theta)}|^2_1 - |V_{(\theta)}|^2_1 \,]$$
$$+ \, (\Psi^+_{(\theta)}(u^h_{(\theta)}, v^h_{(\theta)}) - \Psi^+_{(\theta)}(U_{(\theta)}, V_{(\theta)}), 1)^h \leq (\alpha \, u^h_{(\theta)} - \widehat{\mathcal{G}}^h_{u_{(\theta)},v_{(\theta)}} \tfrac{\partial u^h_{(\theta)}}{\partial t}, E^u_{(\theta)})^h$$
$$+ \, (\beta \, v^h_{(\theta)} - \rho \, \widehat{\mathcal{M}}^h_{u_{(\theta)},v_{(\theta)}} \tfrac{\partial v^h_{(\theta)}}{\partial t}, E^v_{(\theta)})^h. \tag{4.55}$$

Similarly on choosing $\chi \equiv E^{u,+}_\theta \pm E^{v,+}_\theta$ in (4.53) and $\chi \equiv u^h \pm v^h$ in (4.54), it follows that

$$\tfrac{\gamma}{2} \, [\, |E^{u,+}_{(\theta)}|^2_1 + |E^{v,+}_{(\theta)}|^2_1 + |U^+_{(\theta)}|^2_1 + |V^+_{(\theta)}|^2_1 - |u^h_{(\theta)}|^2_1 - |v^h_{(\theta)}|^2_1 \,]$$
$$+ \, (\Psi^+_{(\theta)}(U^+_{(\theta)}, V^+_{(\theta)}) - \Psi^+_{(\theta)}(u^h_{(\theta)}, v^h_{(\theta)}), 1)^h \leq (\widehat{\mathcal{G}}^h_{U^-_{(\theta)},V^-_{(\theta)}} \tfrac{\partial U_{(\theta)}}{\partial t} - \alpha \, U^-_{(\theta)}, E^{u,+}_{(\theta)})^h$$
$$+ \, (\rho \, \widehat{\mathcal{M}}^h_{U^-_{(\theta)},V^-_{(\theta)}} \tfrac{\partial V_{(\theta)}}{\partial t} - \beta \, V^-_{(\theta)}, E^{v,+}_{(\theta)})^h. \tag{4.56}$$

Adding (4.56) to (4.55) and noting the $\widehat{\mathcal{G}}^h_{q_1,q_2}$ analogue of (2.23), the $\widehat{\mathcal{M}}^h_{q_1,q_2}$ analogue of (2.31), (4.1a) and (4.32) yields the desired result (4.52). □

THEOREM. 4.4. *Let the assumptions and notation of Lemma 4.2 hold. Then we have that*

$$\begin{aligned}
&\|E^u_{(\theta)}\|^2_{L^2(0,T;H^1(\Omega))} + \|E^u_{(\theta)}\|^2_{L^\infty(0,T;(H^1(\Omega))')} \\
&\qquad\qquad + \|E^v_{(\theta)}\|^2_{L^2(0,T;H^1(\Omega))} + \|E^v_{(\theta)}\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\quad \leq C_b(T)\,[\,\|u_{(\theta)} - u^h_{(\theta)}\|^2_{L^2(0,T;H^1(\Omega))} + \|v_{(\theta)} - v^h_{(\theta)}\|^2_{L^2(0,T;H^1(\Omega))} \\
&\qquad\qquad + \sum_{n=1}^{N} (\tau_n)^2 \mathcal{E}^n_{(\theta)}\,] \\
&\quad \leq C_b(T)\,\tau^2 \;+\; \begin{cases} C_b(T)\,h^{\frac{4}{3}}\,(\ln \frac{1}{h})^{\frac{2(d-1)}{3}} & \textit{if } \theta > 0 \textit{ and } d \leq 2; \\ C_b(T)\,h & \textit{if } \theta > 0,\ d = 3 \\ & \quad \textit{and } b \textit{ is constant}; \\ C_b(T)\,h^2 & \textit{if } \theta = 0. \end{cases}
\end{aligned} \tag{4.57}$$

PROOF. Similarly to (3.48) we have that

$$\begin{aligned}
&|((\widehat{\mathcal{G}}^h_{u_{(\theta)},v_{(\theta)}} - \widehat{\mathcal{G}}^h_{U^-_{(\theta)},V^-_{(\theta)}})\tfrac{\partial U_{(\theta)}}{\partial t}, E^u_{(\theta)})^h| \\
&\quad \leq \tfrac{\gamma}{32}\,[\,|u_{(\theta)} - U^-_{(\theta)}|^2_1 + \|v_{(\theta)} - V^-_{(\theta)}\|^2_1\,] \\
&\qquad + C(\|u_{(\theta)}\|_2, \|v_{(\theta)}\|_2)\,|\tfrac{\partial U_{(\theta)}}{\partial t}|^2_1\,(\widehat{\mathcal{G}}^h_{u_{(\theta)},v_{(\theta)}} E^u_{(\theta)}, E^u_{(\theta)})^h \\
&\quad \leq \tfrac{\gamma}{16}\,[\,|E^u_{(\theta)}|^2_1 + \|E^v_{(\theta)}\|^2_1\,] + C\,[\,|u_{(\theta)} - u^h_{(\theta)}|^2_1 + \|v_{(\theta)} - v^h_{(\theta)}\|^2_1\,] \\
&\qquad + C\,\tau^2\,[\,|\tfrac{\partial U_{(\theta)}}{\partial t}|^2_1 + +\|\tfrac{\partial V_{(\theta)}}{\partial t}\|^2_1\,] \\
&\qquad + C(\|u_{(\theta)}\|_2, \|v_{(\theta)}\|_2)\,|\tfrac{\partial U_{(\theta)}}{\partial t}|^2_1\,(\widehat{\mathcal{G}}^h_{u_{(\theta)},v_{(\theta)}} E^u_{(\theta)}, E^u_{(\theta)})^h,
\end{aligned} \tag{4.58}$$

where we have noted (4.42). Similarly to (3.49) and (4.58) we have that

$$\begin{aligned}
&|((\widehat{\mathcal{M}}^h_{u_{(\theta)},v_{(\theta)}} - \widehat{\mathcal{M}}^h_{U^-_{(\theta)},V^-_{(\theta)}})\tfrac{\partial V_{(\theta)}}{\partial t}, E^u_{(\theta)})^h| \\
&\quad \leq \tfrac{\gamma}{16}\,[\,|E^u_{(\theta)}|^2_1 + \|E^v_{(\theta)}\|^2_1\,] + C\,[\,|u_{(\theta)} - u^h_{(\theta)}|^2_1 + \|v_{(\theta)} - v^h_{(\theta)}\|^2_1\,] \\
&\qquad + C\,\tau^2\,[\,|\tfrac{\partial U_{(\theta)}}{\partial t}|^2_1 + +\|\tfrac{\partial V_{(\theta)}}{\partial t}\|^2_1\,] \\
&\qquad + C(\|u_{(\theta)}\|_2, \|v_{(\theta)}\|_2)\,\|\tfrac{\partial V_{(\theta)}}{\partial t}\|^2_1\,(\widehat{\mathcal{M}}^h_{u_{(\theta)},v_{(\theta)}} E^v_{(\theta)}, E^v_{(\theta)})^h.
\end{aligned} \tag{4.59}$$

On noting (4.42), (3.21), (3.8) and (3.30) we have that

$$(E^{u,-}_{(\theta)}, E^u_{(\theta)})^h \leq \tfrac{\gamma}{16}\,|E^u_{(\theta)}|^2_1 + C\,(\widehat{\mathcal{G}}^h_{u_{(\theta)},v_{(\theta)}} E^u_{(\theta)}, E^u_{(\theta)})^h + C\,\tau^2\,|\tfrac{\partial U_{(\theta)}}{\partial t}|^2_1, \tag{4.60a}$$

$$(E^{v,-}_{(\theta)}, E^v_{(\theta)})^h \leq C\,(\widehat{\mathcal{M}}^h_{u_{(\theta)},v_{(\theta)}} E^v_{(\theta)}, E^v_{(\theta)})^h + C\,\tau^2\,\|\tfrac{\partial V_{(\theta)}}{\partial t}\|^2_1. \tag{4.60b}$$

Combining (4.52), (4.58) and (4.60a,b), noting (3.26), (3.35), (1.6b), (3.15), (3.20), (3.19), (3.30), (2.69), (2.84a,b), (4.20), (4.36), (4.48a,b) and (4.47), and applying a Gronwall inequality yields the first inequality in (4.57). The second follows from (4.49), (3.51) and (3.59). □

Finally, we have the proof our main result.

PROOF OF THEOREM 1.1. The desired result (1.15) follows from combining (3.51) and (4.57). The desired result (1.16), which is optimal, follows from combining (3.59) and (4.57). □

5 Solution of the Discrete Problem

We now consider an algorithm for solving the discrete system at each time level in ($\mathrm{P}^{h,\tau}_{(\theta)}$). This is based on the general splitting algorithm of [19]; see also [14], [4] and [8] where this algorithm has been applied to solve a single Cahn-Hilliard equation with a constant mobility, a Cahn-Hilliard system with a constant mobility matrix and a single Cahn-Hilliard equation with a concentration dependent mobility; respectively.

For n fixed, as $U^n_\theta \pm V^n_\theta \in K^h$, see (4.5), (1.12c) and (1.13c) can be rewritten as

$$\begin{aligned}&\gamma(\nabla(U^n_{(\theta)} \pm V^n_{(\theta)}), \nabla(\chi - (U^n_{(\theta)} \pm V^n_{(\theta)})))\\ &+ \ (\varphi_{(\theta)}(U^n_{(\theta)} \pm V^n_{(\theta)}) - (\,W^n_{(\theta)} \pm Z^n_{(\theta)} + \alpha\, U^{n-1}_{(\theta)} \pm \beta\, V^{n-1}_{(\theta)}\,), \chi - (U^n_{(\theta)} \pm V^n_{(\theta)}))^h\\ &\qquad\qquad \geq 0 \quad \forall\, \chi \in K^h, \end{aligned} \tag{5.1}$$

where $\varphi_\theta(\cdot) \equiv 2\,\theta\,\phi(\cdot)$ if $\theta > 0$ and $\varphi(\cdot) \equiv 0$. Multiplying (5.1) by $\mu > 0$, a 'relaxation' parameter, adding $(U^n_{(\theta)} \pm V^n_{(\theta)}, \chi - (U^n_{(\theta)} \pm V^n_{(\theta)}))^h$ to both sides and rearranging and recalling (1.12a,b) and (1.13a,b); it follows that $\{U^n_{(\theta)}, V^n_{(\theta)}, W^n_{(\theta)}, Z^n_{(\theta)}\} \in [S^h]^4$ are such that $U^n_{(\theta)} \pm V^n_{(\theta)} \in K^h$ and satisfy

$$(\,(I + \mu\,\varphi_{(\theta)})(U^n_{(\theta)} \pm V^n_{(\theta)}) - (Y^n_1 \pm Y^n_2), \chi - (U^n_{(\theta)} \pm V^n_{(\theta)}))^h \geq 0 \quad \forall\, \chi \in K^h, \tag{5.2a}$$

$$(\tfrac{U^n_{(\theta)} - U^{n-1}_{(\theta)}}{\tau_n}, \chi)^h + b^{n-1}(\nabla W^n_{(\theta)}, \nabla \chi) = (\,[b^{n-1} - b(U^{n-1}_{(\theta)}, V^{n-1}_{(\theta)})]\,\nabla W^n_{(\theta)}, \nabla \chi) \quad \forall\, \chi \in S^h, \tag{5.2b}$$

$$\rho\,(\tfrac{V^n_{(\theta)} - V^{n-1}_{(\theta)}}{\tau_n}, \chi)^h + b^{n-1}(Z^n_{(\theta)}, \chi) = (\,[b^{n-1} - b(U^{n-1}_{(\theta)}, V^{n-1}_{(\theta)})]\,Z^n_{(\theta)}, \chi) \quad \forall\, \chi \in S^h; \tag{5.2c}$$

where $(I + \mu\,\varphi_\theta)(s) \equiv s + \mu\,\varphi_\theta(s)$. In the above Y^n_1, $Y^n_2 \in S^h$ are such that for all $\chi \in S^h$

$$(Y^n_1 - U^n_{(\theta)}, \chi)^h := -\mu\left[\gamma(\nabla U^n_{(\theta)}, \nabla \chi) - (W^n_{(\theta)} + \alpha\, U^{n-1}_{(\theta)}, \chi)^h\right], \tag{5.3a}$$

$$(Y^n_2 - V^n_{(\theta)}, \chi)^h := -\mu\left[\gamma(\nabla V^n_{(\theta)}, \nabla \chi) - (Z^n_{(\theta)} + \beta\, V^{n-1}_{(\theta)}, \chi)^h\right] \tag{5.3b}$$

and b^{n-1} is chosen such that $b^{n-1} \in [b^{n-1}_{\max}, b_{\max}]$ with $b^{n-1}_{\max}$ and $b_{\max}$ as defined in Theorem 4.1. We introduce also X^n_1, $X^n_2 \in S^h$ such that for all $\chi \in S^h$

$$(X^n_1 - U^n_{(\theta)}, \chi)^h := \mu\left[\gamma(\nabla U^n_{(\theta)}, \nabla \chi) - (W^n_{(\theta)} + \alpha\, U^{n-1}_{(\theta)}, \chi)^h\right], \tag{5.4a}$$

$$(X_2^n - V_{(\theta)}^n, \chi)^h := \mu \left[\gamma(\nabla V_{(\theta)}^n, \nabla \chi) - (Z_{(\theta)}^n + \beta\, V_{(\theta)}^{n-1}, \chi)^h \right] \tag{5.4b}$$

and note that $X_1^n = 2\,U_{(\theta)}^n - Y_1^n$ and $X_2^n = 2\,V_{(\theta)}^n - Y_2^n$. We use this as a basis for constructing our iterative procedure:
For $n \geq 1$ set $\{U^{n,0}, V^{n,0}, W^{n,0}, Z^{n,0}\} \equiv \{U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1}, W_{(\theta)}^{n-1}, Z_{(\theta)}^{n-1}\} \in [S^h]^4$ where $W_{(\theta)}^0$, $Z_{(\theta)}^0 \in S^h$ are arbitrary if $n = 1$.
For $k \geq 0$ we define $Y_1^{n,k}, Y_2^{n,k} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$(Y_1^{n,k} - U^{n,k}, \chi)^h := -\mu \left[\gamma(\nabla U^{n,k}, \nabla \chi) - (W^{n,k} + \alpha\, U_{(\theta)}^{n-1}, \chi)^h \right], \tag{5.5a}$$

$$(Y_2^{n,k} - V^{n,k}, \chi)^h := -\mu \left[\gamma(\nabla V^{n,k}, \nabla \chi) - (Z^{n,k} + \beta\, V_{(\theta)}^{n-1}, \chi)^h \right]. \tag{5.5b}$$

Then find $\{U^{n,k+\frac{1}{2}}, V^{n,k+\frac{1}{2}}\} \in [S^h]^2$ such that $U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}} \in K^h$ and for all $\chi \in K^h$

$$\begin{aligned}((I + \mu\, \varphi_{(\theta)})(U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}}), \chi - (U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}}))^h& \\ \geq (Y_1^{n,k} \pm Y_2^{n,k}\, \chi - (U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}}))^h&;\end{aligned} \tag{5.6}$$

and find $\{U^{n,k+1}, V^{n,k+1}, W^{n,k+1}, Z^{n,k+1}\} \in [S^h]^4$ such that for all $\chi \in S^h$

$$\begin{aligned}(\tfrac{U^{n,k+1} - U_{(\theta)}^{n-1}}{\tau_n}, \chi)^h + b^{n-1}(\nabla W^{n,k+1}, \nabla \chi)& \\ = ([b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]\, \nabla W^{n,k}, \nabla \chi)&,\end{aligned} \tag{5.7a}$$

$$\begin{aligned}\rho\, (\tfrac{V^{n,k+1} - V_{(\theta)}^{n-1}}{\tau_n}, \chi)^h + b^{n-1}(Z^{n,k+1}, \chi)& \\ = ([b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]\, Z^{n,k}, \chi)&;\end{aligned} \tag{5.7b}$$

$$\begin{aligned}(U^{n,k+1}, \chi)^h + \mu \left[\gamma(\nabla U^{n,k+1}, \nabla \chi) - (W^{n,k+1}, \chi)^h \right]& \\ = (X_1^{n,k+1} + \mu\, \alpha\, U_{(\theta)}^{n-1}, \chi)^h&,\end{aligned} \tag{5.7c}$$

$$\begin{aligned}(V^{n,k+1}, \chi)^h + \mu \left[\gamma(\nabla V^{n,k+1}, \nabla \chi) - (Z^{n,k+1}, \chi)^h \right]& \\ = (X_2^{n,k+1} + \mu\, \beta\, V_{(\theta)}^{n-1}, \chi)^h&;\end{aligned} \tag{5.7d}$$

where $X_1^{n,k+1} := 2\,U^{n,k+\frac{1}{2}} - Y_1^{n,k}$ and $X_2^{n,k+1} := 2\,V^{n,k+\frac{1}{2}} - Y_2^{n,k}$.

Existence and uniqueness of $U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}}$ solving (5.6) and hence of $\{U^{n,k+\frac{1}{2}}, V^{n,k+\frac{1}{2}}\}$ follows from the monotonicity of $\varphi_{(\theta)}$. In fact if $\theta > 0$ one has to solve two decoupled nonlinear equations at each mesh point; that is, for $j = 1 \to J$

$$(I + 2\,\mu\,\theta\,\phi)((U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}})(x_j)) = (Y_1^{n,k} \pm Y_2^{n,k})(x_j). \tag{5.8}$$

Whereas if $\theta = 0$ one has two simple projections at each mesh point; that is, for $j = 1 \to J$

$$(U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}})(x_j) = \min\{\max\{(Y_1^{n,k} \pm Y_2^{n,k})(x_j),\, 0\},\, 1\}. \tag{5.9}$$

It remains to show that there exists a unique solution to (5.7a–d). Introducing $\{R_1^{n,k}, R_2^{n,k}\} \in \Xi^h \times S^h$ such that

$$(R_1^{n,k},\chi)^h = (b(U^{n-1},V^{n-1})\,\nabla W^{n,k},\nabla\chi) \quad \forall\,\chi\in S^h, \tag{5.10a}$$
$$(R_2^{n,k},\chi)^h = (b(U^{n-1},V^{n-1})\,Z^{n,k},\chi) \quad \forall\,\chi\in S^h; \tag{5.10b}$$

it then follows from (5.7a,b), (3.13), (3.16) and (5.7c) with $\chi\equiv 1$ that

$$\begin{aligned} W^{n,k+1} &= (I - ⨍\,)W^{n,k} - [b^{n-1}]^{-1}\widehat{\mathcal{G}}^h(\tfrac{U^{n,k+1}-U^{n-1}_{(\theta)}}{\tau_n} + R_1^{n,k}) \\ &\qquad + ⨍\,(\tfrac{1}{\mu}(U^{n,k+1}-X_1^{n,k+1}) - \alpha\,U^{n-1}_{(\theta)}), \end{aligned} \tag{5.11a}$$
$$Z^{n,k+1} = Z^{n,k} - [b^{n-1}]^{-1}\widehat{\mathcal{M}}^h(\rho\,(\tfrac{V^{n,k+1}-V^{n-1}_{(\theta)}}{\tau_n}) + R_2^{n,k}). \tag{5.11b}$$

Therefore combining (5.7c,d) and (5.11a,b), (5.7a–d) may be written equivalently as find $U^{n,k+1}\in S^h_m$, see (3.37), such that for all $\chi\in S^h$

$$\begin{aligned} &(U^{n,k+1},(I-⨍\,)\chi)^h \\ &\qquad + \mu\Big[\gamma(\nabla U^{n,k+1},\nabla\chi) + [b^{n-1}]^{-1}(\widehat{\mathcal{G}}^h(\tfrac{U^{n,k+1}-U^{n-1}_{(\theta)}}{\tau_n}),\chi)^h\Big] \\ &\quad = (X_1^{n,k+1} + \mu\,(W^{n,k} + \alpha\,U^{n-1}_{(\theta)} - [b^{n-1}]^{-1}\widehat{\mathcal{G}}^h R_1^{n,k}),(I-⨍\,)\chi)^h \end{aligned} \tag{5.12a}$$

and $V^{n,k+1}\in S^h$ such that for all $\chi\in S^h$

$$\begin{aligned} &(V^{n,k+1},\chi)^h + \mu\Big[\gamma(\nabla V^{n,k+1},\nabla\chi) + [b^{n-1}]^{-1}\,\rho\,(\widehat{\mathcal{M}}^h(\tfrac{V^{n,k+1}-V^{n-1}_{(\theta)}}{\tau_n}),\chi)^h\Big] \\ &\quad = (X_2^{n,k+1} + \mu\,(Z^{n,k} + \beta\,V^{n-1}_{(\theta)} - [b^{n-1}]^{-1}\widehat{\mathcal{M}}^h R_2^{n,k}),\chi)^h. \end{aligned} \tag{5.12b}$$

Existence and uniqueness of $\{U^{n,k+1},V^{n,k+1}\}\in S^h_m\times S^h$ satisfying (5.12a,b) follows since they are the Euler-Lagrange equations of the strictly convex minimisation problems

$$\begin{aligned} &\min_{\chi\in S^h_m}\Big\{\,|\chi|_h^2 + \mu\Big[\gamma\,|\chi|_1^2 + \tfrac{1}{b^{n-1}\tau_n}|\nabla\widehat{\mathcal{G}}^h(\chi - U^{n-1}_{(\theta)})|_0^2\Big] \\ &\qquad - 2\,(X_1^{n,k+1} + \mu\,(W^{n,k} + \alpha\,U^{n-1}_{(\theta)} - [b^{n-1}]^{-1}\widehat{\mathcal{G}}^h R_1^{n,k}),\chi)^h\,\Big\}, \\ &\min_{\chi\in S^h}\Big\{\,|\chi|_h^2 + \mu\Big[\gamma\,|\chi|_1^2 + \tfrac{\rho}{b^{n-1}\tau_n}|\widehat{\mathcal{M}}^h(\chi - V^{n-1}_{(\theta)})|_0^2\Big] \\ &\qquad - 2\,(X_2^{n,k+1} + \mu\,(Z^{n,k} + \beta\,V^{n-1}_{(\theta)} - [b^{n-1}]^{-1}\,\widehat{\mathcal{M}}^h R_2^{n,k}),\chi)^h\Big\}; \end{aligned}$$

respectively. Finally $W^{n,k+1}$ and $Z^{n,k+1}$ are uniquely defined by (5.11a,b). Hence the iterative procedure (5.5a,b)–(5.7a–d) is well-defined.

THEOREM. 5.1. *For all $\mu\in\mathbf{R}^+$ and $\{U^{n,0},V^{n,0},V^{n,0},Z^{n,0}\}\in[S^h]^4$ the sequence $\{U^{n,k},V^{n,k},W^{n,k},Z^{n,k}\}_{k\geq 0}$ generated by the algorithm (5.5a,b)–(5.7a–d) satisfies*

$$U^{n,k}\to U^n_{(\theta)}, \qquad V^{n,k}\to V^n_{(\theta)},$$

$$\int_\Omega b(U^{n-1}_{(\theta)}, V^{n-1}_{(\theta)})\,|\nabla(W^{n,k+1} - W^n_{(\theta)})|^2\,\mathrm{d}x \to 0$$

$$\textit{and} \quad \int_\Omega b(U^{n-1}_{(\theta)}, V^{n-1}_{(\theta)})\,|Z^{n,k+1} - Z^n_{(\theta)}|^2\,\mathrm{d}x \to 0 \quad \textit{as} \quad k \to \infty. \tag{5.13}$$

In addition, if $\theta > 0$ *then* $U^{n,k+\frac{1}{2}} \to U^n_\theta$, $V^{n,k+\frac{1}{2}} \to V^n_\theta$ *as* $k \to \infty$.

Proof. It follows from (5.3a,b), (5.4a,b), (5.5a,b), (5.7c,d), the definition of $X_i^{n,k+1}$, for $k \geq 0$, and the appropriate choice of $X_i^{n,0}$, $i = 1, 2$, that for $k \geq 0$

$$U^n_{(\theta)} = \tfrac{1}{2}(X^n_1 + Y^n_1), \;\; U^{n,k} = \tfrac{1}{2}(X^{n,k}_1 + Y^{n,k}_1), \;\; U^{n,k+\frac{1}{2}} = \tfrac{1}{2}(X^{n,k+1}_1 + Y^{n,k}_1); \tag{5.14a}$$

$$V^n_{(\theta)} = \tfrac{1}{2}(X^n_2 + Y^n_2), \;\; V^{n,k} = \tfrac{1}{2}(X^{n,k}_1 + Y^{n,k}_1), \;\; V^{n,k+\frac{1}{2}} = \tfrac{1}{2}(X^{n,k+1}_2 + Y^{n,k}_2). \tag{5.14b}$$

For notational convenience we set $E^k_U := U^{n,k} - U^n_{(\theta)}$, $E^k_V := V^{n,k} - V^n_{(\theta)}$, $E^k_W := W^{n,k} - W^n_{(\theta)}$, $E^k_Z := Z^{n,k} - Z^n_{(\theta)}$, $E^k_{X_i} := X^{n,k}_i - X^n_i$ and $E^k_{Y_i} := Y^{n,k}_i - Y^n_i$, $i = 1, 2$. It follows from (5.7c), (5.4a) and (5.14a) that

$$\begin{aligned}\gamma\,|E^{k+1}_U|^2_1 - (E^{k+1}_W, E^{k+1}_U)^h &= \tfrac{1}{4\mu}\,(E^{k+1}_{X_1} - E^{k+1}_{Y_1}, E^{k+1}_{X_1} + E^{k+1}_{Y_1})^h \\ &= \tfrac{1}{4\mu}\,(\,|E^{k+1}_{X_1}|^2_h - |E^{k+1}_{Y_1}|^2_h\,).\end{aligned} \tag{5.15a}$$

Similarly, it follows from (5.7d), (5.4b) and (5.14b) that

$$\gamma\,|E^{k+1}_V|^2_1 - (E^{k+1}_Z, E^{k+1}_V)^h = \tfrac{1}{4\mu}\,(\,|E^{k+1}_{X_2}|^2_h - |E^{k+1}_{Y_2}|^2_h\,). \tag{5.15b}$$

In addition, we note that subtracting (5.7b) from (5.2c) with $\chi \equiv 1$ yields that

$$\begin{aligned}\rho\,(E^{k+1}_V, 1)^h = {}& \tau_n\,(\,[b^{n-1} - b(U^{n-1}_{(\theta)}, V^{n-1}_{(\theta)})]\,(E^k_Z - E^{k+1}_Z), 1)^h \\ & - \tau_n\,(b(U^{n-1}_{(\theta)}, V^{n-1}_{(\theta)})\,E^{k+1}_Z, 1)^h.\end{aligned} \tag{5.16}$$

Choosing $\chi \equiv U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}}$ in (5.2a) and $\chi \equiv U^n_{(\theta)} \pm V^n_{(\theta)}$ in (5.6) yields

$$\begin{aligned}&\mu\,(\varphi_{(\theta)}(U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}}) - \varphi_{(\theta)}(U^n_{(\theta)} \pm V^n_{(\theta)}), E^{k+\frac{1}{2}}_U \pm E^{k+\frac{1}{2}}_V)^h \\ &\quad + \;|E^{k+\frac{1}{2}}_U \pm E^{k+\frac{1}{2}}_V|^2_h \leq (E^k_{Y_1} \pm E^k_{Y_2}, E^{k+\frac{1}{2}}_U \pm E^{k+\frac{1}{2}}_V)^h.\end{aligned} \tag{5.17}$$

Adding together the $\pm$ versions of (5.17) and noting (5.14a,b) yields that

$$\begin{aligned}&2\,\mu\,(\varphi_{(\theta)}(U^{n,k+\frac{1}{2}} + V^{n,k+\frac{1}{2}}) - \varphi_{(\theta)}(U^n + V^n), E^{k+\frac{1}{2}}_U + E^{k+\frac{1}{2}}_V)^h \\ &\quad + \;2\,\mu\,(\varphi_{(\theta)}(U^{n,k+\frac{1}{2}} - V^{n,k+\frac{1}{2}}) - \varphi_{(\theta)}(U^n_{(\theta)} - V^n_{(\theta)}), E^{k+\frac{1}{2}}_U - E^{k+\frac{1}{2}}_V)^h \\ &\quad + \;|E^{k+1}_{X_1}|^2_h + |E^{k+1}_{X_2}|^2_h \;\leq\; |E^k_{Y_1}|^2_h + |E^k_{Y_2}|^2_h.\end{aligned} \tag{5.18}$$

It follows from (5.7a), (5.2b) and (4.11) that

$$-\,(E^{k+1}_W, E^{k+1}_U)^h = -\,(E^{k+1}_W, (U^{n,k+1} - U^{n-1}_{(\theta)}) + (U^{n-1}_{(\theta)} - U^n_{(\theta)}))^h$$

$$
\begin{aligned}
&= \tau_n \,|\, [b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1}))]^{\frac{1}{2}} \nabla E_W^{k+1} |_0^2 \\
&\quad + \tau_n \left([b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})] \nabla (W^{n,k+1} - W^{n,k}), \nabla E_W^{k+1} \right) \\
&= \tau_n \,|\, [b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} \nabla E_W^{k+1} |_0^2 \\
&\quad + \tfrac{1}{2} \tau_n \Big[\,|\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} \nabla E_W^{k+1} |_0^2 \\
&\quad - |\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} \nabla E_W^{k} |_0^2 \\
&\quad + |\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} \nabla (W^{n,k+1} - W^{n,k}) |_0^2 \Big].
\end{aligned}
\tag{5.19a}
$$

Similarly, it follows from (5.7b), (5.2c) and (4.11) that

$$
\begin{aligned}
- \rho \, (E_Z^{k+1}, E_V^{k+1})^h &= \tau_n \,|\, [b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} E_Z^{k+1} |_0^2 \\
&\quad + \tfrac{1}{2} \tau_n \Big[\,|\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} E_Z^{k+1} |_0^2 \\
&\quad - |\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} E_Z^{k} |_0^2 \\
&\quad + |\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} (Z^{n,k+1} - Z^{n,k}) |_0^2 \Big].
\end{aligned}
\tag{5.19b}
$$

Adding (5.15a,b) and noting (5.18) and (5.19a,b) yields that

$$
\begin{aligned}
&\tau_n \,[\, |\, [b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} \nabla E_W^{k+1} |_0^2 + \rho^{-1} |\, [b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} E_Z^{k+1} |_0^2 \,] \\
&\quad + \tfrac{1}{2} \left(\varphi_{(\theta)}(U^{n,k+\frac{1}{2}} + V^{n,k+\frac{1}{2}}) - \varphi_{(\theta)}(U_{(\theta)}^n + V_{(\theta)}^n), E_U^{k+\frac{1}{2}} + E_V^{k+\frac{1}{2}} \right)^h \\
&\quad + \tfrac{1}{2} \left(\varphi_{(\theta)}(U^{n,k+\frac{1}{2}} - V^{n,k+\frac{1}{2}}) - \varphi_{(\theta)}(U_{(\theta)}^n - V_{(\theta)}^n), E_U^{k+\frac{1}{2}} - E_V^{k+\frac{1}{2}} \right)^h \\
&\quad + \tfrac{1}{4\mu} [\, |E_{Y_1}^{k+1}|_h^2 + |E_{Y_2}^{k+1}|_h^2 \,] + \tfrac{1}{2} \tau_n \,|\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} \nabla E_W^{k+1} |_0^2 \\
&\quad + \tfrac{1}{2} \tau_n \, \rho^{-1} |\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} E_Z^{k+1} |_0^2 + \gamma \,[\, |E_U^{k+1}|_1^2 + |E_V^{k+1}|_1^2 \,] \\
&\leq \tfrac{1}{4\mu} [\, |E_{Y_1}^{k}|_h^2 + |E_{Y_2}^{k}|_h^2 \,] + \tfrac{1}{2} \tau_n \,|\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} \nabla E_W^{k} |_0^2 \\
&\qquad + \tfrac{1}{2} \tau_n \, \rho^{-1} |\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} E_Z^{k} |_0^2 .
\end{aligned}
\tag{5.20}
$$

Therefore noting the monotonicity of $\varphi_{(\theta)}(\cdot)$ we have that

$$
\begin{aligned}
&\Big\{ \tfrac{1}{4\mu} [\, |E_{Y_1}^{k}|_h^2 + |E_{Y_2}^{k}|_h^2 \,] + \tfrac{1}{2} \tau_n \,|\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} \nabla E_W^{k} |_0^2 \\
&\qquad + \tfrac{1}{2} \tau_n \, \rho^{-1} |\, [b^{n-1} - b(U_{(\theta)}^{n-1}, V_{(\theta)}^{n-1})]^{\frac{1}{2}} E_Z^{k} |_0^2 \Big\}_{k \geq 0}
\end{aligned}
\tag{5.21}
$$

is a decreasing sequence which is bounded below and so has a limit. Therefore the desired results (5.13) follow from (5.21), (5.20), (2.14), $E_U^k \in \Xi^h$ and (5.16). Finally if $\theta > 0$, the strict monotonicity of $\phi(\cdot)$, (5.21) and (5.20) yield the desired convergence of $U^{n,k+\frac{1}{2}}$ and $V^{n,k+\frac{1}{2}}$. □

REMARK.

We see from (5.5a,b)–(5.7a–d) that at each iteration k one needs to solve only: (i) Two decoupled nonlinear equations (or two simple projections if $\theta = 0$) for $(U^{n,k+\frac{1}{2}} \pm V^{n,k+\frac{1}{2}})(x_j)$ at each mesh point x_j, $j = 1 \to J$; see (5.8) or (5.9). (ii) A fixed linear system with constant coefficients for $U^{n,k+1}$, see (5.12a). (iii) A fixed linear system with constant coefficients for $V^{n,k+1}$, see (5.12b). On a uniform mesh (ii) and (iii) can be solved efficiently using a discrete cosine transform; see [9, §5], where a problem similar to (ii) is solved.

6 Numerical Experiments

For the first experiment, we considered $(\mathrm{P}_\theta^{h,\tau})$ with the following data: $d = 1$, $\Omega = (0,1)$, $\gamma = 5 \times 10^{-3}$, $\theta = 0.15$, $\rho = 0.08$, $\alpha = 2$, $\beta = 4$, $b(\cdot)$ given by (1.5) with $\sigma = 0.1$ and hence $b_{\max} = (0.5+\sigma)^4$. As no exact solution to (P_θ) is known, a comparison between the solutions of $(\mathrm{P}_\theta^{h,\tau})$ on a coarse uniform mesh, U_θ, with that on a fine uniform mesh, u_θ, was made. For the coarse meshes we chose $h = (J-1)^{-1}$ with $J = 2^p + 1$, $(p = 6, 7, 8, 9)$, and $\tau = 0.25\, h^{\frac{2}{3}}$. In addition T was taken to be $N\,\tau$, where N was the largest integer such that $N\,\tau \leq 4$. For the fine mesh we chose $J = 2^{12}+1$ and τ to be the value closest to $0.25\, h^{\frac{2}{3}}$ so that the corresponding time step on the coarse mesh was an integer multiple of this fine τ. For the iterative method of Section 5, we chose the relaxation parameter $\mu = \frac{1}{6400h}$ and $b^{n-1} \equiv b_{\max}$, $n = 1 \to N$. For a stopping criterion we chose

$$\max\{\|U_\theta^{n,k} - U_\theta^{n,k-1}\|_{0,\infty},\ \|V_\theta^{n,k} - V_\theta^{n,k-1}\|_{0,\infty}\} < 10^{-7}.$$

The initial data $\{u^0, v^0\}$ were taken to be the clamped (complete) cubic splines taking the values

$$\{0.85, 0.85, 0.56, 0.96, 0.60, 0.24, 0.64, 0.64, 0.64\}$$

and

$$\{-0.13, -0.13, 0.23, 0, 0, -0.1, -0.2, 0.34, 0.34\},$$

respectively, at the points $i/8$ $(i = 0 \to 8)$, see Figure 1. Hence we have that

$$u^0,\ v^0 \in H^3(\Omega) \setminus H^4(\Omega) \text{ and } \tfrac{\mathrm{d}u^0}{\mathrm{d}x} = \tfrac{\mathrm{d}u^0}{\mathrm{d}x}(1) = \tfrac{\mathrm{d}v^0}{\mathrm{d}x}(0) = \tfrac{\mathrm{d}v^0}{\mathrm{d}x}(1) = 0.$$

On setting $U^0 \equiv Q_\gamma^h u^0$ and $V^0 \equiv Q_\gamma^h v^0$, then the assumptions on $\{u^0, v^0\}$ and $\{U^0, V^0\}$ of Theorem 1.1 hold with $\delta = \frac{1}{100}$ and $h_0 = \frac{1}{64}$. In addition this choice of initial data ensured that the singularities in ϕ played a role.

We computed three quantities

$$\xi_1 \quad = \quad \left[\frac{1}{N}\sum_{n=1}^{N} \|\pi^h u_{(\theta)}(\cdot, n\tau) - U_{(\theta)}(\cdot, n\tau)\|_1^2\right]^{\frac{1}{2}},$$

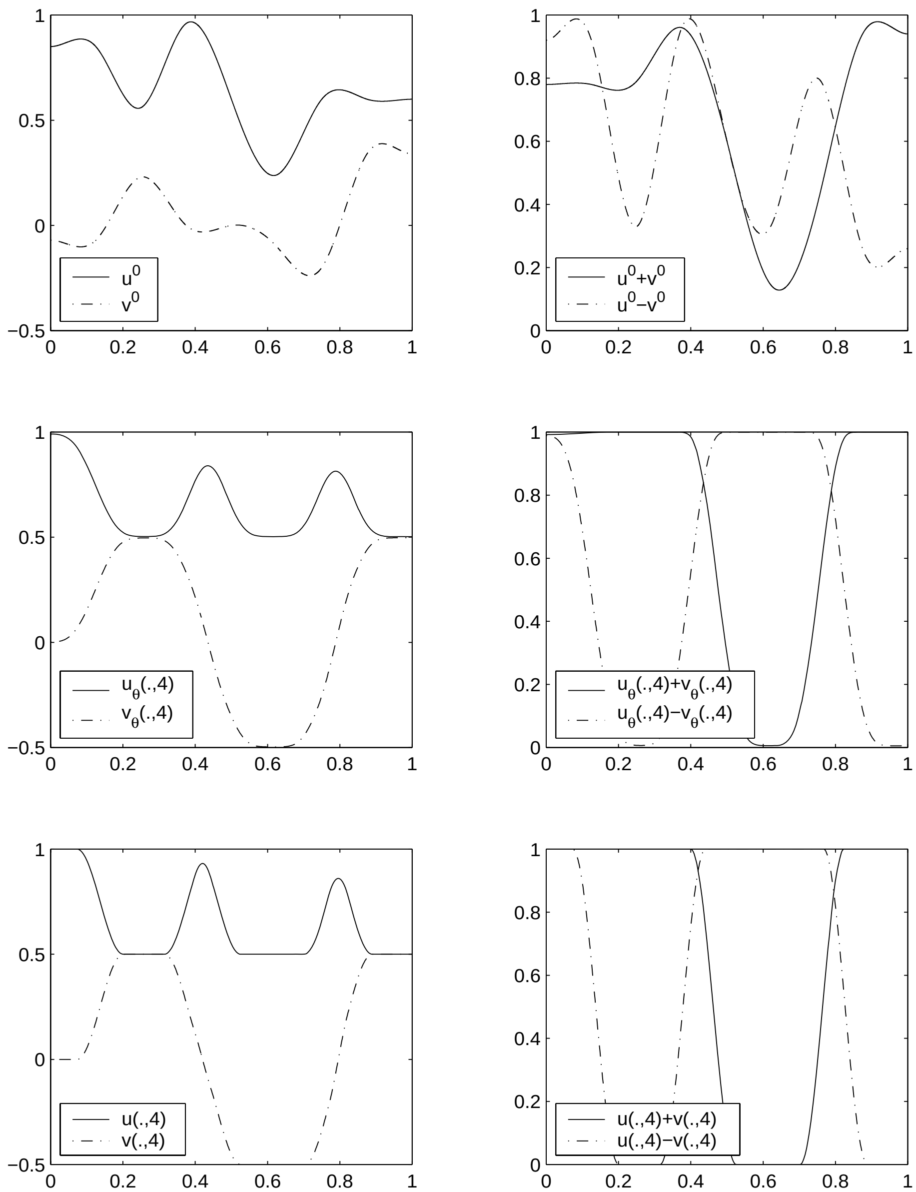

Figure 1: $u_{(\theta)}(\cdot,t)$, $v_{(\theta)}(\cdot,t)$, $u_{(\theta)}(\cdot,t)+v_{(\theta)}(\cdot,t)$ and $u_{(\theta)}(\cdot,t)-v_{(\theta)}(\cdot,t)$ when $t=0$ and 4.

$$\xi_2 = \left[\frac{1}{N} \sum_{n=1}^{N} \|\pi^h v_{(\theta)}(\cdot, n\tau) - V_{(\theta)}(\cdot, n\tau)\|_1^2 \right]^{\frac{1}{2}},$$
$$\xi_3 = \max_{n=1\to N} |\pi^h v_{(\theta)}(\cdot, n\tau) - V_{(\theta)}(\cdot, n\tau)|_0$$

and obtained the following table of values to three significant figures:

J	65	129	257	513
ξ_1^2	6.28×10^{-2}	2.37×10^{-2}	8.36×10^{-3}	2.76×10^{-3}
ξ_2^2	3.86×10^{-2}	1.24×10^{-2}	4.02×10^{-3}	1.26×10^{-3}
ξ_3^2	1.51×10^{-4}	5.57×10^{-5}	1.88×10^{-5}	5.99×10^{-6}

We see that the ratio of consecutive ξ_1^2, ξ_2^2 and ξ_3^2 are between 2.6 and 3.2, which is better than $2^{\frac{4}{3}} = 2.5$ to two significant figures, the rate of convergence proved in Theorem 1.1 for the above choice of $\tau = 0.25\, h^{\frac{2}{3}}$.

Finally, we repeated the above experiment in the deep quench limit. We chose precisely the same data as above except $\tau = h$. Once again all the conditions in Theorem 1.1 hold. We obtained the corresponding table of values:

J	65	129	257	513
ξ_1^2	1.21×10^{-1}	3.14×10^{-2}	7.98×10^{-3}	1.87×10^{-3}
ξ_2^2	1.02×10^{-1}	1.93×10^{-2}	4.25×10^{-3}	9.29×10^{-4}
ξ_3^2	5.53×10^{-5}	1.10×10^{-5}	2.20×10^{-6}	5.15×10^{-7}

The ratio of consecutive ξ_1^2, ξ_2^2 and ξ_3^2 are between 3.9 and 5.3 which are close to $2^2 = 4$, the rate of convergence proved in Theorem 1.1 for the above choice of $\tau = h$.

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